An improved surface radiation condition for high frequency acoustic scattering problems

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Abstract

A new artificial boundary condition for two-dimensional acoustic scattering problems is constructed in this paper. It is designed from the principal symbol of the Dirichlet-to-Neumann map so that both the propagating and evanescent fields are properly modeled. In the high frequency regime, the new artificial boundary condition can be used as an On-Surface Radiation Condition for efficient approximation of the wave field. The accuracy and efficiency of our method are demonstrated in numerical examples.

Key words: Scattering, acoustic, non-reflecting boundary condition, On-Surface Radiation Condition, far-field patterns

1 Introduction

In this paper, we consider the classical scattering problem of a time-harmonic acoustic or electromagnetic wave by an obstacle. The problem can be formulated as a Helmholtz equation in an unbounded domain [13]. A number of
different approaches have been developed to solve this problem. Boundary integral equations can be formulated on the surface of the obstacle and these equations can be discretized by the boundary element method [11,27]. The discretized system can be solved by a Krylov subspace iterative method [29] and the fast multipole method [12,28] can be used to further improve the efficiency. On the other hand, the unbounded domain can be truncated and discretized. The Helmholtz equation can then be solved by the finite element method together with a non-reflecting Artificial Boundary Condition (ABC) [14,18,19,31,32] or a Perfectly Matched Layer (PML) [10,12]. Alternatively, the infinite element method [20] can be used to handle the unbounded domain.

In the high-frequency regime, the wavelength is much smaller than the size of the obstacle and the scattering problem becomes prohibitively expensive to solve by the above methods. In this case, the method of On-Surface Radiation Condition (OSRC) originally developed by Kriegsmann et al. [24] in the middle of the eighties, becomes useful. It is an approximate method that applies local ABCs [7,8,15] directly on the surface of the scatterer, to determine the wave field or its normal derivative on the surface. Since then, the method has been extended and improved by various authors [2,3,5,21,23,22,30]. For frequencies in the middle regime, the most accurate computations are obtained by the second-order symmetric Bayliss, Gunzburger and Turkel (BGT) ABC [5,7,8]. However, when the frequency increases, the method is less accurate, due to inadequate modeling of the surface and creeping rays, the evanescent modes and geometrical singularities [2]. The limitation of the BGT-like ABCs can also be observed when the Helmholtz equation is solved in a domain truncated with an artificial boundary. It is necessary to place the artificial boundary sufficiently far away from the obstacle [14,31], otherwise the solution will not be accurate. In contrast, the PML technique appears to be more effective in truncating the unbounded domain and it has a much wider absorption band of incident angles [12].

In this paper, we develop a new ABC for two-dimensional acoustic scattering problems and use it as an OSRC in the high-frequency regime. In section 2, we consider the canonical problem for which the obstacle is a circular cylinder and observe that the incorrect modeling of the evanescent modes by the BGT-like ABCs produces errors in the solution when the artificial boundary is close to the obstacle. A new ABC is developed in section 3. Following the approach originally developed by Engquist and Majda [15] and the generalizations in [5,19], we develop an ABC from the principal symbol of the Dirichlet-to-Neumann pseudodifferential operator. Details of implementing the ABC as an OSRC are described in section 4. Several numerical examples are performed to demonstrate the accuracy and efficiency of the new method. In section 5, the technique is extended to scattering problems for obstacles covered by a thin coating [6,9,16]. Finally, we conclude our paper and outline investigations under progress in the section 6.
2 Limitations of the standard ABCs

Let us assume that $\Omega$ is a bounded domain in $\mathbb{R}^2$ representing an impenetrable body with boundary $\Gamma$. We denote by $\Omega^e$ the exterior isotropic domain where $\Omega$ is embedded. For an incident time-harmonic (plane) wave $u^{inc}$ illuminating the scatterer $\Omega$, the diffracted acoustic field $u$ satisfies the following exterior Boundary Value Problem (BVP):

\[
\begin{align*}
\Delta u + k^2 u &= 0, \quad \text{in} \quad \Omega^e, \\
u &= g \text{ or } \partial_n u = g, \quad \text{on} \quad \Gamma, \\
\lim_{|x| \to +\infty} |x|^{1/2} (\nabla u \cdot \frac{x}{|x|} - iku) &= 0,
\end{align*}
\]

where $k$ is the wavenumber in the exterior domain, $n$ is the unit normal vector directed to $\Omega$. The boundary condition depends on the physical problem under study. It is given by the Dirichlet boundary condition $g = -u^{inc}$ for a sound-soft scattering problem or the Neumann data $g = -\partial_n u^{inc}$ for the sound-hard scattering problem. Finally, the Sommerfeld radiation condition at infinity is imposed to ensure the uniqueness of the solution.

For numerical computations, it is usually necessary to truncate the exterior domain by an artificial boundary $\Sigma$. The Helmholtz equation is then solved in $\Omega^e_b$ — the domain bounded by $\Gamma$ and $\Sigma$. A suitable boundary condition [18,19,32] on $\Sigma$ is needed to model the behaviour of the wave at infinity. Symmetric BGT-like radiation conditions of various order are given in [5]. The second-order condition is

\[
\partial_n u = -\alpha s \partial_s u + \beta u, \quad \text{on} \quad \Sigma,
\]

where $s$ is the arclength of $\Sigma$ oriented in the counterclockwise direction, $\kappa(s)$ the curvature of $\Sigma$ at point $s$, and

\[
\alpha = \frac{1}{2ik(1 + i\kappa/k)}, \quad \beta = ik - \frac{\kappa}{2} + \frac{i\kappa^2}{8k(1 + i\kappa/k)} + \frac{\partial^2\kappa}{8k^2}.
\]

The half-order condition is exactly the well-known Sommerfeld radiation condition:

\[
\partial_n u = iku, \quad \text{on} \quad \Sigma.
\]

The BVP (1,2,4) can be solved by a finite element method [14,31], but the artificial boundary $\Sigma$ must be placed sufficiently far away from the scatterer. Unfortunately, this leads to larger computational domains and expensive computing costs.
To illustrate, we consider the canonical scattering problem of a circular cylinder (with radius $R_0$) centred at the origin. The incident wave is a plane wave traveling in the $x_1$ direction, i.e.,

$$u^{\text{inc}} = e^{ikx_1},$$

where $x = (x_1, x_2)$. The exact scattered field $u^{\text{ex}}$ can be expressed in the Mie serie as a superposition of the Fourier modes:

$$u^{\text{ex}}(r, \phi) = \sum_{m=0}^{\infty} \epsilon_m (-i)^m a_m^{\text{ex}} H_m^{(1)}(kr) \cos(m\phi), \quad r \geq R_0,$$

where $r$ and $\phi$ are the polar coordinates, $\epsilon_m$ denotes the Neumann function which is equal to 1 for $m = 0$ and 2 otherwise, $H_m^{(j)}$ is the $m$-th order Hankel’s function of the first ($j = 1$) or second ($j = 2$) kind.

We place a circular artificial boundary at a distance $R_1 = R_0 + \delta \lambda$ from the origin, where $\lambda = 2\pi/k$ is the wavelength of the incident wave and $\delta$ is a parameter. An ABC at $r = R_1$ gives rise to a reflected wave field that propagates back to the scatterer. The solution can be written down as

$$u(r, \phi) = \sum_{m=0}^{\infty} \epsilon_m (-i)^m \left[ a_m H_m^{(1)}(kr) + b_m H_m^{(2)}(kr) \right] \cos(m\phi)$$

for $R_0 \leq r \leq R_1$. If the boundary condition at $R_1$ is exact, we have $b_m = 0$ and $a_m = a_m^{\text{ex}}$. Of course, this is not the case when an ABC is used. In Fig. 1, we plot the coefficients $a_m$, $a_m^{\text{ex}}$ and $b_m$ for the second-order symmetric BGT-like ABC (4) and the Sommerfeld radiation condition (6) for $k = 50$, $R_0 = 1$, $\delta = 1/(2\pi)$ and the Dirichlet boundary condition at $r = R_0$. From the first picture, we can see that the propagating modes (low-order harmonics for small and moderate $m$) are properly modeled, but the evanescent modes (high-order harmonics for large $m$) are not exponentially decaying. They are reflected back into the computational domain as shown in the second picture.
Next, we consider

\[ \gamma_m = \frac{1}{u_m} \frac{\partial u_m}{\partial r} \]

where \( u_m \) is the \( m \)-th term in (8). The quantity \( \gamma_m \) can be regarded as the propagation constant of the cylindrical fields [26]. In Fig. 2, we compare \( \gamma_m \) with \( \gamma_m^{\text{ex}} \) (defined similarly from the \( m \)-th term in (7)) at \( r = R_0 \) for the boundary conditions (4) and (6). It is clear that both the half-order and second-order ABCs give good approximations to the propagation constant of the lower-order harmonics. However, both conditions fail to approximate the higher-order harmonics. In fact, the ABCs start to have large errors around \( m = 40 \) which corresponds to a transition from the propagating to the evanescent modes. As the real part of \( \gamma_m \) remains small for large \( m \), the ABCs incorrectly model the evanescent modes as propagating modes. The resulting error can also be observed in the far-field pattern. In Fig. 3, we compare the exact and approximate Radar Cross Section (RCS) given by

\[ \text{RCS}(\vartheta) = 10\log_{10}(2\pi|A(\vartheta)|^2), \]  

where \( \vartheta \) is the angle of diffusion and \( A(\vartheta) \) is the scattering amplitude.

We have demonstrated the limitations of ABCs (4) and (6) for a sound-soft cylinder. Similar results can be observed for the sound-hard scattering problem.

3 An improved ABC for general surfaces

In this section, we develop a new ABC capable of modeling the evanescent modes. The starting point is the leading (square root) symbol of the Dirichlet-to-Neumann (DtN) operator that maps \( u \) to its normal derivative on the artificial boundary. The leading symbol is regularized with a complex wavenumber
and localized with a rational approximation. The resulting ABC has the correct behavior for both the propagating and evanescent modes and it can be efficiently implemented as an OSRC.

The derivation of the new ABC starts with the same first step used in [5]. On an arbitrarily shaped artificial boundary $\Sigma$, the total symbol of the DtN map $\Lambda$ is locally expanded as

$$
\sigma_\Lambda(k, s, \xi) = \sum_{j=-\infty}^{+\infty} \lambda_{-j}(k, s, \xi),
$$

(10)

where $\lambda_{-j}$ is a symbol of homogeneous degree $-j$ according to $(k, \xi)$, with $\xi$ the covariable of the arclength $s$ by Fourier transform. The first four symbols are given in [5]. ABCs of various order can be obtained from Taylor expansions of the symbols assuming $|\xi/k| \ll 1$. For example, the second-order BGT-like ABC corresponds to

$$
\begin{align*}
\lambda_1 &= i \sqrt{k^2 - \xi^2} \approx ik(1 - \frac{\xi^2}{2k^2}) + O\left(\frac{1}{k^3}\right), \\
\lambda_0 &\approx -\frac{\kappa}{2}(1 + \frac{\xi^2}{k^2}) + O\left(\frac{1}{k^3}\right), \\
\lambda_{-1} &\approx \frac{ik^2}{8k} + \frac{i}{2} \partial_s \kappa \frac{\xi}{k} + O\left(\frac{1}{k^3}\right), \\
\lambda_{-2} &\approx \frac{\partial_s^2 \kappa}{8k^2} + \frac{\kappa^3}{8k^2} + O\left(\frac{1}{k^3}\right).
\end{align*}
$$
Here, the square root of a complex number follows the standard definition where the branch cut is the negative real axis. The exact principle symbol $\lambda_1 = i\sqrt{k^2 - \xi^2}$ corresponds to a non-local pseudo-differential operator. The localization process based on the Taylor expansions gives rise to approximants that are valid only for the propagating modes corresponding to $k > |\xi|$. Therefore, even the higher-order ABCs derived in [5] cannot model the evanescent modes.

The expansion (10) is valid for both the propagating modes and the evanescent modes (which correspond to $k < |\xi|$). But it is non-uniform [17] and not valid for the glancing rays corresponding to $k \approx |\xi|$. To illustrate this, we consider the non-local boundary condition for the exact principal symbol:

$$\partial_n u = ik \text{Op}(\sqrt{1 - \xi^2/k^2})u, \quad \text{on } \Sigma,$$

where $\text{Op}(\sigma)$ denotes the pseudo-differential operator of the symbol $\sigma$ given in an inverse Fourier transform representation. Condition (11) is applied to the canonical scattering problem of a sound-soft circular cylinder described in section 2 with the same parameters $R_0 = 1$, $k = 50$ and $\delta = 1/(2\pi)$. The artificial boundary is the circle of radius $R_1$. In this case, $\xi$ can be simply replaced by $m/R_1$. In Fig. 4, the propagation constants $\gamma_m$ (marked by $\varepsilon = 0$) for (11) are compared with the exact $\gamma_m^{\text{ex}}$. Corresponding to the singularity of the square root symbol, there is a singular transition from the propagating modes to the evanescent modes. This singularity induces some oscillations in the far-field pattern as shown in Fig. 5.

Our ABC is constructed from a regularized square root symbol. A positive imaginary part is introduced for the wavenumber $k$ inside the square root. More precisely, we let $k_\varepsilon = k + i\varepsilon$ for $\varepsilon > 0$ and consider the new boundary condition

$$\partial_n u = ik \text{Op}\left(\sqrt{1 - \xi^2/k_\varepsilon^2}\right)u, \quad \text{on } \Sigma.$$

Fig. 4. Propagation constants $\gamma_m^{\text{ex}}$ and $\gamma_m$ for the ABC (12) ($k = 50$ and $\delta = (2\pi)^{-1}$).
This introduces a local regularization for the tangential modes and the new principal symbol \( ik\sqrt{1 - \xi^2/k_0^2} \) is now analytic in \( \xi \). From Fig. 4, we observe that the singularity has been removed for positive values of \( \varepsilon \).

The accuracy of (12) depends crucially on the choice of \( \varepsilon \). To find a proper value of \( \varepsilon \), we again consider the canonical scattering problem of a sound-soft circular cylinder of radius \( R_0 \). When the boundary condition (12) is applied at \( r = R_1 \), the wave field (for \( R_0 < r < R_1 \) can again be expanded in a Mie series like (8), where the coefficients \( a_m \) and \( b_m \) now depend on \( \varepsilon \). From these calculations, we can see that the largest reflection coefficient \( b_m \) is obtained when \( |m| \approx kR_1 \). Therefore, it is natural to calculate the optimal value of \( \varepsilon \) by minimizing \( |b_m| \) for \( |m| \approx kR_1 \) as a function of \( \varepsilon \). Based on some approximations for \( b_m \) \([1,12]\), we obtain the optimal value \( \varepsilon_{\text{opt}} \approx 0.4k^{1/3}R_1^{-2/3} \).

Quite naturally, the result depends on the parameter \( kR_1 \). For a general boundary \( \Gamma \), we formally have

\[
\varepsilon_{\text{opt}} \approx 0.4k^{1/3}\kappa^{2/3},
\]

where \( \kappa \) is the local curvature. From Fig. 5, the magnitude of the incoming modes are compared for \( \varepsilon = 0, 5 \) and \( \varepsilon_{\text{opt}} \). It is clear that smallest reflection is obtained when \( \varepsilon = \varepsilon_{\text{opt}} \). In Fig. 6, we can see that the optimal choice of \( \varepsilon \) leads a satisfactory computation of the far-field pattern.

For the Neumann problem, the results are similar and the optimal value of \( \varepsilon \) given in (13) is still valid.

![Fig. 5. Magnitude of coefficients \( b_m \) and far-field pattern (\( \varepsilon = 0 \)) for the ABC (12) (\( k = 50 \) and \( \delta = (2\pi)^{-1} \)).](image)

The boundary condition (12) involves a non-local pseudodifferential operator that is expensive to evaluate directly. A practical approach is to further approximate the square root symbol by a rational function. The boundary condition can then be evaluated by solving a set of local differential equations defined on \( \Gamma \).

For the square root function, the standard Padé approximant \([25]\) of order \( N \)
\[ \sqrt{1 + z} \approx R_N(z) = 1 + \sum_{j=1}^{N} \frac{a_j z}{1 + b_j z}, \]  
(14)

where \( z = -\xi^2 / k^2 \), \( a_j \) and \( b_j \) are real coefficients given by

\[ a_j = \frac{2}{2N + 1} \sin^2\left(\frac{j\pi}{2N + 1}\right), \quad b_j = \cos^2\left(\frac{j\pi}{2N + 1}\right) \]
(15)

for \( j = 1, \ldots, N \). The Padé approximant converges to the exact value of \( \sqrt{1 + z} \) as \( N \to \infty \), except when \( z \) is real and less than \(-1\), i.e., on the branch cut of the square root function. In our case, since a non-zero \( \varepsilon \) is introduced in \( k_z \), \( z = -\xi^2 / k^2 \) has a small imaginary part. Although the rational approximant converges, a large \( N \) is needed to obtain the desired accuracy. For the model test case shown in Fig. 7, a satisfactory result can only be obtained with \( N = 64 \).

A more accurate approximation is possible if the branch cut is first rotated by some angle. Milinazzo et al. [25] introduced a change of variable, such that the new branch cut is along the ray \( z = -1 + re^{i(\pi + \theta)} \), where \( \theta \) is the angle of
rotation. This gives rise to

\[ \sqrt{1 + z} \approx e^{i\theta/2} R_N(e^{-i\theta} z) = C_0 + \sum_{j=1}^{N} \frac{A_j z}{1 + B_j z}, \]  

(16)

where

\[ C_0 = e^{i\theta/2} R_N(e^{-i\theta} - 1), \quad A_j = \frac{e^{-i\theta/2} a_j}{(1 + b_j(e^{-i\theta} - 1))^2}, \]

\[ B_j = \frac{e^{-i\theta} b_j}{1 + b_j(e^{-i\theta} - 1)}. \]

(17)

The boundary condition (12) is thus approximated by

\[ \partial_n u = i k \mathrm{Op} \left( C_0 - \sum_{j=1}^{N} \frac{A_j \xi^2 / k_z^2}{1 - B_j \xi^2 / k_z^2} \right) u, \quad \text{on } \Sigma. \]

(18)

In Figures 8 and 9, we plot the propagation constant \( \gamma_m \) for \( \theta = -\pi/8 \) and \( -\pi/4 \), respectively. Excellent accuracy is reached at \( \theta = -\pi/4 \) and \( N = 8 \). In this case, we obtain a nearly identical far-field pattern as the exact regularized square root operator (see Fig. 6).

Fig. 8. Propagation constants \( \gamma_m \) for the Padé approximation of the square-root symbol for \( \varepsilon = \varepsilon_{\text{opt}} \) and \( \theta = -\pi/8 \).

Fig. 9. Propagation constants \( \gamma_m \) for the Padé approximation of the square-root symbol for \( \varepsilon = \varepsilon_{\text{opt}} \) and \( \theta = -\pi/4 \).
4 Application as On-Surface Radiation Conditions

The artificial boundary condition (18) can be used as an On-Surface Radiation Condition (OSRC), where the artificial boundary Σ is taken to coincide with the surface of the scatterer Γ. The method of OSRC was originally developed by Kriegsmann et al.’s [24] and it has been extended and improved by several authors [2,3,5,21–23,30]. It is particularly useful for high frequency scattering problems.

When both the wave field \( u \) and its normal derivative are known on the surface \( Γ \), the wave field exterior to the scatterer can be represented by the Helmholtz integral formula:

\[
  u(x) = \int_Γ \left[ \partial_{n(y)} G(x,y)u(y) - G(x,y)\partial_{n(y)}u(y) \right] dΓ(y) \quad \text{for} \quad x ∈ Ω_e, \tag{19}
\]

where \( G \) is the Green’s function of the Helmholtz equation:

\[
  G(x,y) = \frac{i}{4} H^{(1)}_0(k|x-y|).
\]

For a sound-soft obstacle, \( u \) is given on \( Γ \), the OSRC method simply applies (18) to evaluate the normal derivative of \( u \). More precisely, we have

\[
  \partial_n u|_Γ ≈ ik \left( C_0 u + \sum_{j=1}^{N} A_j \phi_j \right) \tag{20}
\]

where \( \phi_j \), for \( j = 1, ..., N \), are auxiliary functions defined on \( Γ \) satisfying

\[
  \left( 1 + \frac{B_j}{k_e^2} \partial_s^2 \right) \phi_j = \frac{1}{k_e^2} \partial_s^2 u \quad \text{on} \quad Γ. \tag{21}
\]

For a sound-hard obstacle, we have the Neumann condition on \( Γ \), i.e., \( \partial_n u = -\partial_n u^{inc} \). In this case, we have to solve \( u|_Γ \) from

\[
  ik \text{ Op} \left( \sqrt{1 - \xi^2/k_e^2} \right) u = \partial_n u \quad \text{on} \quad Γ,
\]

or equivalently

\[
  \left( \partial_s^2 + k_e^2 \right) u = \frac{k_e^2}{ik} \text{ Op} \left( \sqrt{1 - \xi^2/k_e^2} \right) \partial_n u, \quad \text{on} \quad Γ.
\]

Once again, the application of \( \text{Op}(\sqrt{1 - \xi^2/k_e^2}) \) is approximated by the complex Padé approximant (16) and it is reduced to solving a set of differential equations on \( Γ \).
A finite element method is used to discretize the differential equations on the boundary. This follows a suitable variational formulation of the problem [2]. If $\Gamma$ is discretized with $n_h$ points, the resulting cost of solving one equation, such as (21), is $O(n_h)$. The total cost of evaluating the square root operator by its rational approximation is thus $O(N n_h)$. Since $N$ is typically small, the complexity is essentially linear in $n_h$. In comparison, the boundary element method requires $O(n_h^3)$ operations if a direct solver is used. With a suitable preconditioner, the boundary element method can be more efficiently solved by a Krylov subspace method, but each iteration still requires $O(n_h^2)$ operations, or $O(n_h \log n_h)$ operations when a Multilevel Fast Multipole Method [12] is used.

We test our OSRC on three scatterers: the unit circular cylinder $D_1$, the square cylinder $S$ with side length 2 and an elliptical cylinder $E$. The semi-axes of $E$ are $a = 1$ and $b = 0.25$ in the $x_1$ and $x_2$ directions, respectively. All these objects are centered at the origin. The incident wave is a plane wave given by

$$u^{\text{inc}}(x) = \exp(i k \nu \cdot x)$$

where $\nu = (\cos \theta^{\text{inc}}, \sin \theta^{\text{inc}})$ and $\theta^{\text{inc}}$ is the angle of incidence. Reference solutions are obtained from a boundary integral equation solver. Depending on the boundary condition (Dirichlet or Neumann), we use the OSRC method to calculate the unknown surface field ($\partial_n u$ or $u$) and compare it with a reference solution. We also compare the far-field patterns using the RCS formula (9).

In the following computations, we choose the regularizing parameter $\epsilon_{\text{opt}}$ for the OSRC and $\theta = -\pi/4$ and $N = 8$ for the complex Padé approximation. Results for the sound-soft scattering problems are presented in Figures 10-12. These calculations are carried out for relatively high frequencies at $k = 35$ or $k = 40$. While the OSRC method cannot produce scattered fields that are fully accurate, it does give surface fields and far-field patterns that are qualitatively correct. On the other hand, we can see a moderate loss of accuracy in both the surface fields and RCS curves, presumably due to inadequate modeling of the creeping waves on the surface of the scatterers. For the sound-hard problems, the OSRC method gives solutions that have a similar level of accuracy as shown in Fig. 13-15.

To insist on the effect of strong changes in curvatures, we present on Figures 16 the surface fields and RCS computed for the scattering problem by a thin sound-hard elliptical cylinder setting its semi-axes to $a = 1$ and $b = 0.08$. The frequency is fixed to $k = 30$ for an incidence angle equal to 35 degrees. We observe that the accuracy of the surface fields is affected but remains acceptable as well as the corresponding RCS.

A more severe test consists now in considering the scattering problem from a non-convex scatterer. In this challenging case, multiple scattering occurs and
is not \emph{a priori} taken into account by the OSRC approach as shown in [3,4].

The model example consists in the square cylinder $S$ to which we remove the square defined by the four points: $(-1, 0.4)$, $(0, 0.4)$, $(0, -0.4)$ and $(-1, -0.4)$. We choose an incident plane wave of frequency $k = 35$ defined by an incidence angle equal to 45 degrees and consider the Neumann problem. We present on Figure 17 the surface fields and RCS computations. We remark a local loss of accuracy in the cavity due to the lack of modelling of the multiple scattered fields. However, this problem can be partially overcame if one couples the OSRC to another numerical technique (volume finite element method or integral equations for examples) for treating the non-convex part. We refer to [3,4] for some possible coupling algorithms. The repercussion of the deterioration of accuracy on the RCS is also observable.

Despite some limitations on the accuracy, we conclude that the OSRC method can be very useful, because it is more efficient than the boundary integral equation method.

Fig. 10. Sound-soft circular scatterer $D_1$: surface fields and RCS for $k = 40$ and $\theta_{\text{inc}} = 0$ degree.

Fig. 11. Sound-soft elliptical scatterer $E$: surface fields and RCS for $k = 35$ and $\theta_{\text{inc}} = 35$ degrees.
5 Extension to generalized impedance scattering problems

To extend the capacity of the proposed OSRC, we consider the scattering problem of an incident time-harmonic wave $u^{\text{inc}}$ by a sound-hard scatterer $\Omega$ covered by a homogeneous and isotropic layer $\Omega_1$ of thickness $\delta$. This latter domain is characterized by a sound speed $c_1$ and a density $\rho_1$. The interior...
and exterior boundaries of $\Omega_1$ are denoted by $\Gamma_\delta$ and $\Gamma$, respectively. Outside the layer, we have an isotropic and homogeneous unbounded (fluid) medium $\Omega_2$ characterized by a sound speed $c_2$ and a density $\rho_2$. The wave number in the exterior domain $\Omega_2$ is $k_2 = \omega/c_2$. We also have the refractive index $n = c_2/c_1$ and the contrast coefficient $\alpha = \rho_2/\rho_1$. The acoustic field $v$ satisfies the following BVP:
\[ \Delta v_2 + k_2^2 v_2 = 0, \quad \text{in } \Omega_2, \]  
\[ \Delta v_1 + k_1^2 n^2 v_1 = 0, \quad \text{in } \Omega_1, \]  
\[ v_1 = v_2 \quad \text{and} \quad \chi \partial_n v_1 = \chi \partial_n v_2, \quad \text{on } \Gamma, \]  
\[ \partial_n v_1 = 0, \quad \text{on } \Gamma_\delta, \]  
\[ \lim_{|x| \to +\infty} |x|^{1/2} \left[ \nabla (v_2 - u_{\text{inc}}) \cdot \frac{x}{|x|} - i k_2 (v_2 - u_{\text{inc}}) \right] = 0, \]  
where \( \chi \) is equal to 1 in \( \Omega_2 \) and \( \alpha \) in \( \Omega_1 \), \( n \) is the unit normal vector inward to \( \Omega_1 \), \( v_1 \) and \( v_2 \) are restrictions of \( v \) in \( \Omega_1 \) and \( \Omega_2 \), respectively.

When \( \delta \) is small, the above BVP can be simplified with a Generalized Impedance Boundary Condition (GIBC). This method was introduced by Engquist and Nédélec [16] and systematically improved by Bendali and Lemrabet [9]. The original BVP is approximated by the following BVP:

\[ \Delta u_2 + k_2^2 u_2 = 0, \quad \text{in } \Omega_2, \]  
\[ \partial_n u_2 - \mathcal{Z} u_2 = g, \quad \text{on } \Gamma, \]  
\[ \lim_{|x| \to +\infty} |x|^{1/2} \left( \nabla u_2 \cdot \frac{x}{|x|} - i k_2 u_2 \right) = 0, \]  
for some operator \( \mathcal{Z} \) and \( g = - (\partial_n - \mathcal{Z}) u_{\text{inc}} \). We consider the second-order Engquist-Nédélec GIBC:

\[ \mathcal{Z} u = -\partial_x (A \partial_x u) + B u, \]

where \( A = \alpha \delta (1 + \delta \kappa/2) \) and \( B = -\alpha k_2^2 n^2 \delta (1 - \delta \kappa/2) \). In [6], it is shown that the solution of (27)-(29) gives a satisfactory approximation to the solution of the original system (22)-(26) when \( k_2 n \delta \) is small. The system (27)-(29) can be solved by a boundary element method [6,33].

We propose to extend our OSRC method to this problem. Instead of solving the Helmholtz equation of \( u_2 \) in \( \Omega_2 \), we assume that the non-local ABC (12) and its rational approximation (18) are applicable on \( \Gamma \), when \( \kappa \) is replaced by \( k_2 \). More precisely, we have

\[ \partial_n u_2 - i k_2 \text{Op} \left( \sqrt{1 - \xi^2/k_2^2} \right) u_2 = 0, \quad \text{on } \Gamma. \]  

The above equation can be solved together with (28). Alternatively, we can eliminate \( \partial_n u_2 \) and solve the following equation for \( u_2 \):

\[ i k_2 \text{Op} \left( \sqrt{1 - \xi^2/k_2^2} \right) u_2 - \mathcal{Z} u_2 = g, \quad \text{on } \Gamma. \]  

If a direct method is used to solve (31), the auxiliary functions needed in the rational approximation of the square root operator will couple together.
Fortunately, this is not a problem when an iterative method [29] is used. In each iteration, we only need to evaluate the square root operator applied on given functions. This can be efficiently computed as it is reduced to solving a set of differential equations on $\Gamma$. A relatively large error tolerance can be used in the iterative method, since this is an approximate method in the first place.

For numerical examples, we again consider the three scatterers described in section 4. For comparison, reference solutions are calculated using the boundary integral equation method [6]. In all cases, we assume $\alpha = 0.5$ and $n = 1.41$. For the circular cylinder $D_1$, the thin layer $\Omega_1$ has a thickness of $\delta = 0.003$. For the plane incident plane, we assume a zero incidence angle and $k_2 = 35$ (thus $k_2 n \delta = 0.15$). Numerical results for this example are shown in Fig. 18. Exact and approximate jumps of the wave field $u$ through $\Gamma$ and the bistatic RCS are illustrated. A good agreement between the OSRC and boundary integral equation solutions can be observed. For the elliptical cylinder $E$, we consider a plane incident wave with $k_2 = 30$ and $\theta^{\text{inc}} = 215$ degrees. The thickness of $\Omega_1$ is assumed to be $\delta = 1.5 \times 10^{-3}$ (thus, $k_2 n \delta = 0.06$). From the results shown in Fig. 19, we again observe a satisfactory agreement between the approximate and reference solutions. Finally, we consider the square cylinder $S$ covered with a layer of thickness $\delta = 0.02$ (thus, $k_2 n \delta = 0.08$). For the incident wave, we assume a zero incidence angle and $k_2 = 30$. The numerical results are presented in Fig. 20. It is clear that the OSRC method provides a good prediction for both the surface field and the far-field pattern at a very low computation cost.

![Graph](image)

Fig. 18. Sound-hard circular scatterer $D_1$: jump of the field through $\Gamma$ and RCS for $k_2 = 35$ and $\theta^{\text{inc}} = 0$ degree ($n = 1.41$, $\alpha = 0.5$, $\delta = 3. \times 10^{-3}$).

6 Conclusion

In this paper, a new artificial boundary condition is developed for two-dimensional high frequency scattering calculations. It is constructed from a regularized square root symbol and a complex coefficient Padé approximation.
Fig. 19. Sound-hard elliptical scatterer $E$: jump of the field through $\Gamma$ and RCS for $k_2 = 30$ and $\theta^{inc} = 215$ degrees ($n = 1.41$, $\alpha = 0.5$, $\delta = 1.5 \times 10^{-3}$).

Fig. 20. Sound-hard square scatterer $S$: jump of the field through $\Gamma$ and RCS for $k_2 = 30$ and $\theta^{inc} = 0$ degree ($n = 1.41$, $\alpha = 0.5$, $\delta = 2 \times 10^{-3}$).

Unlike the classical second-order BGT-like radiation condition, this ABC has the interesting feature of incorporating both the evanescent and propagating modes and it also models the creeping modes approximately. We have applied this ABC as an On-Surface Radiation Condition to three different obstacles, involving both sound-soft and the sound-hard scatterers. The method is also extended to the diffraction problem of a sound-hard scatterer covered by a thin layer. Numerical results indicate that satisfactory surface fields and far-field patterns can be obtained at a very low computation cost.

The method developed in this paper is readily extended to three-dimensional acoustic scattering problems. Numerical implementation of such an artificial boundary condition for the three-dimensional Helmholtz equation can be based on the edge finite element method developed in [2]. The possibility of extending this method to three dimensional electro-magnetic scattering problems based on the full set of Maxwell’s equations is currently being explored.

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References


