An integral preconditioner for solving the electromagnetic scattering problem by dielectric media using integral equations

Xavier Antoine\textsuperscript{a,b} Yassine Boubendir\textsuperscript{c}

\textsuperscript{a}Institut National Polytechnique de Lorraine (INPL), Ecole Nationale Supérieure des Mines de Nancy, Département de Génie Industriel, Parc de Saurupt, CS 14 234, 54042 Nancy cedex, France. E-mail: Xavier.Antoine@mines.inpl-nancy.fr

\textsuperscript{b}Institut Elie Cartan Nancy (IECN) and INRIA-CORIDA team, Université Henri Poincaré Nancy 1, B.P. 239, F-54506 Vandoeuvre-lès-Nancy Cedex, France. E-mail: Xavier.Antoine@iecn.u-nancy.fr

\textsuperscript{c}School of Mathematics, University of Minnesota, 127 Vincent Hall, 206 Church St., S. E. Minneapolis, MN 55455, USA. E-mail: boubendi@math.umn.edu

Abstract

The solution of the electromagnetic scattering problem by an homogeneous dielectric cylinder of arbitrary shape is considered. The numerical approach is based on a system of integral equations solved by a Krylov subspace method. To accelerate and to improve the convergence of this solver, an efficient and robust preconditioner, based on the Calderón formulae, is developed. Several numerical simulations, for a wide range of physical parameters, validating the choice of this preconditioner are presented.

Key words: electromagnetic scattering, dielectric media, integral equation, Krylov iterative solver, preconditioning techniques, Calderón formulae

1 Introduction

The computational solution of electromagnetic wave scattering problems remains an active and challenging research area for engineers and scientists with numerous important technological applications (11). A widely used and robust formulation for numerically solving this class of problems is provided by the integral equations methods. Since the development of fast matrix-free integral solvers, often based upon the Fast Mutipole Method (11), the solution of
integral equations discretized by boundary element methods is coupled to a Krylov subspace iterative solver (24) as the GMRES among others. Therefore, a crucial question lies in the convergence properties of these iterative schemes. Our approach in this paper consists of developing an efficient preconditioner to accelerate the convergence of the Krylov iterative solver.

In recent years, important efforts have been directed toward the design of efficient and robust algebraic-based preconditioners (ILUT, SPAI,...) for positive definite equations (9; 24). They have been applied to scattering problems solved by volume or boundary finite element methods (7; 19; 21; 26). Unfortunately, even if these preconditioners can yield improvements in some situations, convergence breakdown remains strongly present in the iterative process when e.g. high frequencies or large densities of discretization points per wavelength are considered. For integral equations, we often observe a lack of eigenvalues clustering which explains these behaviors. This is the main reason of the interest to develop operator-based preconditioners (1; 2; 3; 4; 12; 13; 14; 22). One of the most promising directions is related to the Calderón relations explicitly used by Steinbach and Wendland (27) and next by Christiansen and Nédélec (12; 13; 14) in the background of preconditioning techniques. We propose to analyze in this paper the application of this approach for solving the scattering problem by an homogeneous dielectric medium. In this situation, a system of integral equations needs to be solved. The application of the Calderón formulae is not straightforward and requires original developments and understanding in the way of building a suitable block preconditioner. This is the aim of this paper.

We propose to introduce the scattering problem in Section 2 and its integral equation formulation and discretization in Section 3. Section 4 is devoted to the analysis of the problem of convergence when an ILUT-based preconditioner is used and to focus on the different parameters involving in the convergence breakdown. This leads us to introduce in Section 5 a new explicit and robust integral preconditioner based on the Calderón relations and to provide a spectral analysis in the special case of a circular cylinder where the integral operators can be diagonalized. The numerical robustness of this new preconditioner is analyzed in Section 6.

2 The electromagnetic transmission problem

Let us consider an homogeneous isotropic dielectric scatterer $\Omega_1 \subset \mathbb{R}^2$ whose boundary $\Gamma$ is $C^\infty$. The associated domain of propagation is designated by $\Omega_2 = \mathbb{R}^2 \setminus \Omega_1$. Let $u_0$ be a complex-valued time-harmonic incident field defined
in an open neighborhood $\mathcal{V}$ of $\Gamma$ and satisfying the Helmholtz equation

$$
\Delta u_0 + k_2^2 u_0 = 0, \quad \text{in } \mathcal{V},
$$

(1)

where $k_2$ is the wavenumber. A second wavenumber $k_1$ related to the interior domain is defined by $k_1 := k_2 N$, where $N =: \sqrt{\epsilon_r \mu_r}$, $\epsilon_r$ and $\mu_r$ being respectively the relative permittivity and permeability of the problem. If $\sigma \geq 0$ is the conductivity of the interior medium, then $\tilde{\epsilon}_r = \epsilon_r + i\sigma/k_2$ is the complex relative permittivity of $\Omega_1$. Note that the complex number $N$ is generally called the refraction index of the dielectric medium. In the sequel, we denote by $z^{1/2}$ the principal determination of the square root of $z \in \mathbb{C}$ with branchcut along the negative real axis. For a H-polarized (respectively E-polarized) electromagnetic field, we introduce $\alpha = 1/\tilde{\epsilon}_r$ (respectively $\alpha = 1/\mu_r$) as the contrast coefficient of $\Omega_1$.

According to these notations, the two-dimensional transmission electromagnetic scattering problem can be reduced to the computation of the solution $u$ of the following boundary-value problem

$$
\begin{cases}
\Delta u_2 + k_2^2 u_2 = 0, & \text{in } \mathcal{D}'(\Omega_2), \\
\Delta u_1 + k_1^2 u_1 = 0, & \text{in } \mathcal{D}'(\Omega_1), \\
[u] = 0 & \text{in } H^{1/2}(\Gamma) \text{ and } [\chi \partial_n u] = 0 \text{ in } H^{-1/2}(\Gamma), \\
\lim_{|x| \to +\infty} |x|^{1/2} (\nabla (u_2 - u_0) \cdot \frac{x}{|x|} - ik_2 (u_2 - u_0)) = 0,
\end{cases}
$$

(2)

where the piecewise constant function $\chi$ is such that $\chi = 1$ in $\Omega_2$ and $\chi = \alpha$ in $\Omega_1$. Vector $n$ is the outwardly directed unit normal to $\Omega_1$. The restriction of the field $u$ to $\Omega_j$, $j = 1, 2$, is denoted by $u_j := u_{|\Omega_j}$; notation $[.]$ indicates the jump through $\Gamma$ of a distribution defined in $\Omega_1 \cup \Omega_2$ and is given as the difference between the interior and exterior traces: $[\phi] := \phi_{\mid\Gamma} - \phi_{\mid\Gamma}$. If $\mathbf{a}$ and $\mathbf{b}$ are two complex-valued vector fields, their inner product is defined as $\mathbf{a} \cdot \mathbf{b} = \sum_{k=1}^{3} a_k b_k$. The operator $\nabla$ is the gradient operator of a scalar complex field and $\Delta$ designates the Laplace operator. Finally, the last condition in problem (2) is the well-known Sommerfeld radiation condition which ensures the uniqueness of the solution to the boundary-value problem. We refer to (8) for any notation concerning the functional spaces. In a suitable functional setting, the existence and uniqueness of the solution to the above BVP can be proven (5; 6).
3 An integral equation formulation and its boundary element discretization

Let us define by \( G_j(x, y) := \frac{i}{4}H_0^{(1)}(k_j|\mathbf{x} - \mathbf{y}|) \) the Green kernel associated with the Helmholtz equation in \( \Omega_j \), where \( H_0^{(1)} \) is the zeroth-order Hankel function of the first-kind. Consider the two following surface fields \( p_j := u_{j|\Gamma} \) and \( \zeta_j := \chi\partial_n u_{j|\Gamma} \), for \( j = 1, 2 \). Using the interface conditions arising from (2), the determination of the total field can be reduced to the computation of the quantities: \( \zeta := \zeta_j \) and \( p := p_j \). Finally, let us introduce the eight surface integral operators defined on \( \Gamma \) by

\[
V_j\zeta(x) := \int_{\Gamma} G_j(x, y)\zeta(y)d\Gamma(y),
\]

\[
N_jp(x) := -\int_{\Gamma} \partial_n(y)G_j(x, y)p(y)d\Gamma(y),
\]

\[
K_j\zeta(x) := -N_j^T\zeta(x) = \partial_n(x)\int_{\Gamma} G_j(x, y)\zeta(y)d\Gamma(y),
\]

\[
D_jp(x) := -\partial_n(x)\int_{\Gamma} \partial_n(y)G_j(x, y)p(y)d\Gamma(y),
\]

where \( x \in \Gamma \) and \( N_j^T \) indicates the transpose operator of \( N_j \). It can be proven that the integral operators \( V_j \), \( N_j \) and \( K_j \) are pseudodifferential operators of order \(-1\). Moreover, \( D_j \) is a first-order hypersingular operator (see e.g. (9; 15; 16; 20)). Assume that \( Z \) is the product-space \( Z = H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma) \) and define the following integral operators

\[
V := \alpha^{-1}V_1 + V_2, \quad N := N_1 + N_2, \quad D := \alpha D_1 + D_2.
\]

Then, following (6), it can be proven that the solution to the boundary-value problem (2) can be reduced by solving the following symmetrical system of integral equations: find \((\zeta, p) \in Z \) solution to

\[
S\begin{pmatrix} \zeta \\ p \end{pmatrix} = \begin{pmatrix} V & N \\ N^T & -D \end{pmatrix} \begin{pmatrix} \zeta \\ p \end{pmatrix} = \begin{pmatrix} u_{0|\Gamma} \\ -\partial_n u_{0|\Gamma} \end{pmatrix}.
\]

Let \( \Gamma_h \) be a polyhedric surface interpolating \( \Gamma \) and built, using \( NT \) segments, the following triangulation \( T_h = \cup_{i=1}^{NT} K_i \). Each segment \( K_i \) satisfies: \( K_i \cap K_j = \emptyset \) for \( 1 \leq i \neq j \leq NT \). The smallest element size is denoted by \( h \). Let us introduce the linear boundary element space

\[
V_h := \{ q \in C^0(\Gamma_h)/q_K := q_{|K} \in \mathcal{P}_1, \forall K \in T_h \},
\]
where $\mathcal{P}_1$ designates the space of complex-valued polynomials of degree equal to 1. We have $\dim V_h = NT$ and if $Z_h := V_h \times V_h$, the discrete approximation $((\{ \zeta \}_1, \{ p \}_1)) \in Z_h$ is solution to the symmetrical complex-valued and indefinite linear system

$$
\begin{bmatrix}
[S_h]_{1,1} & \{ \zeta \}_1 \\
\{ p \}_1 & \end{bmatrix} =
\begin{bmatrix}
[V_h]_{1,1} & [N_h]_{1,1} \\
[N_h]^T_{1,1} & -[D_h]_{1,1} & \{ \zeta \}_1 \\
[p]_1 & \end{bmatrix} =
\begin{bmatrix}
[M_h]_{1,1} \{ u_0 | \Gamma \}_1 \\
- [M_h]_{1,1} \{ \partial_n u_0 | \Gamma \}_1 & \end{bmatrix},
$$

(7)

where $\{ \phi \}_1$ is the complex-valued vector of a density $\phi$ at the nodal values of $\Gamma_h$ and $[A_h]_{1,1}$ is the matrix resulting from the suitable boundary element approximation of the operator $A$ using a linear polynomial approximation for the unknown and test-functions.

4 Difficulties for the Krylov iterative solution of system (7)

When the frequency increases or/and the mesh is refined, a Krylov subspace iterative solver is generally employed instead of a direct solver since the linear system to solve involves a large number of degrees of freedom. This iterative solver is often coupled to a fast Matrix-Vector product algorithm (like e.g. the Fast Multipole Method (FMM)) to accelerate the overall procedure.

Let us focus on the behavior of the convergence rate of the iterative solution for solving (7) using the GMRES solver. This last solver has been compared to other solvers as the CGS, BiCGStab or QMR algorithms (24). In all these tests, it appears that this choice always leads to the best convergence rate when the convergence occurs. The tolerance has been fixed to $10^{-15}$ in all our examples. We consider a GMRES without or with a restart parameter $\tau$ (then precising GMRES($\tau$)). Moreover, we consider a maximum total number of 500 iterations (more iterations generally corresponding to a breakdown in the convergence process).

A first possible solution for building a simple preconditioner for the system (7) is to consider the mass matrix implicit preconditioner. In our case, this means that we choose the following diagonal preconditioning matrix

$$
P_1 = \begin{bmatrix}
[M_h]_{1,1} & 0 \\
0 & [M_h]_{1,1} \\
\end{bmatrix}
$$

(8)
which is inverted via the solution of a linear system $P_1 \mathbf{v} = \mathbf{y}$, for $\mathbf{y}$ given in $C^{2NT}$.

A second possible solution consists in using an incomplete LU factorization with threshold (ILUT) (24) of the matrix $[S_h]_{1,1}$ involved in (7). This approach proposes the construction of an incomplete LU factorization of a matrix using a dropping tolerance parameter $\text{droptol}$ and a pivot threshold value yielding a sparse approximation of the exact LU factorization. We do not want to give too much details concerning this method which is a priori complicate to expose here and refer to the monograph by Saad (24). In our computations, we will not consider any threshold strategy and precise the value of the dropping tolerance by $\text{ILU-S(}\text{droptol}\text{)}$ for the matrix $[S_h]_{1,1}$. A value of $\text{droptol}$ equal to zero corresponds to the exact LU factorization and a sparser structure is obtained if $\text{droptol}$ is closer to 1. We use in all the simulations the ILU procedure developed in Matlab. Using an incomplete factorization $[\tilde{L}, \tilde{U}]$ as implicit preconditioner requires the solution of two sparsified (according to $\text{droptol}$) linear systems at each iteration. This is the additional cost at each iteration of our iterative solver. If the system is sparse enough, then this computational cost can be admissible otherwise it can be penalizable for our solution. Let us mention that this technique has been used by Sertel & Volakis (26) to precondition various integral equations (the EFIE, MFIE and CFIE) for solving the electromagnetic scattering problem by a perfectly conducting body. One of the conclusions is that the ILU approach can be useful in the background of FMM methods since no fills-in are required unlike the conventional LU decomposition. However, our discussion here will not be focussed on the FMM implementation but rather on the construction of robust preconditioners for solving the scattering problem by a homogeneous dielectric medium, eventually involving an ILU-type technique.

To focus on the different inherent difficulties related to our problem, let us consider a few test cases. We begin by considering the resonator given in Fig. 1 as scatterer. The complex relative permittivity is taken as $\tilde{\varepsilon}_r = 3 + 4i$ and the relative permeability is set to $\mu_r = 1$. We consider an incident plane wave $u^{\text{inc}} = e^{-i\mathbf{k}\cdot\mathbf{x}}$ at a frequency $k_2 = 10$ for an incidence angle equal to $\theta^{\text{inc}} = \pi/4$. The density of discretization points per wavelength $n_\lambda = \lambda/h$ is fixed to $n_\lambda = 12$, where $\lambda = \max(|\lambda_1|, |\lambda_2|) = |\lambda_1|$ (= 0.281 in our situation). We report on Fig. 2 the convergence curves of the GMRES, GMRES(50) and GMRES(30) using the mass matrix and $\text{ILU-S}(5.10^{-2})$ preconditioners. In all these cases, we observe that we get an extremely slow convergence of the solver with the mass matrix preconditioner. This is worst with the ILU-S preconditioner where we see that we have a convergence breakdown. In a real situation, a restart parameter must be chosen then resulting here in the divergence of the iterative process. Moreover, in the case of the ILU-S preconditioner, a sufficiently large value of $\text{droptol}$ must be taken to get a sparsified representation of the preconditioner like the $\text{ILU-S}(1.10^{-1})$ (see
In this case, we observe the divergence of the iterative process in most of the treated cases. This typically shows some instabilities of the ILU process when it is directly used on the linear system to solve. This behavior has been already noticed by Kerchroud et al. (19) in the numerical solution of scattering problems using volume finite element methods in conjunction with an absorbing boundary condition.

Fig. 1. Scattering by a homogeneous conducting dielectric resonator.

We have seen on the first example that the convergence can be difficult to realize due to some instabilities of the preconditioner. Now, we only consider the GMRES(50) solver for the sake of conciseness.

Different parameters get involved in the deterioration of the convergence rate.
Fig. 3. Sparsity patterns of the $\tilde{U}$-matrix generated by ILU-S(5.10$^{-2}$) (left) and ILU-S(1.10$^{-1}$) (right).

Fig. 4. Convergence history of GMRES(50) for solving system (7) using the ILU-S(5.10$^{-2}$) preconditioner for various frequencies and densities of discretization points per wavelength (left) and for various imaginary parts of the complex permittivity $\tilde{\varepsilon}_r$ (right) (for the unit square cylinder).

This is for example the case of the mesh refinement and the consideration of higher frequencies. Let us fix for instance the scattering problem by the unit square cylinder centered at the origin and with sidelength 2. The dielectric constants are the same as in the previous example. We present on Fig. 4 (left) the convergence history for the case of an incident plane wave for $\theta_{\text{inc}} = \pi/4$ and for different densities of discretization points and frequencies. We can see that the convergence is strongly affected by these two parameters. In all these situations, we have furthermore observed a divergence of the mass matrix preconditioner for each case. Finally, comparing Figures 2 and 4 shows us that the geometry itself can be at the root of the convergence breakdown of the solver if interior modes occur. Another parameter is related to the lack of conductivity. This means that the imaginary part of the complex permittivity tends toward zero. To illustrate this problem, we present on Figure 4 (right) the convergence history of GMRES(50) for solving the previous problem at a frequency $k_2 = 10$ and $n_\lambda = 12$. The particularity here is to consider different conductivities. We clearly see that considering some weak conductivities penalizes the
convergence even leading to a divergence when there is no dissipation.

From all these tests, we conclude that the application of the ILU-S preconditioner to the solution of the scattering problem by a dielectric structure often lead to no improvement in the convergence of the solver. We propose a new preconditioner in the next section based on the integral relations of Calderón which yields a fast convergence of the iterative solver.

5 A robust block preconditioner based on Calderón relations

5.1 A few properties of the integral operators and the Calderón relations

We have seen in the previous section that building an efficient preconditioner for the system of integral equations is not an easy task. We propose here to construct an explicit robust integral preconditioner. To answer this problem, we need a few properties and relations associated with the basic integral operators.

Let us consider the integral operators defined by Eq. (3). Then, it is well-known that they define the following continuous mappings (20)

\[
\begin{align*}
V_j & : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma), \\
N_j & : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma), \\
K_j & : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma), \\
D_j & : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma),
\end{align*}
\] (9)

for \( j = 1, 2 \). For a smooth surface, the double-layer operators \( N_j \) are compact from \( H^{1/2}(\Gamma) \) into \( H^{1/2}(\Gamma) \). This is also the case of the transpose operators \( K_j \) while \( V_j \) is compact from \( H^{-1/2}(\Gamma) \) into \( H^{1/2}(\Gamma) \) (15; 16). Finally, they are all bounded operators for the respective associated Sobolev spaces.

Other important relations concerning the integral operators are known as the Calderón relations. They are here the keystone of building an explicit suitable integral preconditioner for our problem. These basic relations are known since a long time (see for example (10; 15)) but have only been used recently by Steinbach and Wendland (27) and next by Christiansen and Nédélec (13; 14) in the construction of preconditioners for integral equations in acoustics and electromagnetism.

**Proposition 1** According to the above notations, the following integral oper-
ators relations hold

\[ V_j D_j = \frac{I}{4} - N_j^2, \quad D_j V_j = \frac{I}{4} - N_j^{T2}, \]
\[ N_j V_j - V_j N_j^T = 0, \quad N_j^T D_j - D_j N_j = 0, \quad \text{(10)} \]

where \( I \) stands for the identity operator throughout the paper and \( j = 1, 2 \). These last relations are known as the Calderón relations.

5.2 Building of the symmetrical block preconditioner

Let us now come back to the problem of the construction of the preconditioner for the system involving \( S \). One of the crucial points of relations (10) is linked to the fact that \( 4D_j \) (respectively \( 4V_j \)) is an explicit pseudo-inverse operator of \( V_j \) (respectively \( D_j \)) up to a compact perturbation of order \(-2\) (which therefore has eigenvalues converging toward the origin). This practically means that the only application of \( 4D_j \) to \( V_j \) is required (and not the inversion of an operator or, from a discrete point of view, a linear system as in the ILUT approach).

Following these remarks, let us build now an explicit block integral preconditioner \( C \) for our problem

\[ C = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}. \quad \text{(11)} \]

Computing the operator product \( CS \), we get the four blocks

\[ CS_1 = C_1 V + C_2 N^T, \quad CS_2 = C_1 N - C_2 D, \]
\[ CS_3 = C_3 V + C_4 N^T, \quad CS_4 = C_3 N - C_4 D. \quad \text{(12)} \]

It appears that two symmetrical solutions seem to be particularly attracting. They are all based on the fact that we fix: \( C_1 = D \) and \( C_4 = -V \). Then, the two solutions differs by the choices: \( C_2 = N \) and \( C_3 = N^T \) for the first one and \( C_2 = N^T \) and \( C_3 = N \) for the second one. In fact, a rapid look at the four Calderón relations (10) leads us to think that the second choice is the best. In fact, from a numerical point of view, this is also something observed (but not given here to save place in the paper). From now, we fix this choice.
Proposition 2 Assume that $C_1 = D$, $C_2 = N^T$, $C_3 = N$ and $C_4 = -V$, then

$$CS = \left(\sqrt{\alpha} + \frac{1}{\sqrt{\alpha}}\right)^2 I + \Delta,$$  \hfill (13)

where $\Delta$ is a compact perturbation.

Proof. Let us analyze the different properties of each coefficient of the resulting matrix $CS$. Let us firstly compute the coefficient $CS_1$. Then, using the Calderón relations, we get

$$CS_1 = \frac{I}{2} + \alpha D_1 V_2 + \alpha^{-1} D_2 V_1 + N_1^T N_2^T + N_2^T N_1^T.$$  \hfill (14)

Moreover, following (12), it can be easily proven that $4D_1$ (respectively $4D_2$) is a pseudo-inverse of $V_2$ (respectively $V_1$) up to a compact perturbation $A_1$ (respectively $A_2$). Therefore, we obtain

$$\alpha D_1 V_2 = \frac{\alpha I}{4} + \alpha A_1, \quad \alpha^{-1} D_2 V_1 = \frac{\alpha^{-1} I}{4} + \alpha^{-1} A_2.$$  \hfill (15)

Let us also notice that $N_1^T N_2^T$ and $N_2^T N_1^T$ are compact as a composition of compact operators (15). This finally shows that we have the following relation

$$CS_1 = \left(\sqrt{\alpha} + \frac{1}{\sqrt{\alpha}}\right)^2 I + \Delta_1,$$  \hfill (16)

where $\Delta_1$ is the compact operator given by

$$\Delta_1 = \alpha A_1 + \alpha^{-1} A_2 + N_1^T N_2^T + N_2^T N_1^T.$$  \hfill (17)

In the same way, we prove that the fourth term $CS_4$ can be written as

$$CS_4 = \left(\sqrt{\alpha} + \frac{1}{\sqrt{\alpha}}\right)^2 I + \Delta_4,$$  \hfill (18)

with $\Delta_4$ a suitable compact operator. If we now consider the off-diagonal term $CS_3$, then, developing the operator products and using the Calderón relations lead to

$$\Delta_3 = CS_3 = \alpha^{-1} (N_2 V_1 - V_1 N_2^T) + (N_1 V_2 - V_2 N_1^T).$$  \hfill (19)
Again, this term is a compact operator as a product of compact operators. Finally, the last term $CS_2$ can be simplified as

$$\Delta_2 = CS_2 = \alpha(D_1N_2 - N_2^TD_1) + (D_2N_1 - N_1^TD_2).$$  \hfill (20)

Since each term is the product of a bounded operator and a compact one, the resulting product is compact (15) showing that $\Delta_2$ is also compact. Moreover, the operator $\Theta_1 = D_1N_2 - N_2^TD_1$ is of negative order. Indeed, in terms of symbols of pseudodifferential operators (28), the principal symbol $\sigma_p(\Theta_1)$ of order 0 of $\Theta_1$ is

$$\sigma_p(\Theta_1) = \sigma_p(D_1N_2) - \sigma_p(N_2^TD_1) = \sigma_p(D_1)\sigma_p(N_2) - \sigma_p(N_2^T)\sigma_p(D_1)$$

$$= \sigma_p(D_1)(\sigma_p(N_2) - \sigma_p(N_2^T)),$$  \hfill (21)

by virtue of the symbolical formula of the product of two pseudodifferential operators (28). However, using the expression of $\sigma_p(N_2) = 0$ in (22) in the two-dimensional case and the property that $\sigma(N_2^T)(x, \xi) = \sigma(N_2)(x, -\xi)$ (setting $\xi$ as the Fourier covariable), we see that the principal symbol of $\Theta_1$ vanishes. (Let us remark that, in the three-dimensional case of acoustics, the principal symbol of $N_2$ is related to the curvature tensor and Gauss curvature at $\Gamma$ according to (22). It still again leads to $\sigma_p(\Theta_1) = 0$.) The same results can be proven for the second term appearing in the right hand side of Eq. (20). This finally shows that $\Delta_2$ is of negative order. In conclusion, we have proven that we have the following expression of $CS$

$$CS = (\sqrt{\alpha} + \frac{1}{\sqrt{\alpha}})^2 I + \Delta,$$  \hfill (22)

where $\Delta$ is the compact perturbation given by

$$\Delta = \begin{pmatrix} \Delta_1 & \Delta_2 \\ \Delta_3 & \Delta_4 \end{pmatrix}$$  \hfill (23)

for the different blocks $\Delta_j, j = 1, \ldots, 4$, defined above.

5.3 The case of the circular cylinder

The robustness of the convergence of the GMRES is directly connected to the condition number of the system and the distribution of eigenvalues. A clustering of eigenvalues is often an interesting characteristic for a good convergence.
In the case of the spreading of the eigenvalues, then the solver can take large times and numerous iterations to convergence.

An explicit situation is given in the case of a circular cylinder. In this particular case, all the integral operators can be diagonalized in the discrete Fourier basis. Let \( L^2[0, 2\pi] \) be the space of periodic square integrable functions endowed with the following scalar product

\[
(v, u)_0 = \frac{1}{2\pi} \int_0^{2\pi} v(\theta)\pi(\theta)d\theta, \quad \forall u, v \in L^2[0, 2\pi].
\] (24)

It is well-known that the family of functions \( \{e^{im\theta}, m \in \mathbb{N}\} \) generates a complete orthonormal basis of \( L^2[0, 2\pi] \). In the case where \( \Gamma \) is the unit circle \( \Gamma = \mathbb{C}_1 \), it constitutes a basis of eigenvectors for the integral operators \( V, N, K \) and \( D \). Therefore, we can explicitly compute the associated eigenvalues. These properties allows us to diagonalize our preconditioned system (17) and to compute its eigenvalues.

**Proposition 3** Let \( m \in \mathbb{N} \) and \( V_{j,m}, N_{j,m}, K_{j,m}, D_{j,m} \), respectively the associated eigenvalues with the elementary integral operators \( V_j, N_j, K_j \) and \( D_j \) given by

\[
V_j e^{im\theta} = \left\{ \frac{i\pi}{2} J_m(k_j)H_m^{(1)}(k_j) \right\}e^{im\theta} = L_{j,m} e^{im\theta},
\]

\[
N_j e^{im\theta} = \left\{ \frac{1}{2} - \frac{i\pi}{2} k_j J'_m(k_j)H_m^{(1)}(k_j) \right\}e^{im\theta} = N_{j,m} e^{im\theta},
\]

\[
K_j e^{im\theta} = -N_j e^{im\theta} = -N_{j,m} e^{im\theta} = K_{m} e^{im\theta},
\]

\[
D_j e^{im\theta} = \left\{ -\frac{i\pi}{2} k_j^2 J'_m(k_j)H_m^{(1)}'(k_j) \right\}e^{im\theta} = D_{j,m} e^{im\theta},
\] (25)

for \( j = 1, 2 \). Functions \( J_m \) and \( H_m^{(1)} \) are respectively the Bessel and first-kind Hankel functions of order \( m \). Their respective derivatives are denoted by adding a prime. The eigenvalues for each mode represented by the index \( m \) of the initial integral system \( S \) are as follows

\[
\lambda_{\pm,m}(S_m) = \frac{(V_{m} - D_{m}) \pm \sqrt{(V_{m} + D_{m})^2 + 4N_{m}^2}}{2},
\] (26)

where \( S = \sum_m S_m e^{im\theta} \), and \( S_m \) the 2 \times 2 matrix defined by

\[
S_m := \begin{pmatrix}
V_{m} & N_{m} \\
N_{m} & -D_{m}
\end{pmatrix},
\] (27)

13
with \( V_m = \alpha^{-1}V_{1,m} + V_{2,m}, \) \( N_m = N_{1,m} + N_{2,m} \) and \( D_m = \alpha D_{1,m} + D_{2,m} \). For the preconditioned system, the eigenvalues are

\[
\lambda_{\pm,m}((CS)_m) = D_m V_m + N_m^2,
\]

where \( CS = \sum_m (CS)_m e^{im\theta} \), and \( (CS)_m \) the \( 2 \times 2 \) matrix defined by

\[
(CS)_m := \begin{pmatrix}
D_m V_m + N_m^2 & 2D_m N_m \\
2N_m V_m & D_m V_m + N_m^2
\end{pmatrix}.
\]

**Proof.** The computation of the eigenvalues given in Equations (25) is done e.g. in (1). The other part of the proposition can then be obtained directly.

To analyse the behavior of the two spectrums, we plot on Fig. 5 the complex eigenvalues distribution of the operators \( S \) and \( CS \) in the circular case. For each system, we report the eigenvalues computed directly from the analytical expressions (26) and (28) and the eigenvalues of the discrete integral equations based on linear boundary elements. The wavenumber is fixed to \( k_2 = 5 \). We consider \( \tilde{\epsilon}_r = 3 + 4i \) and \( \mu_r = 1 \). We fix \( n_\lambda = 10 \) and report \( m_{\text{max}} = 60 \) modes for the analytical formulae. Concerning \( S \) we observe a spreading of the spectrum along a line and the existence of an accumulation point around the origin. This penalizes \textit{a priori} the convergence rate of our solver and explains partially the bad convergence of the iterative solver observed in Section 4. Moreover, the associated condition number grows quickly according to the density of discretization points per wavelength \( n_\lambda \) (or, equivalently to \( m_{\text{max}} \)) and the wavenumber \( k_2 \). The eigenvalues of the preconditioned system present a cluster around \((\sqrt{\alpha} + 1/\sqrt{\alpha})^2/4\) (equal to \(1.28 + 0.96i\) in our test case). This is an important characteristic \textit{a priori} to get an excellent behaviour of an iterative solver. A deeper asymptotic analysis in the spirit of the results stated in (1) in the circular case should highlight the different penalizing parameters observed in Section 4.

5.4 Numerical experiments

To test now the robustness of the preconditioner and its ability for solving the difficulties met in Section 4, we propose to solve the same problems by preconditioning \( S \) by \( C \).

Let us come back to the first example provided in Section 4. We take \( k_2 = 10, n_\lambda = 12, \tilde{\epsilon}_r = 3 + 4i \) and \( \mu_r = 1 \) for the resonator given in Fig. 1. We report on Fig. 6 the history of GMRES(50) using \( C \) to precondition the
system based on $S$. Unlike the results shown in Fig. 2, we observe a very good convergence rate of our solver and no breakdown. This shows the robustness of our preconditioner for solving a complex scattering problem when other methods failed.

In the second test case, we report on Figure 7 (on the left) the convergence history of GMRES(50) according to the frequency $k_2$ and the density of discretization points per wavelength $n_\lambda$. The convergence rate is weakly $k_2$-dependent while the number of iterations is independent of the density. This is due to the property that $CS$ is equal to the identity (modulo a multiplicative constant) up to a compact perturbation. All these convergence results are much better than using the direct ILUT preconditioner (if we compare it to Figure 4). Finally, the right curves on Figure 7 shows that the convergence is not really affected by the conductivity parameter. This is extremely different from the situation reported on Figure 4.
Fig. 7. Convergence history of GMRES(50) preconditioned by the matrix operator $C$ for the scattering problem by the unit square cylinder. The left figure shows that the convergence is independent of the density of discretization points per wavelength $n_\lambda$ and illustrates the dependence according to the wavenumber $k_2$. The right figures gives the variations of the number of iterations according to the conductivity of the medium showing that the convergence weakly depends on $\tilde{\epsilon}_r$.

To end the paper, we consider the scattering problem by a kite-shaped object depicted in Figure 8. The dielectric medium is characterized by $\tilde{\epsilon}_r = 2 + 2i$ and $\mu_r = 1$. We report on Figure 9 (left) the convergence history of GMRES(50) according to the wavenumber $k_2$ and density $n_\lambda$. We can clearly observe the same conclusions as in the previous example. This is again related to the good eigenvalues clustering of the operator $CS$ given on Figure 9 (right) in terms of eigenvalues distribution of the associated finite element matrix approximation for $k_2 = 40$ and $n_\lambda = 10$. The accumulation point is given here by $1.0625 + 0.4375i$.

Fig. 8. The kite-shaped scatterer.
Fig. 9. Convergence history of GMRES(50) preconditioned by the matrix operator $C$ for the scattering problem by the kite-shaped scatterer. The left figure shows that the convergence is independent of the density of discretization points per wavelength $n_\lambda$ and illustrates the dependence according to the wavenumber $k_2$. The right figure reports the eigenvalues distribution for $k_2 = 40$ and $n_\lambda = 10$.

6 Conclusion

The construction of a new integral preconditioner based on the Calderón formulae has been introduced in this paper for solving the scattering problem by a dielectric medium. Its efficiency and robustness has been tested for several difficult situations. This provides new interesting directions for developing preconditioners by using operator based ideas for high frequency scattering problems. The extension to the full Maxwell equations remains to analyse in the case of dielectric media. We can mention that recent interesting theoretical developments have been made by Ossandrón Vélez (23) about this problem.

Open questions are still present. Among others, let us mention the possibility of completely avoiding any dependence of the preconditioner according to the frequency parameters, the problem of scatterers made of several layers of dielectrics or even the problem of the extension to large scale multiple scattering problems often met in applications as for example in photonic crystals (18; 25) used in optical fiber optics design.

Acknowledgements. Parts of this work begun while the first author was Visiting Associate Professor at the Applied and Computational Mathematics Department of the California Institute of Technology, Pasadena, USA. The author wishes to sincerely thank Prof. O.P. Bruno for many useful and interesting discussions during this visit and his friendship support.
References


