ANISOTROPIC REGULARITY AND OPTIMAL RATES OF CONVERGENCE FOR THE FINITE ELEMENT METHOD ON THREE DIMENSIONAL POLYHEDRAL DOMAINS

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Abstract. We consider the model Poisson problem \(-\Delta u = f \in \Omega, u = g \) on \(\partial \Omega\), where \(\Omega\) is a bounded polyhedral domain in \(\mathbb{R}^n\). The objective of the paper is twofold. The first objective is to review the well posedness and the regularity of our model problem using appropriate weighted spaces for the data and the solution. We use these results to derive the domain of the Laplace operator with zero boundary conditions on a concave domain, which seems not to have been fully investigated before. We also mention some extensions of our results to interface problems for the Elasticity equation. The second objective is to illustrate how anisotropic weighted regularity results for the Laplace operator in 3D are used in designing efficient finite element discretizations of elliptic boundary value problems, with the focus on the efficient discretization of the Poisson problem on polyhedral domains in \(\mathbb{R}^3\), following Numer. Funct. Anal. Optim., 28(7-8):775–824, 2007. The anisotropic weighted regularity results described and used in the second part of the paper are a consequence of the well-posedness results in (isotropically) weighted Sobolev spaces described in the first part of the paper. The paper is based on the talk by the last named author at the Congress of Romanian Mathematicians, Brasov 2011, and is largely a survey paper.

Introduction

Let \(\Omega \subset \mathbb{R}^n\) be an open, bounded set. Consider the boundary value problem

\[
\begin{aligned}
\Delta u &= f \quad \text{in } \Omega \\
\frac{\partial u}{\partial \nu} &= g, \quad \text{on } \Omega,
\end{aligned}
\]

defined on a bounded domain \(\Omega \subset \mathbb{R}^n\), where \(\Delta\) is the Laplacian \(\Delta = \sum_{i=1}^{d} \partial_i^2\). When \(\partial \Omega\) is smooth, it is well known that this Poisson problem has a unique solution \(u \in H^{m+1}(\Omega)\) for any \(f \in H^{m-1}(\Omega)\) and \(g \in H^{m+1/2}(\partial \Omega)\) [27, 45, 52]. Moreover, \(u\) depends continuously on \(f\) and \(g\). This result is the classical well-posedness of the Poisson problem on smooth domains.

On the other hand, when \(\Omega\) is not smooth, it is also well known [23, 24, 37, 39, 40] that there exists \(s = s_{\Omega}\) such that \(u \in H^s(\Omega)\) for any \(s < s_{\Omega}\), but \(u \not\in H^{s_{\Omega}}(\Omega)\) in general, even if \(f\) and \(g\) are smooth functions defined in a neighborhood of \(\Omega\). For

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instance, if \( \Omega \) is a polygonal domain in two dimensions, then \( s_{\Omega} = 1 + \pi/\alpha_{\text{MAX}} \), where \( \alpha_{\text{MAX}} \) is the largest interior angle of \( \Omega \) [39]. See also Wahlbin’s paper [54].

In view of applications to the Finite Element Method, we restrict our attention to domains with polyhedral structure. These are natural non-convex generalizations of classical \( n \)-dimensional polyhedra that allow for curved boundaries, cracks (i.e., internal faces), and non-smooth boundary points touching a smooth part of the boundary. We refer to [13] for a precise formulation.

There exist a very large number of papers devoted to boundary value problems on non-smooth domains. While it is impossible to mention them all, let us at least mention the papers of Arnold, Scott, and Vogelius [10], Babuska and Guo [33], Băcuță, Bramble, and Xu [19], Jerison and Kenig [37], Kondratiev [39], Kozlov, Mazya, and Rossmann [40], Mitrea and Taylor [47], Rossmann [51], Verchota [53], and many others. Other results specific to numerical methods for polyhedral domains are contained in the papers of Apel and Dobrowolski [6], Costabel, Dauge, and Nicaise [22], Costabel, Dauge, and Schwab [23], Dauge [24], Demkowicz, Monk, Schwab, and Vardapetyan [25], Elschner [26], Guo and Schwab [35], and many others. Further results and references can be found in the aforementioned papers, as well as in the the monographs of Grisvard [32] as well as the recent book [49].

Regularity for polyhedral domains is useful in designing fast solvers for numerical methods [7, 17]. See also [5, 6, 12, 16, 18, 42, 36, 29, 28, 44, 41] for more applications of these techniques to other types of Partial Differential Equations and numerical methods.

In this paper, we shall review the results from [13, 14, 15, 43], and [46], which make use of the natural stratified space structure on \( \Omega \). This leads, by successive conformal changes of the metric, to a metric for which the smooth part of \( \Omega \) becomes a smooth manifold with boundary whose double is complete. The resulting Sobolev spaces defined by the new metric will lead to spaces on which the Poisson problem is well-posed.

We restrict for simplicity to consider only the Laplace operator in (1). However, all theoretical results presented here extend to scalar, strongly elliptic, linear operators \( P \) with sufficiently regular coefficients, and even to elliptic systems, such as the system of anisotropic elasticity [46]. Furthermore, we can also treat transmission problems, for which the coefficients of \( P \) are allowed to jump across piecewise-smooth hypersurfaces, representing interfaces, under some additional conditions [13, 43]. We briefly discuss these extensions in Subsection 1.3.

For the discretization on polyhedral domains, we build discrete spaces \( S_k \subset H^1_0(\Omega) \) and Galerkin finite element projections \( u_k \in S_k \) that approximate the solution \( u \) of Equation (1) for \( f \in H^{m-1}(\Omega) \) arbitrary. We prove that, by using certain spaces of continuous, piecewise polynomials of degree \( m \), the sequence \( S_k \) achieves quasi-optimal rates of convergence. More precisely we prove the existence of a constant \( C > 0 \), independent of \( k \) and \( f \), such that

\[
\|u - u_k\|_{H^1(\Omega)} \leq C \dim(S_k)^{-m/n} \|f\|_{H^{m-1}(\Omega)}, \quad u_k \in S_k,
\]

where \( n = 2 \) or \( n = 3 \) is the dimension of our polyhedral domain.

The contents of the paper are as follows. In the first section we review well-posedness results in weighted Sobolev spaces on polyhedra domains. These weighted spaces are sometimes called the Babuška-Kondratiev spaces. These results are not sufficient for our applications to the Finite Element Method in three dimensions, so in the second section we review some additional anisotropic regularity results.
These results are used in the third section to construct a sequence of meshes that yields $h^m$-quasi-optimal rates of convergence in three dimensions. Finally, in the last section we discuss some of the main ingredients that enter in the proof and which are of independent interest. These include the Hardy-Poincaré inequality (which guarantees the coercivity of our problem) and a description of the weighted Sobolev (or Babuška-Kondratiev) spaces $K^m_\alpha(\Omega)$, which are the natural spaces for our well-posedness results, as the usual Sobolev spaces for a modified metric on $\Omega$, which nevertheless is conformally equivalent to the old one.

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1. WELL POSEDNESS IN ISOTROPIC WEIGHTED SOBOLEV SPACES

Using the standard notation for partial derivatives, namely $\partial_j = \frac{\partial}{\partial x_j}$ and $\partial^\alpha = \partial^{\alpha_1} \ldots \partial^{\alpha_n}$, for any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_+$, we denote the usual Sobolev spaces on an open set $V \subset \mathbb{R}^n$ by

$$H^m(V) = \{ u : V \to \mathbb{C}, \partial^\alpha u \in L^2(V), |\alpha| \leq m \}.$$

As mentioned in the introduction, the solution of our model Poisson problem (1) has only limited regularity in the spaces $H^m(\Omega)$. The situation changes for the better if one considers weighted Sobolev spaces, though. To define the weighted analogues of these spaces, we need to introduce the notion of singular boundary points of the domain $\Omega \subset \mathbb{R}^n$.

1.1. Weighted Sobolev spaces. Let $\partial^{\text{sing}} \Omega \subset \partial \Omega$ be the set of singular (or non-smooth) boundary points of $\Omega$, that is, the set of points $p \in \partial \Omega$ such $\partial \Omega$ is not smooth in a neighborhood of $p$. In case we consider mixed boundary conditions, the set of singular points includes also the set of points where the boundary conditions change. If, furthermore, interfaces are considered, the set of singular points contains the set of singular points of the interface, as well as the set of points where the interface touches the boundary. We will denote by $r_{\partial \Omega}(x)$ the distance from a point $x \in \Omega$ to the set $\partial^{\text{sing}} \Omega$ and agree to take $r_{\partial \Omega} = 1$ if there are no such points, i.e., if $\partial \Omega$ is smooth. For $\mu \in \mathbb{Z}^+$ and $a \in \mathbb{R}$, we define the weighted Sobolev spaces

$$K^m_\alpha(\Omega) = \{ u \in L^2_{\text{loc}}(\Omega), r_{\Omega}^{\alpha-\alpha} \partial^\alpha u \in L^2(\Omega), |\alpha| \leq \mu \},$$

which we endow with the induced Hilbert space norm. We note that for $n = 3$ for example and $\Omega$ a polyhedral domain in $\mathbb{R}^3$, we have that $r_{\partial \Omega}(x)$ is the distance to the skeleton comprising the union of the closed edges of $\partial \Omega$. Recently, general spaces of this kind were studied by H. Amann [2, 1].

Similar weighted Sobolev spaces are associated to the faces of $\Omega$. By a face, we mean the connected components of the boundary $\partial \Omega$ after the set of singular points is removed. For example for $n = 3$, we define

$$K^m_\alpha(\partial \Omega) = \{ (u_F), r_{\Omega}^{\alpha-\alpha} \partial^\alpha u_F \in L^2(F) \},$$

where $|\alpha| \leq m$ and $F$ ranges through the set of faces of $\partial \Omega$. For $s \in \mathbb{R}^+$, we define the space $K^s_\alpha(\partial \Omega)$ by standard interpolation.
1.2. Well-posedness for the Poisson problem on $n$-dimensional polyhedral domains. The following result is proved in [13]. For simplicity, we shall assume that $\Omega$ has no cracks and that there are no vertices that touch the boundary. (That is, we shall consider only domains $\Omega$ that coincide with the interior of their closure $\overline{\Omega}$.)

**Theorem 1.1.** Let $\Omega \subset \mathbb{R}^n$, be a bounded, curvilinear polyhedral domain and $m \in \mathbb{Z}^+$. Then there exists $\eta_\Omega > 0$ such that $\tilde{\Delta}(u) = (\Delta u, u|_{\partial \Omega})$ defines an isomorphism

$$\tilde{\Delta} : \mathcal{K}_{a}^{m+1}(\Omega) \to \mathcal{K}_{a-1}^{m-1}(\Omega) \oplus \mathcal{K}_{a+1/2}^{m+1/2}(\partial \Omega),$$

for all $|a| < \eta_\Omega$. If $m = 0$, the solution $u$ corresponding to the data $(f, 0) \in \mathcal{K}_{-1}^{-1}(\Omega) \oplus \mathcal{K}_{1/2}^{1/2}(\partial \Omega)$ is also the solution of the associated variational problem.

This theorem amounts to the well-posedness of the boundary value (1) on $n$-dimensional polyhedral domains. For $n = 2$ (in which case $\Omega$ is a polygonal domain), this result is due to Kondratiev [39], in which case $\eta_\Omega = \pi \alpha_{\text{MAX}}$, where $\alpha_{\text{MAX}}$ is the measure in radians of the maximum angle of $\Omega$. For $n = 3$, this result was proved in [14]. For later applications, we shall need the following result.

**Theorem 1.2.** The results of Theorem 1.1 remain true for infinite angles in two dimensions, infinite polyhedral cones in three dimensions, and infinite dihedral angles in three dimensions.

The proof of this theorem proceeds along the lines of the proof of Theorem 1.1 in [14] or [13]. A first difference to remark is that the Hardy-Poincaré inequality does not hold for the whole domain. Then the “desingularization” $\Sigma(\Omega)$ has to involve the directions at infinity also in the case of an angle or a cone. In case of a dihedral angle $D_{\alpha} = \{0 < \theta < \alpha\}$, in cylindrical coordinates $(r, \theta, z)$, one has to consider the also the two point compactification of the edge. In particular,

$$\Sigma(D_{\alpha}) = [0, \alpha] \times [0, \infty] \times [-\infty, \infty].$$

1.3. Extensions. Theorem 1.1 above was extended in several ways. First, the proof applies with almost no change if mixed boundary value problems are considered, provided that no adjacent faces are endowed with Neumann boundary conditions. We do allow, however, different boundary conditions on the same face. We treat the points where the boundary conditions change similarly to the non-smooth boundary points, as solutions exhibit a similar singular behavior in this case.

As already mentioned in the Introduction, we can more generally consider a general uniformly strongly elliptic differential operator of the form

$$Lu = -\sum_{ij} \partial_i (a_{ij} \partial_j u) + cu, \text{ with } c \geq 0.$$  

(Recall that $L$ is uniformly strongly elliptic if, and only if, there exists $C > 0$ such that $\sum_{ij} a_{ij} t_i t_j \geq C \sum_i t_i^2$, for all $(t_i) \in \mathbb{R}^n$.)

We can also include certain transmission or interface problems. More precisely, we now assume that our domain $\Omega$ can be written as a union of curvilinear polyhedral domains $\Omega_j$ with disjoint interiors: $\overline{\Omega} = \bigcup_{j=1}^K \overline{\Omega}_j$. Let $\Gamma := \bigcup_{j=1}^K \partial \Omega_j \setminus \partial \Omega$ be the interface. We assume that $\Gamma$ is smooth and assume further that no adjacent faces of the $\Omega_j$’s are endowed with Neumann boundary conditions. We do allow
Γ to touch the boundary of Ω. We can then extend the result of Theorem 1.1 by using instead the broken weighted Sobolev spaces \( \tilde{K}_m^a(\Omega) \), defined by

\[
\tilde{K}_m^a(\Omega) := \bigoplus_{j=1}^{\mathcal{K}} K_m^a(\Omega_j).
\]

We observe that, if there is no interface, \( \tilde{K}_m^a(\Omega) = K_m^a(\Omega) \). We let \( \partial_D \Omega \) be the part of the boundary with Dirichlet boundary conditions, which we assume to be a closed subset of the boundary, and let \( \partial_N \Omega := \partial \Omega \setminus \partial_D \Omega \). We denote the outer normal vector to \( \Omega \), which is defined a.e. on \( \partial \Omega \), by \( \nu \), and the conormal derivative associated to the operator \( L \) by \( \sum_{ij} \nu_i a_{ij}(x) \partial_j \). Let \( \tilde{L}(u) = (Lu, u|_{\partial_D \Omega}, D^L u|_{\partial_N \Omega}) \). Our most general result in \( n \) dimensions states that for \( m \geq 1 \) \( \tilde{L} \) is an isomorphism (see [13]):

\[
\tilde{L} : D_a \rightarrow \tilde{K}_a^{-1}(\Omega) + K_{a+1}^{m+1/2}(\partial_D \Omega) \oplus K_{a+1/2}^{m-1/2}(\partial_N \Omega),
\]

where

\[
D_a := \{ u \in \tilde{K}_a^{m+1}(\Omega) \cap K_{a+1}(\Omega), \quad u^+ = u^-, \quad D^L_\nu^+ u = D^L_\nu^- u \text{ on } \Gamma \}.
\]

and the subscript \( \pm \) refers to non-tangential limits to each side of the interface. The conormal derivative is defined in the sense of the trace a.e. on \( \partial \Omega \).

Let us mention that the interface \( \Gamma \) will separate different faces where it touches the boundary, and hence we assume that these faces are not both endowed with Neumann boundary conditions.

For elasticity with mixed boundary conditions, a similar result is obtained by Mazzucato and Nistor in [46]. The results in [46] also extend to interface problems under the same assumptions (no adjacent faces with Neumann boundary conditions and a smooth interface) using the methods as in [13] and in [46]. More precisely, we use Korn’s inequality to obtain local regularity results (no weighted spaces). This applies, in particular, to interface problems. There the additional regularity is proved as for the additional regularity at the boundary for smooth domains. See [50] for a proof of the additional regularity at the boundary for systems that extends to interface problems. Once one has the local regularity results, the global regularity results in weighted spaces is proved as in [46] using suitable partitions of unity. The solvability in \( H^1 \) is an immediate consequence of Korn’s inequality and of the Hardy-Poincaré inequality. Combining regularity with solvability in \( H^1 \) yields solvability in higher weighted Sobolev spaces \( K_{a+1}^{m+1}(\Omega) \).

Other regularity results go toward analytic regularity using countably normed spaces as in the work of Babuška-Guo [34, 33], and Costabel, Dauge, and Nicaise [22]. See the Introduction for more references. It would be interesting to extend these results to the de Rham complex [8, 9].

1.3.1. Adjacent Neumann faces and non-smooth interfaces in 2D. The assumption that no vertex \( P \) be the common point of two adjacent faces with Neumann boundary conditions or the assumption that \( \Gamma \) be smooth at any interface point \( P \) are both equivalent to the fact that the function constant equal to one not be a singular function at that singular point \( P \). This assumption is necessary, because, if it is not satisfied, the relevant operator \( \tilde{L} \) is not even Fredholm for the value \( a = 0 \) and it is also not invertible for any \( a \in \mathbb{R} \). However, this assumption is not realistic in practice and, it turns out, not even necessary for designing graded meshes that yield quasi-optimal rates of convergence [43].
To obtain a well-posedness result for interface problems in 2D, we can proceed as follows [43]. Let \( \chi_P \) be a smooth function that is equal to 1 near each singular point \( P \) that is either a point where we have Neumann-Neumann conditions or a non-smooth interface point satisfying respectively Neumann or periodic boundary conditions on the sides at \( P \). This includes points \( P \) that belong to more than two of the subdomains \( \bar{\Omega}_i \) (so called multiple junction points). We assume the \( \chi \)'s have disjoint supports. Let \( W_a \) be the linear span of the functions \( \chi_P \). The choice of boundary conditions or the introduction of additional singular points to a polygonal domain define a polygonal structure on \( \Omega \), see [43] for details.

**Theorem 1.3.** Let \( \Omega \) be a domain with a polygonal structure. The there exists \( \eta > 0 \) such that, for all any \( 0 < a < \eta \) and \( m \in \mathbb{Z}_+ \), the map

\[
\hat{L} : D_a + W_s \to \mathcal{K}_{a-1}^m(\Omega) \oplus \mathcal{K}_{a+1/2}^m(\partial_D \Omega) \oplus \mathcal{K}_{a-1/2}^m(\partial_N \Omega),
\]

with \( D_a \) given in (8), is an isomorphism.

The proof requires the calculation of the index of the operator \( \hat{L} \) acting on \( \mathcal{K}_{a+1/2}^m(\Omega) \cap \mathcal{K}_{a-1}^m(\Omega) \). Note that our result is not valid for \( a = 0 \). We expect a similar result in 3D.

Theorem 1.3 can be used to justify the construction of a sequence of meshes (in 2D) that yields quasi-optimal \( h \)-rates of convergence for transmission problems with non-smooth interfaces (and even with multiple junctions) and problems with adjacent Neumann-Neumann corners in 2D. See [48] for additional issues related to the regularity and numerical methods for interface problems. We notice that the resulting sequence of meshes is the same for all 2D problems on polygonal domains (with or without interfaces or Neumann-Neumann corners), although the theoretical PDE result (or \( a \) priori estimates) are different in these two cases.

1.4. The domain of \( \Delta \) on concave polygons. Let us mention that the method used to obtain Theorem 1.3 can be used to describe the domain \( D(\Delta) \) of the Friedrichs extension of the Laplace operator on \( \Omega \) with zero boundary conditions. First of all, the form associated to \( \Delta \), namely \( B(u,v) = (\nabla u, \nabla v) \), \( u,v \) zero on the boundary, defines the so called energy norm: \( |u|_{H^1(\Omega)} = B(u,u)^{1/2} \). The completion of \( C_c^\infty(\Omega) \) in the energy norm is \( H_0^1(\Omega) \). The proofs in [20, 14] show that \( H_0^1(\Omega) = \mathcal{K}_{1}^1(\Omega) \cap \{u|_{\partial \Omega} = 0\} \), with equivalent norms. The domain of the Friedrichs extension of the Laplacian \( \Delta \) is then

\[
D(\Delta) = \{ u \in H_0^1(\Omega), \Delta u \in L^2(\Omega) \}.
\]

If \( \Omega \) is convex, then it is known that \( D(\Delta) = H^2(\Omega) \cap H_0^1(\Omega) \). This is however not true if \( \Omega \) is concave. To describe \( D(\Omega) \) in the case when \( \Omega \) is concave, let us notice that the map

\[
\Delta : H_0^1(\Omega) \cap \mathcal{K}_{1}^2(\Omega) \to L^2(\Omega)
\]

is Fredholm and its index is the number of re-entrant corners by [39]. Let \( P \) be such a re-entrant corner with angle \( \alpha_P > \pi \). Also, let \( (r, \theta) \) be polar coordinates at \( P \) and consider the function \( \phi_P = r^{\pi/\alpha_P} \sin(\pi \theta/\alpha_P) \chi_P \), where \( \chi_P \) is the function considered in Theorem 1.3. Let \( V_s \) be the space of linear combinations of the functions \( \phi_P \), with \( P \) a re-entrant corner. Then one has that

\[
\Delta : \mathcal{K}_{1}^2(\Omega) \cap H_0^1(\Omega) + V_s \to L^2(\Omega)
\]

has index zero, is injective, and hence bijective. This proves the following result.
Theorem 1.4. The domain of the Friedrichs extension of the Laplace operator with zero boundary conditions on a polygon $\Omega \subset \mathbb{R}^2$ is

$$\mathcal{D}(\Delta) = \mathcal{K}_2^2(\Omega) \cap H_0^1(\Omega) + V_s.$$ 

A similar description is available for other types of boundary conditions. This result immediately leads to a maximal regularity result for the heat equation on polygonal domains.

See also [31] and [30] for related results on Friedrichs extensions of second order elliptic operators on manifolds with conical points.

2. Anisotropic weighted Sobolev spaces and regularity

The well-posedness result of the previous section are not enough to establish quasi-optimal rates of convergence in 3D. We need additional regularity along the edges, as follows. Let $u$ be the solution of problem (1) with $f \in H^{m-1}(\Omega)$ and $g = 0$. We observe that this assumption is stronger than assuming that $f$ is in a weighted Sobolev space of the form $\mathcal{K}_{a-1}^m(\Omega)$ for $|a|$ small. We will need to take advantage of this additional regularity of $f$, which leads to improved regularity for $u$ along the edges. We encode this additional regularity by introducing new anisotropically weighted spaces.

We assume first that the domain $\Omega$ is a dihedral angle with axis along the $z$-coordinate axis, $D_\alpha = \{ 0 < \theta < \alpha \}$, using cylindrical coordinates $(r, \theta, z)$. We further assume that $f \in H^{m-1}(D_\alpha)$. Then $f \in \mathcal{K}_{a-1}^m(D_\alpha)$, and hence

$$u \in \mathcal{K}_{a+1}^{m+1}(D_\alpha)$$

for positive and small enough $a$, by Theorem 1.2. Hence, $\partial_z u \in \mathcal{K}_{a}^m(D_\alpha)$. However, we also have $\Delta \partial_z u = \partial_z \Delta u = \partial_z f \in H^{m-2}(\Omega)$. Then, using Theorem 1.1 which extends to this setting, we also obtain that

$$\partial_z u \in \mathcal{K}_{a+1}^m(\Omega),$$

a better estimate than in Equation (12). These calculations suggest that we introduce a scale of spaces $\mathcal{D}_{a}^m$, $m \in \mathbb{Z}_+$, as follows:

$$\mathcal{D}_{a}^1(D_\alpha) := \mathcal{K}_{1}^{1}(D_\alpha),$$

$$\mathcal{D}_{a}^m(D_\alpha) := \{u \in \mathcal{K}_{a}^{m}(D_\alpha), \partial_z u \in \mathcal{D}_{a}^{m-1}(D_\alpha)\}.$$ 

The spaces $\mathcal{D}_{a}^1$ are thus independent of $a$.

We assume next that the domain $\Omega$ is a cone $\mathcal{C}$ centered at the origin. We let $\rho(x) = |x|$, the distance from $x$ to the origin, and define

$$\mathcal{D}^1_\rho(\mathcal{C}) := \rho^{a-1} \mathcal{K}_{1}^{1}(\mathcal{C}) = \{\rho^{a-1}v, \ v \in \mathcal{K}_{1}^{1}(\mathcal{C})\}.$$ 

To introduce the spaces $\mathcal{D}_{a}^m(\mathcal{C})$ for $m \geq 2$, we shall need to consider the vector field $\rho \partial_\rho := x \partial_x + y \partial_y + z \partial_z$, which is the infinitesimal generator of dilations centered at the vertex of the cone. We then define by induction

$$\mathcal{D}_{a}^{m}(\mathcal{C}) := \{ u \in \mathcal{K}_{a}^{m}(\mathcal{C}), \rho \partial_\rho (u) \in \mathcal{D}_{a}^{m-1}(\mathcal{C})\}, \ m \geq 2.$$ 

For a general bounded polyhedral domain $\Omega$, we define the anisotropic weighted Sobolev spaces $\mathcal{D}_{a}^{m}(\Omega)$ by localizing around vertices and edges, using as models cones and dihedral angles respectively, such that away from the edges these spaces coincide with the usual Sobolev spaces $H^m$. Then, we have the following regularity result [15]:
Theorem 2.1. Let \( f \in H^{m-1}(\Omega) \), with \( m \geq 1 \). Then there exists \( \eta_{\Omega, a} > 0 \) such that the Poisson problem (1) with \( g = 0 \) has a unique solution \( u \in D^{m+1}_{a+1}(\Omega) \) for any \( 0 \leq a < \eta = \eta_{\Omega} \) and
\[
\|u\|_{D^{m+1}_{a+1}(\Omega)} \leq C \eta_{\Omega, a} \|f\|_{H^{m-1}(\Omega)}.
\]

See [11, 21, 22, 38] for related results.

3. Quasi-optimal \( h^m \)-mesh refinement

We describe in this section a strategy to obtain quasi-optimal \( h^m \)-mesh refinement. We follow [15], from where the pictures are taken. The theoretical justification of this construction is based on the anisotropic regularity result of the previous section, Theorem 2.1. Given a bounded polyhedral domain \( \Omega \) and a parameter \( \kappa \in (0, 1/2] \), we will provide a sequence \( T_n \) of decompositions of \( \Omega \) into finitely many tetrahedra, such that, if \( S_n \) is the finite element space of continuous, piecewise polynomials on \( T_n \), then the Lagrange interpolant of \( u \) of order \( m \), \( u_{I,n} \), has “quasi-optimal” approximability properties. The result can be formulated as follows:

Theorem 3.1. Let \( a \in (0, 1/2] \) and \( 0 < \kappa \leq 2^{-m/a} \). Then there exists a sequence of meshes \( T_n \) and a constant \( C > 0 \) such that, for the corresponding sequence of finite element spaces \( S_n \), we have
\[
|u - u_{I,n}|_{H^{1}(\Omega)} \leq C 2^{-km} \|u\|_{D^{m+1}_{a+1}(\Omega)},
\]
for any \( u \in D^{m+1}_{a+1}(\Omega) \), \( u|_{\partial\Omega} = 0 \), and for any \( k \in \mathbb{Z}_+ \).

Theorem (2) is now a direct consequence of Theorem 3.1.

3.1. Refinement Strategy. Our refinement strategy will first generate a sequence of decompositions \( T_n' \) of \( \Omega \) in tetrahedra and triangular prisms, while our meshes \( T_n \) will be obtained by further dividing each prism in \( T_n' \) into three tetrahedra. We now explain how the divisions \( T_n' \) are defined inductively, \( T_n' \) being a refinement of \( T_n \) in which each edge is divided into two (possibly unequal) edges.

To define the way each edge of \( T_n' \) is divided, we need to establish a hierarchy of the nodes of \( T_n' \). Therefore, given a point \( P \in \Omega \), we shall say that \( P \) is of type \( V \) if it is a vertex of \( \Omega \); we shall say that \( P \) is of type \( E \) if it is on an open edge of \( \Omega \). Otherwise, we shall say that it is of type \( S \) (that is, a “smooth” point). The type of a point depends only on \( \Omega \) and not on any partition or meshing. The initial tetrahedralization will consist of edges of type \( VE, VS, ES, EE := E^2 \), and \( S^2 \). We shall assume that our initial decomposition and initial tetrahedralization were defined so that no edges of type \( E^3 \) := \( VV \) are present. The points of type \( V \) will be regarded as more singular than the points of type \( E \), and the points of type \( E \) will be regarded as more singular than the points of type \( S \). All the resulting triangles will hence be of one of the types \( VES, VSS, ESS \). Let us notice that once our initial refinement is fine enough, the edges of our domain will be decomposed into segments of type \( VE \) and \( EE \), and the segments of type \( EE \) will be contained in triangular prisms. Therefore, we can assume that there are no triangles of type \( EES \).

Our refinement procedure depends on the choice of a constant \( \kappa \in (0, 2^{-m/a}) \), where \( a > 0 \) is as in Theorem 2.1 and \( \kappa \leq 1/2 \). We can improve our construction by considering different values of \( \kappa \) associated to different vertices or edges. This
generalization can easily be carry out by the reader. See [43] for example. Let $AB$ be a generic edge in the decompositions $T_n$. Then, as part of the $T_{n+1}$, this edge will be decomposed in two segments, $AC$ and $CA$, such that $|AC| = \kappa |AB|$ if $A$ is more singular than $B$ (i.e., if $AB$ is of type $\text{VE}$, $\text{VS}$, or $\text{ES}$). Except when $\kappa = 1/2$, $C$ will be closer to the more singular point. This procedure is as in [20]. See Figure 3.1.

![Figure 3.1. Edge decomposition](image)

The above strategy to refine the edges induces a natural strategy for refining the triangular faces. If $ABC$ is a triangle in the decomposition $T'_n$, then in $T'_{n+1}$, the triangle $ABC$ will be divided into four other triangles, according with the edge strategy. The decomposition of triangles of type $\text{S}^3$ is obtained for $\kappa = 1/2$. The type $\text{VSS}$ triangle decomposition is described in Figure 3.2 (a). In the case when $ABC$ is of type $\text{VES}$, however, we shall use a different construction. Namely, in this case we remove the newly introduced segment that is opposite to $B$, see Figure 3.2 (b), and divide $ABC$ into two triangles and a quadrilateral. The resulting quadrilateral will belong to a prism in $T'_{n+1}$.

![Figure 3.2. Triangle decomposition, $\kappa = 1/4$](image)

3.2. Divisions in tetrahedra and prisms. We now describe the construction of the sequence of the decompositions $T'_n$ for $n \geq 0$. The required sequence of meshes $T_n$ will be defined by dividing all the prisms in $T'_n$ into tetrahedra. For the first level of semi-uniform refinement of a prism, more details are presented in [15].

We start with an initial division $T'_0$ of $\Omega$ in straight triangular prisms and tetrahedra of types $\text{VESS}$ and $\text{VS}^3$, having a vertex in common with $\Omega$, and an interior region $\Lambda_0$. See Figure 3.2 (a), where we have assumed that our domain $\Omega$ is a tetrahedron. For each of the prisms we choose a diagonal (called mark) which will be used to uniquely define a partition of the triangular prism into tetrahedra. We then divide the interior region $\Lambda_0$ into tetrahedra that will match the marks. Also,
we assume that the marks on adjacent prisms are compatible, so that the resulting meshes are conforming. We can further assume that some of the edge points (as in Figure 3.2 (b)) have been moved along the edges so that the prisms become straight triangular prisms i.e., the edges are perpendicular to the bases.

The decompositions $T'_n$ are then obtained by induction following the Steps 1 through 3 explained next. We assume that the decomposition $T'_n$ was defined and we proceed to define the decomposition $T'_{n+1}$.

**Step 1.** The tetrahedra of type $S^4$ are refined uniformly by dividing along the planes given by $x_i + x_j = k/2^n$, $1 \leq k \leq 2^n$, where $x_j$ are affine barycentric coordinates. This refinement is compatible with the already defined refinement procedure for the faces. See Figure 3.2 (a) for $n = 1$.

**Step 2.** We perform semi-uniform refinement for prisms in our decomposition $T'_n$ (all these prisms will have an edge in common with $\Omega$). This procedure is shown in Figure 3.2 (b).

**Step 3.** We perform non-uniform refinement for the tetrahedra of type $VSS^3$ and $VESS$. More precisely, we divide a tetrahedron of type $VSS^3$ into 12 tetrahedra as...
in the uniform strategy, with the edges through the vertex of type V divided in the ratio \( \kappa \). We thus obtain one tetrahedron of type VS\(^3\) and 11 tetrahedra of type S\(^4\). (At the next step, which yields \( T'_{n+2} \) we iterate this procedure for the small tetrahedron of type VS\(^3\), while the tetrahedra of type S\(^4\) are divided uniformly.) See Figure 3.2 (a). On the other hand, a tetrahedron of type VESS will be divided it into 6 tetrahedra of type S\(^4\), one tetrahedron of type VS\(^3\), and a triangular prism. The vertex of type E of will belong only to the prism. See Figure 3.2 (b). This refinement is compatible with the earlier refinement of the faces.

![Diagram of tetrahedron refinements](image)

(a) Vertex A of type V, B, C, D of type S  
(b) Vertex A of type V, B of type E, C, D of type S and \( D_1D'=\) mark for the prism \( BD_1C_1D'C_1B' \)

**Figure 3.5.** Refinement of tetrahedra of type VS\(^3\) and VESS.

The description of our refinement procedure is now complete.

### 3.3. Intrinsic local refinement

We see that one of the main features of our refinement is that each edge, each triangle, and each quadrilateral that appears in a tetrahedron or prism in the decomposition \( T'_n \) is divided in the decomposition \( T'_{n+1} \) in an intrinsic way that depends only on the type of the vertices of that edge, triangle, or quadrilateral. In particular, the way that a face in \( T'_n \) is divided to yield \( T'_{n+1} \) does not depend on the type of the other vertices of the tetrahedron or prism to which it belongs. This ensures that tetrahedralization \( T_{n+1} \), which is obtained from \( T'_{n+1} \) by dividing each prism in three tetrahedra, is a conforming mesh.

### 4. Hardy-Poincaré inequality and regularity: a glimpse at the proofs

There are two main ingredients for the proofs of the well-posedness results stated in the first section. One is the Hardy-Poincaré inequality, which yields solvability (more precisely well-posedness) in the \( H^1 \)-type spaces and the second one is a regularity result, which allows us then to obtain well-posedness in higher regularity spaces. A third, more technical ingredient, is to describe the trace spaces at the boundary. For this, we use the same ideas as the ones used in the proof of regularity. We now discuss these ingredients.
4.1. The Hardy-Poincaré inequality. Let us denote by \( r_\Omega(x) \) the distance from \( x \) to the set of singular points in the boundary of \( \Omega \). Recall that these singular points consist not just of the edge points, but also of the points where the boundary conditions change and the points where the interface touches the boundary. The following inequality is then proved by induction [13] (see [14] for the three dimensional case, the two dimensional case was well known, see [49] for example).

**Proposition 4.1.** Let \( \Omega \) be a polyhedral domain in \( \mathbb{R}^n \). We assume that either \( \Omega \) is bounded, or that it is a cone or a dihedral angle. Let us assume that the Neumann part of the boundary \( \partial_N \Omega := \partial \Omega \setminus \partial_D \Omega \) contains no adjacent faces of \( \Omega \).

Then there exists a constant \( C_\Omega > 0 \), which depends only on \( \Omega \) and the choice of boundary conditions such that the following Hardy-Poincaré inequality holds:

\[
\int_\Omega \frac{|u|^2}{r_\Omega^2} \, dx \leq C_\Omega \int_\Omega |\nabla u|^2 \, dx
\]

for any function \( u \in H^1(\Omega) \) that is zero on \( \partial_D \Omega \).

Let us assume that \( \Omega \) is bounded. A simple consequence of the Hardy-Poincaré inequality of Proposition 4.1 is that the spaces

\[
H^1_D(\Omega) = \{ u \in H^1(\Omega), u = 0 \text{ on } \partial_D \Omega \}
\]

and

\[
K^1_1(\Omega) \cap \{ u \in H^1_{loc}(\Omega), u = 0 \text{ on } \partial_D \Omega \}
\]

are the same and their respective norms are equivalent. Neither this result nor the Hardy-Poincaré inequality are true if there exist two adjacent faces with Neumann boundary conditions. This is the reason we needed a different approach in Section 1.

4.2. Sobolev spaces and regularity. Our definition of weighted Sobolev spaces, Equation (3), is elementary. However, for the purpose of establishing the needed properties of these spaces, it is convenient to identify them with the usual Sobolev spaces associated to a different metric on \( \Omega \).

To this end, let us recall from [13] that a stratified curvilinear polyhedral domain \( \Omega \) is an open subset of a Riemannian manifold \((M, g)\) of dimension \( d \) together with a stratification of

\[
\Omega = \Omega^{(d)} \supset \Omega^{(d-1)} \supset \ldots \supset \Omega^{(1)} \supset \Omega^{(0)}.
\]

We then define stratified curvilinear polyhedral domains by induction as follows. For \( d = 0 \), \( \Omega \) is just a finite set of points. For \( d = 1 \), \( \Omega \) is a finite set of intervals. The stratum \( S_0 \) for \( d = 1 \) will contain all the boundary points of the intervals, but may contain also other points. For \( d > 1 \), we require our domain \( \Omega \) to satisfy the following conditions: for every point \( p \in \partial \Omega \), there exist a neighborhood \( V_p \subset M \) such that, if \( p \in \Omega^{(l)} \setminus \Omega^{(l-1)} \), \( l = 1, \ldots, d-1 \), then there is a stratified curvilinear polyhedral domain \( \omega_p \subset S^{d-l-1} \), \( \omega_p \neq S^{d-l-1} \), and a diffeomorphism \( \phi_p : V_p \rightarrow B^{d-l} \times B^l \) such that \( \phi_p(p) = 0 \) and

\[
\phi_p(\Omega \cap V_p) = \{ r x', 0 < r < 1, x' \in \omega_p \} \times B^l,
\]

inducing a homeomorphism \( \overline{\Omega} \cap V_p \rightarrow \{ r x', 0 \leq r < 1, x' \in \overline{\omega_p} \} \times B^l \) of stratified spaces that is a diffeomorphism on each stratum.
The set of singular points of $\Omega$ then consists of $\Omega^{(n-2)}$ and is given as part of the definition of $\Omega$, but it must contain all the geometric, intrinsic singular points of $\partial \Omega$. Although we shall not need this definition here, let us mention nevertheless that the desingularization of $\Omega$, denoted $\Sigma(\Omega)$, is obtained by gluing in a natural way all the sets $[0, 1) \times \mathbb{S}^p \times B^1$ as in Equation (17). The resulting set $\Sigma(\Omega)$ is then a manifold with corners that has a natural structure of a Lie manifold with boundary, in the sense of [3]. Then $\Sigma(\Omega) \to \Omega$ is a differentiable map that is a diffeomorphism outside the set of singular points, in $\Sigma(\Omega)$ the set of singular points being the set of points belonging to a face of codimension at least two.

Let $\tilde{r}_0(x) \geq 0$ be the distance from $x$ to the set $\Omega^{(0)}$ if $x$. In general, the function $\tilde{r}_0$ will not be smooth, we therefore replace $\tilde{r}_0$ with an equivalent function $r_0$ that is smooth outside $\Omega^{(0)}$. Therefore, we also have that $r_0(x) > 0$, for $x \notin \Omega^{(0)}$, and that $\tilde{r}_0/r_0$ and $r_0/\tilde{r}_0$ are bounded functions. We shall say that $r_0$ is the smoothed distance to $\Omega^{(0)}$. We replace then the metric $g =: g_0$ with $g_1 := r_0^{-2}g$. We repeat this construction for the remaining non-empty strata in the increasing order of the dimension of the strata, each time measuring distances in the new metric. Thus $r_k$ is the smoothed distance to $\Omega^{(k)}$ in the metric $g_k$, and we let $g_{k+1} := r_k^{-2}g_k$, $k \leq d - 2$. One can prove that $g_{d-1}$ is a compatible metric on the desingularization $\Sigma(\Omega)$ [4, 13] and hence we can use the results on Sobolev spaces from those papers. Let $\rho := r_0r_1 \ldots r_{d-2}$. Let us denote by $\Gamma(\overline{\Omega}, TM)$ the space of restrictions to $\overline{\Omega}$ of smooth vector fields on $M$. The resulting structural Lie algebra of vector fields on $\Sigma(\Omega)$ is simply $V = C^\infty(\Sigma(\Omega))\rho \Gamma(\overline{\Omega}, TM)$. Thus a basis of $V$ over $C^\infty(\Sigma(\Omega))$ is given by $\{\rho \partial_i\}$. The resulting Sobolev spaces are therefore

$$K^m(\Omega) := \{ u, \rho^{|\alpha|-a} \partial^a u \in L^2(\Omega), |\alpha| \leq m \} = \rho^{a-n/2}H^m(\Omega, g_{d-1}),$$

where the space $H^m(\Omega, h)$ is the Sobolev space associated to the metric $h$. Let $r_0(x)$ denote the distance from $x$ to $\Omega^{(d-2)}$. One can prove by induction that $r_0/\rho$ and $\rho/r_0$ are both bounded, so in the above definition of Sobolev spaces we can replace $\rho$ with $r_0$. See [13] for details.

The fact that the Sobolev spaces $K^m(\Omega)$ are associated to a Lie manifold guarantees that Laplacian $\Delta$ satisfies elliptic regularity in the scale of spaces $K^m(\Omega)$. To this end, one also needs to establish that $\rho^2\Delta - \Delta_{g_{d-1}}$ is a lower order differential operator generated by $V$ and $C^\infty(\Sigma(\Omega))$. We also obtain as a byproduct the fact that the traces at the boundary of the spaces $K^m(\Omega)$ can also be described in terms of the Sobolev spaces on $\partial \Omega$ associate to the conformally equivalent metric $h$.

The Hardy-Poincaré inequality can also be interpreted in the setting of the desingularized metric. Indeed, we have that there exists $C > 0$ such that every point of $x$ is at a distance $\leq C$ to the Dirichlet part of the boundary of $\Sigma(\Omega)$ if, and only if, there exist no two adjacent faces with Neumann boundary conditions. Then, once we know that every point is at a distance $\leq C$ to the Dirichlet boundary, we can prove the Hardy-Poincaré inequality in the usual way.

References


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