COMPLEX ANALYSIS
LECTURE NOTES

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Contents

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1. Informal presentation of the content of the class

Real analysis consists in studying *continuity, differentiability, integrals, limits/series* of functions from $I \subseteq \mathbb{R}$ (or $\mathbb{R}^d$; in $d = 1$, usually an interval) to $\mathbb{R}$ (or $\mathbb{R}^d$).

complex analysis: we'll study similar questions for functions $f : U \to \mathbb{C}$ from $U \subseteq \mathbb{C}$ (usually an open set) to $\mathbb{C}$.

1.1. **Derivative.**

Definition 1.1. *$f : U \to \mathbb{C}$ is said to be complex-differentiable at z_0 if*
$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$
exists.

(Identical to the real definition.)
(Stronger than the notion of differentiability of a function $f : \mathbb{R}^2 \to \mathbb{R}^2$; more detail later.)

Complex-differentiable functions have very different properties from real differentiable functions, as we will shortly illustrate here.
Consider two classes of functions

Holomorphic functions	Analytic functions
$f : U \to \mathbb{C}$ s.t. f complex-diff. at each point $z_0 \in U$ (real analogue: bigger class than the class of $\mathcal{C}^1$ functions.)	$f : U \to \mathbb{C}$ s.t. around each point z_0 in U there exists a_n such that $f(z) = \sum_n a_n(z - z_0)^n.$ (real-analogue: smaller class than the class of $\mathcal{C}_\infty$ functions.)

Theorem 1.2. *holomorphic=analytic*

1.2. **Integrals.** Informally, a real integral $\int_a^b f(t)dt$ is the "sum" of the values of a function between a and b (think of Riemann sums).
But in the complex world, what does between a and b mean? Along the segment $[a; b]$? This is one option, but we can do the integral along any *path* from a to b.
(Volume integrals also exist, e.g., in Stoke's theorem, but this is not what were doing here.)
$\to$ we will define path integrals $\int_\gamma f(t)dt$, for any path γ from a to b.

Theorem 1.3. *If $f : U \to \mathbb{C}$ is holomorphic (=analytic) and U a open set "without holes", then $\int_\gamma f(t)dt$ only depends of the endpoints of the path γ.*

$\to$ to compute a complex integral, we can choose a "good" path. Even useful to compute real integrals!

Part A. **Preliminaries**

2. Complex numbers

(The content of this section should be well-known; here to fix notation and show how to do everything formally.)

Definition 2.1. $\mathbb{C} = \{(x, y); x, y \in \mathbb{R}\}$ *with addition*
$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$
and multiplication
$$(x_1, y_1) \cdot (x_2, y_2) = (x_1 \cdot x_2 - y_1 \cdot y_2, x_1 \cdot y_2 + x_2 \cdot y_1).$$

Lemma 2.2. $\mathbb{C}$ *is a field.*

Notation $i := (0, 1)$. Then $(x, y) := x + iy$. We use the injection $\mathbb{R} \to \mathbb{C}$ implicitly $x \mapsto (x, 0)$.

Lemma 2.3. $i^2 = -1$.

Definition 2.4. *Let $z = x + iy$ be a complex number, with $x, y \in \mathbb{R}$, then*
- $\mathrm{Re}(z) := x$
- $\mathrm{Im}(z) := y$
- $|z| = \sqrt{x^2 + y^2}$

- $\overline{z} = x - iy$

Proposition 2.5. *Let $z, w \in \mathbb{C}$. Then*

- $\overline{z+w} = \overline{z} + \overline{w}$;
- $\overline{-z} = -\overline{z}$;
- $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$;
- $\overline{z^{-1}} = (\overline{z})^{-1}$ *(if $z \neq 0$)*;
- $z + \overline{z} = 2\,\mathrm{Re}(z) \quad z - \overline{z} = 2i\,\mathrm{Im}(z)$;
- $z = \overline{z} \Leftrightarrow z \in \mathbb{R}$;
- $z^{-1} = \frac{\overline{z}}{|z|^2}$ *(if $z \neq 0$)*;
- $|z \cdot w| = |z| \cdot |w|$.

Proposition 2.6 (Triangular inequality)**.** *Let $z, w \in \mathbb{C}$*

$$|z + w| \leq |z| + |w|.$$

Proposition 2.7 (Polar form)**.** *Let $z \in \mathbb{C}$. Then there exists $r > 0$ and $\varphi \in \mathbb{R}$ such that $z \neq 0$ and*

$$z = r(\cos\varphi + i\,\sin\varphi). \tag{1}$$

(This proposition will be proved later, when we define properly exp, cos and sin).

Furthermore,

- r is uniquely determined by (1). In fact $r = |z|$ (modulus of z).
- φ is NOT uniquely determined by (1). If (1) holds, φ is called *an* argument ($arg(z) = \varphi$) of z.
- If φ_1 and φ_2 are arguments of z then $\varphi_1 - \varphi_2 \in 2\pi\mathbb{Z}$. As a consequence, any non-zero complex number has a unique argument $\varphi_0 \in]-\pi, \pi]$. Then φ_0 is called *the principal value* of the argument, and denoted by $\mathrm{Arg}(z) = \varphi_0$

Proposition 2.8. *Let $z, w \in \mathbb{C} \setminus \{0\}$. If φ_z and φ_w are arguments of z and w then $\varphi_z + \varphi_w$ is an argument of $z \cdot w$.*

Warning! Even if φ_z and φ_w are principal values of the argument z and w , $\varphi_z + \varphi_w$ is not necessarily the principal value of the argument of $z \cdot w$.
It is customary to represent complex numbers as points in the complex plane.

(Drawing)

3. Topology on $\mathbb{C}$

Observe that if we set $d(z, w) := |z - w|$ for $z, w \in \mathbb{C}$, this defines a distance on $\mathbb{C}$. Indeed, we have $d(z, w) \in \mathbb{R}_+$, the separation property $d(z, w) = 0 \Leftrightarrow z = w$ and the triangular inequality: for $z_1, z_2, z_3 \in \mathbb{C}$,

$$d(z_1, z_3) \leq d(z_1, z_2) + d(z_2, z_3).$$

Hence $\mathbb{C}$ is a metric space. We automatically have the notions of convergent sequences, continuous functions, open/closed sets,...

Notation for balls/disks

$$D(z_0, r) = \{z \in \mathbb{C},\ |z - z_0| < r\}$$
$$\overline{D(z_0, r)} = \{z \in \mathbb{C},\ |z - z_0| \leq r\}$$
$$\partial D(z_0, r) = \{z \in \mathbb{C},\ |z - z_0| = r\}.$$

3.1. Convergent sequences.

Definition 3.1. (convergent) *A sequence $(z_n)_{n\geq 1}$ of complex numbers converges towards $z \in \mathbb{C}$ if for any $\epsilon > 0$ there exists n_0 such that*

$$n \geq n_0 \Rightarrow |z_n - z| \leq \epsilon.$$

Lemma 3.2. *$z_n \xrightarrow{n\to\infty} z$ if and only if*

$$\mathrm{Re}(z_n) \xrightarrow{n\to\infty} \mathrm{Re}(z) \quad \textit{and} \quad \mathrm{Im}(z_n) \xrightarrow{n\to\infty} \mathrm{Im}(z).$$

(As we shall see, this property enables to deduce many topological properties of $\mathbb{C}$ from that of $\mathbb{R}$.)

Proof. Assume $z_n \xrightarrow{n\to\infty} z$. Fix $\epsilon > 0$. Then there exists n_0 such that $n \geq n_0 \Rightarrow |z_n - z| \leq \epsilon$. But $|\operatorname{Re}(z_n) - \operatorname{Re}(z)| = |\operatorname{Re}(z_n - z)| \leq |z_n - z|$ so that for $n \geq n_0 \Rightarrow |\operatorname{Re}(z_n) - \operatorname{Re}(z)| \leq \epsilon$. Similarly for $\operatorname{Im}(z_n)$.

Conversely, assume $\operatorname{Re}(z_n) \longrightarrow \operatorname{Re}(z)$ and $\operatorname{Im}(z_n) \longrightarrow \operatorname{Im}(z)$. Let $\epsilon > 0$ then there exists n_0 such that

$$n \geq n_0 \Rightarrow |\operatorname{Re}(z_n) - \operatorname{Re}(z)| \leq \epsilon \text{ and } |\operatorname{Im}(z_n) - \operatorname{Im}(z)| \leq \epsilon.$$

Then

$$|z_n - z| \leq |\operatorname{Re}(z_n - z)| + |\operatorname{Im}(z_n - z)| = |\operatorname{Re}(z_n) - \operatorname{Re}(z)| + |\operatorname{Im}(z_n) - \operatorname{Im}(z)| \leq 2\epsilon.$$

Therefore, z_n converges to z. ☐

Lemma 3.3. (computational rules) *If $z_n \xrightarrow{n\to\infty} z,\ w_n \xrightarrow{n\to\infty} w$ then*

i) $z_n + z_w \xrightarrow{n\to\infty} z + w$;
ii) $z_n \cdot w_n \xrightarrow{n\to\infty} z \cdot w$;
iii) $\frac{1}{z_n} \xrightarrow{n\to\infty} \frac{1}{z}$ *(if $z_n,\ z \neq 0$).*

Warning! $z_n \xrightarrow{n\to\infty} z$ does NOT imply $\operatorname{Arg}(z_n) \xrightarrow{n\to\infty} \operatorname{Arg}(z)$, where Arg denotes the principal value of the argument.

Definition 3.4. (Cauchy criterion) *The sequence $(z_n)_{n\geq 1} \in \mathbb{C}$ is called a Cauchy-sequence if for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $n, m \geq n_0 \Rightarrow |z_n - z_m| \leq \epsilon$.*

Theorem 3.5. *A sequence $(z_n)_{n\geq 1} \in \mathbb{C}$ is convergent (there exists $z \in \mathbb{C}$ such that $z_n \xrightarrow{n\to\infty} z$) if and only if it is Cauchy. In other terms, $\mathbb{C}$ is a complete space.*

Proof. (Assuming the statement for real sequences) Assume $z_n \xrightarrow{n\to\infty} z$. Let $\epsilon > 0$. Then there exists $n_0 \in \mathbb{N}$ such that $n \geq n_0 \Rightarrow |z_n - z| \leq \epsilon$. $n, m \geq n_0 \Rightarrow |z_n - z_m| \leq |z_n - z| + |z_m - z| \leq 2\epsilon$. Thus (z_n) is a Cauchy sequence.
Assume that (z_n) is a Cauchy sequence

Claim: $(\operatorname{Re}(z_n))_{n\geq 1}$ and $(\operatorname{Im}(z_n))_{n\geq 1}$ are Cauchy sequences.

Proof of the claim: Let $\epsilon > 0$. Then there exists n_0 such that $n, m \geq n_0 \Rightarrow |z_n - z_m| \leq \epsilon$. But

$$n, m \geq n_0 \Rightarrow |\operatorname{Re}(z_n) - \operatorname{Re}(z_m)| = |\operatorname{Re}(z_n - z_m)| \leq |z_n - z_m| \leq \epsilon.$$

Thus $\operatorname{Re}(z_n)$ is a Cauchy sequence. Same proof for $\operatorname{Im}(z_n)$.
But $\operatorname{Re}(z_n)$ and $\operatorname{Im}(z_n)$ are real sequences. Therefore $\operatorname{Re}(z_n)$ and $\operatorname{Im}(z_n)$ are convergent sequences and (Analysis I) there exists x and y such that $\operatorname{Re}(z_n) \longrightarrow x$ and $\operatorname{Im}(z_n) \longrightarrow y$. This implies that $z_n \longrightarrow x + iy$ by Lemma 3.2. ☐

3.2. Open, closed, compact and connected sets.

Definition 3.6. *A subset U of $\mathbb{C}$ is an open set if for any $z_0 \in U$ there exists $r > 0$ such that*

$$|z - z_0| < r \Rightarrow z \in U.$$

A subset F of $\mathbb{C}$ is called a closed set if $\mathbb{C} \setminus F$ is an open set.

Lemma 3.7. *F is closed if and only if for every sequence $(z_n)_{n\geq 1}$ and $z \in \mathbb{C}$ such that $z_n \in F$ and $z_n \xrightarrow{n\to\infty} z$ then $z \in F$.*

Definition 3.8. (compact set) *A subset $K \subseteq \mathbb{C}$ is compact if one of the following equivalent statements hold:*

i) from every covering of K by open sets one can extract a finite covering. That is $K \subset \bigcup_{i\in I} U_i$ with U_i open, then there exists $J \subseteq I$, J finite such that $K \subseteq \bigcup_{i\in J} U_i$.

ii) from any sequence $(z_n)_{n\geq 1} \in K$ *one can extract a convergent subsequence. That is there exists* $(n_k)_{k\geq 1}$ *increasing and* $z \in K$ *such that* $z_{n_k} \xrightarrow{k\to\infty} z$.

iii) K *is closed and bounded.*

Note that bounded means that there exists $M > 0$ such that $z \in K \Rightarrow |z| \leq M$.
The equivalence between $i)$ and $ii)$ holds in all metric spaces. $ii) \Rightarrow iii)$ also but $iii) \Rightarrow ii)$ is some really non-trivial property of $\mathbb{C}$.

Proof. $iii) \Rightarrow ii)$ (assuming the real analogue statement). Let K be a closed and bounded set $K \subseteq \mathbb{C}$. Let $(z_n)_{n\geq 1}$ be a sequence in K. Then there exists M such that for all $n \geq 1, |z_n| \leq M$. But

$$|Re(z_n)| \leq |z_n| \leq M \Rightarrow (\mathrm{Re}(z_n))_{n\geq 1}$$

is bounded (analogue for real sequences) and has a convergent subsequence. I.e. there exists $(n_k)_{k\geq 1}$ increasing and $x \in \mathbb{R}$ such that $\mathrm{Re}(z_{n_k}) \longrightarrow x$ but $|\operatorname{Im}(z_{n_k})| \leq |z_{n_k}| \leq |M| \Rightarrow (\operatorname{Im}(z_{n_k}))_{k\geq 1}$ is bounded and it has a convergent subsequence. I.e. there exists $(k_j)_{j\geq 1}$ increasing and y such that

$$\operatorname{Im}(z_{n_{k_j}}) \xrightarrow{j\to\infty} y. \quad (*)$$

But $(z_{n_{k_j}})$ is a subsequence of $(z_{n_k})_{k\geq 1}$ thus

$$\mathrm{Re}(z_{n_{k_j}}) \xrightarrow{j\to\infty} x. \quad (**)$$

From $(*)$ and $(**)$ and from Lemma 3.2, it follows that $z_{n_{k_j}} \to x+iy$. Thus $(z_{n_{k_j}})$ is a convergent subsequence of (z_n) and its limit $z = x + iy$ is in K because $z_{n_{k_j}} \in K$ and K is closed. ☐

Fix a subset $M \subseteq \mathbb{C}$.

Definition 3.9. *A subset* U *of* M *is open in (open relatively to)* M *if there is an open set* $U_0 \subseteq \mathbb{C}$ *such that* $U = U_0 \cap M$ *(similar definition for closed sets in* M*).*

Definition 3.10. M *is connected if the only subsets of* M *that are at the same time open* <u>*and*</u> *closed in* M *are* $\emptyset$ *and* M *itself. A* **domain** $U \subset \mathbb{C}$ *is an open connected subset of* $\mathbb{C}$.

(This definition of connectedness is very formal.)

Lemma 3.11. $[0, 1]$ *is connected.*

Proof. Take a subset $A \subset [0, 1]$ and assume that A is non-empty, open and closed in $[0, 1]$. The goal is to prove that necessarily, $A = [0, 1]$. By assumption, there exists some a_0 in A. Set $b = \sup\{x \geq a_0 : [a_0, x] \subset A\}$. Since A is closed in $[0, 1]$, and hence close, b is in A. We now use that A is open in $[0, 1]$ to prove $b = 1$. If $b < 1$, there exists $\varepsilon > 0$ s.t. $[b, b + \varepsilon] \subset A$, so that $[a_0, b + \varepsilon] \subset A$, contradicting the maximality of b. We conclude that necessarily $b = 1$, i.e. $[a_0, 1] \subset A$.

A similar proof shows $[0, a_0] \subset A$, so that, necessarily, $A = [0, 1]$. ☐

Lemma 3.12. *The image of a connected set by a continuous function is connected.*

Proof. Let $f : U \to V$ be a continuous function on a connected set U. We can assume w.l.o.g. that f is surjective (otherwise, replace V by $f(V)$). We want to prove that V is connected.

Let $A \subseteq V$ be closed and open. Then $f^{-1}(A)$ is closed and open. Since U connected, this implies $f^{-1}(A) = \emptyset$ or $f^{-1}(A) = U$. Using the surjectivity of f, we conclude that $A = \emptyset$ or $A = V$. ☐

Definition 3.13. (path connected) *A set* $U \subseteq \mathbb{C}$ *is path-connected if for any* $a, b \in U$*, there exists* $\gamma : [0; 1] \to U$ *continuous with* $\gamma(0) = a$ *and* $\gamma(1) = b$.

Proposition 3.14. *Path connectedness implies connectedness.*

Proof. Let $A \subset U$ be a non-empty open and closed subset of U. We want to prove that, necessarily, $A = U$. By assumption A contains some element a_0.

For any b in U, there is a path in $\gamma : [0; 1] \to U$ from a_0 to b. The intersection $A \cap \mathfrak{Im}(\gamma)$ is an open and closed subset of $\mathfrak{Im}(\gamma)$. But, combining the two previous lemmas, we know that $\mathfrak{Im}(\gamma)$ is connected. We conclude that $A \cap \mathfrak{Im}(\gamma) = \mathfrak{Im}(\gamma)$, i.e. $\mathfrak{Im}(\gamma) \subseteq A$. In particular, b is in A.

Since this holds for an arbitrary b in U, we have $A = U$ as wanted. ☐

Example of a non connected set: $A = \{|\Re(z)| \geq 1\}$.

Intuitively, **connected** $\leftrightarrow$ **"in one piece"**.

3.3. Continuous functions.

Definition 3.15. *Let $f : U \to \mathbb{C}$ with $U \subseteq \mathbb{C}$.*

i) f is continuous in $z_0 \in U$ if for any $\epsilon > 0$ there exists $\eta > 0$ such that

$$|z - z_0| \leq \eta \Rightarrow |f(z) - f(z_0)| \leq \epsilon.$$

or equivalently if for any sequence $z_n \in U$ we have

$$z_n \longrightarrow z_0 \Rightarrow f(z_n) \longrightarrow f(z_0).$$

ii) f is continuous on U if f is continuous in each point $z_0 \in U$, or, equivalently, if for any open set $V \subseteq \mathbb{C}$, $f^{-1}(V)$ is open in U.

Proposition 3.16. *Let $f : U \to \mathbb{C}$. Define*

$$\mathrm{Re}(f) : U \to \mathbb{R} \quad z \mapsto \mathrm{Re}(f(z))$$

$$\mathrm{Im}(f) : U \to \mathbb{R} \quad z \mapsto \mathrm{Im}(f(z)).$$

Then f is continuous if and only if $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are continuous.

Proof. Immediate from the sequence characterisation. □

Warning! $f : U \to \mathbb{C}$ continuous implies that for each $x_0, y_0 \in \mathbb{R}$ the functions

$$f_{x_0} := \{y : (x_0, y) \in U\} \to \mathbb{C} \quad y \mapsto f(x_0, y)$$

$$f^{y_0} := \{x : (x, y_0) \in U\} \to \mathbb{C} \quad x \mapsto f(x, y_0)$$

are continuous.

The converse is not true. Being continuous in two variables is not the same as being continuous in each variable.

Example: $z \to \mathrm{Re}(z), z \to \mathrm{Im}(z)$ are continuous but $z \to Arg(z)$ is not.

4. More on convergence

4.1. Convergent and absolutely convergent series.

Definition 4.1. *Take a sequence $(a_n)_{n\geq 1}$ of complex numbers. The sequence of partial sums associated to (a_n) is*

$$s_N := \sum_{n=1}^{N} a_n.$$

We say that the series $\sum_{n=1}^{\infty} a_n$ converges/diverges if $(s_N)_{N\geq 1}$ converges/diverges.

Terminology: diverges = does not converge.

Proposition 4.2. (Cauchy criterion of series) *$\sum_{n=1}^{\infty} a_n$ converges if and only if for every $\epsilon > 0$ there exists N_0 such that for all*

$$N, M \geq N_0 \Rightarrow \left| \sum_{n=N+1}^{M} a_n \right| \leq \epsilon.$$

Proof. Use the Cauchy criterion of $(s_N)_{N\geq 1}$ and observe that if $M > N$ then $s_M - s_N = \sum_{n=N+1}^{M}(a_n)$. □

Lemma 4.3. *$\sum_{n=1}^{\infty} a_n$ converges if and only if*

$$\sum_{n=1}^{\infty} \mathrm{Re}(a_n) \quad \textit{and} \quad \sum_{n=1}^{\infty} \mathrm{Im}(a_n)$$

converge.

Proof. Because s_N converges $\Leftrightarrow$ $\mathrm{Re}(s_N)$ and $\mathrm{Im}(s_N)$ converge. (Lemma 2.2.) □

Corollary 4.4. *If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.*

Proof. Use the Cauchy criterion with $M = N + 1$. □

Definition 4.5. (absolutely convergent) *A series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ is convergent.*

Proposition 4.6. *If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.*

Proof. Use the Cauchy criterion. Let $\epsilon > 0$ then there exists N_0 such that $M > N \geq N_0$ implies $\sum_{n=N+1}^{M} |a_n| \leq \epsilon$. which is exactly the Cauchy criterion for $\sum_{n=1}^{\infty} |a_n|$. But $|\sum_{n=N+1}^{M} a_n| \leq \sum_{n=N+1}^{M} |a_n|$ so that for $M > N \geq N_0$ implies $|\sum_{n=N+1}^{M} a_n| \leq \epsilon$. Considering the Cauchy criterion for $\sum_{n=1}^{\infty} a_n$ we conclude that $\sum_{n=1}^{\infty} a_n$ is convergent. □

Note that the converse is not true. For example take $a_{2n-1} = \frac{1}{n}$ and $a_{2n} = -\frac{1}{n}$ then $s_{2N} = 0$ and $s_{2N-1} = s_{2N-2} + a_{2N-1} = \frac{1}{n}$ thus $s_N \to 0$ and $\sum_{n=1}^{\infty} a_n$ is convergent. But $\sum_{n=1}^{2N} |a_n| = \sum_{n=1}^{N} \frac{2}{n}$ diverges.

Proposition 4.7. (comparison criterion) *Let $(b_n)_{n\geq 1}$ and $(c_n)_{n\geq 1}$ be sequences of non-negative real numbers. Assume $n \geq n_0 \implies b_n \leq c_n$ for some n_0. If $\sum_{n=1}^{\infty} c_n$ converges, then $\sum_{n=1}^{\infty} b_n$ converges.*

Proof. Use the Cauchy criterion for $\sum_{n=1}^{\infty} c_n$. Fix $\epsilon > 0$. Then there exists N_0 such that $M > N \geq N_0$ implies $\sum_{n=N+1}^{M} c_n \leq \epsilon$ but $\sum_{n=N+1}^{M} b_n \leq \sum_{n=N+1}^{M} c_n$ if $N \geq n_0$ so that $M > N \geq max(n_0, N_0)$ implies $\sum_{n=N+1}^{M} b_n \leq \epsilon$. The Cauchy criterion for b_n implies that $\sum_{n=1}^{\infty} b_n$ converges. □

Proposition 4.8. *Let $q > 0$ be a positive real number, then $\sum_{n=0}^{\infty} q^n$ converges if and only if $q < 1$. Let $\alpha \in \mathbb{R}$ then $\sum_{n=0}^{\infty} \frac{1}{n^\alpha}$ converges if and only if $\alpha > 1$.*

Proof. classical. □

Corollary 4.9. (quotient criterion) *Let $(b_n)_{n\geq 1}$ be a sequence of positive real numbers.*

i) Assume there exists $q < 1$ such that $\frac{b_{n+1}}{b_n} \to q$, then $\sum_{n=0}^{\infty} b_n$ converges.

ii) Assume there exists $q > 1$ such that $\frac{b_{n+1}}{b_n} \to q$, then $\sum_{n=0}^{\infty} b_n$ diverges.

4.2. **Cauchy product of absolutely convergent numerical series.** Let $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ be sequences of complex numbers. Define their Cauchy product $(c_n)_{n\geq 0}$ of complex numbers

$$c_n := \sum_{k=0}^{n} a_k b_{n-k}.$$

Proposition 4.10. *Assume* $\sum\limits_{n=0}^{\infty} a_n$ *and* $\sum\limits_{n=0}^{\infty} b_n$ *are both absolutely convergent, then* $\sum\limits_{n=0}^{\infty} c_n$ *is also absolutely convergent and*

$$\sum_{n=0}^{\infty} c_n = \left(\sum_{n=0}^{\infty} a_n\right) \cdot \left(\sum_{n=0}^{\infty} b_n\right).$$

Proof. First assume that $a_n, b_n \geq 0$ (are non-negative) and then consider $\sum_{n=0}^{N} c_n = \sum_{n=0}^{N}(\sum_{k=0}^{n} a_k b_{n-k})$

Claim:

$$\underbrace{\left(\sum_{n=0}^{N/2} a_n\right) \cdot \left(\sum_{n=0}^{N/2} b_n\right)}_{\sum\limits_{k=0}^{N/2}\sum\limits_{j=0}^{N/2} a_k b_j} \leq \underbrace{\left(\sum_{n=0}^{N} c_n\right)}_{\sum\limits_{\substack{k,j\geq 0\\ k+j\leq N}} a_k b_j} \leq \underbrace{\left(\sum_{n=0}^{N} a_n\right) \cdot \left(\sum_{n=0}^{N} b_n\right)}_{\sum\limits_{k=0}^{N}\sum\limits_{j=0}^{N} a_k b_j}$$

But $\left\{k, j \text{ such that } \substack{0\leq k\leq N/2\\ 0\leq j\leq N/2}\right\} \subseteq \left\{k, j \text{ such that } \substack{k,j\geq 0\\ k+j\leq N}\right\} \subseteq \left\{k, j \text{ such that } \substack{k,j\geq 0\\ k\leq N\\ j\leq N}\right\}$ this proves the claim.

$$\left(\sum_{n=0}^{N/2} a_n\right) \cdot \left(\sum_{n=0}^{N/2} b_n\right) \xrightarrow{N\to\infty} \left(\sum_{n=0}^{\infty} a_n\right) \cdot \left(\sum_{n=0}^{\infty} b_n\right)$$

$$\left(\sum_{n=0}^{N} a_n\right) \cdot \left(\sum_{n=0}^{N} b_n\right) \xrightarrow{N\to\infty} \left(\sum_{n=0}^{\infty} a_n\right) \cdot \left(\sum_{n=0}^{\infty} b_n\right)$$

thus

$$\sum_{n=0}^{N} c_n \xrightarrow{N\to\infty} \left(\sum_{n=0}^{\infty} a_n\right) \cdot \left(\sum_{n=0}^{\infty} b_n\right)$$

by sandwich rule.
General case:

$$\left(\sum_{n=0}^{N} a_n\right) \cdot \left(\sum_{n=0}^{N} b_n\right) - \sum_{n=0}^{N} c_n = \sum_{*} a_j b_k$$

where the summation index $*$ is $\left\{j, k \text{ such that } \substack{0\leq j\leq N\\ 0\leq k\leq N\\ j+k>N}\right\}$. So

$$\left|\left(\sum_{n=0}^{N} a_n\right) \cdot \left(\sum_{n=0}^{N} b_n\right) - \sum_{n=0}^{N} c_n\right| \leq \sum_{*} |a_k||b_j|$$

$$\sum_{*} |a_k||b_j| = \left(\sum_{n=0}^{N} |a_n|\right)\left(\sum_{n=0}^{N} |b_n|\right) - \sum_{n=0}^{N} \gamma_n$$

where $\gamma_n = \sum\limits_{k=0}^{n} |a_k||b_k|$. From the non-negative case

$$\left(\sum_{n=0}^{N} |a_n|\right) \cdot \left(\sum_{n=0}^{N} |b_n|\right) - \sum_{n=0}^{N} \gamma_n \xrightarrow{N\to\infty} 0$$

so that

$$\left(\sum_{n=0}^{N} a_n\right) \cdot \left(\sum_{n=0}^{N} b_n\right) - \sum_{n=0}^{N} c_n \xrightarrow{N\to\infty} 0$$

we get

$$\sum_{n=0}^{\infty} c_n = \left(\sum_{n=0}^{\infty} a_n\right) \cdot \left(\sum_{n=0}^{\infty} b_n\right).$$

□

4.3. **Sequences of functions.** In what follows, $U \subseteq \mathbb{C}$ is open, $(f_n)_{n\geq 1}$ is a sequence of functions and $f : U \to \mathbb{C}$ a function.

Definition 4.11. (pointwise convergence) *$f_n \xrightarrow{n\to\infty} f$ pointwise if for all $z \in U$, $f_n(z) \xrightarrow{n\to\infty} f(z)$ or equivalently, if, for every $\epsilon > 0$ there exists $n_0(z)$ such that*

$$n \geq n_0(z) \Rightarrow |f_n(z) - f(z)| < \epsilon.$$

Definition 4.12. (uniform convergence) *If $f_n \xrightarrow{n\to\infty} f$ uniformly if for every $\epsilon > 0$ there exists n_0 such that*

$$n \geq n_0 \Rightarrow \sup_{z\in U} |f_n(z) - f(z)| \leq \epsilon.$$

Lemma 4.13. *If $f_n \xrightarrow{n\to\infty} f$ uniformly then $f_n \xrightarrow{n\to\infty} f$ pointwise.*

Definition 4.14. (Cauchy criterion for uniform convergence) *A sequence $(f_n)_{n\geq 1} : U \to \mathbb{C}$ is Cauchy if for every $\epsilon > 0$ there exists n_0 such that for all*

$$n, m \geq n_0 \Rightarrow \sup_{z\in U} |f_n(z) - f_m(z)| \leq \epsilon.$$

Proposition 4.15. *A sequence (f_n) of functions is uniformly convergent if and only if it is Cauchy.*

Proof. " $\Rightarrow$ " Assume that $f_n \xrightarrow{n\to\infty} f$ uniformly. Let $\epsilon > 0$ then there exists n_0 such that
$\sup_{z\in U} |f_n(z) - f_m(z)| \leq \epsilon$. For $m, n \geq n_0$,

$$\sup_{z\in U} |f_n(z) - f_m(x)| \leq \sup_{z\in U}(|f_n(z) - f(z)| + |f_m(z) - f(z)|) \leq \sup_{z\in U} |f_n(z) - f(z)| + \sup_{z\in U} |f_m(z) - f(z)| \leq 2\epsilon.$$

" $\Leftarrow$ " Assume (f_n) is Cauchy. Fix z_0 in U. Then $(f_n(z_0))$ is Cauchy which implies that $f_n(z_0)$ converges. Call $f(z_0)$ the limit we just defined a function $f : U \to \mathbb{C}$. We know $f_n \xrightarrow{n\to\infty} f$ pointwise. Does it converge uniformly? Let $\epsilon > 0$. Then there exists n_0 such that $n, m \geq n_0$ implies $\sup_{z\in U} |f_n(z) - f_m(z)| \leq \epsilon$ and by definition f_n is Cauchy. Fix $z_0 \in U$ then $|f_n(z_0) - f(z_0)| = \lim_{m\to\infty} |f_n(z_0) - f_m(z_0)| \leq \epsilon$ which is true for all $z_0 \in U$ so that $\sup_{z\in U} |f_n(z) - f(z)| \leq \epsilon$. □

Proposition 4.16. *Let $f_n \xrightarrow{n\to\infty} f$ uniformly and assume that f_n is continuous for all $n \geq 1$. Then f is continuous.*

Proof. Let $z_0 \in U$. Let $\epsilon > 0$. By uniform convergence there exists n_0 such that $n \geq n_0$ implies $\sup_{z\in U} |f_n(z) - f(z)| \leq \epsilon$. But f_{n_0} is continuous in z_0 so that there exists $\eta > 0$ such that
$|z - z_0| \leq \eta \Rightarrow |f_{n_0}(z) - f_{n_0}(z_0)| \leq \epsilon$. Then

$$|z - z_0| \leq \eta \Rightarrow |f(z) - f(z_0)| \leq |f(z) - f_{n_0}(z)| + |f_{n_0}(z) - f_{n_0}(z_0)| + |f_{n_0}(z_0) - f(z_0)| \leq 3\epsilon.$$

□

Note that this is not true for pointwise convergence. For example take $U = \{z, |z| < 1\}$ and $f_n(z) = (1 - |z|)^n$ and let $f(z) = 0$ if $|z| < 1$ and $f(z) = 1$ if $z = 1$. Then f_n is continuous but f is not continuous and $f_n \xrightarrow{n\to\infty} f$ pointwise.

Definition 4.17 (locally uniform convergence)**.** *f_n converges locally uniformly to f if one of the following equivalent conditions holds:*

i) for every $z_0 \in U$ there exists $r > 0$ such that f_n converges uniformly to f on $\{z : |z - z_0| \leq r\}$;
ii) for every compact $K \subset U$, the sequence f_n converges uniformly to f on K.

Lemma 4.18.

(1) Uniform convergence $\Rightarrow$ locally uniform convergence $\Rightarrow$ pointwise convergence.
(2) $f_n \xrightarrow{n\to\infty} f$ locally uniformly and f_n continuous implies f is continuous.

Proof. i) follows directly from the definition.
ii) Let $z_0 \in U$ and $f_n \longrightarrow f$ uniformly on a disk around z_0. This implies that f is continuous on this disk and hence in z_0. But this is true for all $z_0 \in U$ which implies that f is continuous. □

4.4. **Series of functions.**

Definition 4.19. *The series $\sum\limits_{n=1}^{\infty} f_n$ is uniformly/pointwise convergent if the sequence of partial sums $s_N = \sum\limits_{n=1}^{N} f_n$ is uniformly/pointwise convergent.*

We can define normal conv for series of functions as the straight-forward analogue of absolutely conv numerical series, replacing the absolute value by the supremum norm. We'll however be more interested in the following local analogue (beware of terminology!).

Definition 4.20. *The series $\sum\limits_{n=1}^{\infty} f_n$ is (locally) normally convergent if for every $z_0 \in U$ there exists $r > 0$ such that*

$$\sum_{n=1}^{\infty} \|f_n\|_{D(z_0,r)}$$

is convergent.

Remark: $\|f_n\|_M = \sup\limits_{z\in M} |f_n(z)|$

Notation: $f_n\,|_M =$ "f_n restricted to M ".

Proposition 4.21. *If $\sum\limits_{n=0}^{\infty} f_n$ is locally normally convergent, then there exists f such that $\sum\limits_{n=0}^{\infty} f_n \xrightarrow{n\to\infty} f$ locally uniformly.*

Proof. Fix $z_0 \in U$. Then there exists $r > 0$ such that if $D = \{z : |z - z_0| < r\}$, then $\sum_{n=0}^{\infty} \|f_n\|_D < \infty$. We will use the Cauchy criterion for $\sum_{n=0}^{\infty}\|f_n\|_D < \infty :$ for every ϵ there exists N_0 such that $M, N > N_0$ implies $\sum_{n=N+1}^{M}\|f_n\|_D < \epsilon$. But $\left\|\sum_{n=N+1}^{M} f_n\right\|_D \leq \sum_{n=N+1}^{M}\|f_n\|_D$ so that $N, M \geq N_0$ implies $\left\|\sum_{n=N+1}^{M} f_n\right\|_D \leq \epsilon$. By Cauchy criterion for sequences of function the restriction of f_n on D denoted as $\sum_{n=0}^{\infty} f_n\,|_D$ is uniformly convergent. In particular $((f_n(z_0))_{n\geq 1}$ has a limit that we call $f(z_0)$. In other words, f_n converges pointwise to f. We want to show local uniform convergence. For $z_0 \in U$ we have $r > 0$ such that f_n is uniformly convergent on the disk $\{z : |z - z_0| < r\}$ (which was proved already). But of course $f_n|_D$ converges pointwise to $f|_D$ and thus $f_n|_D$ converges uniformly to $f|_D$. □

Part B. Power series and analytic functions

5. Power series

Definition 5.1. (power series) *We call power series a series of functions $\sum\limits_{n=0}^{\infty} f_n$ with $f_n : \mathbb{C} \to \mathbb{C}$ where $z \mapsto a_n z^n$ for some $a_n \in \mathbb{C}$. Shortly a power series is $\sum\limits_{n=0}^{\infty} a_n z^n$.*

5.1. **Radius of convergence.**

Proposition 5.2. (radius of convergence) *Let $\sum\limits_{n=0}^{\infty} a_n z^n$ be a power series. Define*

$$\rho := \sup\left\{r \in \mathbb{R}_+ \text{ such that } (a_n r^n)_{n\geq 0} \text{ is bounded}\right\}$$

i) The series $\sum_{n=0}^{\infty} a_n z^n$ *converges (locally) normally on* $\{z : |z| < \rho\}$.

ii) Fix $z \in \mathbb{C}$ *with* $|z| > \rho$. *The numerical series* $\sum_{n=0}^{\infty} a_n z^n$ *does not converge.*

Remarks:

- the limit $f(z) = \sum_{n\geq 0} a_n z^n$ is a continuous function on the disk $\{z : |z| < \rho\}$.
- for $r = 0, (a_n r^n)$ is trivially bounded which implies that ρ is well defined and $\rho \geq 0$. But $\rho = 0$ or $\rho = \infty$ are possible.
- We dont know what happens for $|z| = \rho$. i.e $\sum_{n=0}^{\infty} a_n z^n$ may converge or not.

Proof. $ii)$ - Let $|r| = z > \rho$. Then the sequence $(a_n r^n)_{n\geq 1}$ is not bounded i.e. $(|a_n|r^n = |a_n z^n|)_{n\geq 1}$ is not bounded. So $a_n z^n$ does not tend to 0. Thus $\sum_{n=0}^{\infty} a_n z^n$ does not converge.
$i)$ - We will prove a slightly stronger statement: If $r < \rho$, then

$$\sum_{n=0}^{\infty} \|a_n z^n\|_{\{z:|z|\leq r\}} < \infty. \qquad (*)$$

Why is $(*)$ stronger than $ii)$?. Each compact subset of $D(0,\rho)$ is contained in $\overline{D(0,r)}$ for some $r < \rho$.

Proof of *: For $|z| \leq r,\ |a_n z^n| \leq |a_n| r^n$ thus $\|a_n z^n\|_{\{z:|z|\leq r\}} \leq |a_n| r^n$. But $r < \rho$ thus there exists $r_0 > r$ such that $(a_n r_0^n)_{n\geq 0}$ is bounded (otherwise $\rho \leq r$ by definition). There exists $M > 0$ such that $|a_n| r_0^n \leq M$ for all $n \geq 0$. But $|a_n| r^n = |a_n| r_0^n \left(\frac{r}{r_0}\right)^n$ so that $|a_n| r^n \leq M \left(\frac{r}{r_0}\right)^n$. Recall that $r < r_0$ (i.e. $\frac{r}{r_0} < 1$) implies $\sum_{n=0}^{\infty} M \left(\frac{r}{r_0}\right)^n < \infty$. Putting everything together $\sum_{n=0}^{\infty} \|a_n z^n\|_{\{z:|z|\leq r\}}$ converges. □

Corollary 5.3. *ρ can be alternatively defined as*

$$\rho = \sup\left\{ r \geq 0 : \sum_{n=0}^{\infty} a_n r^n \ \textit{converges} \right\}$$

or

$$\rho = \sup\left\{ r \geq 0 : \sum_{n=0}^{\infty} a_n r^n \ \textit{converges absolutely} \right\}.$$

Proof. We have seen that for $r > \rho,\ \sum a_n r^n$ does not converge (hence does not converge absolutely). For $r < \rho,\ \sum a_n r^n$ converges absolutely (hence converges). □

Proposition 5.4. (quotient criterion) *Let (a_n) be a sequence of non-zero complex numbers. Assume that* $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = l$ *(l may be 0 or ∞). Then the radius of convergence of* $\sum_{n=0}^{\infty} a_n z^n$ *(corresponding to the power series) is* $\frac{1}{l}$.

Convention: $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$.

Proof.

- Take $r < \frac{1}{l}$. Then $\lim_{n\to\infty} \frac{|a_{n+1} r^{n+1}|}{|a_n r^n|} = r \cdot l < 1$. This implies (from the quotient criterion we have seen for numerical series) that $\sum_{n=0}^{\infty} a_n r^n$ converges.
- Take $r > \frac{1}{l}$ then $\lim_{n\to\infty} \frac{|a_{n+1} r^{n+1}|}{|a_n r^n|} = r \cdot l > 1$. This implies that $\sum_{n\geq 0} a_n r^n$ does not converge.

□

Example 1: Let $\kappa > 0$. Take $a_n = \kappa^n$ then $l = \frac{1}{\kappa}$. Thus the power series $\sum_{n=0}^{\infty} \kappa^n z^n$ converges for $|z| < \kappa$ and the limit is

$$\lim_{N\to\infty} \sum_{n=0}^{N} \kappa^n z^n = \lim_{N\to\infty} \frac{1 - \kappa^{N+1} z^{N+1}}{1 - \kappa z} = \frac{1}{1 - \kappa z}.$$

Example 2: $a_n = \frac{1}{n!}$. Then $l = 0$. Thus $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges for any $z \in \mathbb{C}$. The limit is by definition $\exp(z)$.

5.2. **Sum and product of power series.** Let $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ be two sequences of complex numbers. Define

$$s_n = a_n + b_n \qquad p_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

Proposition 5.5. *Call ρ_a and ρ_b the radiuses of convergence of $\sum\limits_{n=0}^{\infty} a_n z^n$ and $\sum\limits_{n=0}^{\infty} b_n z^n$. Let $\rho = min(\rho_a, \rho_b)$. Then $\sum\limits_{n=0}^{\infty} s_n z^n$ and $\sum\limits_{n=0}^{\infty} p_n z^n$ have radius of convergence at least ρ and for $|z| < \rho$ the sum*

$$\sum_{n=0}^{\infty} s_n z^n = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=0}^{\infty} b_n z^n$$

and the product

$$\sum_{n=0}^{\infty} p_n z^n = \left(\sum_{n=0}^{\infty} a_n z^n\right) \cdot \left(\sum_{n=0}^{\infty} b_n z^n\right).$$

Proof. Start with the sum.
Let $|z| < \rho$ then $\sum_{n=0}^{N} s_n z^n = \sum_{n=0}^{N} a_n z^n + \sum_{n=0}^{N} b_n z^n$ (partial sum). But then

$$\sum_{n=0}^{N} a_n z^n \xrightarrow{N\to\infty} \sum_{n=0}^{\infty} a_n z^n$$

$$\sum_{n=0}^{N} b_n z^n \xrightarrow{N\to\infty} \sum_{n=0}^{\infty} b_n z^n.$$

So that $\sum_{n=0}^{\infty} s_n z^n$ converges and

$$\sum_{n=0}^{\infty} s_n z^n = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=0}^{\infty} b_n z^n.$$

Thus the radius of convergence of $\sum_{n=0}^{\infty} s_n z^n$ is at least ρ.

Case of products.
Let $|z| < \rho$ then $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$ converges absolutely. Therefore their Cauchy product $(\pi_n)_{n\geq 0}$ converges, where

$$\pi_n = \sum_{k=0}^{n} \left(a_k z^k\right)\left(b_{n-k} z^{n-k}\right) = p_n z^n.$$

Thus $\sum_{n=0}^{\infty} \pi_n = \sum_{n=0}^{\infty} p_n z^n$ converges and

$$\sum_{n=0}^{\infty} p_n z^n = \left(\sum_{n=0}^{\infty} a_n z^n\right) \cdot \left(\sum_{n=0}^{\infty} b_n z^n\right).$$

This implies also that the radius of convergence of $\sum_{n=0}^{\infty} p_n z^n$ is at least ρ. □

6. Analytic functions

6.1. Definitions.

Definition 6.1. (analytic function) *Let U be an open subset of $\mathbb{C}$. A function $f : U \to \mathbb{C}$ is called analytic (on U) if for every $z_0 \in U$ there exists $r > 0$ and a sequence $(a_n)_{n\geq 0}$ of complex numbers such that*

- *i) the radius of convergence of $\sum\limits_{n=0}^{\infty} a_n z^n$ is at least r.*
- *ii) $|z - z_0| < r$ implies that $z \in U$ and $f(z) = \sum\limits_{n=0}^{\infty} a_n (z - z_0)^n$.*

Example 1:

Claim: Let p be a polynomial. $\mathbb{C} \to \mathbb{C}$ $z \mapsto p(z)$ is an analytic function.

Proof. Fix $z_0 \in \mathbb{C}$. You may know that $((z-z_0)^k)_{k\geq 0}$ is a basis of the vector space $\mathbb{C}[z]$ (because there is a polynomial of each degree). There exist complex numbers $(a_n)_{n\geq 1} \in \mathbb{C}$ (depending on z_0) with $a_n = 0$ for n bigger than some d such that p is a finite linear combination $p(z) = \sum_{n=0}^{d} a_n(z-z_0)^n$. This equality of polynomials is true for all $z \in \mathbb{C}$. Thus $p(z)$ is analytic in z_0. One takes $r = \infty$ and $(a_n)_{n\geq 0}$ as above. As this is true for all $z_0 \in \mathbb{C}$, $p(z)$ is analytic. □

Example 2: Let $U = \mathbb{C} \setminus \{1\}$ and define $f : U \to \mathbb{C}$ $z \mapsto \frac{1}{1-z}$.
Claim: f is an analytic function.

Before proving the claim we start by a Lemma.

Lemma 6.2. *For* $|z| < 1$, $\frac{1}{1-z} = \sum\limits_{n=0}^{\infty} z^n$.

Proof. Take the partial sum $\sum_{n=0}^{N} z^n = \frac{1-z^{N+1}}{1-z}$ which is the sum of the geometric series. As $|z| < 1, \lim_{N\to\infty} z^{N+1} = 0$. Thus $\sum_{n=0}^{\infty} z^n$ converges and $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$. □

Proof of the claim. As for $|z| < 1$, $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$, f is analytic in 0. (Take $r = 1, a_n = 1$ for all $n \geq 0$). We have to prove that f is analytic in every $z_0 \in U$. Fix $z_0 \in U$. Write

$$f(z) = \frac{1}{1-z} = \frac{1}{1-z_0-(z-z_0)} = \frac{1}{1-z_0} \cdot \frac{1}{1-\frac{z-z_0}{1-z_0}}.$$

If $|z-z_0| < |1-z_0|$ by Lemma 6.2,

$$f(z) = \frac{1}{1-z_0} \sum_{n=0}^{\infty} \left(\frac{z-z_0}{1-z_0}\right)^n = \sum_{n=0}^{\infty} \left(\frac{1}{1-z_0}\right)^{n+1} (z-z_0)^n.$$

Take $r = |1-z_0|$ and $a_n = \left(\frac{1}{z-z_0}\right)^{n+1}$ This proves analyticity of f in z_0. Since x_0 is generic in U, this proves that f is analytic on U. □

(Note that the coefficients a_n in Definition 6.1 may depend on z_0.)
(To prove that something is analytic, we must prove the existence of an expansion in each z_0 in U, not only in $z_0 = 0$. In particular, it is not clear at this point whether sums of power series are analytic.)

6.2. Algebra of analytic functions in U.

Proposition 6.3. *Let $f, g : U \to \mathbb{C}$ be analytic functions (U open), take $\lambda \in \mathbb{C}$ then $\lambda f, f+g, f\cdot g$ are analytic functions.*

Note that the sum and product of functions is pointwise. I.e. $(f+g)(z) = f(z)+g(z)$ and
$(f \cdot g)(z) = f(z) \cdot g(z)$.

Proof. Fix $z_0 \in U$. Then there exists $r, r' > 0$ and some sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ such that $|z-z_0| < r$ implies $z \in U$ and $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$. $|z-z_0| < r'$ implies $z \in U$ and $g(z) = \sum_{n=0}^{\infty} b_n(z-z_0)^n$.

- But $\lambda \cdot f(z) = \sum_{n=0}^{\infty} \lambda \cdot a_n(z-z_0)^n$ for $|z-z_0| < r$ so that $\lambda \cdot f$ is analytic in z_0.
- Take z with $|z-z_0| < \min(r, r')$ then by Proposition 5.5 $f(z)+g(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + b_n(z-z_0)^n$ i.e. $f(z)+g(z) = \sum_{n=0}^{\infty}(a_n+b_n)(z-z_0)^n$ so that $f+g$ is analytic in z_0.
- Take z with $|z-z_0| < \min(r, r')$. Then $f(z) = \sum p_n(z-z_0)^n$ with $p_n = \sum_{k=0}^{\infty} a_k b_{n-k}$ by Proposition 5.5 so that $f \cdot g$ is analytic in z_0. □

Note that this is true for $z_0 \in U$, so that $\lambda \cdot f, f+g, f \cdot g$ are analytic on U.

Corollary 6.4. *The set of analytic functions from U to $\mathbb{C}$ is a subalgebra of the algebra of functions from U to $\mathbb{C}$. (non-empty because $f(z) = 0$ for z in U is analytic).*

Remark: By complicated sum manipulation, it is possible to prove:

- that the composition of two analytic functions, when defined, is analytic.

- that the sum of a power series with positive radius of convergence $\rho > 0$ is analytic on the open disk $\{z : |z| < \rho\}$

For that we will see simpler proofs later (using the equiv. with holomorphic functions mentioned in the intro).

Terminology convention: Let $z_0 \in \mathbb{C}$. We say that a property (p) holds in a neighbourhood of z_0 if there exists $r > 0$ such that $|z - z_0| < r$ implies that (p) holds for z.

6.3. Isolated zeros and analytic continuation.

Proposition 6.5. *Let $f(z) = \sum_{n\geq 0} a_n z^n$ with positive radius of convergence ρ. If one of the a_n is non-zero, then $f(z) \neq 0$ for $z \neq 0$ in a neighbourhood of 0. In other terms, there exists $r > 0$ such that $|z| < r$ and $z \neq 0$ implies $f(z) \neq 0$.*

Proof. Let p be the smallest integer such that $a_p \neq 0$. Then define

$$g(z) = \sum_{m=0}^{\infty} a_{m+p} z^m.$$

For $r > 0$ and $n = m + p$

$$a_n r^n = a_{m+p} r^{m+p} = r^p (a_{m+p} r^m)$$

so that $(a_n r^n)_{n\geq 0}$ is bounded if and only if $(a_{m+p} r^m)$ is bounded. Then g has radius of convergence ρ. For $|z| < \rho$ change the index of the sum such that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=p}^{\infty} a_n z^n$$

as $a_n = 0$ for $n < p$. Now change the sum index $m := n - p$ then

$$f(z) = \sum_{m=0}^{\infty} a_{m+p} z^{m+p} = z^p \left(\sum_{m=0}^{\infty} a_{m+p} z^m \right) = z^p g(z).$$

But g is a continuous function on $\{z : |z| < \rho\}$ and $g(0) = a_p \neq 0$ which implies that there exists a neighbourhood of 0 on which g does not vanish. But $f(z) = z^p g(z)$ so that f does not vanish either on this neighbourhood except possibly in 0 if $p > 0$. ◻

(Warning: $f(z) = \sum a_n z^n$ can vanish at the origin (in fact $f(0) = a_0$), but not in a neighbourhood of the origin.)

Corollary 6.6. *Let $f : U \to \mathbb{C}$ be analytic and U open. Fix $z_0 \in U$. Then the family of coefficients $(a_n)_{n\geq 0}$ of the power series expansion $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ of f in z_0 is uniquely determined.*

(Existence of this expansion is just the definition of analyticity, here we want to prove uniqueness.)

Proof. Assume that f has two expansions around $z_0 \in U$.

- There exists $r > 0$ and $(a_n)_{n\geq 0}$ such that $|z - z_0| < r$ implies that $z \in U$ and $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$.
- There exists $r' > 0$ and $(b_n)_{n\geq 0}$ such that $|z - z_0| < r'$ implies that $z \in U$ and $f(z) = \sum_{n=}^{\infty} b_n (z - z_0)^n$.

For $|z - z_0| = \min(r, r')$ then $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n = \sum_{n=0}^{\infty} b_n (z - z_0)^n$ thus $\sum_{n=0}^{\infty} (a_n - b_n)(z - z_0)^n = 0$ for $|z - z_0| < \min(r, r')$. From Proposition 6.5. we have that $(a_n - b_n) = 0$ for all $n \geq 0$. So $a_n = b_n$, which proves the corollary. ◻

(Warning: we recall that a_n may depend on z_0. The radius r is also not unique; in fact, if some r fits in Definition 6.1, each $r' < r$ also fits.)

Definition 6.7. (limit point) *Let S be a subset of $\mathbb{C}$. Then some $a \in \mathbb{C}$ is called a limit point of S if there exists a sequence $(s_n)_{n\geq 0}$ with $s_n \in S$, $s_n \neq a$ and $\lim_{n\to\infty} s_n = a$.*

Note that a may or may not belong to S.

Example: 0 is a limit point of $\left\{\frac{1}{n},\ n \in \mathbb{N}_*\right\}$.
Notation: $f \equiv g$ on U means that $f(z) = g(z)$ for *all* $z \in U$.

Theorem 6.8. (isolated zeros) *Let $f : U \to \mathbb{C}$ be an analytic function on a* **domain** *U. Assume that $f(s) = 0$ for every s in a set S which has a limit point in U. Then $f \equiv 0$.*

Before we proceed with the proof of Theorem 6.8. I want to stress that the way it is proven is done by a standard procedure. Define a set $B = \{b \in U : f(z) = 0 \text{ on a neighbourhood of } b\} \subseteq U$.

The outline of the proof will look like this:

i) prove that $B \neq \emptyset$.
ii) prove that B is open in U.
iii) prove that B is closed in U.

If we prove i),ii) and iii) from the definition of connectedness we have that $B = U$.

Proof. i) - By hypothesis there exists $a \in U$ and a sequence $(s_n)_{n\geq 0}$ in S with $\lim_{n\to\infty} s_n = a$ by Definition 6.7. f is analytic in a, so that f is the sum of a power series $\sum_{n=0}^{\infty} a_n(z-a)^n$ on a neighbourhood of a for all n. Assume that some a_n is non-zero. Proposition 6.5 implies $f(z) \neq 0$ for $z \neq a$ on the neighbourhood $\{z : |z-a| < r\}$ of a. But $f(s_n) = 0$ so that $|s_n - a| \geq r$ (recall that $s_n \neq a$). This is a contradiction with $\lim_{n\to\infty} s_n = a$. Thus $a_n = 0$ for all $n \geq 0$ and $f(z) = 0$ on a neighbourhood of a. i.e. $a \in B$.

ii) - Let $b \in B$. We want to prove that there exists $\epsilon > 0$ such that $|z-b| < \epsilon$ implies that $z \in B$. Since $b \in B$ there exists $r > 0$ such that $|w-b| < r$ implies $f(w) = 0$ by definition of B. Take z such that $|z-b| < \frac{r}{2}$. For w with $|w-z| < \frac{r}{2}$ we have $|w-b| \leq |z-b| + |w-z| \leq r \Rightarrow f(w) = 0$. Thus $z \in B$.

(Small picture to illustrate w,b, r and the various disks)

iii) - We want to prove that, if $b_n \in B$ and there exists $b \in U$ such that $\lim_{n\to\infty} b_n = b$, then $b \in B$. But $b_n \in B$ implies that $f(b_n) = 0$. With the same proof as in i) we get that $b \in B$.

In conclusion B is a non-empty open and closed subset of U. Since U is connected $B = U$. □

Rephrasement of Theorem 6.8: If f is a non-zero analytic function $f : U \to \mathbb{C}$ (U connected) then its set of zeros has no limit points in U.

Remark: The same is not true for real C^∞ functions. For instance, let

$$f(x) = \begin{cases} 0 & \text{for} \quad x \leq 0; \\ e^{-\frac{1}{x}} & \text{for} \quad x > 0, \end{cases}$$

then $f : \mathbb{R} \to \mathbb{R}$ is $C^\infty(\mathbb{R})$ (this is not obvious in zero, but it is indeed C^∞. Its set of zero is $\mathbb{R}_- = \{x : x \leq 0\}$. Each point in $\mathbb{R}_-$ is a limit point, but the function is not identically zero.

Corollary 6.9. (identity theorem) *Let $f, g : U \to \mathbb{C}$ be analytic functions. Assume U to be a connected open set. If $f \equiv g$ on a set S with a limit point in U, then $f \equiv g$ on U.*

Proof. Apply Theorem 6.8 to $h = f - g$. □

Definition 6.10. (analytic continuation) *Let $U \subseteq V$ be open subsets of $\mathbb{C}$. Consider an analytic function $f : U \to \mathbb{C}$. An analytic continuation of f to V is an analytic function $g : V \to \mathbb{C}$ such $g\,|_U = f$.*

Corollary 6.11. *U non-empty. If V is connected, then, if an analytic continuation of f to V exists, it is unique.*

Proof. Take g_1, g_2 two analytic continuations $g_1|_U = g_2|_U = f$. But $U \neq \emptyset$ there exists $a \in U$ and there exists $r > 0$ such that $|z-a| \leq r$ implies $z \in U$. This implies that a is a limit point of U. Using Corollary 6.9. we get $g_1 = g_2$ on V. □

Note that there is not always an analytic continuation from U to V.

(**Warning:** the existence of analytic continuations to V_1 and V_2 does not imply the existence of an analytic continuation of $V_1 \cup V_2$!)

6.4. **Exponential and π.**

Definition 6.12. *For $z \in \mathbb{C}$, we set*

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Remarks:

- $\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \frac{1}{n+1} \xrightarrow{n\to\infty} 0$, the power series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ has radius of convergence $\rho = \infty$ (quotient criterion), so that $\exp(z)$ is well defined and defines a continuous function on $\mathbb{C}$.
- For real x, you know that $\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ (Taylor formula) so that this "new" definition is coherent.

Lemma 6.13. $a, b \in \mathbb{C}$. *Then* $\exp(a+b) = \exp(a) \cdot \exp(b)$.

Reminder: **Newton binomial formula**

$$(a+b)^n = \sum_{k=0}^{\infty} \binom{n}{k} a^k b^{n-k}$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Proof. $\exp(a)\cdot\exp(b) = \left(\sum_{n=0}^{\infty} \frac{a^n}{n!}\right)\cdot\left(\sum_{n=0}^{\infty} \frac{b^n}{n!}\right)$. Both series converge absolutely (because $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges normally on $\mathbb{C}$) and we can use Theorem on Cauchy products of absolutely convergent series.

$$\left(\sum_{n=0}^{\infty} a_n\right) \cdot \left(\sum_{n=0}^{\infty} b_n\right) = \sum_{n=0}^{\infty} c_n$$

with $c_n = \sum_{k=0}^{n} a_k b_{n-k}$. Set $a_n = \frac{a^n}{n!}$ and $b_n = \frac{b^n}{n!}$ we get

$$\left(\sum_{n=0}^{\infty} \frac{a^n}{n!}\right) \cdot \left(\sum_{n=0}^{\infty} \frac{b^n}{n!}\right) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{a^k}{k!} \frac{b^{n-k}}{(n-k)!}\right)$$

but

$$\sum_{k=0}^{n} \frac{a^k b^{n-k}}{k!(n-k)!} = \frac{1}{n!} \sum_{k=0}^{n} \frac{n! a^k b^{n-k}}{k!(n-k)!} = \frac{(a+b)^n}{n!}.$$

Finally

$$\left(\sum_{n=0}^{\infty} \frac{a^n}{n!}\right) \cdot \left(\sum_{n=0}^{\infty} \frac{b^n}{n!}\right) = \sum_{n=0}^{\infty} \frac{(a+b)^n}{n!}$$

hence $\exp(a+b) = \exp(a) \cdot \exp(b)$. ☐

A few simple properties:

- $\exp(0) = 1, \exp(-z) \cdot \exp(z) = 1$ thus for all $z \in \mathbb{C}$ $\exp(z) \neq 0$ and $\exp(-z) = \frac{1}{\exp(z)}$. Since all coefficients in Definition 6.12 are real, x real $\Rightarrow \exp(x)$ real. $z \in \mathbb{C} \Rightarrow \exp(\overline{z}) = \overline{\exp(z)}$.
- Since all coefficients in Definition 6.12. are positive, the function $[0, +\infty[\to \mathbb{R}$ $x \mapsto \exp(x)$ is strictly increasing. Moreover $\exp(x) \geq x$ so that $\lim_{x\to\infty} \exp(x) = \infty$. Recalling that $\exp(-x) = \frac{1}{\exp(x)}$, we also have that $\lim_{x\to-\infty} \exp(x) = 0$ and that $x \mapsto \exp(+x)$ is increasing on $]-\infty, 0]$.
- If t is a real number $\exp(-it) = \frac{1}{\exp(it)}$ but also $\exp(-it) = \exp(\overline{it}) = \overline{\exp(it)}$ i.e. $\exp(it) \cdot \overline{\exp(it)} = 1 = |\exp(it)|^2$. In conclusion: if t is real then $|\exp(it)| = 1$. As a consequence $z = x + iy,\ x, y \in \mathbb{R}$ then $|\exp(z)| = |\exp(x)| \cdot |\exp(iy)| = \exp(x)$.

Notation: Define $e := \exp(1)$. Then for integers $p, q \geq 1$,

$$\exp(p) = \exp(1)^p = e^p.$$

Moreover $\exp\left(\frac{p}{q}\right)^q = \exp(p) = e^p$. But $\exp\left(\frac{p}{q}\right)$ is a positive real number so that $\exp\left(\frac{p}{q}\right) = e^{\frac{p}{q}}$. In conclusion: $\exp(x) = e^x$ for rational numbers x.

Convention: Denote $e^z = \exp(z)$ for all complex numbers.

Definition 6.14. *For $z \in \mathbb{C}$, we set*

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}; \qquad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}.$$

This is coherent with what you know for real numbers. Observe

- if x is a **real number**, $\cos(x) = \mathrm{Re}(e^{ix})$ and $\sin(x) = \mathrm{Im}(e^{ix})$.
- $e^z = e^x e^{iy} = e^x(\cos(y) + i\sin(y))$ with $z = x + iy$ $x, y \in \mathbb{R}$ but e^x is a positive real number and y is a real number, so that y is an argument of e^z, as defined in Proposition 1.7.

(Figure to show how lines parallel to the axis are transformed by the exponential map.)

Lemma 6.15. *There exists a real number $0 < c < 2$ such that $e^{ic} = i$.*

Definition 6.16. *$\pi = 2 \cdot c.$, where c is the minimal value satisfying Lemma 6.15.*

Reminder: Let $(a_n)_{n\geq 0}$ be a non-increasing sequence of non-negative real numbers with $\lim\limits_{n\to\infty} a_n = 0$. Then $\sum\limits_{n=0}^{\infty}(-1)^n a_n$ converges (not necessarily absolutely) and for all $N \geq 0$

$$\sum_{n=0}^{2N+1}(-1)^n a_n \leq \sum_{n=0}^{\infty}(-1)^n a_n \leq \sum_{n=0}^{2N}(-1)^n a_n.$$

Proof. By definition, $\cos(2) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{(2n)!}$. The sequence $\left(\frac{2^{2n}}{(2n)!}\right)_{n\geq 1}$ is non-increasing (consider quotients of successive entries) so that, using the above reminder,

$$\cos(2) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{(2n)!} \leq 1 - \frac{2^2}{2!} + \frac{2^4}{4!} = -\frac{1}{3} < 0.$$

But $\cos(0) = 1$ and $t \mapsto \cos(t)$ is a continuous function on $[0, 2]$. The intermediate value Theorem implies that there exists c such that $\cos(c) = 0$.

Recall that $\exp(ic)$ has modulus 1. Its real part $\cos(c)$ being equal to 0, we must have $\sin(c) = +1$ or $\sin(c) = -1$. But $\sin(c) = \sum_{n=0}^{\infty} \frac{(-1)^n c^{2n+1}}{(2n+1)!}$. For $c \in (0, 2)$ one can check that $\left(\frac{c^{2n+1}}{(2n+1)!}\right)_{n\geq 0}$ is non-increasing, so that, using the above reminder $\sin(c) \geq c - \frac{c^3}{3!}$. For $c \in (0, 2)$, we trivially have that $c - \frac{c^3}{3!} = c(1 - \frac{c^2}{6}) > 0$. Finally $\sin(c) = 1$ and $e^{ic} = i$. □

Observe that we can choose c to be minimal such that $\cos(c) = 0$ so that for $0 < t < c$, $\cos(t) > 0$. Moreover for $t < c$, $\sin(t) \geq t - \frac{t^3}{6} > 0$.

(We will see later, when discussing path and their length, that this π is indeed the semi-length of a circle of radius 1, as we usually define it in elementary school!)

We finally discuss image sets and fibers of the exponential function.

Proposition 6.17.

i) If $w \in \mathbb{C}$ and $|w| = 1$. Then there exists $t \in \mathbb{R}$ such that $e^{it} = w$.
ii) If $z \in \mathbb{C}, z \neq 0$ then there exists $z' \in \mathbb{C}$ such that $e^{z'} = z$.

Proof. $i)$ Let $w \in \mathbb{C}$ with $|w| = 1$. First assume that $w = u + iv$ with $u, v \geq 0$. Cosine is a continuous function with $\cos(0) = 1$ and $\cos(c) = 0$. But $u = \mathrm{Re}(w) \leq |w| = 1$. So by the intermediate value theorem, there exists $0 \leq t \leq c$ such that $\cos(t) = u$. But $\cos^2(t) + \sin^2(t) = 1 = |w|^2 = u^2 + v^2$ which implies that $\sin^2(t) = v^2$. So $\sin(t) = v$ or $\sin(t) = -v$. But we proved that $\sin(t) \geq 0$ and assumed $v \geq 0$. Finally $\sin(t) = v$ and $e^{it} = w$.
Let's look at the case where $u < 0, v \geq 0$ then $-iw = -i(u + iv) = v - iu$ has modulus 1 and non-negative real and imaginary parts. The first case implies that there exists $0 < t < c$ such that $-iw = e^{it}$ and $w = ie^{it} = e^{i(t+\pi/2)}$.
Lets look at the case $v < 0$. Then $-w = -u - iv$ has modulus 1 and a non-negative imaginary part but then the first case implies that there exists t such that $-w = e^{it} \Rightarrow w = e^{i(t+\pi)}$.

ii) Let $z \neq 0$. As exp:$\mathbb{R} \to \mathbb{R}$ is increasing and has $\lim_{x\to\infty} e^x = \infty$ and $\lim_{x\to-\infty} e^x = 0$, there exists x such that $e^x = |z|$. Then apply *i)* to $\frac{z}{|z|}$, there exists y such that $\frac{z}{|z|} = e^{iy}$ which implies that
$z = |z| \cdot e^{iy} = e^x e^{iy} = e^{x+iy}$. □

Proposition 6.18. *Let z and z' be complex numbers, with $e^z = e^{z'}$ then $\frac{z-z'}{2\pi i}$ is an integer.*

Proof. Observe that $\frac{e^z}{e^{z'}} = e^{z-z'}$ thus $e^z = e^{z'}$ implies that $e^{z-z'} = 1$. We can assume w.l.o.g. that $z' = 0$ i.e. $e^z = 1$. Write $z = x + iy$ with $x, y \in \mathbb{R}$. Then $|e^z| = e^x = 1$. The only real number x with $e^x = 1$ is $x = 0$ since $x \mapsto e^x$ is strictly increasing.
Consider now y. There exists a unique integer k such that $2\pi k \leq y < 2\pi(k+1)$ (k is an integer part of $y/2\pi$). Recall that $e^{iy} = 1$. But $e^{2\pi i k} = \left(e^{2\pi i}\right)^k = 1$ so that $e^{i(y-2\pi k)} = 1$. Setting $y' = y - 2\pi k$, we have $e^{iy'} = 1$ with $0 \leq y' < 2\pi$. We write $e^{\frac{iy'}{4}}$ as $u + iv$ with $u, v \in \mathbb{R}$: since $\frac{y'}{4} < \frac{\pi}{2}$, we have $u = \cos\left(\frac{y'}{4}\right) > 0$ and $v = \sin\left(\frac{y'}{4}\right) \geq 0$. Then

$$1 = e^{iy'} = \left(e^{\frac{iy'}{4}}\right)^4 = (u+iv)^4 = u^4 + v^4 - 6u^2v^2 + 4iuv(u^2 - v^2). \tag{2}$$

Thus $4iuv(u^2 - v^2) = 0$ which implies that $v = 0$ or $u^2 - v^2 = 0$. But $u^2 + v^2 = 1$ so that $v = 0$ or $u^2 = v^2 = \frac{1}{2}$. If $u^2 = v^2 = \frac{1}{2}$ then $u^4 + v^4 - 6u^2v^2 = -1$, which is in contradiction with (2). In conclusion is $v = 0$. Thus $u = 1$ and $y' = 0$. Finally $y = 2\pi k$, which is what we wanted to prove. □

Part C. **Holomorphic Functions**

7. Complex differentiability

7.1. **Definition and basic properties.** Terminology: "(P) holds on a neighbourhood of z_0"="there exists an open set U containing z_0 such that (P) holds on U"

Definition 7.1. (complex-differentiable) *U open, $U \subseteq \mathbb{C}, z_0$ in U and $f : U \to \mathbb{C}$ a function. We say that f is complex-differentiable (or holomorphic) in z_0 if one of the following equivalent assertions hold:*

i) there exists
$$a = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$
ii) there exists $b \in \mathbb{C}$ and α defined on a neighbourhood of 0 such that
$$f(z) = f(z_0) + b(z - z_0) + |z - z_0|\alpha(z - z_0) \quad (*)$$
on a neighbourhood of z_0 with $\lim\limits_{h\to 0} \alpha(h) = 0$.

If i) and ii) hold then $a = b$ and is denoted by $f'(z_0)$.

Proof of the equivalence. ii) ⇒ i) easy. *i) ⇒ ii)* Set
$b = \lim \frac{f(z)-f(z_0)}{z-z_0}$ and $\alpha(h) = \left(\frac{f(z_0+h)-f(z_0)}{h} - b\right)\frac{h}{|h|}$ so that $(*)$ holds. But $\frac{f(z_0+h)-f(z_0)}{h} - b$ tends to 0 when h tends to 0 and $\left|\frac{h}{|h|}\right| = \frac{|h|}{|h|} = 1$. Hence $\alpha(h)$ tends to 0. □

Definition 7.2. (holomorphic) *We say that f is holomorphic on U if f is holomorphic (or complex-differentiable) in every $z_0 \in U$ and the function f' is continuous on U.*

Remark: In the literature you may find the same definition without the hypothesis "f' is continuous". We will see later, that both definitions are equivalent. In other terms, if f is holomorphic in every $z_0 \in U$, then f' is automatically continuous.

Examples:

- $z \mapsto z^2$ is holomorphic on $\mathbb{C}$. Fix $z_0 \in \mathbb{C}$. $\lim\limits_{z\to z_0} \frac{z^2-z_0^2}{z-z_0} = \lim\limits_{z\to z_0} z + z_0 = 2z_0$.
- $z \mapsto \frac{1}{z}$ is holomorphic of $\mathbb{C} \setminus \{0\}$. Fix $z_0 \in \mathbb{C}$. $\lim\limits_{z\to z_0} \frac{\frac{1}{z} - \frac{1}{z_0}}{z-z_0} = \lim\limits_{z\to z_0} -\frac{1}{z_0^2}$.

Proposition 7.3.

i) *Let $f, g : U \to \mathbb{C}, z_0 \in U$. Assume f and g are holomorphic in z_0. Let $\lambda \in \mathbb{C}$. Then $f+g$, $\lambda \cdot g$, $f \cdot g$ are holomorphic in z_0 and $(f+g)'(z_0) = f'(z_0) + g'(z_0)$, $(\lambda \cdot f)'(z_0) = \lambda \cdot f'(z_0)$, $(f \cdot g)'(z_0) = f'(z_0) \cdot g(z_0) + f(z_0) \cdot g'(z_0)$.*

ii) *Let $f : U \to \mathbb{C}$ and $g : V \to \mathbb{C}$. Assume g takes values in U. Let $z_0 \in V$. Assume g holomorphic in z_0 and f holomorphic in $g(z_0)$, then the composition $(f \circ g)'(z_0) = f'(g(z_0)) \cdot g'(z_0)$.*

Proof. Similar to the real case. □

Corollary 7.4. *All polynomials are holomorphic on $\mathbb{C}$, as well as all rational functions on their domain of definition. Moreover, their complex-derivative coincides with the real one.*

7.2. Cauchy-Riemann equations.

Let $U \subseteq \mathbb{R}^n$.

Definition 7.5. *A function $f : U \to \mathbb{R}^m$ is real differentiable in $t_0 \in U$ if there exists a linear map $\ell : \mathbb{R}^n \to \mathbb{R}^m$ and a function α defined on a neighbourhood of $0 \in \mathbb{R}^n$ such that $f(t_0 + h) = f(t_0) + \ell(h) + \|h\|\alpha(h)$ for h in a neighbourhood of 0 and $\lim_{h \to 0} \alpha(h) = 0$.*

Particular cases:

- $n = 1$, $f : \mathbb{R} \to \mathbb{R}^m$, $h \in \mathbb{R}$ then $\ell(h) = h \cdot a$ for some $a \in \mathbb{R}^m$. In this case denote (the vector) $a = f'(t_0)$.
- $n = 2$, $f : \mathbb{R}^2 \to \mathbb{R}^m$, $h = (k, l) \in \mathbb{R}^2$ then $\ell(k, l) = k \cdot a + l \cdot b$ for some $a, b \in \mathbb{R}^m$. Denote $a = \frac{\partial f}{\partial x}(t_0)$ and $b = \frac{\partial f}{\partial y}(t_0)$.

Take $U \subseteq \mathbb{C}$ open, $f : U \to \mathbb{C}$. We have two notions of differentiability.

- Complex differentiability
$$f(z_0 + h) = f(z_0) + bh + |h|\alpha(h)$$
- Real differentiability (using the canonical isomorphism $\mathbb{C} \cong \mathbb{R}^2$)
$$f(z_0 + h) = f(z_0) + \ell(h) + |h|\alpha(h)$$
ℓ is an $\mathbb{R}$-linear map $\mathbb{C} \to \mathbb{C}$.

But it is easy to see that the map
$$\begin{array}{ccc} \mathbb{C} & \to & \mathbb{C} \\ h & \mapsto & b \cdot h \end{array}$$
is an $\mathbb{R}$-linear map with matrix $\begin{pmatrix} u & -v \\ v & u \end{pmatrix}$.

Complex-differentiability is therefore **stronger** than real differentiable. The next statement explains precisely what should be added to real differentiability to get complex differentiability.

Theorem 7.6. (Cauchy-Riemann equations) *Take $f : U \to \mathbb{C}$ and $z_0 = x_0 + iy_0 \in U$. Denote*
$$P(x, y) := \mathrm{Re}(f(x + iy))$$
$$Q(x, y) := \mathrm{Im}(f(x + iy))$$
so that with identification $\mathbb{C} \cong \mathbb{R}^2$, the function f is identified with (P, Q). Then the following are equivalent:

i) *f is complex-differentiable in z_0*

ii) *(P, Q) is real-differentiable in (x_0, y_0) and*
$$\frac{\partial P}{\partial x}(x_0, y_0) = \frac{\partial Q}{\partial y}(x_0, y_0), \qquad \frac{\partial P}{\partial y}(x_0, y_0) = -\frac{\partial Q}{\partial x}(x_0, y_0)$$

Proof. $i) \Rightarrow ii)$ Assume f is complex differentiable in z_0, i.e. there exists $b \in \mathbb{C}$ and $\alpha(h) \xrightarrow{h\to 0} 0$ such that

$$f(z_0+h) = f(z_0) + bh + |h|\alpha(h) \quad (*)$$

but $h \mapsto bh$ is an $\mathbb{R}$-linear map of the form $(k,l) \mapsto (uk - vl, ul + vk)$ with $u, v \in \mathbb{R}$. Thus $f = (P,Q)$ is real differentiable in $z_0 = (x_0, y_0)$ and in (x_0, y_0), we have

$$\frac{\partial P}{\partial x} = u, \frac{\partial P}{\partial y} = -v, \frac{\partial Q}{\partial x} = v, \frac{\partial Q}{\partial y} = u$$

. Indeed, taking the real part of $(*)$

$$P(x_0+k, y_0+l) = P(x_0,y_0) + uk - vl + \|(k,l)\|\alpha(k,l).$$

Similar for the imaginary part. Thus Cauchy-Riemann equations are satisfied.

$ii) \Rightarrow i)$ Assume(P,Q) is real-differentiable in (x_0,y_0) i.e.

$$P(x_0+k, y_0+l) = P(x_0,y_0) + k \cdot \frac{\partial P}{\partial x}(x_0,y_0) + l \cdot \frac{\partial P}{\partial y}(x_0,y_0) + \|(k,l)\|\alpha(k,l).$$

$$Q(x_0+k, y_0+l) = Q(x_0,y_0) + k \cdot \frac{\partial Q}{\partial x}(x_0,y_0) + l \cdot \frac{\partial Q}{\partial y}(x_0,y_0) + \|(k,l)\|\beta(k,l),$$

With $\lim_{h\to 0}\alpha(h) = \lim_{h\to 0}\beta(h) = 0$. Set $u = \frac{\partial P}{\partial x}(x_0,y_0) = \frac{\partial Q}{\partial y}(x_0,y_0), v = \frac{\partial P}{\partial y}(x_0,y_0) = -\frac{\partial Q}{\partial x}(x_0,y_0)$. Then

$$\begin{aligned} f(z_0+h) &= P(x_0+k,y_0+l) + iQ(x_0+k,y_0+l) \\ &= (P(x_0,y_0) + iQ(x_0,y_0)) + (ku + lv) + i(-kv + lu) + \|(k,l)\|(\alpha(k,l) + i\beta(k,l)) \\ &= f(z_0) + (u - iv)(k + il) + |h|\gamma(h) \end{aligned}$$

with $\gamma(h) = \alpha(k,l) + \beta(k,l)$ and thus $\lim_{h\to 0}\gamma(h) = 0$. This is the definition of complex differentiability with $b = u - iv$. ☐

8. Holomorphy of analytic functions

8.1. **Power series.**

Proposition 8.1. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be the sum of a power series with positive radius of convergence $\rho > 0$. Define $g(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$. Then g has the same radius of convergence as f and for $z_0 \in \mathbb{C}$ with $|z_0| < \rho, f$ is complex-differentiable in z_0 and $f'(z_0) = g(z_0)$.*

(We already knew that f is continuous on $D(0,\rho)$, we now see that it is holomorphic in each point of this open disk.)

(In general, differentiating under an integral/sum sign needs extra hypothesis e.g., the derivatives to be dominated by some integrable function. The above statement says that we can differentiate term by terms power series without extra asumptions.)

Proof. $i)$ We want to prove that g has the same radius of convergence as f. Call ρ_g the radius of convergence of g. Assume that $r > 0$ such that $\left(n a_n r^{n-1}\right)_{n\geq 1}$ is bounded. Then $a_n r^n \leq r(n a_n r^{n-1})$ for $n \geq 1$ so that $(a_n r^n)_{n\geq 1}$ is bounded. Thus $r < \rho_g$ implies $r \leq \rho$, so that $\rho_g \leq \rho$.

Assume that $r > 0$ such that $(a_n r^n)_{n\geq 0}$ is bounded. Then for each $r_0 < r$ the sequence $\left(n a_n r_0^{n-1}\right)_{n\geq 1}$ is bounded. Indeed $n a_n r_0^{n-1} = \frac{n}{r_0}\left(\frac{r_0}{r}\right)^n (a_n r^n)$. Let $r_0 < \rho$. Then we can find r with $r_0 < r < \rho$. Thus $(a_n r^n)$ is bounded and $\left(a_n r_0^{n-1}\right)_{n\geq 1}$ is also bounded. This is true for any $r_0 < \rho$ so that $\rho \leq \rho_g$. In conclusion $\rho = \rho_g$.

$ii)$ We want to prove that $f'(z_0) = g(z_0)$.

Fix $z_0 \in \mathbb{C}$ with $|z_0| < \rho$. Chose r with $|z_0| < r < \rho$. For $|h| < r - |z_0|$ we want to show that $\frac{f(z_0+h)-f(z_0)}{h} - g(z_0)$ tends to zero.

$$\frac{f(z_0+h)-f(z_0)}{h} - g(z_0) = \sum_{n=1}^{\infty} a_n \left(\frac{(z_0+h)^n - z_0^n}{h} - n z_0^{n-1} \right)$$

and

$$\frac{(z_0+h)^n - z_0^n}{h} = \left((z_0+h)^{n-1} + (z_0+h)^{n-2} z_0 + \cdots + (z_0+h) z_0^{n-2} + z_0^{n-1} \right).$$

Thus

$$\frac{f(z_0+h)-f(z_0)}{h} - g(z_0) = \sum_{n=0}^{\infty} u_n(h)$$

where $u_n(h) = ((z_0+h)^{n-1} + \cdots + z_0^{n-1} - n z_0^{n-1})$ for $h = r - |z_0|$. We have $|u_n(h)| \leq 2 n a_n r^{n-1}$ using $z_0 \leq r$ and $|z_0 + h| \leq r$. But $\sum_{n=0}^{\infty} 2 n a_n r^{n-1}$ is convergent on $\{h : |h| < r - |z_0|\}$ so that $h \mapsto \sum_{n=0}^{\infty} u_n(h)$ is a continuous function. But $u_n(0) = 0$ so that $\sum_{n=0}^{\infty} u_n(0) = 0$. Thus $\lim_{n\to\infty} \sum_{n=0}^{\infty} u_n(h) = 0$ (because it is continuous). Ie. $\lim_{h\to\infty} \frac{f(z_0+h)-f(z_0)}{h} = g(z_0)$. □

Corollary 8.2. *Sum of power series are infinitely many times complex-differentiable on their disk of convergence.*

Proof. By Proposition 8.1, the sum of a power series is complex-differentiable on its disk of convergence and the derivative is the sum of a power series. Iterating proves the corollary □

Proposition 8.3. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with positive radius of convergence. Then for each $n \geq 0$,*

$$a_n = \frac{f^{(n)}(0)}{n!}.$$

Proof. Let $k \geq 0$. Then the k-th derivative of f is $f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1) \cdot \cdots \cdot (n-k+1) \cdot a_n z^{n-k}$ (immediate induction). Thus $f^{(k)}(0) = k(k-1) \cdot \cdots \cdot 1 \cdot a_k$ (all terms with $n > k$ vanish for $z = 0$). □

(Reminiscent of Taylor expansion, but here we have an infinite sum, not a finite sum + a remainder.)

8.2. **Analytic functions.**

Theorem 8.4. *Analytic functions are infinitely many times complex-differentiable. In particular, they are holomorphic.*

Proof. Let $f : U \to \mathbb{C}$, U open, f analytic. Fix z_0 in U. Then f is equal to the sum of a power series on a neighbourhood of z_0 which implies that f is infinitely many times differentiable on a neighbourhood of z_0. This holds for all z_0 in U, so that f is infinitely many times differentiable on U. □

Proposition 8.5. *If $f : U \to \mathbb{C}$ analytic and $z_0 \in U$ then by definition $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ in a neighbourhood of z_0 then $a_n = \frac{f^{(n)}(z_0)}{n!}$.*

Proof. $f(z) = g(z - z_0)$ with $g(w) = \sum_{n=0}^{\infty} a_n w^n$ from the case of power series $a_n = \frac{g^{(n)}(0)}{n!}$. But $f^{(n)}(z) = g^{(n)}(z - z_0)$. □

8.3. **Examples.**

- (Power series) Exponential function

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

(sum of power series with $\rho = \infty$). From Proposition 8.1 exp is complex-differentiable in any $z_0 \in \mathbb{C}$ and $\exp'(z_0) = \sum\limits_{n=0}^{\infty} n \cdot \frac{1}{n!} z_0^{n-1} = \sum\limits_{n=0}^{\infty} \frac{1}{(n-1)!} z_0^{n-1} = \sum\limits_{m=0}^{\infty} \frac{1}{m!} z_0^m = \exp(z_0)$. This implies immediately that $\exp^{(n)}(z) = \exp(z)$ and in particular $\frac{\exp^{(n)}(0)}{n!} = \frac{1}{n!}$. This is indeed the n-th coefficient of the series, as asserted by Proposition 8.3.

- (Analytic function) $f(z) = \frac{1}{1-z}$.
We have seen that f is analytic and

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{(1-z_0)^{n+1}} (z - z_0)^n$$

for $|z - z_0| < |1 - z_0|$. We also know that f is complex-differentiable (composition of $z \mapsto \frac{1}{z}$ and $z \mapsto 1 - z$) with $f'(z) = \frac{1}{(1-z)^2}$.
From Proposition 8.1, we can also infer that, $|z - z_0| < |1 - z_0|$,

$$f'(z) = \sum_{n=1}^{\infty} \frac{n}{(1-z_0)^{n+1}} (z - z_0)^{n-1}.$$

Comparing both expressions, we get a power series expansion for $\frac{1}{(1-z)^2}$ around z_0.
Let us consider further derivatives. Since f is a rational function, f is infinitely many times complex-differentiable on its domain of definition with $f^{(n)}(z) = \frac{n!}{(1-z)^{n+1}}$. We see that $\frac{f^{(n)}(z_0)}{n!} = \frac{1}{(1-z_0)^{n+1}}$ is indeed the n-th coefficient in the power series expansion around z_0, as asserted by Proposition 8.5 above.

9. Cauchy formula

9.1. **Statement.**

Theorem 9.1. (Cauchy formula) *Let $f : U \to \mathbb{C}$, $U \subseteq \mathbb{C}$ open, f holomorphic. Take $z_0 \in U$ and $r > 0$ such that $|z - z_0| \leq r$ implies $z \in U$. Then for $z \in \mathbb{C}$ with $|z - z_0| < r$*

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f\left(z_0 + re^{i\theta}\right) re^{i\theta}}{z_0 + re^{i\theta} - z} d\theta.$$

(schema showing the center z_0 and the circle $z_0 + re^{i\theta}$; the formula expresses the value of f in any point z inside the disk in terms of the values on the circle, it is quite surprising that such a formula exists!)

Comment: Cauchy formula is one of the **main pieces of complex analysis**; there are many different versions, with various levels of generality and various presentations, we will see a few in this lecture.

9.2. **Reminders/Preliminaries.**

- The formula $(f \circ g)'(x) = f'(g(x)) \cdot g(x)$ also holds for
$g : \mathbb{R} \supseteq I \to \mathbb{C}$ $f : U \to \mathbb{C}$ holomorphic in $g(x)$. Then $f \circ g : I \to \mathbb{C}$ is real-differentiable and $(f \circ g)'(x) = f'(g(x)) \cdot g(x)$.

- Dominated convergence theorem
If $f_n \xrightarrow{n\to\infty} f$ pointwise, and $(f_n, f : \mathbb{R} \supseteq I \to \mathbb{C})$ and $\int_I \sup\limits_{n\in\mathbb{N}} |f_n| < \infty$, then $\int_I f_n \xrightarrow{n\to\infty} \int_I f$.

– special case:
If $I := [a,b]$ is a closed bounded interval and f_n uniformly bounded (there exists M such that $\forall x \in I, \forall n \geq 0$ $|f_n(x)| \leq M$) then $\int_I f_n \xrightarrow{n\to\infty} \int_I f$ if $f_n \xrightarrow{n\to\infty} f$ pointwise.
– special case of the special case:
f_n continuous and $f_n \xrightarrow{n\to\infty} f$ uniformly on I then $\int_I f_n \xrightarrow{n\to\infty} \int_I f$. In particular if $\sum_{n=0}^{\infty} g_n$ converges uniformly then $\sum_{n=0}^{\infty} \int_I g_n = \int_I \sum_{n=0}^{\infty} g_n$ (use previous statement for $f_N = \sum_{n=0}^{N} g_N$).

- Differentiability under an integral sign.
Consider a function $f : I \times [a,b] \to \mathbb{C}$. Assume that $\frac{\partial f}{\partial x}$ exists on $I \times [a,b]$. Assume also f and $\frac{\partial f}{\partial x}$ continuous on $I \times [a;b]$. Set $g(x) = \int_a^b f(x,t)dt$. Then g is differentiable on every $x \in I$ and $g'(x) = \int_a^b \frac{\partial f}{\partial x}(x,t)dt$.

9.3. **Proof.**

Proof of Theorem 9.1. We can assume w.l.o.g. that $z_0 = 0$. (otherwise set $\tilde{f}(z) = f(z+z_0)$). Fix z with $|z| < r$. Define

$$g(\lambda) = \int_0^{2\pi} \frac{f\left((1-\lambda)z + \lambda re^{i\theta}\right) re^{i\theta}}{re^{i\theta} - z} d\theta.$$

We want to differentiate g. Since we assumed f' continuous (in Definition 7.2), we can apply the above recalled result and we get

$$g'(\lambda) = \int_0^{2\pi} \frac{re^{i\theta}}{re^{i\theta} - z} \cdot f'\left((1-\lambda)z + \lambda re^{i\theta}\right) \cdot \left(-z + re^{i\theta}\right) d\theta = \int_0^{2\pi} re^{i\theta} f'\left((1-\lambda)z + \lambda re^{i\theta}\right) d\theta.$$

Consider $h(\theta) = f\left((1-\lambda)z + \lambda re^{i\theta}\right)$ then $h'(\theta) = f'\left((1-\lambda)z + \lambda re^{i\theta}\right) \cdot i\lambda re^{i\theta}$. Thus, for $\lambda \neq 0$, we have

$$g'(\lambda) = \int_0^{2\pi} \frac{1}{i\,\lambda} h'(\theta) d\theta = \frac{1}{i}\left(h(2\pi) - h(0)\right) = 0.$$

because $h(2\pi) = h(0) = f((1-\lambda)z + \lambda r)$. Thus g is constant and $g(0) = g(1)$. But

$$g(0) = \int_0^{2\pi} f(z) \frac{re^{i\theta}}{re^{i\theta} - z} d\theta = f(z) \cdot \left(\int_0^{2\pi} \frac{re^{i\theta}}{re^{i\theta} - z} d\theta\right).$$

Let us set

$$J := \int_0^{2\pi} \frac{re^{i\theta}}{re^{i\theta} - z} d\theta = \int_0^{2\pi} \frac{1}{1 - \frac{z}{re^{i\theta}}} d\theta.$$

But recall that $|z| < r$ so that $\left|\frac{z}{re^{i\theta}}\right| < r$ and we have that $\frac{1}{1-\frac{z}{re^{i\theta}}} = \sum_{n=0}^{\infty} \left(\frac{z}{re^{i\theta}}\right)^n$ (because of Lemma 6.2, for any $r_0 < 1$, $\sum_{n=0}^{\infty} w^n = \frac{1}{1-w}$ uniformly on $\{w :$ such that $|w| \leq r_0\}$). Call $r_0 = \frac{|z|}{r}$ then $\left|\frac{z}{re^{i\theta}}\right| = r_0$ so that the convergence $\frac{1}{1-\frac{z}{re^{i\theta}}} = \sum_{n=0}^{\infty} \left(\frac{z}{re^{i\theta}}\right)^n$ is uniform on θ. Using the above recalled result to exchange infinite sums and integrals, we get that

$$J = \sum_{n=0}^{\infty} \int_0^{2\pi} \left(\frac{z}{re^{i\theta}}\right)^n.$$

If $n = 0$, $\int_0^{2\pi} \frac{z}{re^{i\theta}} d\theta = \int_0^{2\pi} 1 d\theta = 2\pi$. If $n \geq 1$

$$\int_0^{2\pi} \left(\frac{z}{re^{i\theta}}\right)^n d\theta = \int_0^{2\pi} z^n r^{-n} e^{-in\theta} d\theta = z^n r^{-n} \int_0^{2\pi} e^{-in\theta} d\theta = z^n r^{-n} \left[-\frac{1}{in} e^{-in\theta}\right]_0^{2\pi} = 0.$$

Finally $J = 2\pi$ and $g(0) = 2\pi f(z)$.

On the other hand, by definition

$$g(1) = \int_0^{2\pi} \frac{f(re^{i\theta})re^{i\theta}}{re^{i\theta} - z} d\theta.$$

But, recall that since $g'(\lambda)=0$, we have $g(0)=g(1)$, that is

$$f(z)=\frac{1}{2\pi}\int_0^{2\pi}\frac{f(re^{i\theta})re^{i\theta}}{re^{i\theta}-z}d\theta.$$

□

9.4. **Analyticity of holomorphic functions.**

Theorem 9.2. *Let $f:U\to\mathbb{C}$, U open f holomorphic. Fix $z_0\in U$. Call ρ the biggest real such that $|z-z_0|<\rho$ implies $z\in U$ (we may have $\rho=\infty$). Then f is equal to the sum of a power series on $\{z:|z-z_0|<\rho\}$.*

Combined with Theorem 8.4, we get

Corollary 9.3. *A function $f:U\to\mathbb{C}$ is holomorphic if and only if it is analytic.*

Remarks:

- ρ is well defined since $\{r>0:|z-z_0|<r\Rightarrow z\in U\}$ is non-empty (since U is open) and closed.
- Definition of analyticity: there exists a disk around z_0 on which f is equal to a power series (inner disk of picture). By Theorem 8.4 f is equal to a power series on the outer disk. This is a priori stronger than analyticity. Theorem 8.4 implies that it is true for every analytic function, since analytic implies holomorphic.

(schema with two concentric disks, the exterior one being maximal included in U, to illustrate the second remark.)

Proof. Fix $z_0\in U$. Define ρ as in Theorem 8.4. Take $r<\rho$. From Theorem 9.1,

$$f(z)=\frac{1}{2\pi}\int_0^{2\pi}\frac{f\left(z_0+re^{i\theta}\right)\cdot re^{i\theta}}{z_0+re^{i\theta}-z}d\theta=\frac{1}{2\pi}\int_0^{2\pi}f\left(z_0+re^{i\theta}\right)\frac{1}{1-\frac{z-z_0}{re^{i\theta}}}d\theta.$$

But we know that $\sum_{n=0}^{\infty}\left(\frac{z-z_0}{re^{i\theta}}\right)^n=\frac{1}{1-\frac{z-z_0}{re^{i\theta}}}$ with uniform convergence when $\theta\in[0;2\pi]$. Since $f(z_0+re^{i\theta})$ is bounded, the convergence

$$\sum_{n=0}^{\infty}f\left(z_0+re^{i\theta}\right)\left(\frac{z-z_0}{re^{i\theta}}\right)^n=\frac{f\left(z_0+re^{i\theta}\right)}{1-\frac{z-z_0}{re^{i\theta}}}$$

is also uniform for $\theta\in[0;2\pi]$. We can therefore exchange sum and integral:

$$2\pi f(z)=\sum_{n=0}^{\infty}\int_0^{2\pi}\frac{f(z_0+re^{i\theta})}{r^ne^{in\theta}}(z-z_0)^nd\theta=\sum_{n=0}^{\infty}\left(\int_0^{2\pi}\frac{f(z_0+re^{i\theta})}{r^ne^{in\theta}}d\theta\right)(z-z_0)^n.$$

We have proved that for $|z-z_0|<r$, $f(z)=\sum_{n=0}^{\infty}a_n(r)(z-z_0)^n$ with $a_n(r)=\frac{1}{2\pi r^n}\int_0^{2\pi}f(z_0+re^{i\theta})e^{-in\theta}d\theta$. By Corollary 6.6 the power series expansion is unique and the coefficients $a_n(r)$ do not depend on r. Denote $a_n=a_n(r)$ for any $r<\rho$ so that $f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n$ holds for $|z-z_0|<\rho$. □

A consequence of the above proof is the following.

Corollary 9.4 (Cauchy formula for derivatives)**.** *$f:U\to\mathbb{C}$ holomorphic, $z_0\in U$. Assume $r>0$ such that $|z-z_0|\le r\Rightarrow z\in U$. For $n\ge 0$,*

$$\frac{f^{(n)}(z_0)}{n!}=\frac{1}{2\pi r^n}\int_0^{2\pi}f\left(z_0+re^{i\theta}\right)e^{-in\theta}d\theta.$$

Proof. Both are the n-th coefficient of the power series expansion of f around z_0 which is unique by Corollary 6.6 □

We now get easily two results on analytic functions that were announced in Chapter 6, see Remark p. 14.

Proposition 9.5. *The sum of a power series is analytic on its disk of convergence.*

Proposition 9.6. *The composition of analytic functions is analytic.*

Proof of Propositions 9.5 and 9.6. We know them for holomorphic functions, but holomorphic and analytic are equivalent. □

10. First main theorems on holomorphic functions

10.1. Cauchy's inequalities.

Proposition 10.1. *$f : U \to \mathbb{C}$ analytic, $z_0 \in U$. Choose r such that $|z - z_0| \le r \Rightarrow z \in U$. Then*

$$\left|f^{(n)}(z_0)\right| \le n!\, r^{-n} \sup_{\theta \in [0;2\pi]} \left|f\left(z_0 + re^{i\theta}\right)\right|.$$

Note that the bound can be also rewritten as

$$\left|f^{(n)}(z_0)\right| \le n!\, r^{-n} \sup_{z \in \partial D(z_0,r)} |f(z)|.$$

(Indeed, when θ runs over $[0;2\pi]$, the quantity $z_0 + re^{i\theta}$ runs over the circle $\partial D(z_0, r)$.)

Proof. Immediate from Cauchy formula for derivatives. Indeed,

$$\left|\int_0^{2\pi} f\left(z_0 + re^{i\theta}\right) e^{-in\theta} d\theta\right| \le 2\pi \sup_{\theta \in [0;2\pi]} \left|f\left(z_0 + re^{i\theta}\right)\right|.$$

□

Theorem 10.2. *$f_k, f : U \to \mathbb{C}$ holomorphic f function $(k \ge 0)$. Assume that $(f_k)_{k\ge1}$ converges to f locally uniformly on U. Then, for any $n \ge 1$, $\left(f_k^{(n)}\right)_{k\ge1}$ converges locally uniformly to $f^{(n)}$.*

Warning! here n is fixed and $k \to \infty$.

Comment: this is a very surprising statement. In real analysis, one can interchange uniform limits and integrals but certainly not derivatives!

Proof. Fix $z_0 \in U$. There exists $r > 0$ such that $\sup_{|w-z_0|<r} |f_k(w) - f(w)| \xrightarrow{k\to\infty} 0$. Take z such that $|z - z_0| < \frac{r}{2}$. Apply Proposition 10.1 with

- holomorphic function $f_k - f$
- point $z \in U$
- radius $\frac{r}{2}$

$$\left|f_k^{(n)}(z) - f^{(n)}(z)\right| \le n! \left(\frac{r}{2}\right)^{-n} \sup_{\theta\in[0;2\pi]} \left|(f_k - f)\left(z + \frac{r}{2}e^{i\theta}\right)\right|$$

if $w = z + \frac{r}{2}e^{i\theta}$ then $|w - z_0| < \underbrace{|w - z|}_{\frac{r}{2}} + \underbrace{|z - z_0|}_{<\frac{r}{2}} < r$. Thus

$$\sup_{\theta\in[0;2\pi]} \left|(f_k - f)\left(z + \frac{r}{2}e^{i\theta}\right)\right| \le \sup_{|w-z_0|<r} |(f_k - f)(w)|.$$

Then

$$\left|f_k^{(n)}(z) - f^{(n)}(z)\right| \le n! \left(\frac{r}{2}\right)^{-n} \sup_{|w-z_0|<r} |f_k(w) - f(w)|$$

for z such that $|z - z_0| < \frac{r}{2}$. The right hand side does not depend on z and tends to 0. Therefore $\left|f_k^{(n)} - f^{(n)}(z)\right|$ tends to 0 uniformly on z for $|z - z_0| < \frac{r}{2}$. □

Remark: We will see in the next chapter that the hypothesis "f holomorphic" is redundant, i.e. a local uniform limit of holomorphic functions is always holomorphic.

10.2. **Liouville theorem.**

Definition 10.3. (entire function) *A holomorphic function $f : \mathbb{C} \to \mathbb{C}$ is called an entire function.*

Theorem 10.4. (Liouville) *Any bounded entire function is constant.*

Proof. Let f be an entire bounded function. Let $n \geq 1$. Use Cauchy's inequality for $z_0 = 0,\ r > 0$. We can choose any r because $U = \mathbb{C}$.

$$\left|f^{(n)}(0)\right| \leq \frac{n!}{r^n} \sup_{\theta \in [0;2\pi]} \left|f(re^{i\theta})\right|.$$

But f is bounded which implies that there exists $M > 0$ for all $z \in \mathbb{C}$ $|f(z)| \leq M$. In particular, $\sup_{\theta\in[0;2\pi]} |f(re^{i\theta})| \leq M$ i.e. $|f^{(n)}(0)| \leq \frac{n!}{r^n} \cdot M$ for any $r > 0$. Thus $|f^{(n)}(0)| = 0$ (make $r \longrightarrow \infty$). The power series expansion around 0 writes for $z \in \mathbb{C}$,

$$f(z) = \sum_{n\geq 0} \frac{f^{(n)}(0)}{n!} z^n = f(0).$$

Indeed we have seen that analytic functions are equal to their power series expansion on any disk contained in the domain of definition (here $\mathbb{C}$). □

Corollary 10.5. (fundamental theorem of algebra) *Let P a non-constant polynomial (with complex coefficients). Then there exists $z \in \mathbb{C}$ such that $P(z) = 0$.*

Proof. (By contradiction.) Assume that for any $z \in \mathbb{C}$, $p(z) \neq 0$. Consider $\frac{1}{P}$. Since $\mathbb{C} \to \mathbb{C}$, $z \mapsto P(z)$ and $\mathbb{C} \setminus \{0\} \to \mathbb{C}\, z \mapsto \frac{1}{z}$ are both holomorphic, $z \mapsto \frac{1}{P(z)}$ is holomorphic. By hypothesis, P is non-constant, i.e. $P(z) = a_d z^d + \sum_{i=0}^{d-1} a_i z^i$ with $a_d \neq 0\,(d \geq 1)$. Then $|p(z)| \geq |a_d||z|^d - \sum_{i=0}^{d-1} |a_i||z|^i$ (triangular inequality). Because $|a_d| > 0$, the polynomial $|a_d|x^d - \sum_{i=0}^{d-1} |a_i|x^i \xrightarrow{x\to\infty} \infty$. There exists $M > 0$ such that if $x > M$ implies $|a_d|x^d - \sum_{i=0}^{d-1} |a_i|x^i > 1$. Thus $|z| > M$ implies $|P(z)| \geq 1$. Or equivalently $\frac{1}{|P(z)|} \leq 1$. Moreover $\frac{1}{P}$ is a continuous function and thus bounded by K on the compact set $\{z : |z| \leq M\}$. Finally for any $z \in \mathbb{C}$, $\frac{1}{|P(z)|} \leq \max(1, k)$. In particular, $\frac{1}{P}$ is bounded. Theorem 10.4 implies that $\frac{1}{P}$ is constant which is a contradiction with the hypothesis "P non-constant". □

10.3. **Open mapping theorem.**

Definition 10.6. *A function $f : U \to \mathbb{C}$ is called open if the image of any open set is open.*

(Defined more generally when both the domain of definition and the image set are topological spaces.)

Warning: this is different than *continuous*.

Example of a continuous non-open function: $f : \mathbb{R} \to \mathbb{R},\ x \mapsto x^2$ is NOT an open function. Indeed $f((-1;1)) = [0,1)$ is not an open set.

Theorem 10.7. (open mapping theorem) *A non-constant holomorphic function on a connected open set is open.*

Proof. (By contradiction.) Assume that $f : U \to \mathbb{C}$ is a holomorphic function that U is open and connected, and that $V \subseteq U$ is open such that $f(V)$ **is not open**. By def, $f(V)$ not open means that there exists $w \in V$ and a sequence $(\alpha_n)_{n\geq 1}$ with $\alpha_n \to f(w); \alpha_n \notin f(V)$. Choose some radius $r > 0$ such that $|z - w| \leq r \Rightarrow f(z) \neq f(w)$ or $z = w$ (Theorem 6.8 applied to $h(z) = f(z) - f(w)$).

(Draw a picture with w, $\partial D(w,r)$ and their image by f; $f(w)$ is on the boundary of $f(U)$ and approached by a sequence α_n.)

(Idea: for $z \in \partial D(w,r)$, $f(z) - \alpha_n$ is far from zero for n big enough.) Indeed, $|f(z) - f(w)|$ has a minimum $\epsilon > 0$ on the compact set $\{z : |z - w| = r\}$. On the other hand, for n big enough, $|f(w) - \alpha_n| \leq \varepsilon/2$. This implies

$$\inf_{z\in\partial D(w,r)} |f(z) - \alpha_n| \geq \varepsilon/2.$$

On the opposite we know that $|f(w) - \alpha_n|$ tends to 0 so that for n big enough. Therefore, for n big enough,

$$|f(w) - \alpha_n| < \inf_{z\in\partial D(w,r)} |f(z) - \alpha_n|.$$

Taking the multiplicative inverse, we get a contradiction with Cauchy's inequality applied to $g_n(z) = \frac{1}{f(z)-\alpha_n}$ for $z \in V$, which asserts that

$$\frac{1}{|f(w)-\alpha_n|} \leq \sup_{z\in\partial D(w,r)} \frac{1}{|f(z)-\alpha_n|}.$$

(Since $\alpha_n \notin V$, the denominator does not vanish for $z \in V$ and g_n is indeed a holomorphic function on V.) ☐

10.4. **Maximum modulus principle.** Let $f : U \to \mathbb{C}$, U open. We say that $z_0 \in U$ is a local maximum (resp. strict local maximum) of $|f|$ if there exists a neighbourhood V of z_0 such that $|f(z)| \leq |f(z_0)|$ for z in V (resp. $|f(z)| < |f(z_0)|$ for z in $V \setminus \{0\}$).

Theorem 10.8. (Maximum modulus principle; local version) *Let $f : U \to \mathbb{C}$ holomorphic, U open and connected. Assume that z_0 is a local maximum of $|f|$. Then f is constant.*

Comment: that we can not have a strict local max of $|f|$ is trivial from Cauchy inequality; the above statement with (non-necessarily strict) local max is more subtle, but is an easy consequence of the open mapping theorem.)

Proof. There exists a neighbourhood of V of z_0 such that for $z \in V$, $|f(z)| \leq |f(z_0)|$. This means $f(V) \subseteq \{w : |w| \leq f(z_0)\}$. But for any $r > 0$, $D(f(z_0), r) \not\subseteq \{w : |w| \leq f(z_0)\}$. Thus $f(V)$ is not open. The open mapping theorem implies that f must be constant. ☐

For the global version, we need the following reminder.

The closure of a set $U \subseteq \mathbb{C}$ is (both definitions are equivalent)
- the minimum closed set containing U;
- the set of $l \in \mathbb{C}$ which are limits of a sequence in U.

Theorem 10.9. (Maximum modulus principle; global version) *Let U be a connected open bounded set. Let $\overline{U}$ be its closure. Assume that $f : \overline{U} \to \mathbb{C}$ is a continuous function such that $f|_U$ is holomorphic. Let $M = \max_{z\in\overline{U}|_U} f(z)$. Then, for any $z \in \overline{U}, |f(z)| \leq M$. Furthermore, if $|f(z_0)| = M$ for some $z_0 \in U$, then f is constant.*

Proof. Call $M' = \max_{z\in\overline{U}} |f(z)|$. Two cases:
- There exists some $z_0 \in U$ such that $|f(z_0)| = M'$. Then z_0 is a local maximum of $|f|$ which implies that f is constant in U. $\Rightarrow f$ is constant on $\overline{U}$ and $M = M'$. The statement holds trivially.
- There exists no $z_0 \in U$ with $|f(z_0)| = M'$. Thus there exists $z_0 \in \overline{U} \setminus U$ with $|f(z_0)| = M'$ so that $M = M'$ and for all $z \in \overline{U}$, $|f(z)| \leq M' = M$ (nothing to prove for the second statement). ☐

Example: Consider $f : \overline{D(0,1)} \to \mathbb{C}$, $z \mapsto 1 - z^2$, the maximum is reached at $-i$ and i for which $f(-i) = f(i) = 2$. These points indeed lie on the boundary of $D(0,1)$.

If we look at its real restriction $f_{\mathbb{R}}$, that is $[-1,1] \to \mathbb{R}$, $x \mapsto 1-x^2$, it reaches its maximum in 0 (value:1), which is in the interior of the interval of definition. (There is no maximum principle in real analysis!)

Part D. Path integrals

11. Basics

11.1. Definition and examples.

Terminology: Piecewise-C^1 means that there exists a finite subdivision $a = t_0 < t_1 < \cdots < t_l = b$ of $[a,b]$ so that for $1 \leq i \leq l$, $\gamma|_{[t_{i-1},t_i]}$ is C^1.

Definition 11.1. (path) *A path γ is a continuous piecewise-C^1 function from a real bounded closed interval $[a,b]$ to $\mathbb{C}$.*

(Note that this definition of path is slightly different from the one used to define "path-connectedness"; here, we assume more regularity, i.e. the path to be piecewise C^1.)

Examples:

- Circle of center z_0 and radius r. Abuse of notation: the *set* $\partial D(z_0, r) = \{z : |z - z_0| = r\}$ also stands for the *path* $\gamma : [0, 2\pi] \to \mathbb{C}$
  $\theta \mapsto z_0 + re^{i\theta}$

  (Draw the path.)

- Segment $[A; B]$ $\quad A, B \in \mathbb{C}$. Stands for $\gamma : [0, 1] \to \mathbb{C}$
  $t \mapsto (1 - t)A + tB$

  (Draw the path.)

- Triangles $[A; B; C; A]$ $\quad A, B, C \in \mathbb{C}$. Stands for $\gamma : [0; 3] \to \mathbb{C}$

$$t \mapsto \begin{cases} (1-t)A + tB & \text{for} \quad 0 \le t \le 1 \\ (2-t)B + (t-1)C & \text{for} \quad 1 \le t \le 2 \\ (3-t)C + (t-2)A & \text{for} \quad 2 \le t \le 3 \end{cases}$$

  (Draw the path.)

Definition 11.2. (path integral) *Let $f : U \to \mathbb{C}$, U open, f continuous. Let γ be a path with $\Im m(\gamma) \subseteq U$. Then*

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

To remember it: think of the change of variable formula and set $z = \gamma(t)$.

Remark: unrelated to volume integral in $\mathbb{R}^n$.

Is it always well-defined? γ' may not be defined at the point t_i appearing in the definition of being piecewise C^1 (if the left-derivative at t_i is different from the right-derivative). Nevertheless, $t \mapsto f(\gamma(t))\gamma'(t)$ is well-defined, continuous and bounded on each interval (t_{i-1}, t_i). This implies that the integral is well-defined. We have

$$\int_a^b f(\gamma(t))\gamma'(t)dt = \sum_{i=1}^{l} \int_{t_{i-1}}^{t_i} f(\gamma(t))\gamma'(t)dt.$$

Examples: f holomorphic $f : U \to \mathbb{C}$, $z_0 \in U$. Choose $r > 0$ such that $\{z : |z - z_0| \le r\} \subseteq U$. Fix z with $|z - z_0| < r$

$$\int_{\partial D(z_0,r)} \frac{f(w)}{w - z} dw \overset{\text{Definition 11.2}}{=} \int_0^{2\pi} \frac{f\left(z_0 + re^{i\theta}\right)}{z_0 + re^{i\theta} - z} ire^{i\theta} d\theta \overset{\text{Theorem 9.1}}{=} 2\pi i\, f(z).$$

$$\int_{\partial D(z_0,r)} \frac{f(w)}{(w - z_0)^{n+1}} dw \overset{\text{Definition 11.2}}{=} \int_0^{2\pi} \frac{f\left(z_0 + re^{i\theta}\right)}{(z_0 + re^{i\theta} - z_0)^{n+1}} ire^{i\theta} d\theta =$$

$$\frac{i}{r^n} \int_0^{2\pi} f\left(z_0 + re^{i\theta}\right) e^{-in\theta} d\theta \overset{\text{Corollary 9.4}}{=} 2\pi i \frac{f^{(n)}(z_0)}{n!}.$$

11.2. Some computation rules.

Proposition 11.3. *Let $f : U \to \mathbb{C}$ continuous with $\Im m(\gamma) \subseteq U$. Let γ be a path and $\varphi : [a'; b'] \to [a; b]$ an increasing C^1-bijection. Then $\gamma \circ \varphi$ is a path and,*

$$\int_\gamma f(z)dz = \int_{\gamma\circ\varphi} f(z)dz.$$

Proof. $\gamma \circ \varphi$ is as a composition of continuous functions continuous and piece-wise C^1. Then

$$\int_{\gamma\circ\varphi} f(z)dz = \int_{a'}^{b'} f(\gamma(\varphi(t)))(\gamma\circ\varphi)'(t)dt = \int_{a'}^{b'} f(\gamma(\varphi(t)))\gamma'(\varphi(t))\varphi'(t)dt.$$

With the change of variables $u = \varphi(t);\ a = \varphi(a');\ b = \varphi(b');\ du = \varphi'(t)dt$ we get

$$\int_a^b f(\gamma(u))\gamma'(u)du = \int_{\gamma\circ\varphi} f(z)dz = \int_\gamma f(z)dz.$$

□

Graphical interpretation: Several "paths" (ie. function from $[a; b] \to \mathbb{C}$) may have the same picture. For example $[0; 2\pi] \to \mathbb{C}\ \theta \mapsto z_0 + re^{i\theta}$ and $[0; 4\pi] \to \mathbb{C}\ \theta \mapsto z_0 + re^{\frac{i\theta}{2}}$ describe the same circle (but not traveled at the same speed). Proposition 11.3 states that integrals on these two paths are the same.

Comment: $\gamma \circ \varphi$ is a "change of parametrization" of γ. When we consider path, we often think implicitly at paths, up to *change of parametrization.* Thanks to the above proposition, path integrals are well-defined, when considering such equivalence classes of path. Moreover, in such an equivalence class, there always exists a path $\gamma : [0, 1] \to \mathbb{C}$, i.e. with $a = 0, b = 1$. We will sometimes assume this without loss of generality.

Warning! The integral may depend on the orientation of the path.

Proposition 11.4. *Let $f : U \to \mathbb{C}$ continuous with $\Im m(\gamma) \subseteq U$. Let $\gamma : [a; b] \to \mathbb{C}$.*
Define $\gamma^* : [a; b] \to \mathbb{C}$, $t \mapsto \gamma(a + b - t)$. Then γ^* is a path and

$$\int_{\gamma^*} f(z)dz = -\int_\gamma f(z)dz.$$

Proof. γ^* is obviously continuous and piecewise C^1.

$$\int_{\gamma^*} f(z)dz = \int_a^b f(\gamma(a+b-t))(\gamma^*)'(t)dt = \int_a^b f(\gamma(a+b-t))(-\gamma'(a+b-t))dt$$

with a change of variable $u = a + b - t$ ($t = a \leftrightarrow u = b; t = b \leftrightarrow u = a; du = -dt$) we get

$$\int_{\gamma^*} f(z)dz = \int_a^b f(\gamma(u))\gamma'(u)(-du) = -\int_a^b f(\gamma(u))\gamma'(u)du = -\int_\gamma f(z)dz.$$ □

Proposition 11.5. *Let $f : U \to \mathbb{C}$ continuous with $\Im m(\gamma) \subseteq U$.Let γ_1 and γ_2 be paths such that the end point of γ_1 is the starting point of γ_2. W.l.o.g. we can assume $\gamma_1 : [0; 1] \to \mathbb{C}$ and $\gamma_2 : [0; 1] \to \mathbb{C}$. Then our condition writes $\gamma_1(1) = \gamma_2(0)$. The concatenation $\gamma = \gamma_1 * \gamma_2$ of γ_1 and γ_2 is by definition*

$$\gamma : [0; 2] \to \mathbb{C}, \quad t \mapsto \begin{cases} \gamma_1(t) & \text{if } 0 \le t \le 1; \\ \gamma_2(t-1) & \text{if } 1 \le t \le 2. \end{cases}$$

Then γ is a path and

$$\int_\gamma f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz.$$

Note that the definition of γ is coherent because $\gamma_1(1) = \gamma_2(0)$. To be perfectly formal, since we assume that both γ_1 and γ_2 are defined on $[0, 1]$, we rather define the concatenation of two paths, *up to change of parametrization.* The concatenation operation is associative, again *up to change of parametrization.*

(Picture showing two paths and their concatenation.)

Example: $[A;B;C;A] = [A,B] * [B;C] * [C,A]$.

Proof. γ is continuous and piece-wise C^1 because $\gamma|_{[0;1]}$ and $\gamma|_{[1;2]}$ are continuous and piece-wise C^1 (γ may not be differentiable in 1).

$$\int_\gamma f(z)dz = \int_0^2 f(\gamma(t))\gamma'(t)dt = \int_0^1 f(\gamma_1(t))\gamma_1'(t)dt + \int_1^2 f(\gamma_2(t))\gamma_2'(t)dt = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz. \quad \square$$

Definition 11.6. *Let $\gamma : [a;b] \to \mathbb{C}$ be a path. Then its length $L(\gamma)$ is*

$$L(\gamma) := \int_a^b |\gamma'(t)|dt.$$

Remark: With the same hypothesis as in Proposition 11.3 above, we have $L(\gamma \circ \varphi) = L(\gamma)$. Indeed,

$$L(\gamma \circ \varphi) = \int_{a'}^{b'} |(\gamma \circ \varphi)'(t)|dt = \int_{a'}^{b'} |\gamma'(\varphi(t))|\varphi'(t)dt = \int_a^b |\gamma'(u)|du = L(\gamma).$$

Proposition 11.7. (standard estimate) *Let $\gamma : [a;b] \to \mathbb{C}$ be a path. Let $f : U \to \mathbb{C}$ continuous with $\mathfrak{Im}(\gamma) \subseteq U$. Then*

$$\left|\int_\gamma f(z)dz\right| \leq \sup_{z \in \mathfrak{Im}(\gamma)} |f(z)| \cdot L(\gamma)$$

or equivalently

$$\left|\int_\gamma f(z)dz\right| \leq \sup_{t \in [a;b]} |f(\gamma(t))| \cdot L(\gamma).$$

Proof. $\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt$. Set $M = \sup_{z\in\mathfrak{Im}(\gamma)} |f(z)| = \sup_{t\in[a;b]} |f(\gamma(t))|$, then $|f(\gamma(t))\gamma'(t)| \leq M \cdot |\gamma'(t)|$. This implies

$$\Big|\int_\gamma f(z)dz\Big| \leq \int_a^b M \cdot |\gamma'(t)| = M \cdot \int_a^b |\gamma'(t)| = M \cdot L(\gamma). \quad \square$$

Corollary 11.8. *Let $f_n, f : U \to \mathbb{C}$ continuous function ($n \geq 1$). Assume that $f_n \xrightarrow{n\to\infty} f$ locally uniformly. Let $\gamma : [a;b] \to \mathbb{C}$ be a path with $\mathfrak{Im}(\gamma) \subseteq U$. Then*

$$\lim_{n\to\infty} \int_\gamma f_n(z)dz = \int_\gamma f(z)dz.$$

Proof.

$$\left|\int_\gamma f_n(z) - f(z)dz\right| = \left|\int_\gamma (f_n - f)(z)dz\right| \leq \sup_{z\in\mathfrak{Im}(\gamma)} |(f_n - f)(z)| \cdot L(\gamma).$$

It's enough to show that $\sup_{z\in\mathfrak{Im}(\gamma)} |(f_n - f)(z)| \xrightarrow{n\to\infty} 0$. Since $\mathfrak{Im}(\gamma)$ is compact, this follows directly from the local uniform convergence. $\square$

11.3. **Anti-derivative and integrals.**

Definition 11.9. (anti-derivative) *Let $f : U \to \mathbb{C}$ be continuous, U open. An anti-derivative of f is a function $F : U \to \mathbb{C}$ which is complex differentiable in any $z_0 \in U$ and such that $F'(z_0) = f(z_0)$ for all $z_0 \in U$.*

Proposition 11.10. *Let $F : U \to \mathbb{C}$ be holomorphic, set $F' = f$. Let $\gamma : [a;b] \to \mathbb{C}$ be a path with $\mathfrak{Im}(\gamma) \subseteq U$. Then*

$$\int_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(a)).$$

Proof. $\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt = \int_a^b (F \circ \gamma)'(t)dt = F(\gamma(b)) - F(\gamma(a))$. $\square$

Definition 11.11. (closed path) *A path $\gamma : [a;b] \to \mathbb{C}$ is called closed if $\gamma(a) = \gamma(b)$.*

Corollary 11.12. *Let $f : U \to \mathbb{C}$ be continuous. Assume f has an anti-derivative. Let γ be a closed path with $\mathfrak{Im}(\gamma) \subseteq U$. Then,*

$$\int_\gamma f(z)dz = 0.$$

Notation When γ is a closed path (or a cycle; see later), the path integral is sometimes denoted $\oint_\gamma f(w)dw$ instead of $\int_\gamma f(w)dw$.

12. Path integrals and holomorphic functions

12.1. **Holomorphy criteria via integrals.** To explain the content of this section, we start by two informal remarks.

i) If f has an anti-derivative then f is holomorphic. Indeed, F is complex-differentiable in any point and $F' = f$ is continuous. Then by definition F is holomorphic which implies that $F' = f$ is also holomorphic.
ii) In real analysis, integrals are used to construct anti-derivative via $F(t) := \int_a^t f(u)du$.

$\to$ to prove that something is holomorphic, we can try to construct an anti-derivative via a similar formula as above. For this we need the following assumption on the domain.

Definition 12.1. (star-shaped) *An open set $U \subseteq \mathbb{C}$ is called star-shaped if there exists $a \in U$ such that for any $z \in U, [a; z] \subseteq U$.*

Example: (draw a star-shaped domain U.)

Remark: Convex sets are star-shaped (with any choice of a).

Proposition 12.2. *Let U be a star-shaped open set. Assume that for any triangle $[A; B; C; A]$ included in U*

$$\int_{[A;B;C;A]} f(z)dz = 0.$$

Then f has an anti-derivative, and, hence, is holomorphic.

Proof. Let $a \in U$ as in Definition 12.1. Then for $z \in U$, define $F(z) = \int_{[a;z]} f(w)dw$ which is well defined since $[a; z] \subseteq U$. Let $z_0 \in U$. There exists $r > 0$ such that $|z - z_0|$ implies that $z \in U$. For $|z - z_0| < r$

$$F(z) - F(z_0) = \int_{[a;z]} f(w)dw - \int_{[a;z_0]} f(w)dw$$

by definition. By hypothesis

$$\int_{[a;z;z_0;a]} f(w)dw = \int_{[a;z]} f(w)dw + \int_{[z;z_0]} f(w)dw + \int_{[z_0;a]} f(w)dw = 0$$

(Picture showing the triangle $[a; z; z_0; a]$ and the various integration paths.
We therefore have

$$F(z) - F(z_0) = \int_{[z_0;z]} f(w)dw = \int_0^1 f((1-t)z_0 + tz)(z - z_0)dt.$$

$$\frac{F(z) - F(z_0)}{z - z_0} - f(z_0) = \int_0^1 [f((1-t)z_0 + tz) - f(z_0)]dt.$$

We use the continuity of f in z_0 : Fix $\varepsilon > 0$ then there exists $r' > 0$ such that $|z' - z_0| < r' \Rightarrow |f(z') - f(z_0)| \le \varepsilon$ for $|z - z_0| < \min(r, r')$, then for each $0 \le t \le 1$,

$$|(1-t)z_0 + tz - z_0| \le t|z - z_0| < r'$$

thus

$$|f((1-t)z_0 + tz) - f(z_0)| \le \varepsilon$$

integrating over t between 0 and 1,

$$\left|\frac{F(z) - F(z_0)}{z - z_0} - f(z_0)\right| \le \varepsilon.$$

i.e.

$$\lim_{z\to z_0} \frac{F(z)-F(z_0)}{z-z_0} - f(z_0) = 0.$$

This proves that $F'(z_0) = f(z_0)$. Since this holds for any z_0 in U, we have proved that F is an anti-derivative of f; the existence of such an anti-derivative implies that f is holomophic. $\square$

Remarks:

- It is enough to assume that $\int_{[a;B;C;a]} f(z)dz = 0$ where a is a fixed point of U such that $z \in U \Rightarrow [a;z] \subseteq U$.
- There are two natural notions for a triangle included in U.
  - $[A;B;C;A] \subseteq U$ (only the sides of any triangle assumed in U).
  - We say that a triangle is completely included in U if $[A;B;C;A] \subseteq U$ and its interior is also included in U. If U is star-shaped, then Δ included in U if and only if Δ completely included in U.

We now drop the "star-shape" hypothesis.

Proposition 12.3. (Morera's criterion) *Let $f : U \to \mathbb{C}$ continuous, U open. Assume that, for all triangles $[A;B;C;A]$ that are completely included in U,*

$$\int_{[A;B;C;A]} f(z)dz = 0.$$

Then f is holomorphic.

Proof. Let $z_0 \in U$. Then there exists $r > 0$ such that $|z - z_0| < r \Rightarrow z \in U$. Consider the restriction $f|_{D(z_0,r)}$.

- $f|_{D(z_0,r)}$ is a continuous function.
- Defined on $D(z_0, r)$ which is a convex open set, in particular it is star-shaped.
- for any triangle $[A;B;C;A]$ included in $D(z_0,r)$, $[A;B;C;A]$ is completely included in $D(z_0,r)$, thus in U.

Therefore by hypothesis,

$$\int_{[A;B;C;A]} f(z)dz = 0.$$

Thus $f|_{D(z_0,r)}$ is holomorphic for all $z_0 \in U \Rightarrow f$ holomorphic on U. $\square$

(Informally, the proof says that every open set is locally star-shaped. Since being holomorphic is a local property, we do not need the "star-shape hypothesis". On the opposite, "having an anti-derivative" is not a local property; so we cannot conclude here that f has an anti-derivative.)

12.2. **Complex derivatives are automatically continuous.** The following is a kind of converse to the results of the previous section. Integrals of holomorphic functions on triangles do vanish.

Proposition 12.4. (Goursat) *Let $f : U \to \mathbb{C}$, U open, f continuous. Assume f is complex differentiable in all points of U, but possibly one (say p). Then for any triangle $[A;B;C;A]$ completely included in U, we have*

$$\int_{[A;B;C;A]} f(z)dz = 0.$$

In particular, f is holomorphic.

Comment: In other terms, if we assume f complex differentiable in all points of an open set U but one, then f is automatically complex-differentiable in this last point and f' is automatically continuous.

Proof. Let Δ be a triangle completely included in U. We split the proof in four different cases.

First assume p is outside Δ. Consider the midpoint subdivision of Δ as shown in Fig. 1 below.

Claim: We have $\int_\Delta f(z)dz = \sum_{i=1}^4 \int_{\Delta_i} f(z)dz$.

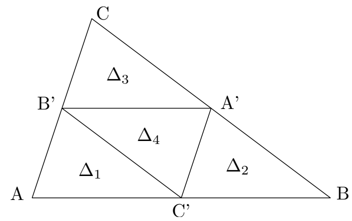


FIGURE 1. The midpoint subdivision of Δ.

Proof of the claim: Denote C' as midpoint of $[A;B]$, A' as midpoint of $[C;B]$, B' as midpoint of $[A;C]$, then

$$\int_{\Delta_1} f(z)dz = \int_{[A;C']} f(z)dz + \int_{[C';B']} f(z)dz + \int_{[B';A]} f(z)dz.$$

Similarly for $\int_{\Delta_2} f(z)dz, \int_{\Delta_3} f(z)dz, \int_{\Delta_4} f(z)dz$. When we sum $\int_{[C';B']} f(z)dz$ and $\int_{[B';C']} f(z)dz$ cancel each other (same for $[B';A']$ and $[A';C']$). We get

$$\sum_{i=1}^{4} \int_{\Delta_i} f(z)dz = \int\limits_{[A;C']} f(z)dz + \int\limits_{[B';A]} f(z)dz + \int\limits_{[C';B]} f(z)dz + \int\limits_{[B;A']} f(z)dz + \int\limits_{[A';C]} f(z)dz + \int\limits_{[C;B']} f(z)dz = \int\limits_{\Delta} f(z)dz.$$

The claim is proved.

$|\int_\Delta f(z)dz| \le \sum_{i=1}^4 |\int_{\Delta_i} f(z)dz|$ implies that there exists $1 \le i \le 4$ such that $|\int_{\Delta_i} f(z)dz| \ge \frac{|\int_\Delta f(z)dz|}{4}$. Call $\Delta^{(1)}$ the triangle Δ_i as above. Subdivide $\Delta^{(1)}$ as shown in Fig. 2.

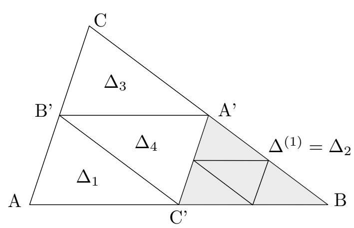


FIGURE 2. The midpoint subdivision of $\Delta^{(1)}$.

Then $\int_{\Delta^{(1)}} f(z)dz = \sum_{i=1}^4 \int_{\Delta_i^{(1)}} f(z)dz$. There exists $1 \le i \le 4$ such that $|\int_{\Delta_i^{(1)}} f(z)dz| \ge \frac{1}{4}|\int_{\Delta^{(1)}} f(z)dz|$. Call this triangle $\Delta^{(2)}$. Iterating the idea, we construct a sequence $\Delta^{(1)}, \Delta^{(2)}, \Delta^{(3)}, \dots$ such that

$$\left|\int_{\Delta^{(n)}} f(z)dz\right| \ge \frac{1}{4^n}\left|\int_\Delta f(z)dz\right| \quad (*)$$

$L(\Delta^{(n)}) = \frac{1}{2^n}L(\Delta)$. (length of a triangle seen as path = perimeter of a triangle). Call $\overline{\Delta^n}$ = triangle $\Delta^{(n)}$ with its interior. By construction $\overline{\Delta^1} \supseteq \overline{\Delta^2} \supseteq \overline{\Delta^3} \supseteq \cdots \supseteq \overline{\Delta^n} \supseteq \dots$. The diameter $\operatorname{diam}(\overline{\Delta^n})$ tends to 0 (for a triangle, $\operatorname{diam}(\overline{\Delta}) \le \frac{1}{2}L(\Delta))$), so that, by a classical theorem of analysis, the intersection $\bigcap_{n\ge1} \overline{\Delta^n}$ is non-empty and reduced to a single point $\{z_0\}$.

But f is complex differentiable in $z_0 \in U$ (because we assumed that $\overline{\Delta} \subseteq U$ and p is outside Δ). There exists $r > 0$ such that

$$|z - z_0| < r \Rightarrow f(z) = f(z_0) + f'(z_0)(z - z_0) + (z - z_0)\alpha(z)$$

with $|\alpha(z)| < \varepsilon$. For n sufficiently large, $\operatorname{diam}(\overline{\Delta^n}) \le r$,

$$\int_{\Delta^{(n)}} f(z)dz = \int_{\Delta^{(n)}} f(z_0)dz + \int_{\Delta^{(n)}} f'(z_0)(z - z_0)dz + \int_{\Delta^{(n)}} (z - z_0)\alpha(z)dz.$$

But $\int_{\Delta^{(n)}} f(z_0)dz = 0$ (closed path, $z \mapsto f(z_0)$ has an anti-derivative $z \mapsto f(z_0) \cdot z$)

$$\int_{\Delta^{(n)}} f'(z_0)(z - z_0)dz = 0$$

for the same reason. Recall that for $z \in \Delta^{(n)}$,

$$|z - z_0| \leq \operatorname{diam}\left(\Delta^{(n)}\right) \leq \frac{1}{2^{n+1}} \cdot L(\Delta).$$

Hence we get

$$\left|\int_{\Delta^{(n)}} f(z)dz\right| \leq \int_{\Delta^{(n)}} |(z - z_0)||\alpha(z)|dz \leq \frac{1}{2^{n+1}} \cdot L(\Delta) \cdot \varepsilon \cdot L\left(\Delta^{(n)}\right) \leq \frac{1}{4^n}\frac{L(\Delta)^2}{2} \cdot \varepsilon.$$

Comparing with (*), we have

$$\frac{1}{4^n}\left|\int_{\Delta} f(z)dz\right| \leq \frac{1}{4^n}\frac{L(\Delta)^2}{2} \cdot \varepsilon,$$

which is true for any $\varepsilon > 0$, thus $\int_\Delta f(z)dz = 0$.

Assume now p is a vertex of Δ. We subdivide Δ as in Fig. 3.

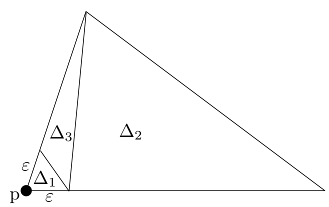


FIGURE 3. Subdivision of Δ is three triangles, where Δ_1 is a triangle of side ε around p.

Claim: $\int_\Delta f(z)dz = \sum_{i=1}^{3} \int_{\Delta_i} f(z)dz$

The claimed is proved exactly as the above claim for midpoint subdivision. From the previous case (p is outside the triangle), we know that $\int_{\Delta_2} f(z)dz = \int_{\Delta_3} f(z)dz = 0$. Thus

$$\left|\int_{\Delta} f(z)dz\right| = \left|\int_{\Delta_1} f(z)dz\right| \leq L(\Delta_1) \sup_{z\in\Delta_1} |f(z)|. \qquad (**)$$

But $L(\Delta_1) \leq 4\varepsilon$. Since f is a continuous function, $\sup_{z\in\Delta_1} |f(z)| \leq M$ for ε sufficiently small. (**) is true for any $\varepsilon > 0$, so that $\int_\Delta f(z)dz = 0$.

Assume now p is on an edge of the triangle Δ. We subdivide Δ as in Fig. 4.

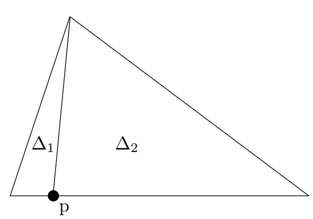


FIGURE 4. Subdivision of Δ into two triangles, such that p is a vertex of Δ_1 and Δ_2.

Claim (proved as before): $\int_\Delta f(z)dz = \int_{\Delta_1} f(z)dz + \int_{\Delta_2} f(z)dz$.
From the previous case (where p is a vertex of the triangle), $\int_{\Delta_1} f(z)dz = \int_{\Delta_2} f(z)dz = 0$, so that $\int_\Delta f(z)dz = 0$, as wanted.

Finally, assume that p is inside the triangle. We subdivide Δ as in Fig. 5 and apply the previous case. $\square$

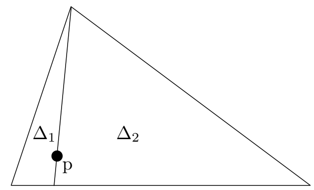


FIGURE 5. Subdivision of the triangle in two parts such that p is on an edge of the triangles Δ_1 and Δ_2.

Corollary 12.5. *Let $f_n, f : U \to \mathbb{C}$, U open. Assume f_n holomorphic for $n \geq 1$. Assume $f_n \xrightarrow{n\to\infty} f$ locally uniformly on U. Then f is holomorphic.*

Proof. Take a triangle Δ completely included in U. Then $\int_\Delta f_n(z)dz = 0$ for $n \geq 1$ by Proposition 12.4. But, from Corollary 11.8, we know that $\int_\Delta f_n(z)dz \xrightarrow{n\to\infty} \int_\Delta f(z)dz$ so that $\int_\Delta f(z)dz = 0$. Proposition 12.3 implies that f is holomorphic. □

Remarks on the proof: In general, Goursat's theorem and Morera's criterium are very convenient to prove that a function defined as a limit, an integral, ... is holomorphic!

Remarks on the statement:

- Not true in real analysis (a uniform limit of differentiable functions might not be differentiable).
- We saw in the previous chapter that this also implies $f_n^{(k)} \xrightarrow{n\to\infty} f^{(k)}$ for $k \geq 1$, locally uniformly.

12.3. Summary on holomorphic functions.

Theorem 12.6. *$f : U \to \mathbb{C}$, $U \subseteq \mathbb{C}$ open set. Then the following are equivalent:*

i) f is holomorphic.
ii) f is complex-differentiable on any $z \in U$ and f' is continuous.
iii) f satisfies Cauchy's formula

$$f(z) = \int_{\partial D(z_0,r)} \frac{f(w)}{w-z} dw.$$

iv) For each $z_0 \in U$, the function f admits a power series expansion on the biggest open disk centred in z_0 included in U.
v) f is analytic.
vi) f is infinitely many times complex-differentiable.
vii) f is continuous on U and f is complex-differentiable in any point of U (but eventually one).
viii) For any triangle completely included in U

$$\int_{[A;B;C;A]} f(z)dz = 0.$$

ix) If $V \subseteq U$ and V is star-shaped, then $f|_V$ has an anti-derivative.
x) If $V \subseteq U$ and V is star-shaped, then

$$\int_\gamma f(z)dz = 0$$

for any closed path with $\Im m(\gamma) \subseteq V$.

Proof. $(i) \Leftrightarrow (ii)$ by definition. (in some textbooks $(i) \Leftrightarrow (vii)$ by definition)
$(ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (vi)$ (previous chapter)
$(vi) \Rightarrow (ii)$ (trivial)
$(ii) \Rightarrow (vii)$ (trivial)
$(vii) \Rightarrow (viii) \Rightarrow (ix) \Rightarrow (ii)$ (this chapter)
$(ix) \Rightarrow (x)$ Corollary 11.12
$(x) \Rightarrow (ix)$ Proposition 12.2 □

Remark: in (x) "star-shaped" is crucial: below, we give two examples of non-zero integrals of an holomorphic function along a closed path, in non star-shaped domains. We will see later in the lecture that "V star-shaped" can be replaced by the less restrictive assumption "V *simply connected*".

Example 1: $U = \mathbb{C} \setminus \{0\},\ f : z \mapsto \frac{1}{z}$ holomorphic on U and

$$\int_{\partial D(0,1)} \frac{1}{z} dz = \int_0^{2\pi} \frac{ie^{i\theta}}{e^{i\theta}} d\theta = 2\pi i.$$

Example 2: $f : U \to \mathbb{C}$ holomorphic. Fix $z \in U$, consider $g(w) = \frac{f(w)}{w-z}$ holomorphic on $U \setminus \{z\}$ and

$$\int_{\partial D(z_0,r)} g(w)dw = f(z) \qquad by (iii).$$

$f(z)$ is "in general" non-zero, "because" $U \setminus \{z\}$ is not star-shaped.

13. Winding numbers

Goal of this section: generalize Cauchy's formula to other closed paths than circles. This needs the notion of winding number.

13.1. **Definition and properties.** Throughout this section, $\gamma : [\alpha; \beta] \to \mathbb{C}$ is a **closed path** and we define $\Omega = \mathbb{C} \setminus \mathfrak{Im}(\gamma)$.

(Draw a closed path, and show what Ω is; insist that it is typically not connected.)

Definition 13.1. (winding number) *For $z \in \Omega$, define*

$$n_\gamma(z) = \frac{1}{2\pi i} \int_\gamma \frac{1}{w - z} dw.$$

$n_\gamma(z)$ is called winding number of γ around z.

(Since $z \in \Omega$, the denominator in the above integrand does not vanish for $w \in \mathfrak{Im}(\gamma)$.)

Remark: We will see later that $n_\gamma(z)$ can be interpreted as "number of times γ is turning (or winding) around z".

Proposition 13.2. *For any closed path γ, the function n_γ is a continuous function on Ω.*

Proof. By definition,

$$2\pi\, i\, n_\gamma(z) = \int_\gamma \frac{1}{w - z} dw = \int_\alpha^\beta \frac{\gamma'(t)}{\gamma(t) - z} dt = \sum_{i=0}^{l-1} \int_{t_i}^{t_{i+1}} \frac{\gamma'(t)}{\gamma(t) - z} dt$$

where γ is C^1 on each segment $[t_i; t_{i+1}]$ $\quad (t_0 = \alpha; t_l = \beta)$. Then, for each $i \leq l-1$, the function $(t, z) \mapsto \frac{\gamma'(t)}{\gamma(t)-z}$ is continuous and bounded on $[t_i; t_{i+1}] \times K$ for each compact $K \subseteq \Omega$. Thus

$$z \mapsto \int_{t_i}^{t_{i+1}} \frac{\gamma'(t)}{\gamma(t) - z} dt$$

is continuous on K. We conclude that n_γ is continuous on Ω. □

Proposition 13.3. *For any closed path γ, the function n_γ only takes integer values on Ω.*

Remark: here, the fact that γ is closed is crucial. In general, **do not** write n_γ for non-closed paths (even if the above integral makes sense).

Proof. Assume w.l.o.g. $\alpha = 0,\ \beta = 1$. Introduce

$$\varphi(t) = \exp\left(\int_0^t \frac{\gamma'(s)}{\gamma(s) - z} ds\right)$$

z fixed.

- $\varphi(0) = \exp(0) = 1$
- $\varphi(1) = \exp\left(\int_0^1 \frac{\gamma'(s)}{\gamma(s)-z} ds\right) = \exp\left(\int_\gamma \frac{1}{w-z} dw\right) = \exp\left(2\pi i n_\gamma(z)\right)$
- Look at φ' :. For t in $[\alpha, \beta] \setminus \{t_1, \cdots, t_{k-1}\}$ (the t_i are the points appearing in the subdivision wrt which γ is piece-wise C^1), we have
$$\frac{\varphi'(t)}{\varphi(t)} = \frac{\gamma'(t)}{\gamma(t) - z},$$
and therefore,
$$\left(\frac{\varphi(t)}{\gamma(t) - z}\right)' = \frac{\varphi'(t)(\gamma(t) - z) - \gamma'(t)\varphi(t)}{(\gamma(t) - z)^2} = 0.$$
The function $t \mapsto \frac{\varphi(t)}{\gamma(t)-z}$ is continuous on $[\alpha, \beta]$ and has derivative zero, except possibly at finitely many points. Thus it is constant.
In particular, $\frac{\varphi(1)}{\gamma(1)-z} = \frac{\varphi(0)}{\gamma(0)-z}$. But γ is a closed path, i.e. $\gamma(0) = \gamma(1)$, implying that $\varphi(0) = \varphi(1)$.

Combining the three items, we get $\exp\left(2\pi i\, n_\gamma(z)\right) = 1$, which proves that $n_\gamma(z)$ is an integer. □

Corollary 13.4. *For any* **connected** *subset V of Ω, the function n_γ is constant on V.*

Proof. By Lemma 3.12 and the continuity of n_γ, we know that $n_\gamma(V)$ is connected. Moreover, it is included in $\mathbb{Z}$.

But, in $\mathbb{Z}$, each singleton $\{i\} \subseteq \mathbb{Z}$ is open ($\{i\} = (i-1, i+1) \cap \mathbb{Z}$) and closed ($\{i\} = \{i\} \cap \mathbb{Z}$). Thus the connected subsets of $\mathbb{Z}$ are singletons and $\emptyset$.

We conclude that $n_\gamma(V)$ is either empty (which can only happen if V is empty) or a singleton. I.e. n_γ is constant on V. □

13.2. **Cauchy formula with general paths in star-shaped domains.** The following proposition justifies why we introduced the winding number.

Proposition 13.5. *$U \subseteq \mathbb{C}$ star-shaped open set, $f : U \to \mathbb{C}$ holomorphic. γ closed path with $\Omega := \Im m(\gamma) \subseteq U$*
Then, for any fixed $z \in U \setminus \Omega$, we have
$$\int_\gamma \frac{f(w)}{w - z} dw = 2\pi i n_\gamma(z) f(z).$$

Proof. Define the auxiliary function
$$g(w) = \begin{cases} \frac{f(w)-f(z)}{w-z} & \text{for} \quad w \in U \setminus \{z\} \\ f'(z) & \text{for} \quad w = z \end{cases}$$

Clearly, g is continuous on U and holomorphic on $U \setminus \{z\}$. From the results of the previous chapter, g is holomorphic on U. Thus $\int_\gamma g(w) = 0$ because U is star-shaped and
$$\int_\gamma \frac{f(w)}{w - z} dw = f(z) \int_\gamma \frac{1}{w - z} dw = 2\pi i n_\gamma(z) f(z).$$ □

13.3. **Computation of winding number.**

Preliminary: decomposition of a set into connected components

Let Ω be a subset of $\mathbb{C}$ (what follows is valid more generally in any topological space). We consider the **maximal connected** subsets of Ω (maximal for inclusion). Call them $(\Omega_i)_{i \in I}$ (I might be finite or infinite). This forms a partition of Ω. I.e.

- they are disjoint (if $\Omega_i \cap \Omega_j \neq \emptyset$, then it is easy to prove that the union $\Omega_i \cup \Omega_j$ is connected, contradicting the maximality of Ω_i).
- and their union is Ω (for each x in Ω, the set $\{x\}$ is connected, and thus is either strictly contained in some Ω_i, or $\{x\}$ is maximal connected and is itself one of the Ω_i).

The $(\Omega_i)_{i\in I}$ are called *connected components* of Ω.

Back to winding numbers:
Recall that the winding number is constant on the connected subsets of Ω, so in particular on its connected components. On one of them, its value is trivially determined.

Lemma 13.6. *Ω $(= \mathbb{C} \setminus \mathfrak{Im}(\gamma))$ has exactly one unbounded connected component Ω_∞ and, for z in Ω_∞, $n_\gamma(z) = 0$.*

Proof. By definition, γ is continuous on a compact set, and therefore bounded. I.e. $\exists M > 0$ such that $\mathfrak{Im}(\gamma) \subseteq D(0, M)$. Consider $\overline{D(0, M)}^c := \{z : |z| > M\}$. We have $\overline{D(0, M)}^c \subseteq \Omega$.

Claim: $\overline{D(0, M)}^c$ is path-connected, and hence connected.

(Proof of the claim by a picture.)

Therefore, $\overline{D(0, M)}^c$ is included in one connected component of Ω. Call this connected component Ω_∞. Clearly Ω_∞ is unbounded. Take another connected component Ω_i. Then $\Omega_i \cap \Omega_\infty = \emptyset$. Since Ω_∞ contains $\overline{D(0, M)}^c$, this implies that $\Omega_i \subseteq \overline{D(0, M)}$; in particular, Ω_i is bounded. We have proved the existence of a unique unbounded connected component.

From Corollary 13.4, n_γ is constant on Ω_∞. But $n_\gamma(z) = \frac{1}{2\pi i}\int_\gamma \frac{1}{w-z}dw$. Thus by standard estimate,

$$|n_\gamma(z)| \leq \frac{1}{2\pi i} \cdot L(\gamma) \cdot \sup_{w\in\mathfrak{Im}(\gamma)} \left|\frac{1}{w - z}\right|.$$

For $|z| > M,\ w \in \mathfrak{Im}(\gamma),\ |w - z| \geq |z| - |w| \geq |z| - M$. Since $|w| \leq M,\ \sup_{w\in\mathfrak{Im}(\gamma)} \left|\frac{1}{w-z}\right| \leq \frac{1}{|z|-M}$. Finally

$$|n_\gamma(z)| \leq \frac{1}{2\pi i} L(\gamma) \frac{1}{|z| - M}.$$

When $|z| \longrightarrow \infty$ this upper bound tends to 0. But $n_\gamma(z)$ is constant on Ω_∞. So $n_\gamma(z) = 0$ on Ω_∞. $\square$

Here is an interesting consequence, that we shall use later.

Corollary 13.7. *U star-shaped domain. γ path with $\mathfrak{Im}(\gamma) \subseteq U$. Let $b \notin U$. Then $n_\gamma(b) = 0$.*

Proof. By definition of star-shaped, we can take $a \in U$ such that $z \in U$ implies $[a; z] \in U$. Recall that $b \notin U$. This implies that for all $\lambda \geq 0$, $b + \lambda(b - a) \notin U$ (otherwise b would be in U; make a picture).

Now, for any $M > 0$, the points b and $b+M(b-a)$ are connected by a path in $\mathbb{C}\setminus U$ and thus in Ω (the path can be chosen as the line segment between them). They are therefore in the same connected component of Ω. Since M may be chosen arbitrary large, this must be the unbounded component Ω_∞. From the previous Lemma, $n_\gamma(b) = 0$. $\square$

Case of circles:
Let $z_0 \in \mathbb{C}, r > 0$ and $\gamma = \partial D(z_0, r)$. Let $\Omega = \mathbb{C}\setminus\partial D(z_0, r)$. Ω has two connected components $\{z : |z-z_0| > r\}$ (unbounded) and $\{z : |z - z_0| < r\}$ (bounded).
(These two sets are open and closed in Ω, and trivially path-connected, so that they are indeed the connected components of Ω.)

(Draw a circle and color the two connected components of its complement.)

Proposition 13.8. *With the notation above*

$$n_{\partial D(z_0,r)}(z) = \begin{cases} 0 & \textit{if} \quad |z - z_0| > r \\ 1 & \textit{if} \quad |z - z_0| < r. \end{cases}$$

Proof. $\{z : |z - z_0| > r\}$ is the unbounded connected component of Ω. This implies the first case. For the second case, we can assume w.l.o.g. that $z = z_0$ (we know $n_{\partial D(z_0,r)}$ is constant on $\{z : |z - z_0| < r\}$). Then

$$2\pi i n_\gamma(z_0) = \int_{\partial D(z_0,r)} \frac{1}{w - z_0} dw = \int_0^{2\pi} \frac{ire^{i\theta}}{re^{i\theta}} d\theta = 2\pi i.$$

$\Rightarrow n_\gamma(z_0) = 1.$ $\square$

Remarks:

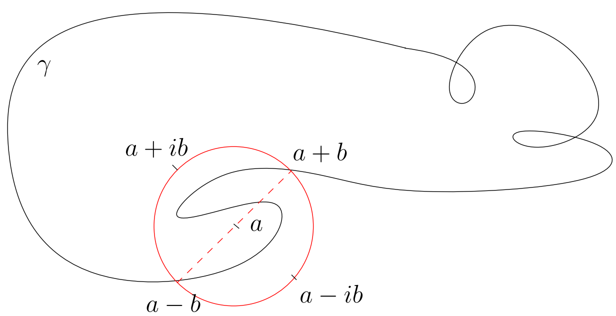
- Proposition 13.5 for circles writes
$$\int_{\partial D(z_0,r)} \frac{f(w)}{w-z} dw = \begin{cases} 2\pi i\, f(z) & \text{if } z \text{ is inside the circle} \\ 0 & \text{if } z \text{ is outside the circle.} \end{cases}$$
This indeed contains Cauchy formula.
- If we take γ^* (path γ taken in the other direction $\gamma^*(t) = \gamma(1-t)$)), then
$$\int_{\gamma^*} \frac{1}{w-z} dw = -\int_{\gamma} \frac{1}{w-z} dw.$$
Therefore, $n_{\gamma^*}(z) = -n_\gamma(z)$. In particular, the winding number of a circle in clockwise direction is 0 or -1.

More general closed paths (the wall crossing theorem)

Theorem 13.9. (wall crossing theorem) *We assume that $\gamma : [\alpha; \beta] \to \mathbb{C}$ closed path. and that there exists u, v with $\alpha < u < v < \beta$ and $a, b \in \mathbb{C}$ such that*

- *$\gamma(u) = a - b$;*
- *$\gamma(v) = a + b$;*
- *$|\gamma(t) - a| < |b|$ if and only if $u < t < v$;*
- *$|\gamma(t) - a| = b$ if and only if $t = u$ or $t = v$.*

Then $n_\gamma(a + ib) = n_\gamma(a - ib) + 1$.



Proof. Call γ_{int} and γ_{ext} the parts of the path γ that are inside and outside the disk $D(a, |b|)$ respectively (red disk on the above picture). We also consider the semi circle C_1 (resp. C_2) of center a and radius $|b|$ going from $a + b$ to $a - b$ (resp. from $a - b$ to $a + b$) in the trigonometric direction. Finally let z_+ (resp. z_-) be some point inside the disk $D(a, |b|)$ in the same connected component of Ω as $a + ib$ (res. $a - ib$). In particular,
$$n_\gamma(z_+) = n_\gamma(a + ib), \quad n_\gamma(z_-) = n_\gamma(a - ib).$$
We use $+$ for path concatenation and $*$ for the path reversing operation. We note that:

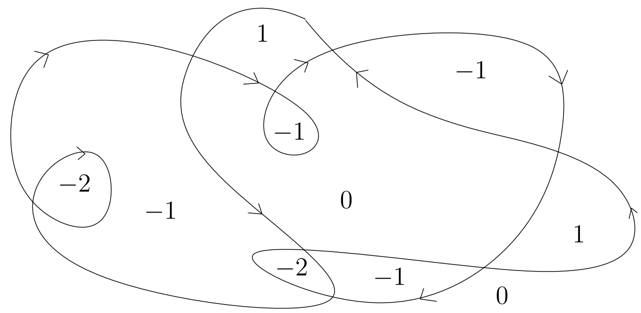
- z_- is in the unbounded component of the complement of the closed path $\gamma_{\text{int}} + C_1$, so that
$$n_{\gamma_{\text{int}}+C_1}(z_-) = 0.$$
- z_+ is in the unbounded component of the complement of the closed path $\gamma_{\text{int}} + C_2^*$, so that
$$n_{\gamma_{\text{int}}+C_2^*}(z_+) = 0.$$
- z_+ and z_- are inside the circle $\partial D(a, |b|) = C_1 + C_2$, so that
$$n_{C_1+C_2}(z_+) = n_{C_1+C_2}(z_-) = 1.$$
- z_+ and z_- are in the same connected component of the closed path $\gamma_{\text{ext}} + C_2$, so that
$$n_{\gamma_{\text{ext}}+C_2}(z_+) = n_{\gamma_{\text{ext}}+C_2}(z_-).$$

We conclude as follows

$$\begin{aligned} n_\gamma(z_+) &= n_{\gamma_{\text{int}}+\gamma_{\text{ext}}}(z_+) = n_{\gamma_{\text{int}}+C_2^*}(z_+) + n_{\gamma_{\text{ext}}+C_2}(z_+) = n_{\gamma_{\text{ext}}+C_2}(z_-) \\ &= n_{\gamma_{\text{ext}}+\gamma_{\text{int}}}(z_-) + n_{C_1+C_2}(z_-) - n_{\gamma_{\text{int}}+C_1}(z_-) = n_\gamma(z_-) + 1. \end{aligned} \qquad \square$$

Remarks:

- Informally, the theorem says "when we cross a wall, we increase or decrease the winding number by 1, depending on the orientation of the path" → from this and the fact that $n_\gamma(z) = 0$ on Ω_∞ we can compute winding numbers for "nice" paths. The winding number of a given z then is, at its name suggests, the (algebraic) number of times the path γ turns around z in the trigonometric direction. For example,



- Jordan's theorem states that the complement of a simple closed path (i.e. without self-intersections) has exactly two connected components, one bounded and one unbounded. The value of the winding number in the bounded region is then ± 1 (the sign depends on the orientation of the path), as can be proved in most cases with wall-crossing theorem. In general, what are these two components is quite obvious, and we use that the winding number is ± 1 inside and 0 outside without formal proof.

14. General Cauchy formula

Goal of the section: give a Cauchy formula that holds also in non star-shaped domains (but as seen before, we need some hypothesis on the path). It will more convenient to consider formal linear combinations of paths (called chains and cycles) instead of single paths.

14.1. **The statement.**

Definition 14.1. (chain) *A chain Γ is a finite linear combination of paths with integer coefficients that is*

$$\Gamma = \sum_{i=1}^{d} \alpha_i \gamma_i \qquad \alpha_i \in \mathbb{Z},\ \gamma_1, \dots, \gamma_d \text{ are paths}$$

By convention, if

$$\Gamma = \sum_{i=1}^{d} \alpha_i \gamma_i \qquad \mathfrak{Im}(\Gamma) := \bigcup_{i=1}^{d} \mathfrak{Im}(\gamma_i)$$

if $f : U \to \mathbb{C}$ continuous with $\mathfrak{Im}(\Gamma) \subseteq U$,

$$\int_\Gamma f(w)dw = \sum_{i=1}^{d} \alpha_i \int_{\gamma_i} f(w)dw.$$

We often implicitly consider chains up to replacing a formal combination $\gamma_1 + \gamma_2$ by their concatenation (when it's well defined). This does not change the integral! This allows to assume that each path γ_i in a chain is C^1 (and not only piecewise C^1; indeed each path is a concatenation of finitely many C^1 paths).

Definition 14.2. (cycle) *A cycle is a chain $\Gamma = \sum \alpha_i \gamma_i$ such that each $a \in \mathbb{C}$ appears as many times as starting point of γ_i as as endpoints of γ_i (counted with multiplicity α_i).*

Examples:

- $\Gamma = [A;B]+[B;C]+[C;A]$ (formal sum of paths, not the concatenation). A is once a starting point (of $[A;B]$) and once and endpoint (of $[C;A]$). Same for B and $C \Rightarrow \Gamma$ is a cycle.
- $\Gamma = 2[A;B]+[B;C]-[A;C]+\gamma$ where γ is any path from B to A. A is starting point of $[A;B]$ (with multiplicity 2) and $[A;C]$ (with multiplicity -1), endpoint of γ. B and C are also as many times starting points than endpoints. $\Rightarrow \Gamma$ is a cycle.
- Every linear combination of closed paths is a cycle.

If Γ is a cycle and $z \notin \mathfrak{Im}(\Gamma)$ define

$$n_\Gamma(z) = \frac{1}{2\pi i}\int_\Gamma \frac{1}{w-z}dw.$$

Note that $n_\Gamma(z) = \sum_{i=1}^{d} \alpha_i n_{\gamma_i}(z)$ if $\Gamma = \sum_{i=1}^{d} \alpha_i \gamma_i(z)$ with γ_i closed path (do not write this for non closed paths).

Lemma 14.3. *Let Γ be cycle. Then its winding number n_Γ is a continuous function on $\Omega := \mathbb{C} \setminus \mathfrak{Im}(\Gamma)$, taking integer values. Thus n_Γ is constant on connected components of Ω. Moreover $n_\Gamma(z) = 0$ for z in the unbounded connected component.*

Proof. Same proof that for closed paths. □

Theorem 14.4 (general Cauchy formula)**.** *Let U be an open set . Let Γ be a cycle in U, $z \in U$, $z \notin \mathfrak{Im}(\Gamma)$. Assume that, for any $\alpha \notin U$, $n_\Gamma(\alpha) = 0$. Then, for $f : U \to \mathbb{C}$ holomorphic,*

$$\int_\Gamma \frac{f(w)}{w-z}dw = 2\pi i n_\Gamma(z) f(z).$$

Remark: We **do not** assume U to be star shaped. This hypothesis is somehow replaced by the asumption "for any $\alpha \notin U$, $n_\Gamma(\alpha) = 0$".

Let us compare these two assumptions. If U star-shaped and γ closed path in U, $\alpha \notin U$, we have that $n_\gamma(\alpha) = 0$ because α is in the unbounded component of $\mathbb{C} \setminus \mathfrak{Im}(\gamma)$. This also holds for a cycle Γ instead of the closed path γ. In conclusion, the assumption of the theorem holds as soon as U is star-shaped. So this theorem is a generalisation of Cauchy formula for star-shaped sets.

14.2. **The proof.** Define

$$g(z,w) = \begin{cases} \frac{f(w)-f(z)}{w-z} & \text{if} \quad w \neq z \\ f'(z) & \text{if} \quad w = z. \end{cases}$$

We want to prove that $\int_\Gamma g(z,w)dw = 0$ for fixed $z \in U \setminus \mathfrak{Im}(\Gamma)$. Indeed, if this holds, the same proof as for star-shaped domains will prove Cauchy's formula.

Define two auxiliary functions

$$h_1 : U \to \mathbb{C} \quad z \mapsto \int_\Gamma g(z,w)dw$$
$$h_2 : \Omega_0 \to \mathbb{C} \quad z \mapsto \int_\Gamma \frac{f(w)}{w-z}dw$$

where Ω_0 is the open set $\{\alpha \in \mathbb{C} \setminus \mathfrak{Im}(\Gamma) : \ n_\Gamma(\alpha) = 0\}$. (By assumption Ω_0 contains $\mathbb{C} \setminus U$, i.e. $\Omega_0 \cup U = \mathbb{C}$).
If $z \in U \cap \Omega_0$, then

$$h_1(z) = \int_\Gamma \frac{f(w)-f(z)}{w-z}dw = \int_\Gamma \frac{f(w)}{w-z}dw - \underbrace{\int_\Gamma \frac{f(z)}{w-z}dw}_{=2\pi i n_\Gamma(z) f(z) = 0} = h_2(z).$$

We can therefore define the following function on $U \cup \Omega_0 = \mathbb{C}$.

$$h(z) = \begin{cases} h_1(z) & \text{for} \quad z \in U \\ h_2(z) & \text{for} \quad z \in \Omega_0. \end{cases}$$

Claim: (proved below) h_1 and h_2 are holomorphic functions, and hence, h is an entire function (holomorphic on $\mathbb{C}$).

We will prove that h is bounded, by considering its behaviour when $|z| \longrightarrow \infty$.

First note that for $|z|$ large enough, z is in the unbounded connected component of $\mathbb{C} \setminus \mathcal{I}m(\Gamma)$ which implies $n_\Gamma(z) = 0$, i.e. $z \in \Omega_0$. Therefore, for $|z|$ large enough,

$$|h(z)| = |h_2(z)| = \left| \int_\Gamma \frac{f(w)}{w-z} \right| \leq L(\Gamma) \cdot \sup_{w \in \mathcal{I}m(\Gamma)} \frac{|f(w)|}{|w-z|}.$$

But $\mathcal{I}m(\Gamma)$ is a compact subset of $\mathbb{C}$. Therefore it is bounded by some M_1 and $|f(w)|$, for w in $\mathcal{I}m(\Gamma)$, is bounded by some M_2. Then for $w \in \mathcal{I}m(\Gamma)$, we have

$$\frac{|f(w)|}{|w-z|} \leq \frac{M_2}{|z|-|w|} \leq \frac{M_2}{|z|-M_1}.$$

Combining both inequalities, for $|z|$ large enough,

$$|h(z)| \leq L(\Gamma) \frac{M_2}{|z|-M_1}.$$

This upper bound tends to 0 when $|z| \longrightarrow \infty$, so that $\lim_{|z| \to \infty} h(z) = 0$. In particular, h is bounded. (Indeed there exists $M_3 > 0$ such that if $|z| \geq M_3$, then $|h(z)| \leq 1$. But h is bounded on the compact set $\overline{D(0, M_3)}$, so that h is bounded on $\mathbb{C}$.)

By Liouville's theorem (Theorem 10.4), the function h is constant. Since it tends to 0 for $|z| \to \infty$, it must be identically 0, so that for all $z \in \mathbb{C}$, $h(z) = 0$. Thus $h_1(z) = 0$ for z in U, which is (stronger than) what we wanted to prove. This ends the proof, up to proving the above claim, that h_1 and h_2 are holomorphic.

Proof of the claim: We first focus on h_1. We shall need the following lemma.

Lemma 14.5. *g is a continuous function on U^2.*

Remark (skipped in class). By definition, g is clearly continuous on a neighbourhood of (z_0, w_0) with $z_0 \neq w_0$. We can also see that for all z_0, the function $w \mapsto g(z_0, w)$ is continuous and for all w_0, the function $z \mapsto g(z, w_0)$ also, but this is not sufficient to prove the continuity as a function of two variables (z, w).

Proof. (skipped in class) Observe that

$$g(z, w) = \int_0^1 f'[((1-t)z + tw)]dt$$

for $z, w \in U$. Indeed if $z = w$,

$$\int_0^1 f'[((1-t)z + tw)]dt = \int_0^1 f'(z)dt = f(z)$$

if $z \neq w$,

$$\int_0^1 f'[((1-t)z + tw)]dt = \left[\frac{1}{w-z} f((1-t)z + tw) \right]_0^1 = \frac{1}{w-z} \left(f(w) - f(z) \right).$$

The function $(t, z, w) \to f'[(1 - tz + tw)]$ is continuous on $[0; 1] \times U \times U$ so that

$$\int_0^1 f'[((1-t)z + tw)]dt$$

is continuous in (z, w). $\square$

We first prove that h_1 is continuous on U. If $\Gamma = \sum_{i=1}^{d} \alpha_i \gamma_i$, then

$$h_1(z) = \int_\Gamma g(z, w)dw = \sum_{i=1}^{d} \alpha_i \int_{\gamma_i} g(z, w)dw = \sum_{i=1}^{d} \alpha_i \int_0^1 g(z, \gamma_i(t))\gamma_i'(t)dt.$$

Assume w.l.o.g. $\gamma_i : [0; 1] \to \mathbb{C}. \gamma_i$ is C^1 (and not only piece-wise C^1). Then, for each i, the function $(z, t) \mapsto g(z, \gamma_i(t))\gamma_i'(t)$ is continuous. Hence, its integral over t in the compact interval $[0, 1]$ is continuous in z. Finally h_1 is continuous in z.

We now apply Morera's criterium to prove that h_1 is holomorphic on U. Let Δ be a triangle completely included in U. We have

$$\int_\Delta h_1(z)dz = \int_\Delta \left(\int_\Gamma g(z,w)dz \right) dw.$$

Second claim:

$$\int_\Delta \left(\int_\Gamma g(z,w)dz \right) dw = \int_\Gamma \left(\int_\Delta g(z,w)dw \right) dz.$$

Proof of this second claim: We write $\Gamma = \sum \alpha_i \gamma_i$ and $\Delta = \delta_1 + \delta_2 + \delta_3$, where the γ_i and the δ_j are C^1 on $[0,1]$. Clearly it is enough to prove the claim, replacing Γ and Δ by γ_i and δ_j, respectively. We consider the two following integrals

$$\int_{\gamma_i} \left(\int_{\delta_j} g(z,w)dw \right) dz = \int_0^1 \left(\int_0^1 g(\delta_j(t)\gamma_i(u)) {\delta_j}'(t){\gamma_i}'(u)du \right) dt$$

$$\int_{\delta_j} \left(\int_{\gamma_i} g(z,w)dz \right) dw = \int_0^1 \left(\int_0^1 g(\delta_j(t)\gamma_i(u)) {\delta_j}'(t){\gamma_i}'(u)dt \right) du.$$

But $g(\delta_j(t)\gamma_i(u)){\delta_j}'(t){\gamma_i}'(u)$ is continuous on $[0;1] \times [0;1]$ and hence bounded. Therefore

$$\int_0^1 \int_0^1 \left| g(\delta_j(t)\gamma_i(u)){\delta_j}'(t){\gamma_i}'(u) \right| dtdu < +\infty$$

and Fubini's theorem asserts that both integrals are equal.

Back to the proof that h_1 is holomorphic.

$$\int_\Delta h_1(z)dz = \int_\Gamma \left(\int_\Delta g(z,w)dz \right) dw$$

but for a fixed w, $z \mapsto g(z,w)$ is holomorphic (continuous and complex-differentiable in $U \setminus \{w\}$) hence $\int_\Delta g(z,w)dz = 0$. Thus $\int_\Delta h_1(z)dz = \int_\Gamma 0dw = 0$.

The proof that h_2 is holomorphic is similar (in fact, slightly simpler, there is no continuity issue of the integrand). ☐

Remark: it is important to remember (and be able to re-use) the above strategy. If you want to prove that an integral with some parameter define a holomorphic function of this parameter, use Morera's criterium and exchange the order of integration with Fubini's theorem. This works easily in many situations, so that we won't give a general theorem of this kind in this lecture.

14.3. Some direct corollaries.

Corollary 14.6. *U open, Γ cycle in U such that for any $\alpha \notin U$, $n_\Gamma(\alpha) = 0$. For all holomorphic $g : U \to \mathbb{C}$,*

$$\int_\Gamma g(w)dw = 0.$$

Proof. Chose arbitrarily $z \in U \setminus \mathfrak{Im}(\Gamma)$. Set $f(w) = g(w)(w-z)$, this is a holomorphic on U. From the general Cauchy formula (Theorem 14.4), we have that

$$\int_\Gamma \frac{f(w)}{w-z} dw = 2\pi i n_\Gamma(z) f(z).$$

But $\frac{f(w)}{w-z} = g(w)$ and $f(z) = 0$, so that the above formula specializes to $\int_\Gamma g(w)dw = 0$. ☐

Corollary 14.7. *Let U be a domain, Γ_1 and Γ_2 closed paths (or cycles) in U such that for any $\alpha \notin U$, $n_{\Gamma_1}(\alpha) = n_{\Gamma_2}(\alpha)$. For all holomorphic $g : U \to \mathbb{C}$,*

$$\int_{\Gamma_1} g(w)dw = \int_{\Gamma_2} g(w)dw.$$

Proof. Apply Corollary 14.6 to $\Gamma := \Gamma_1 - \Gamma_2$. Since $n_\Gamma(\alpha) = n_{\Gamma_1}(\alpha) - n_{\Gamma_2}(\alpha) = 0$ for all $\alpha \notin U$, we have

$$\int_\Gamma g(w)dw = \int_{\Gamma_1} g(w)dw - \int_{\Gamma_2} g(w)dw = 0.$$ ☐

15. Homotopy and simply connected sets

In this section, we use the following non-standard terminology.

Definition 15.1. (curve) *A curve in $U \subseteq \mathbb{C}$ is a continuous function $\gamma : [0;1] \to U$. A curve is closed if $\gamma(0) = \gamma(1)$.*

Remark: With this definition, a path is a piece-wise-C^1 curve. (Warning: in some contexts, paths are not assumed to be piece-wise C^1.)

15.1. Homotopy and path integrals.

Definition 15.2. (homotopy) *Let $U \subseteq \mathbb{C}$ open, $\gamma_0, \gamma_1 : [0;1] \to U$ continuous. Then γ_0 and γ_1 are said to be $U-$homotopic if there exists a function $H : [0;1]^2 \to U$ continuous such that*

$$\begin{aligned} H(s,0) &= \gamma_0(s) \quad \forall s \in [0;1] \\ H(s,1) &= \gamma_1(s) \quad \forall s \in [0;1] \\ H(0,t) &= H(1,t) \quad \forall t \in [0;1]. \end{aligned}$$

We usually denote in this case $\gamma_t(s) := H(s,t)$. From the above definition, for all t, we have:

- $\gamma_t(0) = \gamma_t(1)$;
- γ_t is continuous (since H is continuous).

In particular γ_t is a closed curve. H corresponds to a continuous deformation of γ_0 into γ_1, using only closed curves.

(Picture)

Theorem 15.3. *Let $U \subseteq \mathbb{C}$ open. $\gamma_0, \gamma_1 : [0;1] \to U$ paths. Assume that they are $U-$homotopic. Then, for all $\alpha \notin U$,*

$$n_{\gamma_0}(\alpha) = n_{\gamma_1}(\alpha).$$

We start with a lemma.

Lemma 15.4. *Let $\delta_1, \delta_2 : [0;1] \to \mathbb{C}$ be closed paths. Take $\alpha \notin \mathfrak{Im}(\delta_1) \cup \mathfrak{Im}(\delta_2)$. We assume that, for all $0 \le s \le 1$, we have $|\delta_1(s) - \delta_2(s)| < |\alpha - \delta_1(s)|$. Then*

$$n_{\delta_1}(\alpha) = n_{\delta_2}(\alpha).$$

Proof. Consider the closed path $\delta(s) = \frac{\delta_2(s)-\alpha}{\delta_1(s)-\alpha}$. Then we have

$$|1 - \delta(s)| = \left| \frac{\delta_1(s) - \delta_2(s)}{\delta_1(s) - \alpha} \right| < 1,$$

which implies $n_\delta(0) = 0$ (since 0 is in the unbounded component of $\mathbb{C} \setminus \mathfrak{Im}(\gamma)$). But

$$n_\delta(0) = \frac{1}{2\pi i} \int_0^1 \frac{\delta'(s)}{\delta(s)} ds = \frac{1}{2\pi i} \int_0^1 \left(\frac{\delta_2'(s)}{\delta_2(s) - \alpha} - \frac{\delta_1'(s)}{\delta_1(s) - \alpha} \right) ds = n_{\delta_2}(\alpha) - n_{\delta_1(\alpha)}.$$

We conclude that $n_{s_2}(\alpha) = n_{s_1}(\alpha)$, as wanted. ☐

Comment. This allows to define the winding number of a curve γ around a point z: take any path $\tilde{\gamma}$ homotopic to γ in $\mathbb{C} \setminus \{z\}$ and set $n_\gamma(z) := n_{\tilde{\gamma}}(z)$ (one has to show that such a $\tilde{\gamma}$ exists and that $n_{\tilde{\gamma}}(z)$ does not depend on the choice of $\tilde{\gamma}$).

Proof of Theorem 15.3. Let $\gamma_0, \gamma_1 : [0;1] \to U$ and $H : [0;1]^2 \to U$ be a homotopy between γ_0 and γ_1. Fix $\alpha \in U$. To simplify, we first assume that γ_t is a path for all $t \in [0;1]$ (i.e. γ_t piece-wise C^1). Idea: Construct a sequence $(t_0, t_1, \dots, t_n)$ with $0 = t_0 < t_1 < \dots < t_n = 1$ such that, for all $i \le n-1$ and all $s \in [0,1]$,

$$\left| \gamma_{t_i}(s) - \gamma_{t_{i+1}}(s) \right| < |\alpha - \gamma_{t_i}(s)| . \tag{3}$$

If we can construct such a sequence, by Lemma 15.4, we have $n_{\gamma_{t_i}}(\alpha) = n_{\gamma_{t_{i+1}}}(\alpha)$ (the winding numbers are well-defined because the γ_{t_i} are paths). This implies

$$\underbrace{n_{\gamma_{t_0}}(\alpha)}_{n_{\gamma_0}(\alpha)} = n_{\gamma_{t_1}}(\alpha) = \dots = \underbrace{n_{\gamma_{t_n}}(\alpha)}_{n_{\gamma_1}(\alpha)}.$$

How to establish (3)? H is continuous on a compact set ($[0;1]^2$). Thus $H([0;1]^2)$ is compact. Moreover, it does not contain α (indeed, $\alpha \notin U$, while $H([0;1]^2) \subseteq U$). Thus there exists $\varepsilon > 0$ such that for all $0 \leq s,t \leq 1$,

$$|H(s,t) - \alpha| \geq \varepsilon. \tag{4}$$

Besides, since H is continuous on the compact $[0;1]^2$, it is uniformly continuous. Thus, there exists $\delta > 0$ such that, for all $s, , s', t'$ in $[0,1]$,

$$\|(s,t) - (s',t')\| \leq \delta \Rightarrow |H(s,t) - H(s',t')| < \varepsilon.$$

Assume w.l.o.g. that $\delta = \frac{1}{n}$ and set $t_i = \frac{i}{n}$. Then we have $\|(s,t_i) - (s,t_{i+1})\| = \frac{1}{n}$ for all $i \leq n-1$ and all $s \in [0,1]$; this implies

$$|\underbrace{H(s,t_i)}_{\gamma_{t_i}(s)} - \underbrace{H(s,t_{i+1})}_{\gamma_{t_{i+1}(s)}}| < \varepsilon.$$

Comparing with (4), this proves

$$\left|\gamma_{t_i}(s) - \gamma_{t_{i+1}}(s)\right| < |\alpha - \gamma_{t_i}(s)|,$$

as wanted. □

This proves the theorem, with the additional asumption that, for each t, the curve γ_t is piecewise C^1. To get rid of this extra hypothesis, we can either define the winding number for curves and extend Lemma 15.4 to curves (as suggested in the above comment) or approximate each γ_t with a path $\tilde{\gamma}_t$ as done below.

Complete proof of Theorem 15.3 (skipped in class). Replace in the previous proof $\gamma_{\frac{i}{n}}$ by $\widetilde{\gamma}_{\frac{i}{n}}$ such that

- $\widetilde{\gamma}_{\frac{i}{n}}(\frac{j}{n}) = \gamma_{\frac{i}{n}}(\frac{j}{n})$ for all $0 \leq j \leq n$.
- $\widetilde{\gamma}_{\frac{i}{n}}$ is affine on $[\frac{j}{n}; \frac{j+1}{n}]$ for all $0 \leq j \leq n-1$.

There exists $\varepsilon > 0$ such that for all $(s,t) \in [0;1]^2$ $|H(s,t) - \alpha| \geq \varepsilon$. But H is uniformly continuous. There exists n such that

$$\sup(|s-s'|, |t-t'|) \leq \tfrac{1}{n} \Rightarrow |H(s,t) - H(s',t')| \leq \tfrac{\varepsilon}{4}. \quad (*)$$

We want to find a bound such that $\sup\limits_{0\leq s\leq 1} \left|\widetilde{\gamma}_{\frac{i}{n}} - \widetilde{\gamma}_{\frac{i+1}{n}}\right|$ for $\frac{j}{n} \leq s \leq \frac{j+1}{n}$ $\quad j := \llcorner s \cdot n \lrcorner$ (integral value).

$$\left|\widetilde{\gamma}_{\frac{i}{n}}(s) - \gamma_{\frac{i}{n}}\left(\tfrac{j}{n}\right)\right| \leq \left|\gamma_{\frac{i}{n}}\left(\tfrac{j+1}{n}\right) - \gamma_{\frac{i}{n}}\left(\tfrac{j}{n}\right)\right| \leq \tfrac{\varepsilon}{4}$$

Similarly,

$$\left|\widetilde{\gamma}_{\frac{i+1}{n}}(s) - \gamma_{\frac{i+1}{n}}\left(\tfrac{j}{n}\right)\right| \leq \tfrac{\varepsilon}{4}.$$

But

$$\left|\gamma_{\frac{i}{n}}\left(\tfrac{j}{n}\right) - \gamma_{\frac{i+1}{n}}\left(\tfrac{j}{n}\right)\right| \leq \tfrac{\varepsilon}{4} \quad \textit{from } (*).$$

Finally for any $0 \leq s \leq 1$

$$\left|\widetilde{\gamma}_{\frac{i}{n}}(s) - \widetilde{\gamma}_{\frac{i+1}{n}}(s)\right| \leq \tfrac{3\varepsilon}{4}.$$

We would like

$$\left|\widetilde{\gamma}_{\frac{1}{n}}(s) - \alpha\right| > \left|\widetilde{\gamma}_{\frac{i}{n}}(s) - \widetilde{\gamma}_{\frac{i+1}{n}}(s)\right| \quad (**)$$

in order to conclude that $\left|\widetilde{\gamma}_{\frac{i}{n}}(s) - \widetilde{\gamma}_{\frac{i+1}{n}}(s)\right|$ have the same winding number. But

$$\left|\widetilde{\gamma}_{\frac{i}{n}}(s) - \alpha\right| \geq \underbrace{\left|\gamma_{\frac{i}{n}}\left(\tfrac{j}{n}\right) - \alpha\right|}_{>\varepsilon} - \underbrace{\left|\gamma_{\frac{i}{n}}\left(\tfrac{j}{n}\right) - \widetilde{\gamma}_{\frac{i}{n}}(s)\right|}_{\leq\frac{\varepsilon}{4}} > \tfrac{3\varepsilon}{4}.$$

Finally $(**)$ is proved and $n_{\widetilde{\gamma}_{\frac{i}{n}}} = n_{\widetilde{\gamma}_{\frac{i+1}{n}}}$ for all $0 \leq i \leq n-1$. Thus

$$n_{\widetilde{\gamma}_0(\alpha)} = n_{\widetilde{\gamma}_{\frac{1}{n}}}(\alpha) = \ldots = n_{\widetilde{\gamma}_1}(\alpha).$$

But again

$$|\gamma_0(s) - \widetilde{\gamma}_0(s)| \leq \left|\gamma_0(s) - \gamma_0\left(\tfrac{j}{n}\right)\right| + \left|\gamma_0\left(\tfrac{j}{n}\right) - \widetilde{\gamma}_0\left(\tfrac{j}{n}\right)\right| + \left|\widetilde{\gamma}_0\left(\tfrac{j}{n}\right) - \widetilde{\gamma}_0(s)\right| \leq \tfrac{\varepsilon}{2} \leq |\alpha - \gamma_0(s)|.$$

Thus $n_{\gamma_0}(\alpha) = n_{\widetilde{\gamma}_0}(\alpha)$. Similarly $n_{\gamma_1}(\alpha) = n_{\widetilde{\gamma}_1}(\alpha) \Rightarrow n_{\gamma_0}(\alpha) = n_{\gamma_1}(\alpha)$. □

Homotopy gives a sufficient condition so that two paths have the same winding numbers around all points α not in U, allowing to apply the theorems of the previous section. In particular,

Corollary 15.5. *Let $\gamma_0, \gamma_1 : [0;1] \to U$ be closed paths (U open $\subseteq \mathbb{C}$). Assume that γ_0 and γ_1 are U-homotopic. For any holomorphic $f : U \to \mathbb{C}$,*

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz.$$

Proof. Combine Corollary 14.7 and Theorem 15.3. □

15.2. **Simply connected sets.**
A particularly interesting case to apply the above corollary is when γ_1 is a constant path, i.e. $\gamma_1(t) = z_0$ for some fixed z_0 in U and all $0 \leq t \leq 1$. If γ_0 is $U-$homotopic to the constant path γ_1, then $\int_{\gamma_0} f(z)dz = 0$. Indeed

$$\int_{\gamma_1} f(z)dz = \int_0^1 f(\gamma_1(t))\gamma_1'(t)dt = 0.$$

Definition 15.6. (simply connected) *An open subset $U \subseteq \mathbb{C}$ is called simply connected if it is path-connected and if any closed curve is homotopic to a constant curve.*

(Picture of a simply connected non star-shaped set and a non simply connected one. Explain intuitively the terminology "simply connected".)

Proposition 15.7. *Every star-shaped set U is simply connected.*

Proof. Let $U \subseteq \mathbb{C}$ and $a \in U$ such that $z \in U$ implies $[a;z] \subseteq U$. (The existence of a is the definition of U being star-shaped). Let $\gamma : [0;1] \to U$ be a closed curve. Define

$$H(s,t) = ta + (1-t)\gamma(s)$$

$H(s,0) = \gamma(s)$, $\quad H(s,1) = a$, $\quad H(0,t) = H(1,t)$. H is continuous in (s,t). I.e. γ is $U-$homotopic to the constant path with value a. We have shown that U is simply connected. □

(The converse is not true: see above picture.)

Theorem 15.8 (Vanishing integrals and Cauchy formula for simply connected sets)**.** *Let U be a simply connected, $f : U \to \mathbb{C}$ be holomorphic and $\gamma : [0;1] \to U$ be a closed path. Then we have*

$$\int_\gamma f(w)dw = 0;$$

$$\text{for } z \notin \mathfrak{Im}(\gamma), \quad \int_\gamma \frac{f(w)}{w-z}dw = 2\pi i\, n_\gamma(z)\, f(z)$$

Proof. Since U is simply connected, γ is U-homotopic to a constant path. Thus $\int_\gamma f(w)dw = 0$.

For the second part, note that, for any α not in U, since γ is U-homotopic to the constant path γ_1, Theorem 15.3 gives us $n_\gamma(\alpha) = n_{\gamma_1}(\alpha) = 0$ (α is in the unbounded component of γ_1). We conclude with general Cauchy's formula (Theorem 14.4). □

<u>Remark:</u> We already knew both formulas for U star-shaped.

Corollary 15.9. *A holomorphic function on a simply connected open set has an anti-derivative.*

Proof. Assume U is simply conntected and take $f : U \to \mathbb{C}$ holomorphic. Fix $a \in U$. For any $z \in U$, there is a path γ_z from a to z (U path connected). Then define

$$F(z) = \int_{\gamma_z} f(w)dw.$$

But is it well defined? (Ie. does $F(z)$ depend on γ_z or only on z?) Let γ_z, γ'_z two paths from a to z. Then consider $\gamma_z + \gamma_z'^*$ (path concatenation).

(Picture)

Then

$$\int_{\gamma_z + \gamma_z'^*} f(w)dw = 0.$$

But

$$\int_{\gamma_z + \gamma_z'^*} f(w)dw = \int_{\gamma_z} f(w)dw + \int_{\gamma_z'^*} f(w)dw = \int_{\gamma_z} f(w)dw - \int_{\gamma'_z} f(w)dw.$$

In conclusion F is well-defined.

We want to prove that for $z_0 \in U,\ F'(z_0) = f(z_0)$. Take a path γ_{z_0} from a to z_0 such that

$$F(z_0) = \int_{\gamma_{z_0}} f(w)dw.$$

There exists $r > 0$ such that $|z - z_0| < r$ implies $z \in U$. Let z with $|z - z_0| < r$. Then $\gamma_{z_0} + [z_0, z]$ (path concatenation) is a path from a to z.

$$F(z) = \int_{\gamma_{z_0} + [z_0, z]} f(w)dw = \int_{\gamma_{z_0}} f(w)dw + \int_{[z_0;z]} f(w)dw = F(z_0) + \int_{[z_0,z]} f(w)dw.$$

Then the proof is the same as in Proposition 12.2. ☐

Take home message: To sum up, the hypothesis "U is star-shaped" can be replaced by "U simply connected" in the summary of Section 12.3 and in Proposition 13.5.

Finally, we mention the following result without proof.

Theorem 15.10 (Riemann's mapping theorem)**.** *For every non-empty simply connected open proper subset U of $\mathbb{C}$, there exists a biholomorphic map $f : U \to D(0, 1)$ (i.e. a bijective holomorphic map f, whose inverse is holomorphic as well).*

(Not true for $U = \mathbb{C}$ because of Liouville's theorem.)

Corollary 15.11. *Let U be an open subset of $\mathbb{C}$. Then U is simply connected if and only if it is homeomorphic to $D(0, 1)$.*

Proof. Assume U simply connected. Either $U = \mathbb{C}$, in which case it is easy to construct a homeomorphism to $D(0, 1)$ (but not a holomorphic one!), or $U \subsetneq \mathbb{C}$, and we can apply Riemann's mapping theorem.

Conversely, if U is homeomorphic to $D(0, 1)$, one can easily show that U is simply connected (and even contractible; see exercise sheet), since $D(0, 1)$ is. ☐

15.3. **A connection to topology: the fundamental group.** (This section is not really about complex analysis. The goal is to explain how the concept of homotopy is used in *algebraic topology.* We therefore do not include all details.)

Fix a domain $U \subseteq \mathbb{C}$. Denote $\gamma_1 \sim \gamma_2$ if γ_1 and γ_2 are $U-$homotopic.

Lemma 15.12. *$\sim$ is an equivalence relation.*

Lemma 15.13. *Fix z_0 in U. Let $\gamma_1, \gamma_2, \delta_1, \delta_2$ be closed paths in U. starting and ending at z_0. If $\gamma_1 \sim \gamma_2$ and $\delta_1 \sim \delta_2$ then $\gamma_1 + \delta_1 \sim \gamma_2 + \delta_2$.*

As a consequence, the path concatenation operation can be projected to an operation + on equivalence classes of closed paths.

(Picture illustrating the above lemma.)

Lemma 15.14. *Equivalence classes of closed paths "at z_0" with the + operation is a group denoted* $\Pi_1^{z_0}(U)$.

The neutral element is the constant path in z_0 ($\gamma_1 + "z_0" \backsim \gamma_1$). The inverse of γ is the "reverse path" γ^*.

Warning! The notation + might be misleading in that this group is not abelian in general. (In topology, path concatenation is often denoted multiplicatively.)

Lemma 15.15. *If $z_1, z_2 \in U$, then we have a group isomorphism* $\Pi_1^{z_1}(U) \simeq \Pi_1^{z_2}(U)$.

Definition 15.16. *Let $U \subseteq \mathbb{C}$ open. The group $\Pi_1^{z_0}(U)$, for some fixed z_0 in U, is denoted $\Pi_1(U)$ and called fundamental group of U. (It is well-defined, i.e. independent from z_0, up to isomorphism)*

Remarks:

- By definition, U is simply connected if and only if $\Pi_1(U)$ is the one-element group.
- The definition generalizes to any topological space U (not only subsets of $\mathbb{C}$).

Lemma 15.17. *If U and V topological spaces and $f : U \to V$ is a homeomorphism (i.e. a continuous bijection with f^{-1} continuous), then* $\Pi_1(U) \simeq \Pi_1(V)$.

(In particular, U is simply connected if and only V is.)

Sketch of proof: If $\gamma : [0; 1] \to U$ is a closed path in U, then $f \circ \gamma : [0; 1] \to V$ is a closed path in V. The map $\gamma \mapsto f \circ \gamma$ is "compatible" with homotopy, thus defines a map $\Pi_1(U) \to \Pi_1(V)$. Its inverse is $\gamma \mapsto f^{-1} \circ \gamma$ which maps $\Pi_1(V)$ to $\Pi_1(U)$. □

An object, such as $\Pi_1(U)$, associated with a topological space U in such a way that it is invariant by homeomorphisms, is called a "topological invariant". Topological invariants might be used to distinguish topological spaces, i.e. show that they are not isomorphic (if $\Pi_1(U) \not\simeq \Pi_1(V)$ as groups, then $U \not\simeq V$ as topological spaces). Here is an example of such a result, using the fundamental group.

Proposition 15.18. *For $m \geq 3$, there does not exist a homeomorphism* $f : \mathbb{R}^2 \to \mathbb{R}^m$.

Sketch of proof: (by contradiction) Let $m \geq 3$ and f be a homeomorphism $\mathbb{R}^2 \to \mathbb{R}^m$.
Set $U := \mathbb{R}^2 \setminus \{(0, 0)\} \subset \mathbb{R}^2 \simeq \mathbb{C}$ and $V = \mathbb{R}^m \setminus \{f(0, 0)\} \subset \mathbb{R}^m$. Then f restricts to a homeomorphism from U to V and we should have $\Pi_1(U) \simeq \Pi_1(V)$.

Consider $\Pi_1(U)$. To this end, we recall that

$$\int_{\partial D(0,1)} \frac{dz}{z} = 2\pi i.$$

Therefore the path $\partial D(0, 1)$ is not homotopic to a constant path in U. Thus $\Pi_1(U)$ is not reduced to a single element.

On the opposite, we can show that V is simply connected (quite intuitive, a bit technical to show), which yields a contradiction. □

Comment: It is quite easy to show (see Linear Algebra I) that the existence of a *linear* bijective map $\ell : \mathbb{R}^n \to \mathbb{R}^m$ implies $n = m$. I.e. the dimension is a well-defined *linear* invariant.

But what if we only assume the existence of a *homeomorphism* $f : \mathbb{R}^n \to \mathbb{R}^m$? Does this imply $n = m$? I.e. is the dimension a topological invariant?

If $n = 1$, it is easy to prove that we must have $m = 1$. Indeed $\mathbb{R} \setminus \{0\}$ is not connected so that $f(\mathbb{R} \setminus \{0\}) = \mathbb{R}^m \setminus \{f(0)\}$ should also not be connected, which implies $m = 1$. If $n = 2$, the same holds by replacing "connected" by "simply connected" (see above). For $n, m \geq 3$, it is still true that necessarily $n = m$, but we need more complicated topological invariants to prove it. To learn about that, you should attend an algebraic topology class…

16. Complex logarithms and m-th roots

16.1. Logarithms.

Definition 16.1. (logarithm) *Let $z \in \mathbb{C} \setminus \{0\}$. We call a logarithm of z any complex number w such that*

$$\exp(w) = z.$$

Reminder: Exists for any $z \in \mathbb{C}^*$ but it is not unique: If $\exp(w) = \exp(w') = z$, then $w - w' \in 2\pi i\mathbb{Z}$. More explicitly, if $z = \rho e^{i\theta},\ \rho > 0$ then the complex logarithms of z are $\log(\rho) + i\theta + 2\pi i k$ for $k \in \mathbb{Z}$.

Definition 16.2. (determination of the logarithm) *Let $U \subseteq \mathbb{C}$ be open with $0 \notin U$. Then a function $f : U \to \mathbb{C}$ is called a determination of the logarithm if*

i) f is continuous.
ii) for any $z \in U$, $\exp(f(z)) = z$.

Existence and uniqueness:

- It does not always exists. For some U (e.g. $U = \mathbb{C} \setminus \{0\}$; see below), there is no determination of the logarithm $U \to \mathbb{C}$.
- If $f : U \to \mathbb{C}$ is a determination of the logarithm then $f + 2\pi i k \quad (k \in \mathbb{Z})$ also is.
- Conversely, if U is **connected**, f and $\tilde{f}$ are two determinations of logarithm on U, then there exist $k \in \mathbb{Z}$ such that $f - \tilde{f} \equiv 2\pi i k$.
  (Indeed, $f - \tilde{f}$ is then a continuous function on a compact set, that takes values in the discrete set $\{2\pi i k, k \in \mathbb{Z}\}$, hence, it must be constant.)

Here is a first existence result.

Proposition 16.3. *Let $U_\pi = \mathbb{C} \setminus \mathbb{R}_-$. Then define*

$$f : U_\pi \to \mathbb{C}$$

by

$$f(x+iy) = \log(\sqrt{x^2+y^2}) + i \begin{cases} \sin^{-1}\left(\frac{y}{\sqrt{x^2+y^2}}\right) & \text{if} \quad x \geq 0; \\ \pi - \sin^{-1}\left(\frac{y}{\sqrt{x^2+y^2}}\right) & \text{if} \quad x \leq 0, y > 0; \\ -\pi - \sin^{-1}\left(\frac{y}{\sqrt{x^2+y^2}}\right) & \text{if} \quad x \leq 0, y < 0; \end{cases}$$

where

$$\sin^{-1} : [-1;1] \to \left[-\frac{\pi}{2}; \frac{\pi}{2}\right].$$

(Note that $f(x+iy)$ is not defined for $x \leq 0$ and $y = 0$). Then, if $-\pi < \theta < \pi$ and $\rho > 0$,

$$f(\rho e^{i\theta}) = \log(\rho) + i\theta \qquad (*)$$

and f is a determination of logarithm on U_π.

This function f is called the principal determination of the logarithm.

only sketched in class.
- First prove that f is well-defined.
  - If $x = 0$ and $y > 0$, then first and second formulas coincide.

$$\sin^{-1}\left(\frac{y}{\sqrt{x^2+y^2}}\right) = \sin^{-1}(1) = \frac{\pi}{2}$$

and

$$\pi - \sin^{-1}\left(\frac{y}{\sqrt{x^2+y^2}}\right) = \pi - \sin^{-1}(1) = \frac{\pi}{2}.$$

– If $x = 0$ and $y < 0$, then first and third formulas coincide.

$$\sin^{-1}\left(\frac{y}{\sqrt{x^2+y^2}}\right) = \sin^{-1}(-1) = -\frac{\pi}{2}$$

and

$$-\pi - \sin^{-1}\left(\frac{y}{\sqrt{x^2+y^2}}\right) = -\pi - \sin^{-1}(-1) = -\frac{\pi}{2}.$$

- Second let us prove $(*)$.
  – If $-\pi < \theta \leq -\frac{\pi}{2}$, let $z = \rho e^{i\theta} = \rho(\cos(\theta) + i\sin(\theta))$. Let $x = \rho\cos(\theta),\ y = \rho\sin(\theta)$. Then $\rho = \sqrt{x^2+y^2}$. $x \leq 0,\ y < 0$,

$$f(z) = f(x+iy) = \log(\rho) + i[-\pi - \sin^{-1}(\sin(\theta))] = \log(\rho) + i\theta.$$

  Similar arguments for the cases $x \geq 0$ and $x \leq 0,\ y > 0$.
- Finally let us prove that it is a determination of the logarithm. If $z \in U_\pi$, then

$$z = \rho e^{i\theta},\ \rho > 0,\ -\pi < \theta < \pi$$

  and

$$\exp(f(z)) = \exp(\log(\rho) + i\theta) = \rho e^{i\theta} = z.$$

  It is continuous on $\{x \geq 0\}$, $\{x \leq 0,\ y > 0\}$, $\{x \leq 0,\ y < 0\}$ and it is well-defined on the intersection, therefore it is continuous on the union U_π.

□

Proposition 16.4. *Let $\theta_0 \in \mathbb{R}$. Let $U_{\theta_0} = \mathbb{C} \setminus \{\rho e^{i\theta},\ \rho \geq 0\}$. Then there is a determination f_{θ_0} of logarithm on U_{θ_0} such that if $\rho > 0,\ \theta_0 < \theta < \theta_0 + 2\pi$, then*

$$f_{\theta_0}\left(\rho e^{i\theta}\right) = \log(\rho) + i\theta.$$

Proof. The case $\theta_0 = -\pi$ corresponds to the previous proposition. Let $\theta_0 \in \mathbb{R}$. Define

$$f_{\theta_0}(z) = f_{-\pi}\left(ze^{-i(\pi+\theta_0)}\right) + i(\pi + \theta_0).$$

If $z \in U_{\theta_0}$, then $ze^{-i(\pi+\theta_0)} \in U_{-\pi}$. Indeed, if $z' = ze^{-i(\pi+\theta_0)} \in \mathbb{R}_-$ then $z = z'e^{i(\pi+\theta_0)} = (-z')e^{i\theta_0} \notin U_{\theta_0}$. Thus f_{θ_0} is defined on U_{θ_0}. Clearly f_{θ_0} is continuous and

$$\exp(f_{\theta_0}(z)) = \exp(f_{-\pi}(ze^{-i(\pi+\theta_0)}))\exp(i(\pi+\theta_0)) = z.$$ □

Summary: Don't write $\log(z)$ for $z \notin \mathbb{R}_+^*$ but say "let's consider the determination $\ell(z)$ of the logarithm on U_{θ_0} (precising θ_0)."

We now characterize complex logarithms through their derivative.

Proposition 16.5. *Let $U \subseteq \mathbb{C}$ be a domain (open connected set) not containing* 0. *Then the following assertions are equivalent:*

i) $f : U \to \mathbb{C}$ is a holomorphic determination of the logarithm.
ii) $f : U \to \mathbb{C}$ is holomorphic, $f'(z) = \frac{1}{z}$ and for some z_0 in U, $\exp(f(z_0)) = z_0$.

Proof. $(i) \Rightarrow (ii)$ We have $\exp(f(z)) = z$ for $z \in U$. Taking derivatives, we get $f'(z)\exp(f(z)) = z\,f'(z) = 1$ so that $f'(z) = \frac{1}{z}$.
$(ii) \Rightarrow (i)$ Define $V = \{z \in U \text{ such that } \exp(f(z)) = z\}$. Then we have:

- V is non-empty, since it contains z_0.
- V is closed, since $V = h^{-1}(\{0\})$ where $h(z) = e^{f(z)} - z$ is a continuous function on U.
- V is an open set. Define $g(z) = z\exp(-f(z))$ then

$$g'(z) = \exp(-f(z)) - zf'(z)\exp(-f(z)) = (1 - zf'(z))\exp(-f(z)) = 0.$$

  for all $z \in U$. We want to show that V is an open set. Let $z_1 \in V$. There exists r such that $D(z_1, r) \subseteq U$. For $z \in D(z_1, r)$

$$g(z) - g(z_1) = \int_{[z;z_1]} g'(w)dw = 0.$$

But $g(z_1) = 1$ since $z_1 \in V \Rightarrow g(z) = 1 \Rightarrow z \in V$.

Since U is connected, we conclude that $V = U$. □

Corollary 16.6. *Let U simply connected $0 \notin U$. Then there exists a holomorphic determination of the logarithm on U.*

Proof. Let g be an anti-derivative of $z \mapsto \frac{1}{z}$ on U (exists as U is simply connected). Fix $z_0 \in U$ and w_0 a logarithm of z_0. Define

$$f(z) = g(z) + w_0 - g(z_0).$$

Clearly $f'(z_0) = g'(z_0) = \frac{1}{z}$, $f(z_0) = w_0 \Rightarrow \exp(f(z_0)) = z_0$. From the previous proposition, f is a holomorphic determination of logarithm. □

Corollary 16.7. *Let $U \subseteq \mathbb{C}$ open, not containing 0 (and not necessarily simply connected). All determinations of the logarithm on U (if any) are holomorphic.*

Proof. Fix $z_0 \in U$. There exists $r > 0$ such that $D(z_0, r) \subseteq U$. $D(z_0, r)$ is convex, and hence simply connected (convex⇒ star-shaped⇒ simply connected). Thus, there exists $h : D(z_0, r) \to \mathbb{C}$ a holomorphic determination of the logarithm. But, if f is a determination of logarithm on U, then $f\mid_{D(z_0,r)}$ is a determination of logarithm on $D(z_0, r)$, and, on $D(z_0, r)$, the difference $h - f$ is constant, equal to $2\pi ik$ for some $k \in \mathbb{Z}$. We conclude that $f \equiv h + 2\pi ik$ is holomorphic on $D(z_0, r)$. This holds for any $z_0 \in U$, so that f holomorphic on U. □

In particular, the above defined function $f_{\theta_0} : U_{\theta_0} \to \mathbb{C}$ is holomorphic.

Proposition 16.8. *There exists no determination of logarithm on $\mathbb{C} \setminus \{0\}$.*

Proof. (By contradiction) Let $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ such a determination of logarithm. Then f is holomorphic. Proposition 16.5 implies that $f'(z) = \frac{1}{z}$. But the function $\frac{1}{z}$ has no anti-derivative on $\mathbb{C} \setminus \{0\}$ since $\int_{\partial D(0,1)} \frac{1}{z} dz = 2\pi i \neq 0$. □

Proposition 16.9. *Let*

$$f(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (z-1)^n$$

for $|z - 1| < 1$, then f is a determination of logarithm on $D(1, 1)$.

Proof. The power series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} w^n$$

has radius of convergence 1. Thus f is holomorphic on $D(1, 1)$ and the derivative is

$$f'(z) = \sum_{n=1}^{\infty} (-1)^{n-1}(z-1)^{n-1} = \sum_{m=0}^{\infty} (1-z)^m = \frac{1}{1-(1-z)} = \frac{1}{z}.$$

Moreover, $\exp(f(1)) = \exp(0) = 1$. We conclude by Proposition 16.5. □

<u>Warning!</u> Let $f : U \to \mathbb{C}$ be a determination of logarithm. $z, z' \in U$. Then, in general,

$$f(z \cdot z') \neq f(z) + f(z').$$

<u>Example:</u> Let $U = U_\pi = \mathbb{C} \setminus \mathbb{R}_-$, f principal determination of logarithm, and $z = z' = e^{\frac{3\pi i}{4}}$. Then

$$f(z) = f(z') = \tfrac{3\pi i}{4}.$$

But

$$f(z \cdot z') = -\tfrac{\pi i}{2} \quad (\neq \frac{3\pi i}{2}).$$

16.2. **Complex powers and m-th roots.** $a, b \in \mathbb{C},\ a \neq 0$. What is a^b? For real numbers $a^b := \exp(b\log(a))$.

Natural answer: Choose a logarithm ℓ_a of a; define $a^b := \exp(b\ell_a)$.

- if $b \notin \mathbb{Z},\ \exp(b\ell_a)$ depends on the logarithm ℓ_a (not well-defined).
- To use complex powers, we need to specify a logarithm (or a determination of logarithm).

Warning! In general, $(aa')^b \neq a^b(a')^b$. It is however true that, for any $a \in \mathbb{C}^*,\ b, b' \in \mathbb{C}$ and any choice of logarithm of a, one has

$$a^{b+b'} = \exp\left([b+b']\ell_a\right) = \exp(b\ell_a)\cdot\exp(b'\ell_a) = a^b \cdot a^{b'}.$$

Always refer to the exp-log expression to know which computation rules can be used or not!

Particular case: $b = \frac{1}{m},\ m \in \mathbb{N}$.

Definition 16.10. (m-th root) *Let $a \in \mathbb{C}^*$. Then we call a m-th root of a any $b \in \mathbb{C}$ such that $b^m = a$.*

Remarks:

- It does always exist. Take $b = \exp(\frac{1}{m}\ell_a)$ where ℓ_a is a logarithm of a.
- It is not unique. Let b and b' be two m-th roots of a.

$$\frac{b^m}{b'^m} = \left(\frac{b}{b'}\right)^m = 1 \Rightarrow \frac{b}{b'} = e^{2\pi ik/m}$$

  for some $k \in \{0, \dots, m-1\}$ which implies that each non-zero complex number has exactly m m-th roots.

Definition 16.11. (determination of the m-th root) *$U \subseteq \mathbb{C}$ with $0 \notin U$. We call determination of the m-th root a continuous function $f : U \to \mathbb{C}$ such that for all $z \in U$,*

$$f(z)^m = z.$$

Remarks:

- The existence of determination of logarithm implies the existence of a determination of the m-th root. Take $f(z) = \exp(\frac{1}{m}\ell(z))$.
- Uniqueness: Let $f_1, f_2 : U \to \mathbb{C}$ be a determination of the m-th root. Then for and $z \in U$,

$$\frac{f_1(z)}{f_2(z)} = \exp\left(\frac{2\pi ik(z)}{m}\right).$$

  Assume furthermore that U connected. Then the image of U by $z \mapsto \frac{f_1(z)}{f_2(z)}$ is a connected subset of $\{e^{\frac{2\pi ik}{m}};\ k \in \{0, \dots, m-1\}\}$. Therefore, it is a singleton, i.e. $\frac{f_1(z)}{f_2(z)} = e^{\frac{2\pi ik}{m}}$ for some $k \in \{0, \dots, m-1\}$.

Corollary 16.12.

i) If U simply connected, there exists a holomorphic determination of the m-th root.
ii) Any determination of the m-th root on any open set U (not assumed to be simply connected) is holomorphic.

Proof. (i) follows from the existence of a holomorphic determination of the logarithm $\ell : U \to \mathbb{C}$. (ii) Same proof as for the determination of the logarithm. □

16.3. **An application: Local normal form and biholomorphic function.**

Lemma 16.13. (local normal form) *Let $f : U \to \mathbb{C}$ be a holomorphic function. Fix $z_0 \in U$ and assume f non constant around z_0. Then there exists $m \geq 1$ and $g : V \to \mathbb{C}$ (where V is a neighbourhood of 0) such that $g(0) \neq 0$ and $f(z_0 + h) = f(z_0) + (h \cdot g(h))^m$ for every $h \in V$ with $z_0 + h \in U$.*

Remark: $h \cdot g(h)$ has a non-zero derivative in zero. Thus, by the local inversion theorem, $h \mapsto h\,g(h)$ is locally invertible.

Proof. Consider the power series expansion of f around 0. For h in some neighbourhood W of 0,

$$f(z) = f(z_0 + h) = \sum_{n=0}^{\infty} a_n h^n = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Recall $a_0 = f(z_0)$, $f(z_0 + h) = f(z_0) + \sum_{n=1}^{\infty} a_n h^n$. Let $m > 0$ be the smallest integer such that $a_m \neq 0$ (which exists since f is not constant). We have

$$\sum_{n=1}^{\infty} a_n h^n = \sum_{n=m}^{\infty} a_n h^n = \sum_{p=0}^{\infty} a_{m+p} h^{m+p} = h^m \cdot \left(\sum_{p=0}^{\infty} a_{m+p} h^p \right).$$

But $\left(\sum_{p=0}^{\infty} a_{m+p} h^p\right)$ has same radius of convergence that $\sum_{n=0}^{\infty} a_n h^n$ (easy), so that $\sum_{p=0}^{\infty} a_{m+p} h^p$ defines a function $g_1(h)$ on a neighbourhood of 0. Note that $g_1(0) = a_m \neq 0$. Thus there exists a neighbourhood V of 0 such that $g_1(V) \subseteq D(a_m, |a_m|)$. Since $D(a_m, |a_m|)$ is simply connected and does not contain 0, there exists a determination of the m-th root on $D(a_m, |a_m|)$. Call it γ_m. Finally

$$f(z_0 + h) = f(z_0) + \sum_{n=1}^{\infty} a_n h^n = f(z_0) + h^m g_1(h)$$

for $h \in V$, $g_1(h) \in D(a_m, |a_m|)$, $g_1(h) = \gamma_m(g_1(h))^m$ which implies

$$f(z_0 + h) = f(z_0) + (h \cdot \gamma_m(g_1(h)))^m.$$

□

Remark: m is the smallest integer such that $a_m \neq 0$, that is $f^{(m)}(z_0) \neq 0$. In particular, $m > 1 \Leftrightarrow f'(z_0) = 0$.

Corollary 16.14. *$f : U \to \mathbb{C}$ holomorphic. Assume that there exists $z_0 \in U$ such that $f'(z_0) = 0$. Then f is not injective.*

Remark: Analogue in real analysis does not hold. $f : \mathbb{R} \to \mathbb{R} \quad x \mapsto x^3$ is bijective, but $f'(0) = 0$.

Proof. We write f in local normal form at z_0: $f(z_0 + h) = f(z_0) + (hg(h))^m$ for $h \in V$ with $z_0 + h \in U$. Moreover, $m > 1$ (since $f'(z_0) = 0$). The function $h \mapsto h \cdot g(h)$ is holomorphic and non-constant. Its image W is open, hence it contains a neighbourhood of 0. In particular, for $r > 0$ small enough, we have $D(0, 2r) \subset W$, which implies that r and $re^{\frac{2\pi i}{m}}$ are in W. Namely, there exists h_1, h_2 such that $h_1(g(h_1)) = r$ and $h_2 g(h_2) = re^{\frac{2\pi i}{m}}$ (here $h_1, h_2 \in V$ with $z_0 + h_1$ and $z_0 + h_2$ in U). As $m > 1$, $r \neq re^{\frac{2\pi i}{m}} \Rightarrow h_1 \neq h_2$. But $f(z_0 + h_1) = f(z_0) + r^m = f(z_0 + h_2)$. Hence f is not injective. □

Theorem 16.15. *$f : U \to V$ holomorphic and bijective. U, V open $\subseteq \mathbb{C}$. Then f^{-1} is holomorphic.*

Proof. The image of an open set by f is an open set (open mapping theorem). In other words the pre-image of an open set by f^{-1} is an open set. I.e. f^{-1} is continuous.

Let $z_0 \in V$. We want to prove that f^{-1} is complex-differentiable in z_0.

$$\frac{f^{-1}(z) - f^{-1}(z_0)}{z - z_0} = \frac{f^{-1}(z) - f^{-1}(z_0)}{f(f^{-1}(z)) - f(f^{-1}(z_0))} = \left(\frac{f(f^{-1}(z)) - f(f^{-1}(z_0))}{f^{-1}(z) - f^{-1}(z_0)} \right)^{-1}.$$

But when $z \longrightarrow z_0$, $f^{-1}(z) \longrightarrow f^{-1}(z_0)$ (since f^{-1} continuous) so that

$$\frac{f(w) - f(w_0)}{w - w_0} \longrightarrow f'(w_0),$$

where $w_0 = f^{-1}(z_0)$, $w = f^{-1}(z)$. Finally, using the fact that $f'(w_0) \neq 0$ (previous corollary), we have

$$\frac{f^{-1}(z) - f^{-1}(z_0)}{z - z_0} = \left(\frac{f(w) - f(w_0)}{w - w_0} \right)^{-1} \longrightarrow f'(f^{-1}(z_0))^{-1}.$$

In particular, f^{-1} is complex differentiable in z_0. This holds for every $z_0 \in V$, proving that f^{-1} holomorphic. □

Terminology: Such functions f are called biholomorphic.

Part E. Isolated singularities and the residue theorem

17. Laurent's expansions

Throughout this section we fix R_1 and R_2 with $0 \leq R_1 < R_2 \leq \infty$.

Definition 17.1. (annulus) *The annulus $A(R_1, R_2)$ is the set $\{z \in \mathbb{C},\ R_1 < |z| < R_2\} = D(0, R_2) \setminus \overline{D(0, R_1)}$.*

(Picture)

Our goal is to study holomorphic functions on $A(R_1, R_2)$. Here is a recipe to construct such functions.

- Take $(a_n)_{n\geq 0} \in \mathbb{C}$ such that $\sum_{n=0}^{\infty} a_n z^n$ has a radius of convergence $\rho_a \geq R_2$. Then $z \mapsto \sum_{n=0}^{\infty} a_n z^n$ defines a holomorphic function on $D(0, \rho_a)$, thus on $A(R_1, R_2)$.
- Note: $|z| > R_1 \Leftrightarrow |\frac{1}{z}| < \frac{1}{R_1}$. Take $(b_n)_{n\geq 1} \in \mathbb{C}$ such that $\sum_{n=1}^{\infty} b_n w^n$ has radius of convergence $\rho_b \geq \frac{1}{R_1}$. Then $w \mapsto \sum_{n=1}^{\infty} b_n w^n$ is holomorphic for $|w| < \frac{1}{R_1}$. Setting $w = \frac{1}{z}$, we see that $z \mapsto \sum_{n=1}^{\infty} b_n \left(\frac{1}{z}\right)^n$ is holomorphic for $\left|\frac{1}{z}\right| < \frac{1}{R_1}$ i.e. for $|z| > R_1$. In particular, it is holomorphic on $A(R_1, R_2)$.
- Let a_n and b_n as above. We can sum the two holomorphic function above: then $\sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \left(\frac{1}{z}\right)^n$ defines a holomorphic function on $A(R_1, R_2)$.

Notation: Usually we denote $a_{-n} = b_n$ for $n \geq 1$ such that $(a_n)_{n\in\mathbb{Z}}$ is a sequence induced by $\mathbb{Z}$; and

$$\sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \left(\frac{1}{z}\right)^n = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} a_{-n} z^{-n} = \sum_{n\in\mathbb{Z}} a_n z^n.$$

(You can either think of $\sum_{n\in\mathbb{Z}} a_n z^n$ as the sum of the $\mathbb{Z}$-indexed summable family $(a_n z^n)$ or as a notation for $\sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} a_{-n} z^{-n}$.)

Terminology: $\sum_{n\in\mathbb{Z}} a_n z^n$ is called a Laurent series. (Warning: sometimes, in particular for formal power series, Laurent series are assumed to have finitely many terms with negative exponent; not in complex analysis!)

Summary: Let $(a_n)_{n\in\mathbb{Z}} \in \mathbb{C}$ (sometimes called "bi-infinte" sequence). Assume $\sum_{n=0}^{\infty} a_n z^n$ has a radius of convergence at least R_2 and that $\sum_{n=1}^{\infty} a_{-n} w^n$ has radius of convergence at least $\frac{1}{R_1}$, then $\sum_{n\in\mathbb{Z}} a_n z^n$ defines a holomorphic function on $A(R_1, R_2)$.

We will see a converse.

Theorem 17.2 (Laurent's expansion theorem)**.** *Any holomorphic function on an annulus is equal to the sum of a Laurent series. More precisely, if $f : A(R_1, R_2) \to \mathbb{C}$ holomorphic, then for any $z \in A(R_1, R_2)$,*

$$f(z) = \sum_{n\in\mathbb{Z}} a_n z^n,$$

where

$$a_n = \frac{1}{2\pi i} \int_{\partial D(0,r)} f(w) w^{-n-1} dw$$

for any $R_1 < r < R_2$.

We start with a lemma.

Lemma 17.3. *f holomorphic on $A(R_1, R_2)$. Let $R_1 < r < R_2$. Then $\int_{\partial D(0,r)} f(w)dw$ does not depend on r.*

Proof. Let $R_1 < r_1 < r_2 < R_2$. We want to show that $\int_{\partial D(0,r_1)} f(w)dw = \int_{\partial D(0,r_2)} f(w)dw$. But these two paths are homotopic in $A(R_1, R_2)$ via the homotopy

$$H : \begin{array}{ccc} [r_1, r_2] \times [0, 2\pi] & \to & A(R_1, R_2); \\ (r, \theta) & \mapsto & re^{i\theta}. \end{array}$$

Corollary 15.5 concludes the proof. ∎

Proof of Laurent's expansion theorem. (Note: the following proof is reminiscent of the proof of the analyticity of complex-differentiable functions, based on Cauchy formula; see Section 9.4.)

Fix $z \in A(R_1, R_2)$. Choose r_1, r_2 with $R_1 < r_1 < |z| < r_2 < R_2$. Denote γ_1, γ_2 as in the previous proof. Since $\partial D(0, r_1)$ and $\partial D(0, r_2)$ are homotopic in $A(R_1, R_2)$, we have $n_{\partial D(0,r_1)}(\alpha) = n_{\partial D(0,r_2)}(\alpha)$ for any $\alpha \notin A(R_1, R_2)$. Consider the cycle $\Gamma = \partial D(0, r_1) - \partial D(0, r_2)$. Then $n_\Gamma(\alpha) = 0$ for any $\alpha \notin A(R_1, R_2)$. and we can apply Theorem 14.4 (general Cauchy formula):

$$n_\Gamma(z) f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w - z} dw.$$

But $n_\Gamma(z) = n_{\partial D(0,r_1)}(z) - n_{\partial D(0,r_2)}(z) = -1$, using the computation of winding numbers for circles (Proposition 13.8; z is outside $\partial D(0, r_1)$, but inside $\partial D(0, r_2)$) Therefore

$$f(z) 2\pi i = -\int_\Gamma \frac{f(w)}{w-z} dw = \underbrace{\int_{\partial D(0,r_2)} \frac{f(w)}{w-z} dw}_{I_2} - \underbrace{\int_{\partial D(0,r_1)} \frac{f(w)}{w-z} dw}_{I_1}.$$

Compute I_1 and I_2 separately.

$$I_2 = \int_{\partial D(0,r_2)} \frac{f(w)}{w(1 - \frac{z}{w})} = \int_{\partial D(0,r_2)} \frac{f(w)}{w} \left(\sum_{n \geq 0} \left(\frac{z}{w}\right)^n \right) dw$$

because $\left|\frac{z}{w}\right| = \frac{|z|}{r_2} < 1$. Recall that the convergence $\sum_{n \geq 0} \left(\frac{z}{n}\right)^n = \frac{1}{1 - \frac{z}{w}}$ is uniform on $\partial D(0, r_2) = \{w : |w| = r_2\}$, so that we can exchange sum and integral. We get

$$I_2 = \sum_{n \geq 0} \left(\int_{\partial D(0,r_2)} \frac{f(w)}{w} \left(\frac{z}{w}\right)^n dw \right) = \sum_{n \geq 0} \left(\int_{\partial D(0,r_2)} f(w) w^{-n-1} dw \right) z^n.$$

Consider now I_1. We write

$$I_1 = \int_{\partial D(0,r_1)} \frac{-f(w)}{z(1 - \frac{w}{z})} dw = \int_{\partial D(0,r_1)} \frac{-f(w)}{z} \left(\sum_{m \geq 0} \left(\frac{w}{z}\right)^m \right) dw$$

(If $w \in \Im m(\partial D(0, r_1))$, i.e. $|w| = r_1$, then $\left|\frac{z}{w}\right| = \frac{|z|}{r_1} > 1$, so $\sum_{n \geq 0} \left(\frac{z}{w}\right)^n$ does not converge and we cannot proceed as for I_2.) Again we have uniform convergence, thus we can exchange sum and integral.

$$\begin{aligned} -I_1 &= \sum_{m \geq 0} \left(\int_{\partial D(0,r_1)} \frac{f(w)}{z} \left(\frac{w}{z}\right)^m dw \right) = \sum_{m \geq 0} \left(\int_{\partial D(0,r_1)} f(w) w^m dw \right) z^{-m-1} \\ &= \sum_{n=-1}^{-\infty} \left(\int_{\partial D(0,r_1)} f(w) w^{-n-1} dw \right) z^n. \end{aligned}$$

From Lemma 17.3,

$$\int_{\partial D(0,r_1)} f(w) w^{-n-1} dw = \int_{\partial D(0,r_2)} f(w) w^{-n-1} dw.$$

Thus

$$f(z) = \frac{1}{2\pi i}(I_2 - I_1) = \frac{1}{2\pi i} \sum_{n \in \mathbb{Z}} \left(\int_{\partial D(0,r_2)} f(w) w^{-n-1} dw \right) z^n$$

□

<u>Examples:</u>
$f(z) = \frac{1}{z(1-z)}$ holomorphic on $\mathbb{C} \setminus \{0, 1\}$.

- Look at f on $A(0, 1) = D(0, 1) \setminus \{0\}$. Then

$$f(z) = \frac{1}{z} \cdot \frac{1}{1-z} = \frac{1}{z} \left(\sum_{m=0}^{\infty} z^m \right) = \sum_{m=0}^{\infty} z^{m-1} = \sum_{n=-1}^{\infty} z^n.$$

- We now consider the same function f on $A(1,\infty) = \mathbb{C} \setminus \overline{D(0,1)}$.
  Warning! $\sum_{m=0}^{\infty} z^m$ is not convergent on $A(1,\infty)$, ($|z| > 1$) so the previous formula does not hold. The expansion of f on $A(1,\infty)$ is *different* from the one on $A(0,1)$.
  Namely, we have
  $$f(z) = \frac{1}{z} \cdot \frac{\frac{1}{z}}{\frac{1}{z} - 1} = \frac{-1}{z^2} \cdot \frac{1}{1 - \frac{1}{z}} = -\frac{1}{z^2} \left(\sum_{m=0}^{\infty} \left(\frac{1}{z}\right)^m \right) = \sum_{m=0}^{\infty} -z^{m-2} = \sum_{n=-2}^{\infty} -z^n.$$
- Consider $g(h) = f(1+h)$ for $h \in A(0,1)$. i.e. look at f on $\{0 < |z-1| < 1\}$. We have
  $$g(h) = \frac{1}{1+h} \cdot \frac{1}{-h} = \frac{-1}{h} \cdot \frac{1}{1-(-h)} = -\frac{1}{h} \sum_{m=0}^{\infty} (-h)^m.$$
  (The last sum converges sice $|h| < 1$.) Rewriting it further,
  $$g(h) = \sum_{m=0}^{\infty} (-1)^{m-1} h^{m-1} = \sum_{n=-1}^{\infty} (-1)^n h^n.$$
  In terms of f, we have, for any z in $D(1,1)$,
  $$f(z) = \sum_{n=-1}^{\infty} (-1)^n (z-1)^n.$$

Conclusion: as power series expansions, Laurent series expansions depend around which point we are studying the function. But, even around the same point, they depend on which annulus we consider! Here we have three different Laurent series expansions of the same function: two on different annuli around 0 and one in an annulus around 1.

In the following, we see a uniqueness result, when the point on which we are working and the annulus are both fixed.

Corollary 17.4. (Laurent separation theorem) *Let $f : A(R_1, R_2) \to \mathbb{C}$ be a holomorphic function. Then there exists a unique pair (f_C, f_D) such that*

i) f_C is a holomorphic function on $\{z : |z| < R_2\}$
ii) f_D is a holomorphic function on $\{z : |z| > R_1\}$ and $\lim_{|z|\to\infty} f_D(z) = 0$.
iii) $f(z) = f_C(z) + f_D(z)$ for $z \in A(R_1, R_2)$.

Proof. Existence. We know that $f(z) = \sum_{n\in\mathbb{Z}} a_n z^n$ on $A(R_1, R_2)$. Then define
$$f_C(z) = \sum_{n\geq 0} a_n z^n$$
and
$$f_D(z) = \sum_{n=-1}^{\infty} a_n z^n = \sum_{m=0}^{\infty} a_{-m-1} z^{-m-1} = \frac{1}{z} \left(\sum_{m=0}^{\infty} a_{-m-1} \left(\frac{1}{z}\right)^m \right).$$

Reminder: By definition $\sum_{n\in\mathbb{Z}} a_n z^n$ converges on $A(R_1, R_2)$ means that both $\sum_{n\geq 0} a_n z^n$ converges on $A(R_1, R_2)$ and $\sum_{m=0}^{\infty} a_{m-1} \left(\frac{1}{z}\right)^m$ converges on $A(R_1, R_2)$.

- Let us prove that f_C is holomorphic on $\{z : |z| < R_2\}$, $\sum_{n\geq 0} a_n z^n$ is convergent on $A(R_1, R_2)$. Hence radius of convergence of $\sum a_n z^n$ is at least R_2. This implies that $\sum_{n\geq 0} a_n z^n$ converges for $|z| < R_2$.
- Let us prove that f_D is convergent on $\{z : |z| > R_1\}$, $\sum_{m=0}^{\infty} a_{m-1} \left(\frac{1}{z}\right)^m$ is convergent for $R_1 < |z| < R_2$, so that $\sum_{m=0}^{\infty} a_{-m-1} w^m$ is convergent for $\frac{1}{R_2} < |w| < \frac{1}{R_1}$ Thus the radius of convergence of $\sum_{m=0}^{\infty} a_{-m-1} w^m$ is at least $\frac{1}{R_1}$. This implies that $\sum_{m=0}^{\infty} a_{-m-1} w^m$ converges for $|w| < \frac{1}{R_1}$, *i.e.* $\sum_{m=0}^{\infty} a_{-m-1} \left(\frac{1}{z}\right)^m$ converges for $|z| > R_1$.
- Let us show $\lim_{|z|\to\infty} f_D(z) = 0$. We have $f_D(z) = \frac{1}{z} \left(\sum_{m=0}^{\infty} a_{-m-1} \left(\frac{1}{z}\right)^m \right)$. Sum of power series define continuous functions so that $\lim_{w\to 0} \sum_{m=0}^{\infty} a_{m-1} w^m = a_{-1}$, i.e. $\lim_{|z|\to\infty} \sum_{m=0}^{\infty} a_{m-1} w^m = a_{-1}$. We conclude that $\lim_{|z|\to\infty} f_D(z) = 0$, as wanted.
- Obviously $f(z) = f_D(z) + f_C(z)$.

Uniqueness. Assume that

$$f(z) = f_C(z) + f_D(z) = g_C(z) + g_D(z)$$

on $A(R_1, R_2)$ where f_C, f_D, g_C, g_D fulfill the conditions in Corollary 17.4. Note that $f_C(z) - g_C(z)$ is holomorphic for $\{z : |z| < R_2\}$, while $g_D(z) - f_D(z)$ is holomorphic for $|z| > R_1$. Moreover, both functions coincide on the intersection $A(R_1, R_2)$. We can therefore define

$$h(z) = \begin{cases} f_C(z) - g_C(z) & \text{if} \quad |z| < R_1; \\ g_D(z) - f_D(z) & \text{if} \quad |z| > R_2 \end{cases}$$

The function h is entire. Moreover $\lim_{|z|\to\infty} h(z) = \lim_{|z|\to\infty} g_D(z) - f_D(z) = 0$, which implies that h is bounded. Theorem 10.4 (Liouville) implies that h is constant and since $\lim_{|z|\to\infty} = 0$, we have $h \equiv 0$. Therefore, $f_C(z) = g_C(z)$ and $g_D(z) = f_D(z)$, proving the uniqueness. □

Corollary 17.5. *The coefficients $(a_n)_{n\in\mathbb{Z}}$ in Laurent expansions are unique. In other words if*

$$\sum_{n\in\mathbb{Z}}^{\infty} a_n z^n = \sum_{n\in\mathbb{Z}}^{\infty} b_n z^n$$

on $A(R_1, R_2)$ then $a_n = b_n$ for any $n \in \mathbb{Z}$.

Proof. From Corollary 17.4, we have $\sum_{n\geq 0} a_n z^n = \sum_{n\geq 0} b_n z^n$ for $|z| < R_2$ (both are f_C, but f_C is unique). Using uniqueness of power series expansion (Corollary 6.6), we have $a_n = b_n$ for any $n \geq 0$. A similar argument using f_D and setting $w = \frac{1}{z}$ proves that $a_n = b_n$ for any $n < 0$. □

Summary: A homomorphic function f on the annulus $A(R_1, R_2)$ has a unique Laurent series expansion $\sum_{n\in\mathbb{Z}} a_n z^n$. As for power series expansion, the coefficients can be expressed as integrals

$$a_n = \frac{1}{2\pi\, i} \int_{\partial D(0,r)} f(w)\, w^{-n-1} dw,$$

for any $r \in (R_1, R_2)$ (the integral does not depend on r).

The coefficients a_n however do not correspond to derivatives. Keep also in mind that they depend on the annulus you are working on!

18. Isolated singularities

Framework of this section: $U \subseteq \mathbb{C}$ is an open set, z_0 is in U and f is a holomorphic function on $\boldsymbol{U \setminus \{z_0\}}$.

(Note: since we do not assume f continuous on U, we cannot conclude that f is also holomorphic in z_0 as we did several times already in the lecture; in fact, we are now in the context where f might not even be defined in z_0.)

Terminology: z_0 is called isolated singularity of f.

Examples:

- $f_1(z) = \frac{e^z - 1}{z}$ holomorphic on $\mathbb{C} \setminus \{0\}$; $U = \mathbb{C}$, $z_0 = 0$.
- $f_2(z) = \frac{1}{z(1-z)}$ holomorphic on $\mathbb{C} \setminus \{0, 1\}$; $U = \mathbb{C} \setminus \{1\}$, we look at it around $z_0 = 0$.
- $f_3(z) = \exp(\frac{1}{z})$ holomorphic on $\mathbb{C} \setminus \{0\}$; $U = \mathbb{C}$, $z_0 = 0$.

We will **classify isolated singularities in 3 categories**.

Basic tool: Laurent expansion theorem

Consider $g : h \mapsto f(z_0 + h)$. Since U is open, there exists $R > 0$ with $D(z_0, R) \subseteq U$. Then g is holomorphic on $A(0, R)$. Laurent's expansion theorem (Theorem 17.2) tells us that $f(z_0 + h) = \sum_{m\in\mathbb{Z}} a_n h^n \quad (a_n \in \mathbb{C})$.

18.1. **Removable singularity.**

Definition 18.1. (removable singularity) *Let $U \subseteq \mathbb{C}$ be an open set, take $z_0 \in U$, and assume that $f : U \setminus \{z_0\} \to \mathbb{C}$ is a holomorphic function. Then z_0 is called removable singularity if there exists $g : U \to \mathbb{C}$ continuous with $g\mid_{U\setminus\{z_0\}} = f$.*

Remarks:

- Such a g is automatically holomorphic on U (holomorphic on $U \setminus \{z_0\}$, continuous on U).
- removable means that we can remove the singularity i.e. define f in z_0.
- usually, we also call f the extension g.

Lemma 18.2. *Let z_0 be an isolated singularity of a holomorphic function $f : U \setminus \{z_0\} \to \mathbb{C}$ and consider the Laurent expansion of f around z_0,*

$$f(z_0 + h) = \sum_{n\in\mathbb{Z}} a_n h^n.$$

Then z_0 is a removable singularity if and only if $a_n = 0$ for any $n < 0$.

Proof. " $\Leftarrow$ " We have $f(z_0 + h) = \sum_{n\geq 0} a_n h^n$ for h in $D(0, R) \setminus \{0\}$ for some $R > 0$. Then f can be extended on z_0 by setting $g(z_0 + h) = \sum_{n\geq 0} a_n h^n$ for h in $D(0, R)$. and $g(z) = f(z)$ for $z \in U \subset \overline{D(0, R/2)}$. (The two formulas coincide on the intersection of the domains.)
" $\Rightarrow$ " Consider the power series expansion of g around z_0 (g is defined in z_0).

$$g(z_0 + h) = \sum_{n\geq 0} b_n h^n = \sum_{n\in\mathbb{Z}} a_n h^n = f(z_0 + h)$$

for $h \in A(0, R)$. Corollary 17.5 (uniqueness of Laurent expansion) implies that, for $n \geq 0,\ b_n = a_n$ and for $n < 0,\ a_n = 0$. □

Lemma 18.3. *Let z_0 be an isolated singularity of a holomorphic function $f : U \setminus \{z_0\} \to \mathbb{C}$. Then z_0 is a removable singularity if and only if f is bounded on a neighbourhood of z_0.*

Proof. " $\Rightarrow$ " When $z \xrightarrow{z\neq z_0} z_0$, then $f(z) \longrightarrow g(z_0)$ (indeed, $f(z) = g(z)$ for $z \neq z_0$ and g is continuous). Thus there is a neighbourhood V of z_0 such that $z \in V \setminus \{z_0\}$ implies $|f(z) - g(z_0)| \leq 1 \Rightarrow |f(z)| \leq |g(z_0)| + 1$ i.e. f is bounded on $V \setminus \{z_0\}$.
" $\Leftarrow$ " Assume f is bounded on $V \setminus \{z_0\}$ where V is an open set containing z_0. We set

$$\tilde{f}(z) = \begin{cases} f(z)(z - z_0) & \text{for} \quad z \neq z_0; \\ 0 & \text{for} \quad z = z_0 \end{cases}$$

for $z \in U$. Clearly, $\tilde{f}$ is holomorphic on $U\setminus\{z_0\}$ as a product of holomorphic functions. Also, when $z \xrightarrow{z\neq z_0} z_0$, we have $\tilde{f}(z) \longrightarrow 0$ (since $f(z)$ is bounded and $z - z_0 \longrightarrow 0$), i.e. $\tilde{f}$ is continuous in z_0. Therefore, $\tilde{f}$ is holomorphic on U (holomorphic on $U \setminus \{z_0\}$, continuous on U). We look at its power series expansion around z_0: $\tilde{f}(z_0 + h) = \sum_{n\geq 0} b_n h^n$. Since $b_0 = \tilde{f}(z_0) = 0$, we can start the sum at $n = 1$ and write

$$\tilde{f}(z_0 + h) = \sum_{n\geq 1} b_n h^n = \sum_{m=0} b_{m+1} h^{m+1} = h\left(\sum_{m=0}^{\infty} b_{m+1} h^m\right).$$

But, by definition $\tilde{f}(z_0 + h) = f(z_0 + h)\, h$ so that, for h in a neighbourhood of 0 with $h \neq 0$, we have:

$$f(z_0 + h) = \sum_{m=0}^{\infty} b_{m+1} h^m.$$

From the previous lemma, z_0 is a removable singularity. □

Comment: The implication " $\Leftarrow$ " is surprising and has no analogue in real analysis. Consider for example $f(z) = \sin(\frac{1}{z})$, which is real-analytic on $\mathbb{R} \setminus \{0\}$ and bounded on a neighbourhood of 0. Clearly, f cannot be extended in a continuous way in 0.

Example: $f_1(z) = \frac{e^z - 1}{z}$ holomorphic on $\mathbb{C} \setminus \{0\}$. But $\lim_{z\to 0} f_1(z) = 1$. I.e. f_1 can be extended as a holomorphic function on $\mathbb{C}$ by setting $f_1(0) = 1$. Its Laurent expansion in 0 is

$$f_1(h) = \frac{\left(\sum_{n=0}^{\infty} \frac{h^n}{n!} - 1\right)}{h} = \frac{\left(\sum_{n=1}^{\infty} \frac{h^n}{n!}\right)}{h} = \sum_{n=1}^{\infty} \frac{h^{n-1}}{n!} = \sum_{m=0}^{\infty} \frac{h^m}{(m+1)!}$$

and indeed does not contain negative powers of h.

18.2. **Poles.**

Definition 18.4. (pole) *Let $U \subseteq \mathbb{C}$ be an open set, take $z_0 \in U$, and assume that $f : U \setminus \{z_0\} \to \mathbb{C}$ is a holomorphic function. Then z_0 is called a pole if it is not a removable singularity, but there exists an integer $m > 0$ such that $(z - z_0)^m f(z)$ has a removable singularity in z_0. The smallest integer $m > 0$ such that $(z - z_0)^m f(z)$ has a removable singularity in z_0 is called the order of the pole.*

Terminology:

- single pole = pole of order 1.
- double pole = pole of order 2.
- multiple pole = pole of order > 1.

Lemma 18.5. *Let z_0 be an isolated singularity of a holomorphic function $f : U \setminus \{z_0\} \to \mathbb{C}$ and consider the Laurent expansion of f around z_0:*

$$f(z_0 + h) = \sum_{n\in\mathbb{Z}}^{\infty} a_n h^n.$$

i) Then z_0 is a pole if and only if there is at least one, but finitely many $n < 0$ with $a_n \neq 0$.
ii) More precisely, z_0 is a pole of order m if and only if $a_{-m} \neq 0$ and $a_n = 0$ for $n < -m$.

Proof. *ii)*" $\Leftarrow$ " $a_{-m} \neq 0$ implies that f does not have a removable singularity in z_0. Set $g(z) = (z - z_0)^m f(z)$.

$$g(z_0 + h) = h^m f(z_0 + h) = h^m \left(\sum_{n\in\mathbb{Z}}^{\infty} a_n h^m\right) = h^m \left(\sum_{n\geq -m}^{\infty} a_n h^m\right) = \sum_{n\geq -m}^{\infty} a_n h^{n+m} = \sum_{p=0}^{\infty} a_{p-m} h^p.$$

This expansion does not contain any negative power of h, so that g has a removable singularity in z_0.
Let us prove that m is minimal such that $(z - z_0)^m f(z)$ has a removable singularity in z_0 (as in the definition of pole). Let $m_2 < m$. Consider $g_2(z) = (z - z_0)^{m_2} f(z)$,

$$g_2(z_0 + h) = h^{m_2} \left(\sum_{n\in\mathbb{Z}} a_n h^n\right) = \sum_{n\in\mathbb{Z}} a_n h^{n+m_2}.$$

The coefficient of h^{-m+m_2} in its expansion is $a_{-m} \neq 0$. But $-m + m_2 < 0$ so that the Laurent expansion of $g_2(z_0 + h)$ contains negative powers of h, i.e. z_0 is not a removable singularity of g_2. Thus the pole z_0 has indeed order m as claimed.
ii)" $\Rightarrow$ " Assume that f has a pole of order m in z_0. We consider the Laurent expansion of f around z_0:

$$f(z_0 + h) = \sum_{n\in\mathbb{Z}} a_n h^n$$

Since $h^m f(z_0 + h)$ has a removable singularity in $z = z_0$, we have $h^m f(z_0 + h) = \sum_{p\geq 0} b_p h^p$ (no negative powers by Lemma 18.2). We compare both expression

$$f(z_0 + h) = \sum_{n\in\mathbb{Z}} a_n h^n = h^{-m} \sum_{p\geq 0} b_p h^p = \sum_{p\geq 0} b_p h^{p-m} = \sum_{n\geq -m} b_{m+n} h^n.$$

By the uniqueness of the Laurent expansion $a_n = 0$ for $n < -m$ and $a_n = b_{m+n}$ for $n \geq -m$.
It remains to prove that $a_{-m} \neq 0$. By contradiction assume $a_{-m} = 0$, that is $a_n = 0$ for $n < -(m-1)$. Using the proof of " $\Leftarrow$ ", this implies that $(z - z_0)^{m-1} f(z)$ has a removable singularity in z_0. This is a contradiction with the minimality of m.
i)" $\Rightarrow$ " Assume z_0 is a pole. Call m its order. By ii), $a_{-m} \neq 0$ so that there is at least one $n < 0$ with $a_n \neq 0$. We also have $a_n = 0$ for $n < -m$, which proves that there are only finitely many $n < 0$ with $a_n \neq 0$.

i)" $\Leftarrow$ " Call m the biggest integer $m > 0$ such that $a_{-m} \neq 0$ (exists since there is at least one, but finitely many, such positive integers). By maximality, $a_n = 0$ for $n < -m$. Using ii), this implies that z_0 is a pole of order m. □

Intuition of pole: If f has a pole of order m, then $(z-z_0)^m f(z)$ can be extended in a holomorphic function $g(z)$ defined on U. Thus f can be writen as $f(z) = \frac{g(z)}{(z-z_0)^m}$, for some holomorphic function g.

Warning: if $f(z) = \frac{g(z)}{(z-z_0)^m}$, this does not necessarily mean that f has a pole of order m. This is the case only if $g(z_0) \neq 0$. Indeed, assume, for the sake of contradiction, that $g(z_0) = 0$. Then

$$g(z_0 + h) = \sum_{n\geq 1} b_n h^n = h\left(\sum_{m\geq 0} b_{m+1} h^m\right)$$

i.e. $\frac{g(z_0+h)}{h} = \frac{g(z)}{z-z_0}$ is holomorphic on U. Thus $f(z)(z-z_0)^{m-1} = \frac{g(z)}{z-z_0}$ has a removable singularity in z_0. This is a contradiction with minimality of m.

Lemma 18.6. *$f : U \setminus \{z_0\} \to \mathbb{C}$ holomorphic. Then z_0 is a pole if and only if $\lim_{z\to z_0} |f(z)| = \infty$.*

Proof. " $\Rightarrow$ " For $z \neq z_0$, $f(z) = \frac{g(z)}{(z-z_0)^m}$ with $g(z_0) \neq 0$ (see above discussion). When $z \longrightarrow z_0$, we have $|g(z)| \longrightarrow |g(z_0)|$ and $|(z-z_0)^m| \longrightarrow 0$, which entails $|f(z)| \longrightarrow \infty$.
" $\Leftarrow$ " Consider $\tilde{f}(z) = \frac{1}{f(z)}$ (defined on a neighbourhood V of z_0 since $\lim_{z\to z_0} |f(z)| = \infty$ and therefore f does not vanish on a neighbourhood of z_0). Since $\lim_{z\to z_0} |f(z)| = \infty$, we have $\lim_{z\to z_0} |\tilde{f}(z)| = 0$ and $\tilde{f}$ is bounded on a neighbourhood of z_0. By Lemma 18.3, z_0 is a removable singularity of $\tilde{f}$. Considering the Laurent's expansion, we have $\tilde{f}(z_0 + h) = \sum_{n\geq 0} b_n h^n$ (no negative powers from Lemma 18.2).

For $h \neq 0$, $\tilde{f}(z_0+h) \neq 0$, so that at least one b_n is non-zero. Call m the smallest integer such that $b_m \neq 0$. Note that $\tilde{f}(z_0 + h) \xrightarrow{h\to 0} 0$, so that $b_0 = 0$ and $m > 0$.

$$\tilde{f}(z_0 + h) = \sum_{n\geq m} b_n h^n = h^m \left(\sum_{p=0} b_{m+p} h^p\right)$$

$$h^m f(z_0 + h) = \frac{h^m}{\tilde{f}(z_0 + h)} = \frac{1}{\sum\limits_{p=0}^{\infty} b_{p+m} h^p}.$$

But $\sum\limits_{p=0}^{\infty} b_{p+m} h^p$ is holomorphic and does not vanish on a neighbourhood of 0 (it is a convergent power series with constant term $b_m \neq 0$). Its inverse $h^m f(z_0 + h)$ is holomorphic on a neighbourhood of 0, i.e. $(z-z_0)^m f(z)$ has a removable singularity in z_0. Since z_0 is not a removable singularity of f (f is not bounded around z_0), we conclude that z_0 is a pole of f. □

Example: Consider $f_2(z) = \frac{1}{z(1-z)}$ for $z \in \mathbb{C} \setminus \{0, 1\}$. The singularity in 0 is clearly not removable. The function $zf_2(z) = \frac{1}{1-z}$ can be extended to a holomorphic on $\mathbb{C} \setminus \{1\}$. Thus 0 is a (simple) pole of f_2. We recall that the Laurent expansion on f_2 on $A(0,1)$ is $f_2(z) = \frac{1}{z} + \sum_{n\geq 0} z^n$ (see examples in the previous chapter). It has exactly one term with a negative power of z; this is consistent with Lemma 18.5. Besides, we indeed have $\lim_{z\to 0} |f_2(z)| \to +\infty$, as expected from Lemma 18.6.

Warning! f_2 admits another Laurent expansion "with center 0", namely $f_2(z) = \sum_{n=-2}^{-\infty} -z^n$ for $|z| > 1$. This expansion has infinitely many terms with negative powers of z, even though 0 is a pole of f_2. To classify isolated singularity with Laurent's expansion, it is important to consider the Laurent expansion on an annulus of the kind $A(0, R)$, as done throughout this chapter.

18.3. **Essential singularities.**

Definition 18.7. (essential singularity) *Let z_0 be an isolated singularity of a holomorphic function $f : U \setminus \{z_0\} \to \mathbb{C}$. Then z_0 is called an essential singularity if it is neither a removable singularity nor a pole.*

Lemma 18.8. *$f : U \setminus \{z_0\} \to \mathbb{C}$ holomorphic. Consider its Laurent expansion*

$$f(z_0 + h) = \sum_{n \in \mathbb{Z}} a_n h^n.$$

Then z_0 is an essential singularity if and only if there are infinitely many $n < 0$ with $a_n \neq 0$.

Proof. Direct consequence of the analogous lemmas for removable singularities and poles. □

Example: $f_3(z) = \exp\left(\frac{1}{z}\right)$. For $z \neq 0$ (i.e. $z \in A(0, \infty)$), then

$$\exp\left(\frac{1}{z}\right) = \sum_{p \geq 0} \frac{\left(\frac{1}{z}\right)^p}{p!} = \sum_{p \geq 0} \frac{z^{-p}}{p!} = \sum_{n=0}^{-\infty} \frac{z^n}{(-n)!}$$

infinitely many negative powers of z. $\Rightarrow$ essential singularity.

What about the behaviour of $f(z)$ near z_0? From the results on removables singularities and poles, we know that,

- f is not bounded around z_0 (otherwise z_0 is a removable singularity);
- $|f(z)| \not\longrightarrow \infty$ (otherwise z_0 is a pole).

We can prove more.

Theorem 18.9 (Cassorati-Weierstrass)**.** *Let z_0 be an isolated singularity of a holomorphic function $f : U \setminus \{z_0\} \to \mathbb{C}$. Then z_0 is an essential singularity if and only if for any neighbourhood V of z_0 with $V \subseteq U$, one has $\overline{f(V \setminus \{z_0\})} = \mathbb{C}$.*

Proof. " $\Leftarrow$ " Assume that for any neighbourhood V of z_0 with $V \subseteq U$, one has $\overline{f(V \setminus z_0)} = \mathbb{C}$.

- Then z_0 is not removable since f is not bounded on a neighbourhood of z_0 (otherwise $\overline{f(V \setminus z_0)} \subseteq \overline{D(0, M)}$).
- Moreover, z_0 is not a pole since $|f(z_0)|$ does not tend to $+\infty$ (indeed if $|f(z)| \xrightarrow{z \to z_0} \infty$, there exists a neighbourhood V of z_0 such that $|f(z)| \geq 1$ on V, i.e. $\overline{f(V \setminus z_0)} \subseteq \{w : |w| \geq 1\}$).

We conclude that z_0 is an essential singularity.
" $\Rightarrow$ " (by contradiction) Assume $a \notin \overline{f(V \setminus z_0)}$ for some neighbourhood V of z_0. Consider $g(z) = \frac{1}{f(z) - a}$ for $z \in V \setminus \{z_0\}$. Note that, for z in V, we have $f(z) - a \neq 0$ since $a \notin f(V \setminus \{z_0\})$; g is therefore holomorphic on $V \setminus \{z_0\}$. But a being in the open set $\mathbb{C} \setminus \overline{f(V \setminus \{z_0\})}$ means that there exists $\delta > 0$ such that $D(a, r) \cap f(V \setminus \{z_0\}) = \emptyset$. In other words for $z \in V \setminus \{z_0\}$,

$$f(z) \notin D(a, \delta) \Leftrightarrow |f(z) - a| \geq \delta.$$

Thus $|g(z)| \leq \frac{1}{\delta}$ i.e. g bounded on $V \setminus \{z_0\}$. From Lemma 18.3, g has a removable singularity in z_0, that is g can be extended to a holomorphic function $V \to \mathbb{C}$, that we will abusively also denote by g. But $f(z) = \frac{1}{g(z)} + a$ for $z \in V \setminus \{z_0\}$. We now distinguish two cases:

- either $g(z_0) \neq 0$,

$$f(z) \xrightarrow{z \to z_0} \frac{1}{g(z_0)} + a$$

  i.e. z_0 is a removable singularity of f.
- or $g(z_0) = 0$, and then

$$|f(z)| \xrightarrow{z \to z_0} \infty,$$

  so that z_0 is a pole (Lemma 18.6).

In both cases, we have a contradiction.

We conclude that, for any neighbourhood V of z_0, we have $\overline{f(V \setminus \{z_0\}} = \mathbb{C}$, as claimed. □

Example: Recall $f_3(z) = \exp\left(\frac{1}{z}\right)$ has an essential singularity in 0. We want to understand $f_3(V \setminus \{0\})$. To simplify the discussion, we first consider $f_3(\mathbb{C} \setminus \{0\})$.

- Note that $0 \notin f_3(\mathbb{C} \setminus \{0\})$.

- Take $w \neq 0$. Is there $z \neq 0$ such that $\exp\left(\frac{1}{z}\right) = w$? We know that, if ℓ_w is a logarithm of w,
$$\exp\left(\frac{1}{z}\right) = w \Leftrightarrow \frac{1}{z} = \ell_w + 2\pi ik \Leftrightarrow z = \frac{1}{\ell_w + 2\pi ik}$$
for some $k \in \mathbb{Z}$ such that $\ell_w + 2\pi ik \neq 0$. Hence $f_3(\mathbb{C} \setminus \{0\}) = \mathbb{C} \setminus \{0\}$.

But what about $f_3(V \setminus \{0\}$ in general? Fix $w \neq 0$. We have $\lim_{k\to\infty} \frac{1}{\ell_w+2\pi ik} = 0$, so, for k big enough, $\frac{1}{\ell_w+2\pi ik}$ lies in $V \setminus \{0\}$. Thus $w \in f_3(V \setminus \{0\})$. Since this holds for all $w \neq 0$, we have $f_3(V \setminus \{0\}) = \mathbb{C} \setminus \{0\}$. In particular $\overline{f_3(V \setminus \{0\})} = \mathbb{C}$, as asserted by Cassorati-Weiertstrass theorem.

Finally, we mention the following theorem without proof, which improves Cassorati-Weierstrass theorem.

Theorem 18.10 (Picard)**.** *If z_0 is an essential singularity of f and V a neighbourhood of z_0, then*
$$|\mathbb{C} \setminus f(V \setminus \{z_0\})| \leq 1.$$

Informally, f is almost surjective: it "misses" at most one value. This is optimal as shown by the example $f(z) = \exp(1/z)$ (it "misses" 0).

18.4. **Summary of isolated singularities.** Here is the picture to keep in mind on isolated singularities.

type of singularity	*Laurent expansion*	*behaviour around z_0*
removable	no negative powers	bounded
pole	finitely many neg. powers	tends to ∞
essential	infinitely many neg. powers	image dense in $\mathbb{C}$

In the third column, it is not a priori obvious that all cases are covered. In particular, the results of this section tell us that it is impossible to have e.g., $\overline{f(V)} = \{\Re(z) \geq 0\}$ for all neighbourhood V of z_0.

18.5. **Meromorphic functions.**

Definition 18.11. (meromorphic function) *A meromorphic function f on an open set U is a set P and a holomorphic function $f : U \setminus P \to \mathbb{C}$ such that*

i) P is a discrete subset of U (i.e. a subset without limit points).
ii) each $p \in P$ is a pole of f (since P is discrete; note that each p in P is automatically an isolated singularity of f).

Warning! We say "meromorphic function on U", but f is a function on $U \setminus P$.

Remark: If $A \subseteq U$, A discrete set, U open. $f : U \setminus A \to \mathbb{C}$ holomorphic. Assume f has no essential singularities. Then $A = R \sqcup P$, where R denotes the set of removable singularities and P the set of poles. Then we can extend f to a holomorphic function on $(U \setminus A) \cup R = U \setminus P$. This extension fulfills the definition of meromorphic function, i.e. f can be seen as a meromorphic function on U.

Terminology: We say that f is meromorphic in z_0 if either f is holomorphic in z_0, or f has a pole in z_0.

Examples:

i) holomorphic functions on U are meromorphic on U ($P = \emptyset$).
ii) Let U be a connected open set, $g : U \to \mathbb{C}$ holomorphic with $g \not\equiv 0$. Then $f = \frac{1}{g}$ is meromorphic on U with $P = \{z \in U,\ g(z) = 0\}$ (discrete because of the isolated zero principle). Indeed, $f : U \setminus P \to \mathbb{C}$ holomorphic and for $p \in P$,
$$|f(z)| = \frac{1}{|g(z)|} \xrightarrow{z \to p} \infty$$
implies p is a pole.

The second example in fact generalizes to any fraction of holomorphic functions.

Proposition 18.12. *Let U be a connected open set and $g_1, g_2 : U \to \mathbb{C}$ be holomorphic functions, non identically equal to* 0*. Then $f = \frac{g_1}{g_2}$ is a meromorphic function on U whose set P of poles is included in $\{z \in U : g_2(z) = 0\}$.*

Proof. Let $A = \{z \in U : g_2(z) = 0\}$. A is a discrete set because of isolated zero principle. f holomorphic on $U \setminus A$. We want to prove that each $a \in A$ is either a pole or a removable singularity of f. Let $a \in A$. Consider the power-series expansion of g_1 and g_2 around a.

$$g_1(a+h) = \sum_{n \geq 0} b_n h^n = h^{m_1} \left(\sum_{p=0}^{\infty} b_{p+m_1} h^p \right)$$

$$g_2(a+h) = \sum_{n \geq 0} c_n h^n = h^{m_2} \left(\sum_{p=0}^{\infty} c_{p+m_2} h^p \right)$$

where m_1 is the smallest integer with $b_{m_1} \neq 0$ and m_2 is the smallest integer with $c_{m_2} \neq 0$ then

$$\frac{g_1(a+h)}{g_2(a+h)} = h^{m_1 - m_2} \cdot \frac{\left(\sum_{p=0}^{\infty} b_{p+m_1} h^p\right)}{\left(\sum_{p=0}^{\infty} c_{p+m_2} h^p\right)}.$$

Taking the limit $h \to 0$, we have,

- if $m_1 < m_2$,

$$|f(a+h)| = \frac{|g_1(a+h)|}{|g_2(a+h)|} \longrightarrow \infty$$

  i.e. a is a pole;
- if $m_1 \geq m_2$,

$$|f(a+h)| = \frac{|g_1(a+h)|}{|g_2(a+h)|} \longrightarrow \begin{cases} 0 & \text{if} \quad m_1 > m_2 \\ \frac{b_{m_1}}{c_{m_2}} & \text{if} \quad m_1 = m_2 \end{cases}$$

  i.e. a is a removable singularity. □

Corollary 18.13. *Any rational function is meromorphic on* $\mathbb{C}$.

We have a converse to the above proposition.

Theorem 18.14. *Any meromorphic function f on an open set U is the quotient of holomorphic functions on U. As a consequence, the field of meromorphic functions on a connected open set U is the fraction field of the field of holomorphic functions on U.*

We only prove this for meromorphic functions with a finite set of poles (i.e. $|P| < \infty$).

Proof (skipped in class). Let $P = \{p_1, \dots, p_r\}$ be the set of poles of f. Let $m_1, \dots, m_r$ be the orders of $p_1, \dots, p_r$. Consider

$$g_1(z) = \prod_{i=1}^{r} (z - p_i)^{m_i} f(z).$$

Clearly, g_1 is holomorphic on $U \setminus P$. Fix $p_i \in P$. Then $(z - p_i)^{m_i} f(z)$ has a removable singularity in p_i (by definition of a pole). But

$$g_1(z) = \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} (z - p_j)^{m_j} \right) \cdot (z - p_i)^{m_i} f(z)$$

On this expression, we see that $g_1(z)$ has a removable singularity in p_i. This is true for any $p_i \in P$, so that g_1 is a holomorphic function on U. We conclude that

$$f(z) = \frac{g_1(z)}{\prod_{i=1}^{r} (z - p_i)^{m_i}}$$

is a quotient of holomorphic functions. □

19. Residue theorem

19.1. The theorem.

Definition 19.1. (residue) *Let $z_0 \in U$, U open. Let $f : U \setminus \{z_0\} \to \mathbb{C}$ holomorphic. Consider its Laurent expansion around z_0,*

$$f(z_0 + h) = \sum_{n\in\mathbb{Z}} a_n h^n.$$

Then we call residue of f in z_0 the coefficient a_{-1}.

Notation: $\operatorname{Res}(f; z_0) := a_{-1}$.

The residue can alternatively be obtained by integration.

Lemma 19.2. *Let z_0 be a point in an open set U and $f : U \setminus \{z_0\} \to \mathbb{C}$ be a holomorphic function. Take $r > 0$ such that $D(z_0, r) \subseteq U$. Then we have*

$$2\pi i \operatorname{Res}(f; z_0) = \int_{\partial D(z_0,r)} f(z)dz.$$

Proof. In the proof of Laurent's expansion theorem, we proved that, for all $n \in \mathbb{Z}$, we have

$$2\pi i a_n = \int_{\partial D(z_0,r)} f(z)z^{-n-1}dz.$$

□

Theorem 19.3. (residue theorem) *Let U be a simply connected open set and $A \subseteq U$ discrete. Take $f : U \setminus A \to \mathbb{C}$ holomorphic. Consider a closed path γ with $\mathfrak{Im}(\gamma) \subseteq U \setminus A$. Then*

$$\int_\gamma f(z)dz = 2\pi i \cdot \sum_{a\in A} \operatorname{Res}(f; a)\, n_\gamma(a). \tag{5}$$

Comments:

- If f is meromorphic on U, then it is holomorphic on $U \setminus A$ for some discrete set A, so that we can apply the theorem. The theorem is however also applicable with essential singularities.
- It also holds for a cycle Γ. We can also relax the hypothesis that U is simply connected by assuming $\alpha \notin U \Rightarrow n_\gamma(\alpha) = 0$.
- Recall that, for any closed path γ, the set $\{w \in \mathbb{C}$ such that $n_\gamma(w) \neq 0\}$ is bounded (indeed, the winding number is 0 on the unbounded component). Besides, adding the condition $n_\gamma(\alpha) \neq 0$ in the summation index on the right hand side of (5) does not change anything. But $A \cap \{w \in \mathbb{C}$ such that $n_\gamma(w) \neq 0\}$ is finite. Thus the residue theorem always gives a finite sum.

  (But we will sometimes apply the theorem, for each n, to some path γ_n and take a limit $n \to \infty$, leading to infinite sums; see later.)
- If a is a removable singularity of f, $\operatorname{Res}(f; a) = 0$ (no negative terms in the Laurent expansion). We can forget removable singularity. Any $a' \in U \setminus A$ can be seen as a removable singularity. Hence the RHS of (5) is sometimes written as

$$\sum_{a\in A} \operatorname{Res}(f; a)n_\gamma(a) = \sum_{a\in U} \operatorname{Res}(f; a)n_\gamma(a),$$

  with the convention $\operatorname{Res}(f; a) = 0$ if $a \notin A$.
- Often γ is taken as the "contour" of a bounded set V, in the **counterclockwise direction**. Then $n_\gamma(w) = \mathbf{1}[w \in V]$, so that the residue theorem gives

$$\int_\gamma f(z)dz = 2\pi i \sum_{a\in A\cap V} \operatorname{Res}(f; a).$$

  In this case, to compute $\int_\gamma f(z)dz$ (γ contour of some set V), the recipe is the following:

  i) find isolated singularities;
  ii) determine which singularities are in V;
  iii) compute their residue;

iv) make the sum and **don't forget the factor** $2\pi i$.

First proof of the residue theorem. Write $A = A_0 \sqcup A_1$, where

$$A_0 = \{a \in A; n_\gamma(a) = 0\} \qquad\qquad A_1 = \{a \in A; n_\gamma(a) \neq 0\}.$$

As explained above, A_1 is finite. For each $a \in A_1$, there exists $r_a > 0$ such that $\overline{D(a, r_a)} \setminus \{a\} \subseteq U \setminus A$ (since U open and A is discrete). Consider the cycle Γ defined by

$$\Gamma = \gamma - \sum_{a \in A_1} n_{\gamma_1}(a) \partial D(a; r_a).$$

- Fix $a_1 \in A_1$. We have
$$n_\Gamma(a_1) = n_\gamma(a_1) - \sum_{a \in A_1} n_\gamma(a) n_{\partial D(a, r_a)}(a_1).$$
From our choice of r_a, for $a \neq a_1$, we know that $a_1 \notin D(r, r_a)$ which implies $n_{\partial D(a,r_a)}(a_1) = 0$. Therefore
$$n_\Gamma(a_1) = n_\gamma(a_1) - n_\gamma(a_1) n_{\partial D(a_1, r_{a_1})}(a_1) = 0.$$
- Similarly, for $a_0 \in A_0$, we have $n_{\partial D(a,r_a)}(a_0) = 0$ for all $a \in A_1$ and
$$n_\Gamma(a_0) = n_\gamma(a_0) = 0.$$
- Finally for any $\alpha \notin U$, we have $n_\Gamma(w) = 0$, since U is simply connected.

We conclude that any w not in $U \setminus A$ satisfies $n_\Gamma(w) = 0$. Recall that, by assumption, f is holomorphic on $U \setminus A$. We can apply Corollary 14.6 and we have $\int_\Gamma f(z)dz = 0$. But, using the above Lemma,

$$\int_\Gamma f(z)dz = \int_\gamma f(z)dz - \sum_{a \in A_1} n_\gamma(a) \int_{\partial D(a, r_1)} f(z)dz = \int_\gamma f(z)dz - 2\pi\, i \left(\sum_{a \in A_1} n_\gamma(a) \operatorname{Res}(f; a) \right).$$

This concludes the proof. □

Second proof of the residue theorem (skipped in class). Let $A_1 = \{a_1, \dots, a_p\}$ as in the previous proof (A_1 is finite). For each $i \in \{1, \dots, p\}$, we consider the Laurent expansion of f around a_i:

$$f(z) = \sum_{n \in \mathbb{Z}} c_{n,i} (z - a_i)^n.$$

We use the above coefficients $c_{n,i}$ to introduce

$$g(z) = f(z) - \sum_{i=1}^{p} \left(\sum_{n<0} c_{n,i} (z - a_i)^n \right).$$

From Corollary 17.4 (Laurent separation theorem), $\sum_{n<0} c_{n,i}(z - a_i)^n$ is the principal part of $f(z)$ in some annulus $A(0, r)$ and therefore is convergent (uniformly on compacts) on $|z - a_i| > 0$, i.e. on $\mathbb{C} \setminus \{a_i\}$. Therefore g is holomorphic on $U \setminus A$. Note that,

- for fixed $1 \leq i \leq p$, the function
$$f(z) - \sum_{n<0} c_{n,i}(z - a_i)^n = \sum_{n \geq 0} c_{n,i}(z - a_i)^n$$
has a removable singularity in a_i;
- moreover, for $j \neq i$, $\sum_{n<0} c_{n,j}(z - a_j)^n$ is holomorphic in a_i

Therefore g has a removable singularity in a_i, which proves that g is holomorphic on $U \setminus A_0$. For any w outside $U \setminus A_0$, we have $n_\gamma(w) = 0$ (either by definition of A_0 for $w \in A_0$, or since U is simply connected, for $w \notin U$). We can apply Corollary 14.6 and we get $\int_\gamma g(z)dz = 0$, i.e.

$$\int_\gamma f(z)dz = \int_\gamma \left(\sum_{i=1}^{p} \left(\sum_{n<0} c_{n,i}(z - a_i)^n \right) \right) dz = \sum_{i=1}^{p} \sum_{n<0} \left(c_{n,i} \int_\gamma (z - a_i)^n dz \right). \tag{6}$$

The exchange sum/integrals is justified because the convergence is uniform on compact subsets of $\mathbb{C} \subset A_1$. For $n \neq -1$, the integral $\int_\gamma (z - a_i)^n dz$ vanishes since $z \mapsto (z - a_i)^n$ has an anti-derivative (namely

$\frac{1}{n+1}(z-a_i)^{n+1})$. For $n=-1$, we have $\int_\gamma (z-a_i)^{-1}dz = 2\pi\, i n_\gamma(a_i)$ by definition of the winding number. Putting this back in (6) concludes the proof. □

Corollary 19.4 (General Cauchy formula for derivatives)**.** *Let U be a simply connected domain, z in U, γ a closed path with $n_\gamma(z)=1$ and f a holomorphic function on U. Then, for any $n \geq 0$,*

$$f^{(n)}(z) = \frac{n!}{2\pi\, i}\int_\gamma \frac{f(w)}{(w-z)^{n+1}}dw.$$

(As before, we can assume $\alpha \notin U \Rightarrow n_\gamma(\alpha)=0$ instead of U simply connected and/or take a cycle instead of a closed path.)

Proof. The function $g(w) = \frac{f(w)}{(w-z)^{n+1}}$ is holomorphic on $U \setminus \{z\}$ and has residue $\frac{f^{(n)}(z)}{n!}$ in z. The formula is thus an application of the residue theorem. □

<u>How to compute residues?</u>

Lemma 19.5. *$f : U \setminus \{z_0\} \to \mathbb{C}$ holomorphic, $z_0 \in U$.*

i) If f has a removable singularity in z_0, $\mathrm{Res}(f; z_0) = 0$.

ii) linearity: if $g : U \setminus \{z_0\} \to \mathbb{C}$ holomorphic and $\lambda, \mu \in \mathbb{C}$ then,

$$\mathrm{Res}(\lambda f + \mu g; z_0) = \lambda\,\mathrm{Res}(f; z_0) + \mu\,\mathrm{Res}(g; z_0).$$

iii) If f has a simple pole in z_0,

$$\mathrm{Res}(f; z_0) = \lim_{z\to z_0}(z-z_0)f(z).$$

iv) More generally, if f has a pole of order m in z_0,

$$\mathrm{Res}(f; z_0) = \frac{g^{(m-1)}(z_0)}{(m-1)!}$$

where $g(z) = (z-z_0)^m f(z)$.

v) Let $f_1, f_2 : U \to \mathbb{C}$ be holomorphic functions on U and take z_0 in U. Recall that $f = \frac{f_1}{f_2}$ has then a removable singularity or pole in z_0. We assume that $f_2(z_0) = 0,\ f_2'(z_0) \neq 0$. Then

$$\mathrm{Res}\left(\frac{f_1}{f_2}; z_0\right) = \frac{f_1(z_0)}{f_2'(z_0)}.$$

<u>Remark:</u> None of these help when z_0 is an essential singularity.

Proof.

i) Already discussed above.

ii) Consider the Laurent expansion of f and g around z_0:

$$f(z_0+h) = \sum_{n\in\mathbb{Z}} c_n h^n, \qquad g(z_0+h) = \sum_{n\in\mathbb{Z}} d_n h^n.$$

The expansion of $\lambda f + \mu g$ then is

$$(\lambda f + \mu g)(z_0+h) = \sum_{n\in\mathbb{Z}} (\lambda c_n + \mu d_n)\, h^n.$$

The coefficient of h^{-1} is indeed $\lambda c_{-1} + \mu d_{-1}$, as claimed.

iii) In case of a simple pole, the smallest exponent appearing in the Laurent expansion is -1, i.e. we have $f(z) = \sum_{n\geq -1} c_n(z-z_0)^n$. Multiplying by $z - z_0$ gives

$$(z-z_0)f(z) = \sum_{n\geq -1} c_n (z-z_0)^{n+1} = \sum_{p\geq 0} c_{p-1}(z-z_0)^p.$$

In the limit $z \to z_0$, only the term for $p=0$ vanishes and we have

$$\lim_{z\to z_0}(z-z_0)f(z) = c_{-1} = \mathrm{Res}(f; z_0).$$

iv) Again, we consider the Laurent expansion of f around z_0, in which, this time, the smallest exponent is $-m$: $f(z) = \sum_{n \geq -m} c_n (z-z_0)^n$. Multiplying by $(z-z_0)^m$ gives

$$g(z) = (z-z_0)^m f(z) = \sum_{n \geq -m} c_n (z-z_0)^{n+m} = \sum_{p \geq 0} c_{p-m} (z-z_0)^p.$$

Note that $\mathrm{Res}(f; z_0) = c_{-1}$ is the coefficient of $(z-z_0)^{m-1}$ in this power series expansion of the holomorphic function $g(z)$ in z_0. This coefficient is $\frac{g^{(m-1)}(z_0)}{(m-1)!}$, as claimed.

v) If $f_1(z_0) = 0$, then $\left|\frac{f_1(z)}{f_2(z)}\right| \xrightarrow{z \to z_0} \frac{f_1'(z_0)}{f_2'(z_0)}$ and the singularity in z_0 is removable. The residue is therefore 0, coinciding with the given formula.

On the other hand, if $f_1(z_0) \neq 0$, we have $\left|\frac{f_1(z)}{f_2(z)}\right| \xrightarrow{z \to z_0} \infty$, so that $f(z) = \frac{f_1(z)}{f_2(z)}$ has a pole in z_0. But

$$(z-z_0)\frac{f_1(z)}{f_2(z)} = f_1(z) \left(\frac{f_2(z) - f_2(z_0)}{z - z_0} \right)^{-1} \xrightarrow{z \to z_0} \frac{f_1(z_0)}{f_2'(z_0)}$$

so that $(z-z_0)\frac{f_1(z)}{f_2(z)}$ has a removable singularity in z_0. We conclude that the pole of $\frac{f_1(z)}{f_2(z)}$ in z_0 in **simple** and we can apply iii):

$$\mathrm{Res}\left(\frac{f_1}{f_2}; z_0 \right) = \lim_{z \to z_0} (z-z_0) \frac{f_1(z)}{f_2(z)} = \frac{f_1(z_0)}{f_2'(z_0)}.$$

□

19.2. **Application to computation of real integrals.**

19.2.1. *Functions in* $\cos(t)$ *and* $\sin(t)$. We will present the idea on an example.

Example: Fix a parameter $a > 1$, ans say we want to compute the integral

$$I = \int_0^{2\pi} \frac{1}{a + \sin(\theta)} d\theta.$$

We'll write this as a path integral. To this end, observe that, from the definition of path integrals, we have

$$\int_{\partial D(0,1)} \frac{f(z)}{iz} dz = \int_0^{2\pi} \frac{f(e^{i\theta}) i e^{i\theta}}{i e^{i\theta}} d\theta = \int_0^{2\pi} f(e^{i\theta}) d\theta.$$

Taking $f(z) = \frac{1}{a + \frac{1}{2i}\left(z - \frac{1}{z}\right)}$, we have $f(e^{i\theta}) = \frac{1}{a + \sin(\theta)}$ and

$$I = \int_{\partial D(0,1)} \frac{1}{iz} \frac{1}{a + \frac{1}{2i}\left(z - \frac{1}{z}\right)} dz = \int_{\partial D(0,1)} \frac{2}{2iaz + z^2 - 1} dz.$$

We will compute I through the residue theorem

- First step: find the singularities of the integrand.

  We need to solve $2iaz + z^2 - 1 = 0$: this is a quadratic equation with discriminant

  $$\Delta = -4a^2 + 4 = (2i\sqrt{a^2 - 1})^2.$$

  Thus the singularities we are looking for are $\{z^-, z^+\}$, where

  $$z^{\pm} = \frac{-2ia \pm 2i\sqrt{a^2-1}}{2} = -ia \pm i\sqrt{a^2 - 1}.$$

- Second step: are these singularities inside the integration path?

  Note that $|a + \sqrt{a^2 - 1}| > |a| > 1$ so that the singularity $z^- = -i(a + \sqrt{a^2 - 1})$ is not inside of our integration path. Since $z^+ \cdot z^- = -1$, we have on the other hand $|z^+| = \frac{1}{|z^-|} < 1$, so that z^- is inside our integration path.

- Third step: compute the residue of z^+.
  It is a simple pole (simple root of the denominator, while the numerator does not vanish) so that
$$\operatorname{Res}\left(\frac{2}{2iaz+z^2-1}; z^+\right) = \lim_{z\to z^+} \frac{2(z-z^+)}{2iaz+z^2-1}$$
  Trick: the denominator is $(z-z^-)(z-z^+)$ (monic degree 2 polynomial with root z^+ and z^-), thus
$$\operatorname{Res}\left(\frac{2}{2iaz+z^2-1}; z^+\right) = \lim_{z\to z^+} \frac{2}{z-z^-} = \frac{2}{z^+-z^-} = \frac{1}{i\sqrt{a^2-1}}.$$
- Conclusion (don't forget the factor $2\pi\, i$):
$$I = \frac{2\pi}{\sqrt{a^2-1}}.$$

<u>Remark:</u> The integral of a real-valued/nonnegative function on a real interval is a real-valued/nonnegative number. If you find, e.g., a purely imaginary number, you've done a mistake (maybe forgetting the factor $2\pi\, i$).

<u>General statement:</u> Let $R(x,y) = \frac{P(x,y)}{Q(x,y)} \in \mathbb{C}(x,y)$. Assume that $x^2+y^2=1 \Rightarrow Q(x,y)\neq 0$. Then
$$\int_0^{2\pi} R(cos(\theta), \sin(\theta))d\theta = 2i\pi \sum_{|z_0|<1} \operatorname{Res}(\widetilde{R}; z_0)$$
where
$$\widetilde{R}(z) = \frac{i}{iz} R\left(\frac{1}{2}\left(z+\frac{1}{z}\right); \frac{1}{2i}\left(z-\frac{1}{z}\right)\right)$$

Proof. Similar to example. □

19.2.2. *Integrals of rational functions over* $\mathbb{R}$.

<u>Example:</u> We want to compute
$$I = \int_{-\infty}^{\infty} \frac{1}{1+x^6} dx$$
Note that $\frac{1}{1+x^6} \leq \frac{1}{x^2}$ for $|x|$ sufficiently big, which implies that the integral is indeed convergent. Therefore, $I = \lim_{M\to\infty} \int_{-M}^{M} \frac{1}{1+x^6} dx$.

Difficulty: Residue theorem gives values of integrals over closed paths. Here we want an integral over an interval of size tending to infinity.

General strategy: Recall that $I = \lim_{M\to\infty} \int_{-M}^{M} \frac{1}{1+x^6} dx$. We complete $[-M;M]$ into a closed path, in such a way that the integral on the "completion" tends to zero when M tends to $+\infty$.

In the example, we set $\delta_M(\theta) = Me^{i\theta}$ for $\theta\in[0;\pi]$ (i.e., δ_M is the upper semicircle with center 0 and radius M). Then $\gamma_M = [-M;M] + \delta_M$ is a closed path (+ stands for path concatenation).

(Picture)

We have
$$\int_{\gamma_M} \frac{1}{1+z^6} dz = \int_{[-M;M]} \frac{1}{1+z^6} dz + \int_{\delta_M} \frac{1}{1+z^6} dz$$
and
$$\int_{\delta_M} \frac{1}{1+z^6} dz \leq L(\delta_M) \sup_{c\in\delta_M} \left|\frac{1}{1+z^6}\right| \leq M\pi \frac{1}{M^6-1} \xrightarrow{M\to\infty} 0.$$
Taking the limit $M\to\infty$, we get
$$\int_{-\infty}^{+\infty} \frac{1}{1+z^6} dz = \lim_{M\to\infty} \int_{\gamma_M} \frac{1}{1+z^6} dz. \tag{7}$$

The residue theorem gives:

$$\int_{\gamma_M} \frac{1}{1+z^6} dz = 2\pi i \sum_{z_0 \text{ pole of } \frac{1}{1+z^6}} n_\gamma(z_0) \operatorname{Res}\left(\frac{1}{1+z^6}; z_0\right)$$

- The set of poles of $\frac{1}{1+z^6}$ is $\{z_0 \in \mathbb{C}; z_0^6 + 1 = 0\} = \{e^{(2k+1)\frac{i\pi}{6}} : k \in \{0, 1, \dots, 5\}\}$. They are all simple poles.
- For $M > 1$, the winding numbers of γ around the poles are given by

$$n_\gamma(z_0) = \begin{cases} 1 & \text{for } z_0 \in \left\{e^{\frac{i\pi}{6}}; e^{\frac{3i\pi}{6}}; e^{\frac{5i\pi}{6}}\right\}; \\ 0 & \text{for } z_0 \in \left\{e^{\frac{7i\pi}{6}}; e^{\frac{9i\pi}{6}}; e^{\frac{11i\pi}{6}}\right\}. \end{cases}$$

- We compute the corresponding residues (using, *e.g.*, the formula with the derivative, valid since poles are simple):

$$\operatorname{Res}\left(\frac{1}{1+z^6}; z_0\right) = \frac{1}{6z_0^5} = -\frac{z_0}{6}$$

- Conclusion: the residue theorem implies

$$\int_{\gamma_M} \frac{1}{1+z^6} dz = \frac{-2\pi i}{6}\left(e^{\frac{i\pi}{6}} + e^{\frac{i\pi}{2}} + e^{\frac{5i\pi}{6}}\right).$$

But $e^{\frac{i\pi}{6}} + e^{\frac{5i\pi}{6}} = 2i \sin\left(\frac{\pi}{6}\right) = i$. Finally, for $M > 1$, we have

$$\int_{\gamma_M} \frac{1}{1+z^6} dz = -\frac{2\pi i}{6}(2i) = \frac{2\pi}{3}.$$

Back to (7), we see that

$$\int_{-\infty}^{+\infty} \frac{1}{1+z^6} dz = \frac{2\pi}{3}.$$

General statement: Let $f(z) = \frac{P(z)}{Q(z)}$ be a rational function. Assume $Q(x) \neq 0$ for $x \in \mathbb{R}$ and $\deg(Q) \geq \deg(P) + 2$. Then

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx = 2\pi i \sum_{\substack{\operatorname{Im}(z_0)>0 \\ z_0 \text{ poles of } P/Q}} \operatorname{Res}\left(\frac{P}{Q}; z_0\right)$$

Sketch of proof. The condition $\deg(P) \leq \deg(Q) + 2 \Rightarrow$ ensures that for $|z|$ is sufficiently big, $\left|\frac{P(z)}{Q(z)}\right| \leq \frac{C}{|z|^2}$ (for some constant C) so that the integral is well defined. This upper bound also implies

$$\left|\int_{\delta_M} \frac{P(z)}{Q(z)} dz\right| \xrightarrow{M\to\infty} 0$$

where δ_M is defined as above. The remaining arguments are similar to that used in the above example. □

19.3. **Counting zeroes: the argument principle.** We start with a definition.

Definition 19.6. *Let f be a holomorphic function $U \subseteq \mathbb{C}$ and z_0 be in U. We say that z_0 is a zero of f of order m (m being a positive integer) if*

i) $f(z_0) = f'(z_0) = \dots = f^{(m-1)}(z_0) = 0$, but $f^{(m)}(z_0) \neq 0$.
ii) equivalently, $f(z) = (z - z_0)^m g(z)$ where g is holomorphic on a neighbourhood of z_0 and fulfills $g(z_0) \neq 0$.

Proof of the equivalence in the above definition. $(ii) \Rightarrow (i)$ By induction, it is easy to show the following: for $k \leq m$, we have $f^{(k)}(z) = (z-z_0)^{m-k} h_k(z)$ where h_k is a holomorphic function on U such that $h_k(z_0) \neq g(z_0)$. This implies that $f^{(k)}(z_0) = 0$ if $k < m$, while $f^{(m)}(z_0) = g(z_0) \neq 0$.
$(i) \Rightarrow (ii)$ Consider the power-series expansion of f around z_0: $f(z) = \sum_{n \geq 0} a_n (z - z_0)^n$. We know that

$$a_k = \frac{f^{(k)}(z_0)}{k!} \begin{cases} = 0 & \text{for } k < m; \\ \neq 0 & \text{for } k = m. \end{cases}$$

Therefore, we have

$$f(z) = \sum_{n \geq m} a_n (z - z_0)^n = (z - z_0)^m \left(\sum_{p \geq 0} a_{m+p} (z - z_0)^p \right).$$

The map $g(z) := \sum_{p \geq 0} a_{m+p} (z - z_0)^p$ is a holomorphic function on a neighbourhood of 0 (sum of power series with a positive radius of convergence. Its value in 0 is $g(0) = a_m \neq 0$, as wanted. $\square$

We recall that z_0 is a pole of order m of f if $f(z) = \frac{g(z)}{(z-z_0)^m}$ with g holomorphic around z_0 and $g(z_0) \neq 0$. Comparing with the second characterization of zero of order m, we get that z_0 is a zero of order m of f if and only if z_0 is a pole of order m of $1/f$.

Theorem 19.7 (argument principle)**.** *Let U be a simply connected open set. Let f be a meromorphic function on U and γ be a closed path in U such that $\Im m(\gamma)$ does not contain any poles or zeros of f. Then*

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \sum_{z_0 \text{ zero of } f} n_\gamma(z_0) \, order(z_0) - \sum_{z_0 \text{ pole of } f} n_\gamma(z_0) \, order(z_0).$$

Proof. The key point is that orders of poles and zeroes are residues of f'/f. Indeed, the following holds:

- Assume z_0 is neither a pole nor a zero of f, i.e. f is holomorphic in z_0 and $f(z_0) \neq 0$. Then f'/f holomorphic in z_0 implies $\mathrm{Res}(f'/f; z_0) = 0$.
- Assume that z_0 is a zero of order m of f, i.e. $f(z) = (z - z_0)^m g(z)$ with $g(z) \neq 0$. Then
$$\frac{f'(z)}{f(z)} = \frac{m}{z - z_0} + \frac{g'(z)}{g(z)}.$$
We have trivially $\mathrm{Res}(\frac{m}{z-z_0}; z_0) = m$, while $\mathrm{Res}(g'/g; z_0) = 0$ from the first item. We conclude that $\mathrm{Res}(f'/f; z_0) = m$.
- Assume now that z_0 is a pole of order m of f, i.e. $f(z) = \frac{g(z)}{(z-z_0)^m}$ with $g(z_0) \neq 0$. Then
$$\frac{f'(z)}{f(z)} = \frac{g'(z)}{g(z)} - \frac{m}{z - z_0}.$$
We conclude as above that $\mathrm{Res}(f'/f; z_0) = -m$.

The theorem then follows from the residue theorem applied to f'/f. $\square$

Remarks:

- Both sums are always finite (since $\{w : n_\gamma(w) \neq 0\}$ is bounded and the set of zeros/poles of f is discrete).
- As usual, we can relax the assumption "U simply connected" by only requiring "$\alpha \notin U \Rightarrow n_\gamma(a) = 0$". Similarly, the theorem holds more generally for cycles Γ instead of only closed paths γ.
- The theorem is particularly interesting when γ is the counterclockwise counter of some bounded set V. In this setting, the theorem gives
$$\int_\gamma \frac{f'(z)}{f(z)} dz = 2\pi i \left(\sum_{\substack{z_0 \text{ zero of } f \\ z_0 \in V}} \mathrm{order}(z_0) - \sum_{\substack{z_0 \text{ pole of } f \\ z_0 \in V}} \mathrm{order}(z_0) \right).$$
The right hand-side is then the number of zeroes counted with multiplicities (the multiplicity being the order of the zero, interpreting poles as zeroes of negative multiplicities.)

- The left hand side can be reinterpreted as a winding number. Indeed, by definition of path integral

$$\int_\gamma \frac{f'(z)}{f(z)}dz = \int_a^b \frac{f'(\gamma(t))\gamma'(t)}{f(\gamma(t))}dt = \int_a^b \frac{(f\circ\gamma)'(t)}{(f\circ\gamma)(t)}dt = \int_{f\circ\gamma}\frac{1}{z}dz = 2\pi i\, n_{f\circ\gamma}(0).$$

Corollary 19.8. (Rouché's theorem) *Let U be a simply connected domain and f and g be meromorphic functions on U. Let γ be the counterclockwise contour of some bounded set V such that $\partial V = \Im m(\gamma)$ does not contain zeros or poles of f or g. Assume furthermore that $z \in \Im m(\gamma)$ implies $|f(z) - g(z)| \le |g(z)|$. Then f and g have the same number of zeros in V (where zeroes are counted with multiplicities, poles being zeros of negative multiplicities).*

Proof. We want to prove

$$n_{f\circ\gamma}(0) = \int_\gamma \frac{f'(z)}{f(z)}dz = \int_\gamma \frac{g'(z)}{g(z)}dz = n_{g\circ\gamma}(0).$$

By assumption, for $t \in [a;b]$,

$$|(f\circ\gamma)(t) - (g\circ\gamma)(t)| \le |(g\circ\gamma)(t)|.$$

Lemma 15.4 implies $n_{f\circ\gamma}(0) = n_{g\circ f}(0)$. □

(Skipped in class) To illustrate the last statement, we give yet another proof of the fundamental theorem of algebra, i.e. that a complex polynomial of degree d always have d complex roots (when counted with multiplicities).

Proof. Let d be a positive integer and $p(z) = \sum_{i=0}^d a_i z^i$ be a polynomial of degree d ($a_d \neq 0$). We want to apply Rouch theorem to $f(z) = p(z)$ and its dominant monomial $g(z) = a_d z^d$. We have to check the hypothesis i.e. that

$$\left|p(z) - a_d z^d\right| = \left|\sum_{i=1}^{d-1} a_i z^i\right| \overset{?}{\le} |a_d z^d|.$$

By the triangular inequality, $\left|\sum_{i=1}^{d-1} a_i z^i\right| \le \sum_{i=1}^{d-1}|a_i||z^i|$, while $|a_d z^d| = |a_d||z^d|$. The difference $|a_d||z|^d - \sum_{i=1}^{d-1}|a_i||z^i|$ is a polynomial in $|z|$ with positive dominant coefficient, and therefore tends to $+\infty$ when $|z|$ tends to $+\infty$. Thus there exists R_0 such that

$$|z| \ge R_0 \;\Rightarrow\; |a_d||z|^d > \sum_{i=1}^{d-1}|a_i||z^i|.$$

This implies also $\left|p(z) - a_d z^d\right| < |a_d z^d|$, as wanted.

Take $\gamma = \partial D(0,R_0)$. As justified above, for $z \in \Im m(\gamma)$, we have $|p(z) - a_d z^d| \le |a_d z^d|$. We can therefore apply Rouché's theorem to γ, $f(z) = p(z)$ and $g(z) = a_d z^d$, and conclude that both functions have the same number of zeros in $D(0,R_0)$. But $a_d z^d$ has one zero of multiplicity d in 0. We conclude that $p(z)$ has d zeros (when counted with multiplicities) in $D(0,R_0)$. It is straightforward to check that p has no root outside $D(0,R_0)$. □

(Note: unlike previous proofs, this proof does not only prove the existence of a root, but directly gives their number. It is however easy to show that each complex polynomial has d root, when counted with multiplicity, knowing that all non-constants polynomials have at least one root ρ: write $P(z) = (z-\rho)Q(z)$ and apply the statement inductively to $Q(z)$.)

Another consequence of the argument principle is Hurwitz theorem.

Theorem 19.9 (Hurwitz)**.** *Let f_n a sequence of holomorphic functions on a domain $U \subseteq \mathbb{C}$, tending to f uniformly of compact subsets of U. Assume f is not constant and has a zero of order $m \ge 0$ in z_0. Then for $r > 0$ small enough and n sufficiently large (depending on r), f_n has exactly m zeroes (counted with multiplicity) in the disk $D(z_0, r)$.*

TODO: ADD A PROOF.

Corollary 19.10. *Let f_n a sequence of holomorphic functions on a domain $U \subseteq \mathbb{C}$, tending to f uniformly of compact subsets of U. We assume that f_n does not vanish on U. Then, either f does not vanish on U, or it is identically equal to* 0.

Corollary 19.11. *Let f_n a sequence of* **injective** *holomorphic functions on a domain $U \subseteq \mathbb{C}$, tending to f uniformly of compact subsets of U. Then f is either injective or constant.*

20. A bit of geometry

20.1. **Holomorphic functions and angles.** Let $\gamma_1, \gamma_2 : [0,1] \to \mathbb{C}$ be two curves with the same starting point, i.e. $\gamma_1(0) = \gamma_2(0) = a$. If both are differentiable in 0, then one can define the angle between the two curves, as being the angles between their tangent lines, i.e. an argument of $\gamma_2'(0)/\gamma_1'(0)$. (We're considering here an algebraic angle, i.e. with a sign)

(Picture)

Proposition 20.1. *Let $\gamma_1, \gamma_2 : [0,1] \to \mathbb{C}$ be as above and f be a holomorphic function on an open set containing a and assume $f'(a) \neq 0$. Then the angle between $f \circ \gamma_1$ and $f \circ \gamma_2$ equals the angle between γ_1 and γ_2.*

Informally, holomorphic functions preserve angles (when the derivative does not vanish!!). This is in fact an "if and only if". A real-differentiable function $\mathbb{C} \to \mathbb{C}$ that preserves angles is necessarily complex-differentiable.

20.2. **Riemann-sphere.**

Definition 20.2. *The Riemann-sphere is the set $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ with the following topology: $U \subseteq \mathbb{C}$ is open if and only if*

i) $U \cap \mathbb{C}$ is open in $\mathbb{C}$
ii) if $\infty \in U$, then $\widehat{\mathbb{C}} \setminus U$ is bounded.

Remarks:

- We should check that it defines a topology (union of open set is open, finite intersection of open sets is open, $\emptyset$ is open, $\widehat{\mathbb{C}}$ is open etc.).
- $\widehat{\mathbb{C}}$ is separated (i.e. if $z, w \in \mathbb{C}$ we can find U, V open, $U \cap V = \emptyset$, $z \in U$, $w \in V$).

Proposition 20.3. *$\widehat{\mathbb{C}}$ is compact and is homeomorphic to the 2-dimensional sphere (unit sphere in $\mathbb{R}^3$)*

$$S^2 := \{(u,v,w) \in \mathbb{R}^3 : u^2 + v^2 + w^2 = 1\}.$$

Proof (skipped in class). Let us prove first that $\widehat{\mathbb{C}}$ is compact. Assume that $\widehat{\mathbb{C}} = \bigcup_{i \in I} U_i$ therefore there exists i_0 such that $\infty \in U_{i_0} \Rightarrow \widehat{\mathbb{C}} \setminus U_{i_0}$ is bounded. But $\widehat{\mathbb{C}} \setminus U_{i_0} \subseteq \mathbb{C}$ and closed so that $\widehat{\mathbb{C}} \setminus U_{i_0}$ is compact. Moreover

$$\widehat{\mathbb{C}} \setminus U_{i_0} \subseteq \bigcup_{\substack{i \in I \\ i \neq i_0}} U_i$$

By compactness we can extract the finite covering: i.e. there exists $J \subseteq I$, ($|J| < \infty$) such that

$$\widehat{\mathbb{C}} \setminus U_{i_0} \subseteq \bigcup_{i \in J} U_i \Rightarrow \widehat{\mathbb{C}} \subseteq \bigcup_{i \in J \cup \{i_0\}} U_i$$

this shows that $\widehat{\mathbb{C}}$ is compact.

Let us prove that $S^2 \cong \widehat{\mathbb{C}}$. We define $\varphi : S^2 \to \widehat{\mathbb{C}},\ (u,v,w) \mapsto \begin{cases} \frac{u+iv}{1-w} & \text{if } w \neq 1 \\ \infty & \text{if } w = 1. \end{cases}$

- φ is a bijection. Geometrically $\begin{cases} \varphi^{-1}(\infty) = N \\ \varphi^{-1}(m) = (N,m) \cap S^2 \end{cases}$ (easy with the formula to prove that φ is bijective).

- φ is continuous. From the formula, φ is continuous on $S^2 \setminus \{(0;0;1)\}$. Continuity in $N = (0;0;1)$. We have to prove that $\lim\limits_{(u;v;w)\to(0;0;1)} |\varphi(u;v;w)| = \infty$. But
$$|\varphi(u;v;w)|^2 = \frac{u^2+v^2}{(1-w)^1} = \frac{1-w^2}{(1-w)^2} = \frac{1+w}{1-w}$$
which tends to ∞ when $w \longrightarrow 1$.
- φ^{-1} is continuous. Because S^2 and $\widehat{\mathbb{C}}$ are compact (if $F \subseteq S^2$ is closed, then F is compact so that $\varphi(F) = (\varphi^{-1})^{-1}(F)$ is also closed. I.e. pre-image of closed sets by φ^{-1} are closed, which proves the continuity of φ^{-1}). □

It is natural to consider functions from or to the Riemann sphere. Here is how these notions should be interpreted.

Functions with values in $\widehat{\mathbb{C}}$:

Let f be meromorphic on U. By definition, f is a holomorphic function $U \setminus P \to \mathbb{C}$, for a discrete subset P of U, consisting uniquely of poles of f.

Then we can extend f to a function $U \to \widehat{\mathbb{C}}$ by setting $f(p) = \infty$ for $p \in P$ (since $\lim_{z\to p} |f(z)| = \infty$, this extension $U \to \widehat{\mathbb{C}}$ is a continuous function).

Functions with domain set $\widehat{\mathbb{C}}$

Let U be a subset of $\widehat{\mathbb{C}}$, $\infty \in U$. We want to consider holomorphic and meromorphic on U.

Definition 20.4. *Let $f : U \setminus \{\infty\} \to \mathbb{C}$ holomorphic. We say that f is holomorphic (or meromorphic) at ∞ if $w \mapsto f(\frac{1}{w})$ is holomorphic (or meromorphic) at zero.*

Remark: $f(\frac{1}{w})$ has an isolated singularity in 0 (since U is open and contains ∞, it contains the complement of a bounded set, i.e. $\{w : \frac{1}{w} \in U\}$ contains a neighbourhood of 0). In particular, the results on isolated singularity can be applied. Namely,

- The limit $\rho = \lim |z| \to \infty f(z)$ exits if and only if $|f(z)|$ is bounded outside a compact set.
- Similarly there exists m such that the limit $\lim_{z\to\infty} \frac{f(z)}{z^m}$ exists if and only if $\lim_{|z|\to\infty} |f(z)| \to \infty$.

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