

Caractères irréductibles du groupe symétrique et grands diagrammes de Young

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Outline of the talk

- 1 Rappel de la première séance
- 2 Asymptotique des caractères
 - Equivalents classiques
 - Forme limite de diagrammes de Young

Exact formula for character values

We want to compute the value of the irreducible characters of S_n :

- indexed by $\lambda \vdash n$;
- on a **small** permutation $\pi \in S_k \xhookrightarrow{\iota} S_n$.

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Definition

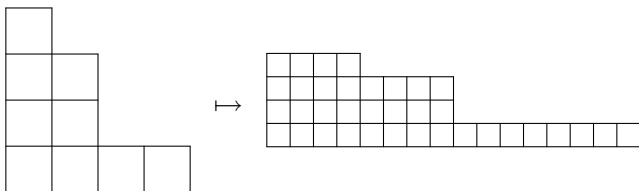
$N_{\sigma_1, \sigma_2}(T)$ is the number of functions $f : \{1, \dots, k\} \rightarrow \lambda$ such that, for all i , $f(i)$ and $f(\sigma_1(i))$ (resp. $f(\sigma_2(i))$) are in the same **row** (resp. **column**).

$$\Sigma_{\pi}(\lambda) := \frac{n \cdot (n-1) \dots (n-k+1) \chi^{\lambda}(\iota(\pi))}{\dim \lambda} = \sum_{\substack{\sigma_1, \sigma_2 \in S_k \\ \sigma_2 \sigma_1 = \pi}} (-1)^{\sigma_2} N_{\sigma_1, \sigma_2}(\lambda).$$

Asymptotics is easy to read on this formula

Model : fix a permutation $\pi_0 \in S_k$ and a partition $\lambda_0 \vdash k$

Consider $\pi = \iota(\pi_0)$ (i.e. we just add fixpoints) and $\lambda = c \cdot \lambda_0 = \lambda$ multiplied by c (i.e. horizontal lengths are multiplied by c)

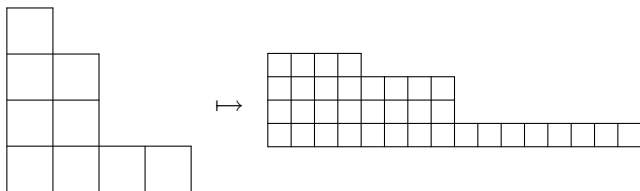


Question : asymptotics of $\frac{\chi^\lambda(\pi)}{\dim \lambda}$?

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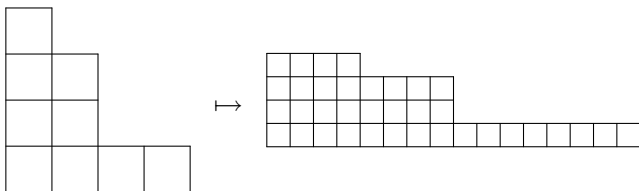
Easy on the N 's : $N_{\sigma_1, \sigma_2}(c \cdot \lambda) = c^{|\mathcal{C}(\sigma_2)|} N_{\sigma_1, \sigma_2}(\lambda)$

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Dominant term of $\Sigma_\pi(\lambda) \sim n^k \chi^\lambda(\pi) / \dim(\lambda)$:

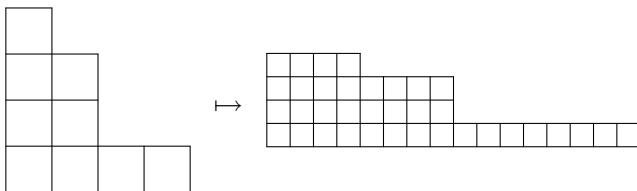
$$N_{\pi, \text{Id}_k}(\lambda) = \prod_{\mu_j \in \text{type}(\pi)} \left(\sum_j \lambda_j^{\mu_j} \right)$$

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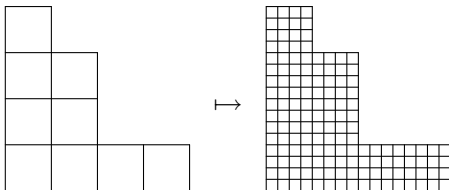
Dominant term of $\Sigma_\pi(\lambda) \sim n^k \chi^\lambda(\pi) / \dim(\lambda)$:

$$N_{\pi, \text{Id}_k}(\lambda) = \prod_{\mu_i \in \text{type}(\pi)} \left(\sum_j \lambda_j^{\mu_i} \right) = \prod_{\mu_i \in \text{type}(\pi)} p_{\mu_i}(\lambda)$$

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Model : fix a permutation $\pi_0 \in S_k$ and a partition $\lambda_0 \vdash k$

Consider $\pi = \iota(\pi_0)$ (i.e. we just add fixpoints) and $\lambda = c \bullet \lambda_0 = \lambda$ dilated by c (i.e. **horizontal and vertical** lengths are multiplied by c)

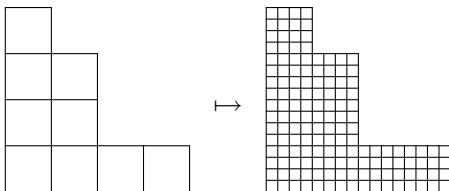


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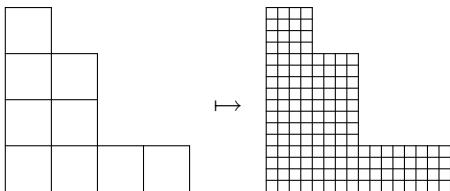
Question : asymptotics of $\frac{\chi^\lambda(\pi)}{\dim \lambda}$?

Easy on the N 's : $N_{\sigma_1, \sigma_2}(c \bullet \lambda) = c^{|\mathcal{C}(\sigma_1)| + |\mathcal{C}(\sigma_2)|} N_{\sigma_1, \sigma_2}(\lambda)$

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Dominant term of $\sum_{\pi} \chi^\lambda(\pi) \sim n^k \chi^\lambda(\pi) / \dim(\lambda)$:

$$\sum_{\substack{\sigma_1 \sigma_2 = \pi \\ |\mathcal{C}(\sigma_1)| + |\mathcal{C}(\sigma_2)| \text{ maximal}}} \pm N_{\sigma_1, \sigma_2}(\lambda)$$

Free cumulants

$$\Sigma_{\pi}(c \bullet \lambda) = \sum_{\substack{\sigma_1 \sigma_2 = \pi \\ |C(\sigma_1)| + |C(\sigma_2)| \text{ maximal}}} \pm N_{\sigma_1, \sigma_2}(\lambda)$$

$$\text{But, } \left\{ \begin{array}{l} \sigma_1 \sigma_2 = \pi \\ |C(\sigma_1)| + |C(\sigma_2)| \text{ maximal} \end{array} \right\} \simeq \prod NC_{\mu_i} \simeq \prod \text{Trees}(\mu_i).$$

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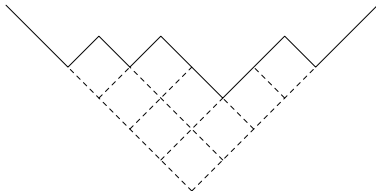
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With generating series, one can prove (Rattan, 2006)

$$rhs = \prod R_{\mu_i+1}(\lambda)$$

R_k : free cumulants defined from the shape ω_{λ} by Biane (1998).



Remarks

Works in more general context than sequences $c \cdot \lambda_0$ and $c \bullet \lambda_0$ (in fact, works as soon as a sequence of Young diagrams has a *limit*)

These results were already known (Vershik & Kerov 81, Biane 98), but :

- we provide unified approach of both cases ;
- our bound for error terms are better.

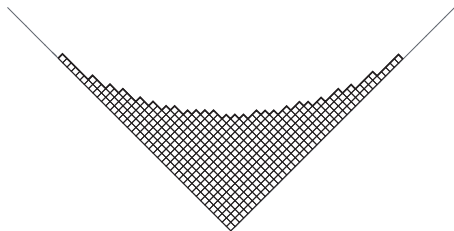
Description of the problem

Consider Plancherel's probability measure on Young diagrams of size n

$$P(\lambda) = \frac{(\dim \lambda)^2}{n!}$$

\exists equivalent description as a Markov process

Question : is there a limit shape for (renormalized rotated) Young diagram taken randomly with Plancherel's measure when $n \rightarrow \infty$?



Normalized character values have simple expectations !

Fix $\pi \in \mathfrak{S}_n$. Let us consider the random variable :

$$X_\pi(\lambda) = \chi^\lambda(\pi) = \frac{\text{tr}(\rho_\lambda(\pi))}{\dim V_\lambda}.$$

Let us compute its expectation :

$$\begin{aligned} \mathbb{E}(X_\pi) &= \frac{1}{n!} \sum_{\lambda \vdash n} (\dim V_\lambda) \cdot \text{tr}(\rho_\lambda(\pi)) \\ &= \frac{1}{n!} \text{tr}\left(\bigoplus_{\lambda \vdash n} V_\lambda^{\dim V_\lambda}\right)(\pi) = \frac{1}{n!} \text{tr}_{\mathbb{C}[\mathfrak{S}_n]}(\pi) \end{aligned}$$

Last expression is easy to evaluate :

$$\mathbb{E}(X_\pi) = \delta_{\pi, \text{Id}_n}$$

Convergence of cumulants

Recall : we proved that $\prod_i R_{k_i+1} \approx \Sigma_{(1 \dots k_1)(k_1+1 \dots k_1+k_2)\dots}$

Thus

$$\mathbb{E}(R_2) \approx \mathbb{E}(n \cdot \chi(\text{Id})) = n$$

$$\mathbb{E}(R_i) \approx \mathbb{E}(n^i \cdot \chi((1 \dots i))) = 0 \text{ if } i > 2$$

$$\begin{aligned} \text{Var}(R_i) = \mathbb{E}(R_i^2) - \mathbb{E}(R_i)^2 &\approx \mathbb{E}(n^{2i} \cdot \chi((1 \dots i)(i+1 \dots 2i))) \\ &\quad - \mathbb{E}(n^i \cdot \chi((1 \dots i)))^2 = 0 \text{ if } i \geq 2 \end{aligned}$$

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Recall : we proved that $\prod_i R_{k_i+1} \approx \Sigma_{(1 \dots k_1)(k_1+1 \dots k_1+k_2)\dots}$

Thus

$$\lim \mathbb{E}(R_2/n) = 1$$

$$\lim \mathbb{E}\left(R_i/\sqrt{n^i}\right) = 0 \text{ if } i > 2$$

$$\lim \text{Var}\left(R_i/\sqrt{n^i}\right) = 0 \text{ if } i \geq 2$$

Easy to make it formal because $R_k \in \text{Vect}(\Sigma_\pi)$.

\Rightarrow Random variables $R_i/\sqrt{n^i}$ converge in probability towards the sequence $(0, 1, 0, 0, \dots)$

General lemma from Kerov :

convergence of cumulants \Rightarrow convergence of Young diagrams

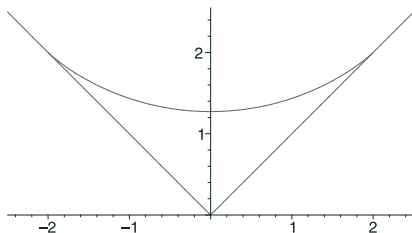
Existence of a limiting curve

Theorem (Logan and Shepp 77, Kerov and Vershik 77)

Let us take randomly (with Plancherel measure) a sequence of Young diagram λ_n of size n . Then, in probability, for the uniform convergence topology on continuous functions, one has :

$$r_{45^\circ}(h_{1/\sqrt{n}}(\lambda_n)) \rightarrow \delta_\Omega,$$

where Ω is an explicit function drawn here :



Convergence of q -Plancherel measure

Case where expectation of character values are big :

- there can not be a limit shape after dilatation.
- we use the first approximation for characters $\Sigma_{\pi}(\lambda) \approx \prod p_{\mu_i}(\lambda)$.

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Example : q -Plancherel measure ($q < 1$)

- defined using representation of Hecke algebras
- one can prove

$$\mathbb{E}_q(\Sigma_\pi) = \frac{(1-q)^{|\mu|}}{\prod_i 1-q^{\mu_i}} n(n-1)\dots(n-|\mu|+1)$$

Thus

$$\mathbb{E}_q(p_k) \approx \frac{(1-q)^k}{\prod_i 1-q^k} n^k \quad \text{Var}_q(p_k) \approx 0$$

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Thus

$$\lim \mathbb{E}_q(p_k/n^k) = \frac{(1-q)^k}{\prod_i 1-q^k} \quad \lim \text{Var}_q(p_k/n^k) = 0$$

Convergence of q -Plancherel measure

Theorem (F., Méliot, 2010)

Let $q < 1$. In probability, under q -Plancherel measure,

$$\forall k \geq 1, \frac{p_k(\lambda)}{|\lambda|^k} \xrightarrow{M_{n,q}} \frac{(1-q)^k}{1-q^k}.$$

Moreover,

$$\forall i \geq 1, \frac{\lambda_i}{n} \xrightarrow{M_{n,q}} (1-q) q^{i-1};$$

We also obtained the second-order asymptotics.

End of the talk

Thank you for listening

Questions ?