Shifted Schur functions

Sahi, Biedenharn-Louck, Okounkov-Olshanski, ... 90's
a result joint with P. Alexandersson at the end

Context: Schur polynomials $s_\lambda(x_1, \dots, x_N)$ $(\lambda \vdash d)$

A - defined as a quotient of determinants
B - as sum over SSYT.

Other Approach developped in the 90's:

consider interpolation symmetric polynomial
$\rightsquigarrow$ see again Schur functions appear...

$\rightarrow$ We will conclude by a new positivity result on those interpolation sym pol.

I Interpolation symmetric polynomial.

Idea: define sym functions by values at specific points
(like interp. pol.)

How many points? to get sym pol. deg $\leq d$,
a priori $|\mathcal{P}^{\leq N}_{\leq d}|$ points

⚠ not any set of points will work. part. size at most d, length at most N

natural choice: $\mathcal{P}^{\leq N}_{\leq d}$

We will add shifts: fix parameters $e_1, \dots, e_N$
let $M_e := \mathcal{P}^{\leq N}_{\leq d} + (e_1, \dots, e_N) = \{(\lambda_1 + e_1, \dots, \lambda_N + e_N);\ |\lambda| \leq d,\ \ell(\lambda) \leq N\}$

Thm: let $d \geq 1$ and $e \in \mathbb{C}^n$.

Assume $\forall i<j$, $e_i - e_j \neq -1, -2, \dots, -\lfloor \frac{d}{i} \rfloor$.

Then for every map $f: M_e \to \mathbb{C}$, there exists a unique symmetric polynomial f of degree $\leq d$ such that $f/M_e = f$.

Proof: Write $f = \sum_{|\lambda| \leq d} a_\lambda m_\lambda$

Condition $f/M_e = f$ is a square system of lin. equations in a_λ

$\Rightarrow$ existence for all f implies uniqueness. ($\Leftarrow$ det square matrix $\neq 0$)

proof of ex by induction on $n+d$

Look for f as

$$f(x_n) = \left[\prod_{i=1}^{N} (x_i - e_N)\right] \cdot h + g \qquad (*) \qquad h, g \text{ sym polynomial}$$

when $x_N = e_N$ (i.e. $\lambda_N = 0$), first term vanish so we want $g/_{M_e \cap \{\lambda_N = 0\}} = f$

$M_e \cap \{\lambda_N = 0\} \simeq P^{\leq N-1}_{\leq d} + (e_1, \dots, e_{N-1})$

Induction $\Longrightarrow$ there exists sym polynomial g s.t $g/M_e \cap \{\lambda_N = 0\} = f$.

(we cheated: I.H. says there exists g sym in $\underline{N-1}$ variables and we want g sym in $\underline{N}$ variables & but simple to create a sym pol. in $\underline{N}$ var from sym pol in $\underline{N-1}$ variables)

when $x_N \neq e_N$, rewrite $(*)$ as $h = \frac{f-g}{\prod (x_i - e_N)}$

we want $f/M_e \cap \{\lambda_N > 0\} = f$ i.e. $h/M_e \cap \{\lambda_N > 0\} = \frac{f-g}{\prod (x_i - e_N)} / M_e \cap \{\lambda_N > 0\}$

determined + non zero denominator

g has been chosen ; we ensured that $f/_{M_e \cap \{\lambda_N = 0\}} = f$

$\rightsquigarrow$ we still need to ensure $f/_{\cancel{M_e \cap \{\lambda_N > 0\}}} = f$ on $M_e \cap \{\lambda_N > 0\}$

i.e. $h/_{\{M_e \cap \{\lambda_N > 0\}\}} = \dfrac{f - g}{\prod (x_i - e_N)}$ $\nwarrow$ non-zero on $M_e \cap \{\lambda_N > 0\}$

$$\mathcal{P}^{\leq N}_{\leq d-N} + (e_1+1, \dots, e_n+1)$$

Induction $\Rightarrow$ there exists ~~sym po~~ such a sym poly. h.

conclude existence of f and the whole proof ∎

Particular case: $e_i = N-i$. Fix μ of size d.

There exists, up to a multiplicative constant, a unique t_μ s.t.

$$t_\mu(\lambda) = 0 \quad \text{if} \quad |\lambda| \leq |\mu| \quad \lambda \neq \mu$$
$$t_\mu(\mu) \neq 0.$$

II Two formulas for t_μ

A. Determinental formula

$(x)_k = x(x-1)\dots(x-k+1)$
$= 0$ if $x < k$ are integers!

Prop : $t_\mu(x_1, \dots, x_N) = *. \dfrac{\det\left((x_i)_{\mu_j + N - j}\right)}{\det\left((x_i)^{\mu_j}\right)}$ $\det((x_i)^{\mu_j}) = \prod_{i<j}(x_i - x_j)$

Proof: RHS sym as quotient of two anti-sym. polynomials good degree

Vanishing of RHS for $x_i = \lambda_i + N - i$ $|\lambda| \leq |\mu|$ $|\lambda| \neq |\mu|$

Assume $\lambda_{i_0} < \mu_{i_0}$ $(x_i)_{\mu_j + N - j} = 0$ if $i \geq i_0$, $j \leq i_0$

Thus $(x_i)_{\mu_j + N - j} = \left(\begin{array}{c|c} & \\ \hline 0 & \end{array} \right) \Rightarrow \det$ is 0. ∎

Cor: $t_\mu = s_\mu$ + smaller degree term

Cor: $t_\mu(\lambda) = 0$ unless $\lambda \supseteq \mu$.
(extra-vanishing property).

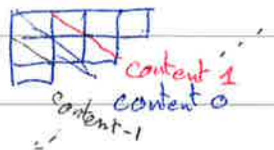
B. Combinatorial formula

Def: A ~~Reverse~~ reverse semi-std Young tableau (RSSYT) of shape λ is

1	1	2	3
2	3	3	
3			

$RSSYT(\lambda, N) :=$ set of RSSYT of shape λ entries at most N

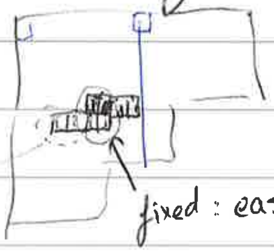
If $\square \in \lambda$, then $c(\square) =$ column index $-$ row index



Prop: $$t_\mu(x_1, \dots, x_N) = \sum_{T \in RSSYT(\lambda; N)} \prod_{\square \in \mu} \left(x_{T(\square)} - n + T(\square) - c(\square) \right)$$

Step 1 RHS sym (exercise)
Hint show that it's symmetric in x_i and x_{i+1}

Focus on boxes containing i and $i+1$



possibilities $\boxed{i \mid i}$ $\boxed{i+1 \mid i+1}$ $\boxed{i+1 \mid i}$
easy to check sum is symmetric

fixed: easy to check its symmetric

Vanishing property

$$\cancel{t_\mu}(\lambda+\delta) = \sum_{T \in RSSYT(\mu)} \prod_{\square \in T} (\lambda_{T(\square)} - c(\square))$$

$$\prod_{\square \in T} (\lambda_{T(\square)} - c(\square)) \neq 0$$

$\Rightarrow \lambda_{T(1,i)} \geq i$ but $T(1,i) \geq \mu'_i$

(for all i) $\Rightarrow \lambda_{\mu'_i} \geq i \Rightarrow \lambda'_i \geq \mu'_i$

$\Rightarrow \lambda \supseteq \mu$.

II A new positivity property

Prop (OO 96) : $t_\mu(\lambda+\delta) \geq 0$. $\forall \mu, \lambda$,

Thm (F., Alexandersson 2015):

$t_\mu(\lambda+\delta) = \cancel{\sum}$ has nonnegative ~~positive~~ coefficients in the basis

$$\left((\lambda_1-\lambda_2)_{a_1} \cdots (\lambda_{N-1}-\lambda_N)_{a_{N-1}} (\lambda_N)_{a_N} \right)_{a_1,\dots,a_N \geq 0}$$

lift to polynomials the positivity property above

$\rightsquigarrow$ α-deformation: shifts $e_i = \frac{i}{\alpha}$

NO ~~EXP~~ DET. FORMULA   But $t_\mu^{(\alpha)} = J_\mu^{(\alpha)} +$ smaller degree terms

We conjecture a similar nonnegativity property.