Dual approach for Jack polynomials and cumulants of almost independent variables

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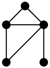
Forum on Probability, Statistics, Algebra and Combinatorics
Nagoya, July 29th, 2012.

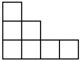
Field of research

Interactions between three branches of mathematics:

- combinatorics: permutations, graphs, Young diagrams.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 2 & 4 & 1 \end{pmatrix}$$





- algebra: representation theory, symmetric functions.
- probability theory: asymptotic behavior of large discrete structures.

The symmetric group

A permutation of size 5: $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 2 & 4 & 1 \end{pmatrix}$.

Permutations of the same size n can be multiplied:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 2 & 4 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 4 & 1 & 2 \end{pmatrix}$$

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Interests:

- simple infinite family of non-commutative groups;
- they act on labelled discrete structures, multivariate polynomials.

Representation theory

Let G be a finite group.

def: a representation of G $=$ a finite-dimensional vector space V

and a morphism $\rho : \ G \to GL(V)$.

Concretely, if we fix a basis of V:

- to each $g \in G$, we associate a matrix $\rho(g)$.
- product in G $\leftrightarrow$ product of matrices.

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Example: geometric representation of S_n

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 2 & 4 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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Interests:

- it gives a concrete representation of elements of G;
- if an operator is invariant by an action of G, its eigenspaces are representations of G (important in theoretical physics).

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Second simplification (theory of characters):

- Most natural questions can be answered knowing only the character of the representation, that is the trace of the matrices.

Representation theory of finite groups

We are interested in

$$\chi^\lambda(\mu) = \mathrm{tr}\Big(\rho^\lambda(g)\Big),$$

where ρ^λ is an irreducible representation and g an element of $\mathcal{C}_\mu$.

Second simplification (theory of characters):

- Most natural questions can be answered knowing only the character of the representation, that is the trace of the matrices.
- it depends only on the conjugacy class $\mathcal{C}_\mu$ of g in G.

$\chi^\lambda(\mu)$ are called irreducible character values.

Representation theory of symmetric groups

Consider the case $G = S_n$.

The quantities $\chi^\lambda(\mu)$ have been studied by G. Frobenius (1900):

- link with symmetric function theory;
- there is a combinatorial formula for them: Murnaghan-Nakayama rule.

Representation theory of symmetric groups

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These numbers are useful to:

- enumerate graphs on surfaces;
- evaluate mixing times (for a deck of cards for example, Diaconis);
- compute matrix integrals (link with representation of unitary groups).

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I am currently studying a more general setting, involving Jack polynomials.
$\rightarrow$ a lot of open problems here!

Outline of the presentation

Partitions

Let $G = S_n$. Irreducible character values of the symmetric group S_n

$$\chi^\lambda(\mu)$$

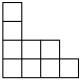
are indexed by partitions λ and μ of size n.

Definition

A partition λ of size n (short notation: $\lambda \vdash n$) and length r is a non-decreasing list of integers

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r > 0 \text{ with } \sum_{i=1}^{r} \lambda_i = n.$$

Example: $(4, 3, 1, 1) \vdash 9$.

Graphical representation:  .

Kerov and Olshanski dual approach

Fix a partition μ (denote $k = |\mu|$). Consider the following function on partitions of any size:

$$\mathrm{Ch}_\mu(\lambda) = \begin{cases} |\lambda|(|\lambda|-1)\dots(|\lambda|-k+1)\frac{\chi^\lambda\left(\mu,1,\dots,1\right)}{\chi^\lambda(1,\dots,1)} & \text{if } |\lambda| \geqslant k; \\ 0 & \text{otherwise.} \end{cases}$$

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- Roughly, its values are the (renormalized) irreducible character values of symmetric groups;
- the novelty here is to see it as a function on all Young diagrams: we consider characters of several symmetric group at the same time;
- Ch_μ has nice analytic properties, but no combinatorial description in the work of Kerov and Olshanski.

Stanley formula

Fix a partition μ of size k and a permutation π in S_k of cycle-type μ.

Theorem (F., Ann. Comb. 2010, conjectured by Stanley)

$$\mathrm{Ch}_\mu = \sum_{\substack{\tau,\sigma \in S_k \\ \tau\sigma = \pi}} (-1)^{|C(\sigma)|} N_{G_{\sigma,\tau}}.$$

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Interest: it gives a combinatorial framework to Kerov's and Olshanski's theory.

The summation index: factorisation of permutations

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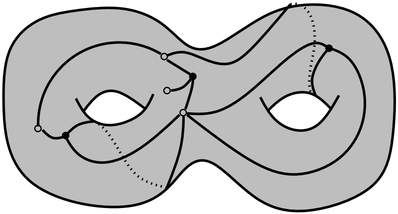
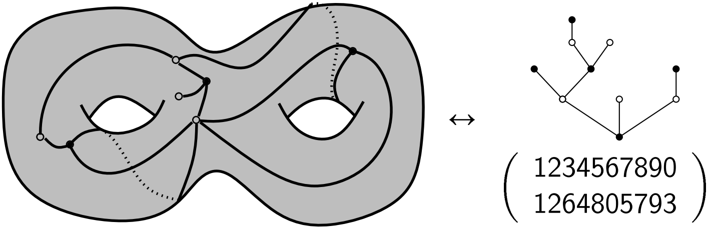
When $\pi = (1\ 2\ \cdots\ k)$, it is equivalent to study unicellular bipartite map with k edges.



map: a connected graph G embedded in a surface S
bipartite: with black and white vertices and no monochromatic edges
unicellular: $S \backslash G$ is homeomorphic to an open disc

A simple model for unicellular maps

With G. Chapuy and É. Fusy, we have found a simple model for maps



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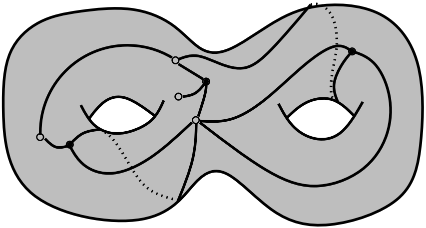

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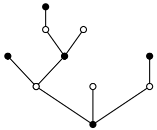

$$\begin{pmatrix} 1234567890 \\ 1264805793 \end{pmatrix}$$

Interests:

- our correspondence preserves a lot of structure;
- trees and permutations are simpler than unicellular maps.

Consequences:

- we can prove in a simple and unified way a lot of formulas;
- our construction also gives a new formula for Ch_μ.

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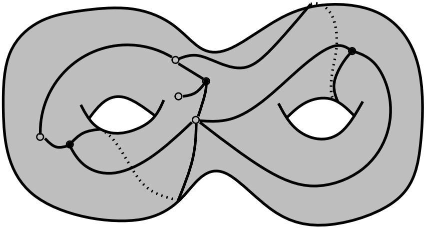

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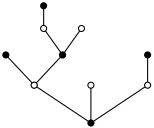

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See Guillaume's talk on Friday morning.

Non-decreasing functions on oriented graphs

We are interested in functions N_G.

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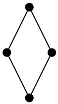
$G =$ 

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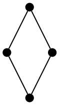
$G =$ 

$$F(G) = \sum_{\varphi} \prod_{v \in G} x_{\varphi(v)}$$

where the sum runs over non-decreasing functions from V_G to $\mathbb{N}$ (*i.e.* $(u \to v) \in E_G \Rightarrow \varphi(u) \leqslant \varphi(v)$).

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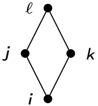
$G =$ 

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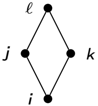
$G =$ 

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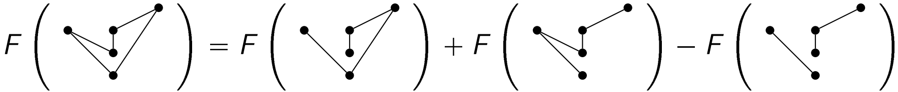
$F(G)$ appears also in P-partition theory, quasi-symmetric function theory...

Cyclic inclusion-exclusion (1/2)

Functions $F(G)$ fulfill the following relation:



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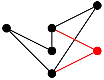
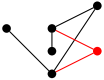
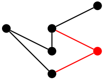
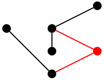
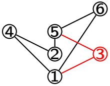
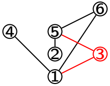
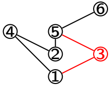
$F\Big($  $\Big) = F\Big($  $\Big) + F\Big($  $\Big) - F\Big($  $\Big)$

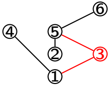
- it is still true if we add the same vertices/edges to all graphs.

Cyclic inclusion-exclusion: proof

$G_0 =$ 

, $G_1 =$ 

, $G_2 =$ 

, $G_3 =$ 

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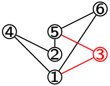
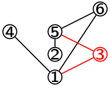
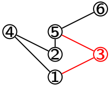
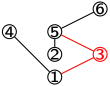
$G_0 =$ 

, $G_1 =$ 

, $G_2 =$ 

, $G_3 =$ 

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All graphs have the same vertex set $V = \{1, \dots, 6\}$. Hence,

$$F(G_i) = \sum_{\varphi : V \to \mathbb{N}} x_{\varphi(1)} \cdots x_{\varphi(6)} \delta_{\varphi, G_i}$$

where

$$\delta_{\varphi, G_i} = \begin{cases} 1 & \varphi \text{ is non-decreasing on } G_i; \\ 0 & \text{otherwise.} \end{cases}$$

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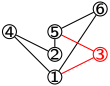
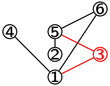
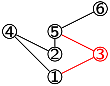
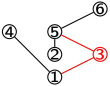
$G_0 =$ 

, $G_1 =$ 

, $G_2 =$ 

, $G_3 =$ 

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We will show that for any φ, its total contribution is 0.

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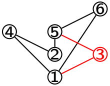
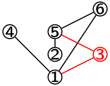
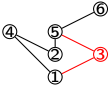
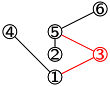
$G_0 =$ 

, $G_1 =$ 

, $G_2 =$ 

, $G_3 =$ 

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All graphs have the same vertex set $V = \{1, \dots, 6\}$. Hence,

$$F(G_i) = \sum_{\varphi: V \to \mathbb{N}} x_{\varphi(1)} \cdots x_{\varphi(6)} \delta_{\varphi, G_i}$$

We will show that for any φ, its total contribution is 0.

- If $\varphi(2) \leqslant \varphi(4)$, then $\delta_{\varphi,G_0} = \delta_{\varphi,G_1}$ and $\delta_{\varphi,G_2} = \delta_{\varphi,G_3}$.

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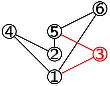
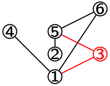
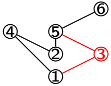
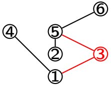
$G_0 =$ 

, $G_1 =$ 

, $G_2 =$ 

, $G_3 =$ 

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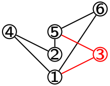
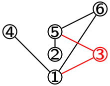
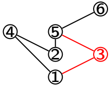
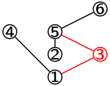
$G_0 =$  , $G_1 =$  , $G_2 =$  , $G_3 =$ 

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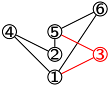
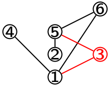
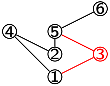
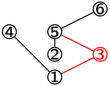
$G_0 =$  , $G_1 =$  , $G_2 =$  , $G_3 =$ 

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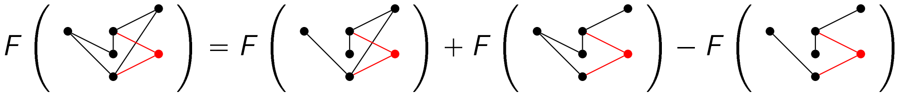
- Same thing if $\varphi(2) > \varphi(5)$ or $\varphi(5) > \varphi(6)$.
- Otherwise $\varphi(2) > \varphi(4) \geqslant \varphi(1) > \varphi(6) \geqslant \varphi(5) \geqslant \varphi(2)$. Impossible.

Cyclic inclusion-exclusion (1/2)

Functions $F(G)$ fulfill the following relation:



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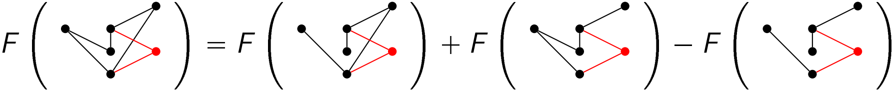

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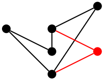
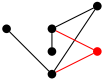
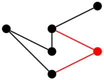
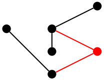
$F\Big($  $\Big) = F\Big($  $\Big) + F\Big($  $\Big) - F\Big($  $\Big)$

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I call these relations *cyclic inclusion-exclusion*.

Cyclic inclusion-exclusion (2/2)

These new family of relations have nice properties

- they are simple local combinatorial operations on the graphs;
- they span the kernel of the application $G \mapsto F(G)$;
- iterating these relations displays surprising properties: confluence, positivity.

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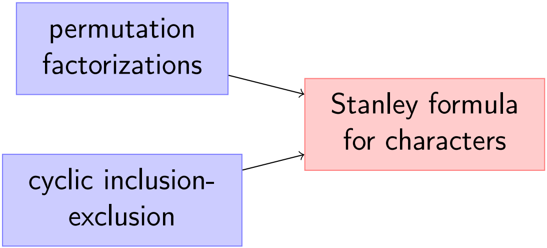
Applications

- they are central in the proof of Kerov's conjecture for Ch_μ;
- with A. Boussicault, we have used them to generalize some identity due to C. Greene;
- their investigation leads to consider new bases of (word) quasi-symmetric functions.

Transition

Stanley formula
for characters

Transition



permutation factorizations
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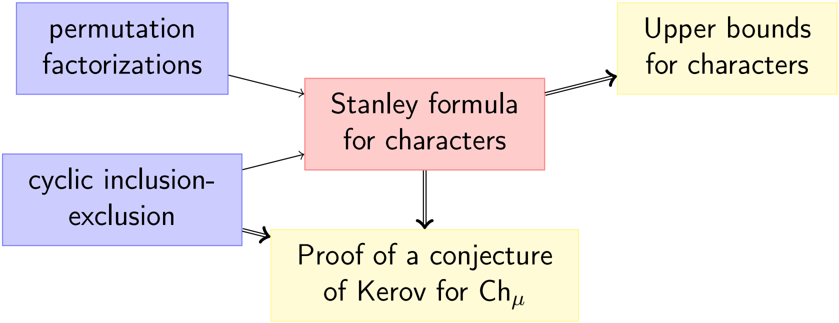

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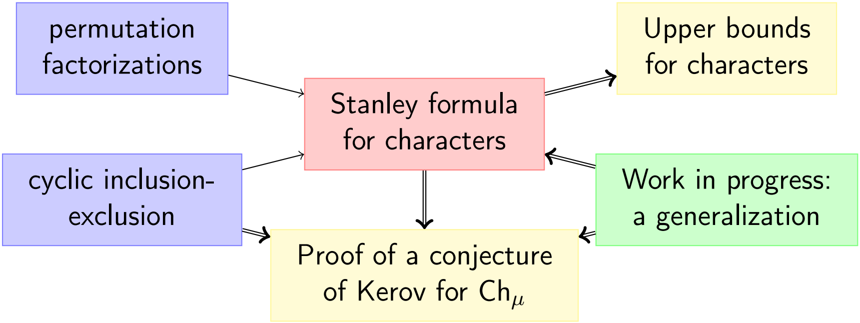

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Upper bounds for characters
Work in progress: a generalization
Proof of a conjecture of Kerov for Ch_μ

Symmetric functions and characters

We consider symmetric polynomials in n variables $x_1, \dots, x_n$.

Power sums:
$$\begin{aligned} &\text{for } k \geqslant 1, \quad p_k(x_1, \dots, x_n) = x_1^k + \cdots + x_n^k; \\ &\text{if } \mu \vdash d, \quad p_\mu(x_1, \dots, x_n) = \prod_{h=1}^{\ell(\mu)} p_{\mu_h}(x_1, \dots, x_n). \end{aligned}$$

Schur functions: $s_\lambda(x_1, \dots, x_n) = \dfrac{\det\left(x_i^{\lambda_j+n-j}\right)_{1 \leqslant i,j \leqslant n}}{\prod_{1 \leqslant i,j \leqslant n}(x_j - x_i)}$

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Theorem (Frobenius, 1900)

For any n, any $x_1, \dots, x_n$ and any partition $\mu \vdash d$, one has:

$$p_\mu(x_1, \dots, x_n) = \sum_{\lambda \vdash d} \chi^\lambda(\mu)\ s_\lambda(x_1, \dots, x_n).$$

Note: this property determines uniquely $\chi^\lambda(\mu)$.

Jack polynomials

$J_\lambda^{(\alpha)}$: one-parameter deformation of Schur functions.
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Define $\chi^{\lambda,(\alpha)}(\mu)$ by: for all $n \geqslant 1$ and $\mu \vdash d$,

$$p_\mu(x_1, \dots, x_n) = \sum_{\lambda \vdash d} \chi^{\lambda,(\alpha)}(\mu) \; J_\lambda^{(\alpha)}(x_1, \dots, x_n).$$

We can also define a one-parameter deformation $\mathrm{Ch}_\mu^{(\alpha)}$ of Ch_μ.

Extension of Stanley formula

M. Lassalle formulated two positivity conjectures (extending Stanley and Kerov's conjecture) on $\mathrm{Ch}_\mu^{(\alpha)}$.

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A solution to Lassalle's conjectures would reveal a continuous interpolation between the orientable and non-orientable settings.

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Besides their combinatorial interests, these conjectures are interesting from a symmetric function point of view.

Our approach

We look for an expression of $\mathrm{Ch}^{(\alpha)}$ in terms of the N_G.

- $\mathrm{Ch}^{(\alpha)} \in \mathrm{Vect}(N_G)$ so such an expression exists but is not unique;
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First step: study the algebra $\mathrm{Vect}(N_G)$ (first preliminary result: it is isomorphic to quasi-symmetric functions).

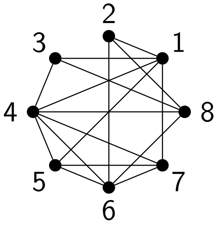
Transition

Second part: cumulants of almost independent variables

A problem in random graphs

Erdös-Rényi model of random graphs $G(n,p)$:

- G has n vertices labelled $1,\dots,n$;
- each edge (i,j) is taken independently with probability p;



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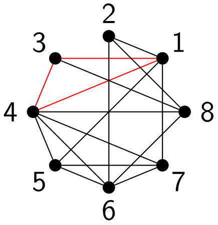

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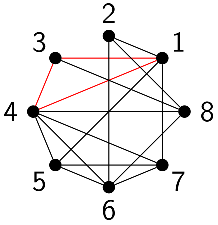

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Answer (Rucińsky, 1988)

The fluctuations are asymptotically Gaussian.

A good tool for that: mixed cumulants

- the r-th mixed cumulant k_r of r random variables is r-linear symmetric. Examples:

$$\kappa_1(X) = \mathbb{E}(X), \quad \kappa_2(X,Y) = \mathrm{Cov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$$

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- if the variables can be split in two mutually independent sets, then the cumulant vanishes.
- if, for each $r \neq 2$, the sequence $\kappa_r(X_n, \dots, X_n)$ converges towards 0 and if $\mathrm{Var}(X_n)$ has a limit, then X_n converges in distribution towards a Gaussian law.

Application to the number of triangles

$$T_n = \sum_{1 \leqslant i,j,k \leqslant n} B_{i,j,k},$$

$$\text{where } B_{i,j,k}(G) = \begin{cases} 1 & \text{if } G \text{ contains the triangle } i,j,k; \\ 0 & \text{otherwise.} \end{cases}$$

By multilinearity,

$$\kappa_\ell(T_n) = \sum_{i_1,j_1,k_1,\dots,i_\ell,j_\ell,k_\ell} \kappa_\ell(B_{i_1,j_1,k_1},\dots,B_{i_\ell,j_\ell,k_\ell}).$$

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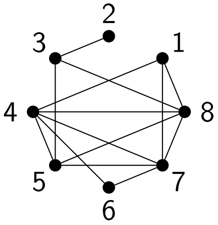
This is a classical approach, formalized by the notion of dependency graphs. (see for example the book of S. Janson, T. Luczack and T. Rucinski)

A slightly different model of random graphs

Erdös-Rényi model of random graphs $G(n, M)$:

- G has n vertices labelled $1, \ldots, n$;
- we choose a set of M edges, uniformly among the $\binom{n}{2}$ 2-element subsets of vertices;



Example : $n = 8, M = 14.$

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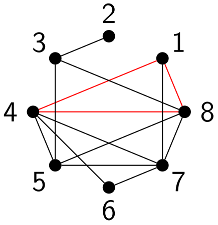

Example : $n = 8, M = 14$.

Question

Fix $p \in]0; 1[$ and let $M(n) = \lfloor p\binom{n}{2} \rfloor$.
Describe asymptotically the fluctuations of the number T_n of triangles in $G(n, M(n))$.

A slightly different model of random graphs

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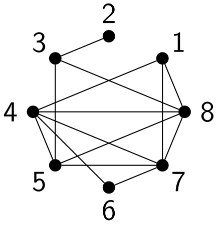

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Solved by Janson (1994): fluctuations are still Gaussian.
We will present a new approach to this problem.

As before, we can write:

$$\kappa_\ell(T_n) = \sum_{i_1,j_1,k_1,\dots,i_\ell,j_\ell,k_\ell} \kappa_\ell(B_{i_1,j_1,k_1},\dots,B_{i_\ell,j_\ell,k_\ell}).$$

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Lemma (F.)

If $i_1, j_1, k_1, \dots, i_\ell, j_\ell, k_\ell$ are pairwise distinct, then

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- this bound on cumulants seems new;
- it can be generalized to non distinct indices;
- it can be used to prove the convergence in distribution of $\widetilde{T_n}$ (T_n after a suitable normalization) towards a Gaussian law.

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- **Random permutations** (with uniform or Ewens distribution): the images of different integers have small cumulants.
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- **Random permutations** (with uniform or Ewens distribution): the images of different integers have small cumulants.
  $\Rightarrow$ We can prove the Gaussian fluctuations of a large class of statistics, called dashed patterns.

- **Random unitary/orthogonal matrices** (distributed with Haar measure):
  Cumulants of powers of entries in different rows and columns (and their conjugate) can be bounded (Collins, Śniady, 2003, 2006).
  $\Rightarrow$ lead still to be explored...

Project

- Define a theory of ε-dependency graph, containing these examples;
- in each framework, try to go as far as possible and compare with existing results, ...;
- study large deviations, local limit laws (this requires a uniform bound, in ℓ and n, on cumulants $\kappa_\ell(X_n)$).