Mod-ϕ convergence II: dependency graphs

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Reminder

Definition (uniform control on cumulants)

A sequence (S_n) admits a uniform control on cumulants with DNA (D_n, N_n, A) and limits σ^2 and L if $D_n = o(N_n)$, $N_n \to +\infty$ and

$$\forall r \geq 2, \ \ |\kappa^{(r)}(S_n)| \leq N_n \, (2D_n)^{r-1} \, r^{r-2} \, A^r;$$

$$\frac{\kappa^{(2)}(S_n)}{N_n \, D_n} = (\sigma_n)^2 \to_{n\to\infty} \sigma^2; \qquad \frac{\kappa^{(3)}(S_n)}{N_n \, (D_n)^2} = L_n \to_{n\to\infty} L.$$

This yields mod-Gaussian convergence of a suited renormalization of S_n, hence deviation probability estimate and a bound on the speed on convergence in a CLT.

Today's talk

See a setting where the inequality above holds: dependency graphs and a weighted variant.

Many applications: subword/subgraph counts in random words/graphs, magnetization in Ising model, linear statistics of Markov chains.

We will also discuss a weaker framework, which is useful to prove CLT, but where we cannot prove mod-Gaussian convergence.

Dependency graphs

Definition (Malyshev, '80, Petrovskaya/Leontovich, '82, Janson, '88)

A graph L with vertex set A is a dependency graph for the family $\{Y_\alpha, \alpha \in A\}$ if

- if A_1 and A_2 are disconnected subsets in L, then $\{Y_\alpha, \alpha \in A_1\}$ and $\{Y_\alpha, \alpha \in A_2\}$ are independent.

Roughly: there are edges between pairs of dependent random variables.

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Example 1

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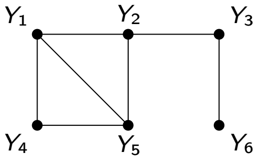

Y_2 and Y_4 are independent;
$\{Y_1, Y_4, Y_5\}$ independent from $\{Y_3, Y_6\}$.

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Example 2 (triangles in $G(n,p)$)

Consider $G = G(n,p)$. Let $A = \{\Delta \in \binom{[n]}{3}\}$ (set of potential triangles) and

$$\{\Delta_1,\ \Delta_2\} \in E_L \text{ iff } \Delta_1 \text{ and } \Delta_2 \text{ share an edge in } G.$$

Then L is a dependency graph for the family $\{\mathbf{1}_{\Delta \subset G}, \Delta \in \binom{[n]}{3}\}$.

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Janson's normality criterion

Setting: for each n,

- $\{Y_{n,i}, 1 \leq i \leq N_n\}$ is a family of bounded random variables; $|Y_{n,i}| < A$ a.s.
- we have a dependency graph L_n with maximal degree $D_n - 1$.
- we set $S_n = \sum_{i=1}^{N_n} Y_{n,i}$ and $\sigma_n^2 = \mathrm{Var}(S_n)$.

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Theorem (Janson, 1988)

Assume that $\left(\frac{N_n}{D_n}\right)^{1/s} \frac{D_n}{\sigma_n} \to 0$ *for some integer* s. *Then* S_n *satisfies a CLT.*

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For triangles, $N_n = \binom{n}{3}$, $D_n = O(n)$, while $\sigma_n \asymp n^2$. (for fixed p)

Corollary

Fix p *in* $(0,1)$*. Then* T_n *satisfies a CLT.*

(also true for $p_n \to 0$ with $np_n \to \infty$; originally proved by Rucinski, 1988).

Applications of dependency graphs to CLT results

- mathematical modelization of cell populations (Petrovskaya, Leontovich, 82);
- subgraph counts in random graphs (Janson, Baldi, Rinott, Penrose, 88, 89, 95, 03);
- Geometric probability (Avram, Bertsimas, Penrose, Yukich, Bárány, Vu, 93, 05 , 07);
- pattern occurrences in random permutations (Bóna, Janson, Hitchenko, Nakamura, Zeilberger, 07, 09, 14).
- m-dependence (Hoeffding, Robbins, 53, ...; now widely used in statistics) is a special case.

(Some of these applications use variants of Janson's normality criterion, which are more technical to state and omitted here...)

A bound for cumulants

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$$\kappa_r(S_n) \leq C_r \, N_n \, D_n^{r-1} \, A^r,$$

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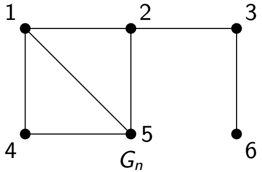
FMN, 2016: one can take $C_r = 2^{r-1} r^{r-2}$,
i.e. we have the uniform bound on cumulants.

A bound for mixed cumulants

How to bound $\kappa_r(S_n)$?

$$\kappa_r(S_n) = \kappa(S_n, \dots, S_n) = \sum_{i_1,\dots,i_r} \kappa(Y_{n,i_1}, \cdots, Y_{n,i_r})$$



$\kappa(Y_{n,i_1}, \cdots, Y_{n,i_r})$ is the mixed cumulants; multilinear version of cumulants.

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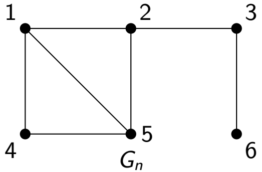

Most terms are zero: $\kappa(Y_{n,i_1}, \cdots, Y_{n,i_r}) = 0$ unless the induced graph $G_n[\{i_1, \dots, i_r\}]$ is connected.

e.g. $\kappa(Y_{n,1}, Y_{n,3}, Y_{n,4}, Y_{n,5}) = 0$

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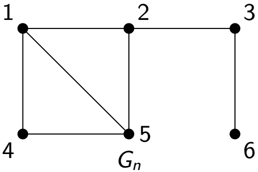

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Usual strategy: bound each term $\kappa(Y_{n,i_1}, \cdots, Y_{n,i_r})$ and the number of non-zero terms.
We prove a bound depending on the induced graph $G_n[\{i_1, \dots, i_r\}]$.

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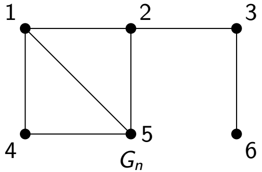

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Proposition (FMN, 2016)

$$|\kappa(Y_{n,i_1}, \cdots, Y_{n,i_r})| \le 2^{\ell-1} A^r \, \mathrm{ST}\left(G_n[\{i_1, \dots, i_r\}]\right),$$

where ST(G) *denotes the number of spanning tree of a graph* G*.*

e.g. $|\kappa(Y_{n,1}, Y_{n,2}, Y_{n,4}, Y_{n,5})| \le 2^3 A^4 \, 8$;

$|\kappa(Y_{n,1}, Y_{n,2}, Y_{n,3}, Y_{n,4})| \le 2^3 A^4 \, 1$.

From mixed cumulants to cumulants

Recall that $\kappa_r(S_n) = \sum_{i_1,\dots,i_r} \kappa(Y_{n,i_1}, \cdots, Y_{n,i_r})$.

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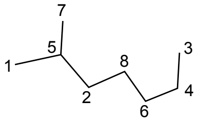
Fix a Cayley tree T. For how many lists $i_1, \dots, i_r$ is it contained in $G_n[\{i_1, \dots, i_r\}]$?

$T =$ 

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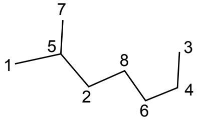
$T =$ 

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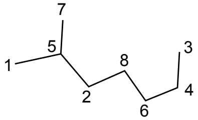
$T =$ 

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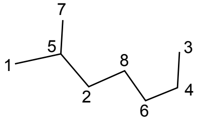
$T =$ 

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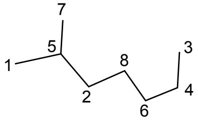
$T =$ 

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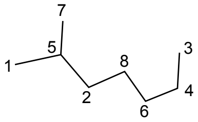
$T =$ 

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- ... $|\kappa_r(S_n)| \leq 2^{r-1} r^{r-2} N_n D_n^{r-1} A^r$

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Theorem (FMN, 2017)

Assume that $\frac{\sigma_n^2}{N_n D_n} \to \sigma^2 > 0$. *Then the error term in the CLT for* $\frac{1}{\sigma_n}(S_n - \mathbb{E}(S_n))$ *is* $O(\sqrt{D_n/N_n})$.
If furthermore $\frac{\sigma_n^2}{N_n D_n} \to L^3 \neq 0$, *the normality zone is* $o\big((D_n/N_n)^{1/6}\big)$.

Proof: uniform bound of cumulants + general results from yesterday.

The speed of convergence was already known from Rinott (1994), through Stein's method.

Application 1: subwords

Consider a random word with i.i.d. letters

$$w = w_1 \dots w_N$$

We are interested in the number of (non-necessarily consecutive) occurrences of a given word u (e.g., for $u = aab$, we count triplet of indices (i, j, k) with $i < j < k$ and $w_i = w_j = a$, $w_k = b$).
A CLT proved by Flajolet, Szpankowski and Vallée, '01.

Proposition

The number of copies of a given word of length k in a random word with i.i.d. letters admits a uniform control on cumulants with DNA $(n^{k-1}, n^k, 1)$.

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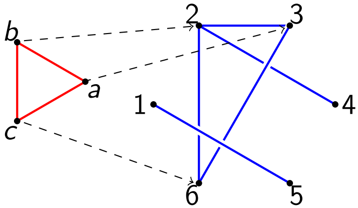
Proof: bounds on cumulant given by an obvious dependency graph, estimate for the second and third cumulants given by a combinatorial expansion.

Application 2: subgraph counts

Copies of F in a random graph G



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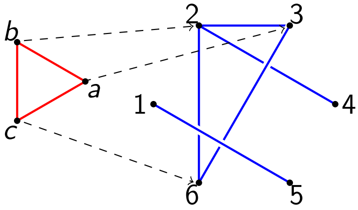

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Proof: as in subword case.

☹ When $p = p_n \to 0$, we do not have a good uniform control on cumulants.

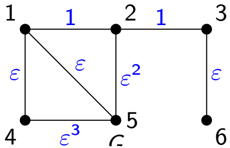
Transition

We will now discuss a weighted analogue of dependency graphs.

Goal: consider sum of pairwise dependent random variables, where the dependencies are asymptotically small.

Uniform weighted dependency graph

Let $S = \sum_{i=1}^{N} Y_i$ as above and let G be an edge weighted graph with vertex set $[N]$ (weights are in $[0,1]$).



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$$\left|\kappa(Y_{i_1}, \dots, Y_{i_r})\right| \le C^r \sum_{T \text{ Cayley tree}} \prod_{\{j,k\} \in E_T} w(\{i_j, i_k\}), \qquad \text{(UWDG)}$$

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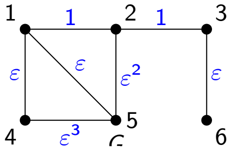

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e.g. $\kappa(Y_{n,1}, Y_{n,3}, Y_{n,4}, Y_{n,5}) = 0;$

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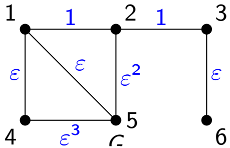

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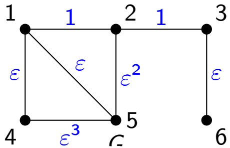

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then $\left|\kappa_r(S)\right| \le C^r \, N \, D^{r-1}$, where $D-1$ is the maximal weighted degree of the graph.

Proof: simple adaptation of the case of dependency graphs (which corresponds to edges of G having weights 0 or 1).
When (UWDG) holds, we say that G is a C-uniform weighted dependency graph for $\{Y_i, 1 \le i \le N\}$.

Linear statistics in Markov chains

Setting:

- Let $(M_i)_{i \geq 0}$ be an irreducible aperiodic Markov chain on a finite space state S;
- Assume M_0 is distributed with the stationary distribution π;
- Set $Y_i = f_i(M_i)$ for some functions f_i uniformly bounded by B.

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Proposition (FMN, 2017, based on Saulis, Statelivičius, 1991)

There exists $\varepsilon > 0$ depending on the transition matrix such that the complete graph on $\mathbb{N}_{\geq 0}$ with weights $w(\{s,t\}) = \varepsilon^{|s-t|}$ is a $4B$-uniform weighted dependency graph for the Y_i's.

The maximal weighted degree of the restriction to $[n]$ is constant!

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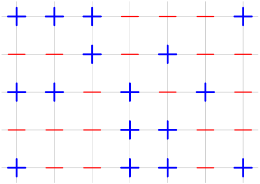
$\rightarrow$ deviation and speed of convergence estimates for linear statistics of Markov chains (for speed of convergence, see Bolthausen, 1980).

Ising model

Statistical mechanics model to modelize ferromagnetism.



$$P(\omega) \propto \exp\left(\beta \sum_{i\sim j} \omega_i \omega_j + h \sum_i \omega_i\right)$$

Defined a priori for finite subsets $\Lambda \Subset \mathbb{Z}^d$, but we can take the "thermodynamic limit" $\Lambda \uparrow \mathbb{Z}^d$ (with $+$ boundary conditions).

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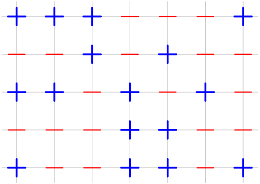

Proposition (Duneau, Iagolnitzer, Souillard, 1974)

In the termodynamic limit, for $h \neq 0$ or $h = 0$ and sufficiently small β, there exists $\varepsilon(d), C(d) > 0$ such that the complete graph on $\mathbb{N}_{\geq 0}$ with weights $w(\{s,t\}) = \varepsilon(d)^{\|s-t\|}$ is a $C(d)$-uniform weighted dependency graph for the σ_i's.

In particular, for disjoint $x_1, \cdots, x_r$,

$$\kappa(\sigma_{x_1}, \ldots, \sigma_{x_r}) = \mathcal{O}_r(\varepsilon^{\ell_T(x_1,\ldots,x_r)}),$$

where $\ell_T(x_1, \ldots, x_r)$ is the smallest length of a tree connecting $x_1, \ldots, x_r$.

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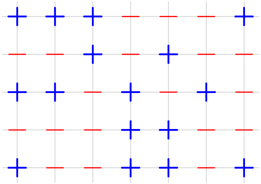

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The same cannot be true for large β ($\beta > \beta_c(d)$).

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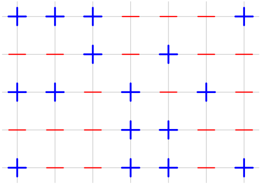

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$\to$ deviation and speed of convergence estimates for the $\sum \sigma_i$.

Transition

(Non-uniform) weighted dependency graphs

(Non-uniform) weighted dependency graphs

Definition (Reminder)

Let C be a positive constant.
We say that G is a C-uniform weighted dependency graph for $\{Y_i, 1 \leq i \leq N\}$ if, for any $i_1, \dots, i_r$ in $[N]$, we have

$$\left|\kappa(Y_{i_1}, \dots, Y_{i_r})\right| \leq C^r \sum_{T \text{ Cayley tree}} \prod_{\{j,k\} \in E_T} w(\{i_j, i_k\}), \qquad \text{(UWDG)}$$

(Non-uniform) weighted dependency graphs

Definition (F. 2016)

Let $\mathbf{C} = (C_r)_{r \geq 2}$ be a sequence of positive numbers.
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The bounds are now non-uniform in r.

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We can replace the sum by a max. Spanning trees of maximal weight can be found efficiently (e.g. Prim's algorithm).

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Lemma

$$\left|\kappa_r\left(\sum Y_i\right)\right| \leq C_r\, N\, D^{r-1}$$

Good to prove CLT, but not for mod-Gaussian convergence.

Applications of non-uniform weighted dependency graphs

- crossings in random pair-partitions;
- subgraph counts in $G(n, M)$;
- random permutations;
- particles in symmetric simple exclusion process;
- linear functional of Markov chain;
- spins in Ising model (with Jehanne Dousse);
- (*) patterns in multiset permutations, in set-partitions (with Marko Thiel);

*in progress. In blue: the ones which are also uniform WDG.

(Some of these applications use a variant of the above definition/lemma, which is more technical to state. . .)

Stability by powers of weighted dependency graphs

Setting:

- Let $\{Y_\alpha, \alpha \in A\}$ be r.v. with weighted dependency graph $\widetilde{L}$;
- fix an integer $m \geq 2$;
- for a multiset $B = \{\alpha_1, \cdots, \alpha_m\}$ of elements of A, denote

$$\boldsymbol{Y}_B := Y_{\alpha_1} \cdots Y_{\alpha_m}.$$

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Proposition (F., 2016)

The set of r.v. $\{\boldsymbol{Y}_B\}$ has a weighted dependency graph $\widetilde{L}^m$, where

$$\mathrm{wt}_{\widetilde{L}^m}(\boldsymbol{Y}_B, \boldsymbol{Y}_{B'}) = \max_{\alpha \in B, \alpha' \in B'} \mathrm{wt}_{\widetilde{L}}(Y_\alpha, Y_{\alpha'}).$$

In short: if we have a dependency graph for some variables Y_α, we have also one for monomials in the Y_α.

☹ No analogue for uniform weighted dependency graphs.

Multilinear statistics in Markov chains

Setting:

- Let $(M_i)_{i \geq 0}$ be an irreducible aperiodic Markov chain on a finite space state S;
- Assume M_0 is distributed with the stationary distribution π;
- Set $Y_{i,s} = \mathbf{1}_{M_i = s}$.

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Proposition

We have a weighted dependency graph $\widetilde{L}$ with $\mathrm{wt}_{\widetilde{L}}\left(\{Y_{i,s}, Y_{j,t}\}\right) = |\varepsilon|^{j-i}$.

We already know that: in fact this is a uniform weighted dependency graph.

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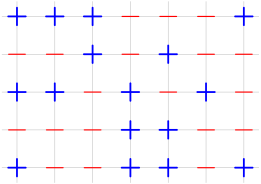
$\rightarrow$ gives a CLT for the number of copies of a given word in $(M_i)_{0\leq i\leq N}$.
(Answers a question of Bourdon and Vallée.)

(But no uniform control on cumulants here ☹)

Back to Ising model



$$\mathbb{P}(\omega) \propto \exp\big[-H(\omega)\big];$$
$$H(\omega) = -\beta \textstyle\sum_{x\sim y} \omega_x\omega_y - h \textstyle\sum_x \omega_x.$$

Theorem

In presence of a magnetic field or at very low or very large temperature, there exists $\varepsilon = \varepsilon(d,h,\beta) > 0$ such that the complete graph on $\mathbb{Z}^d$ with weight $\varepsilon^{\|x-y\|_1}$ on the edge $\{x,y\}$ is a weighted dependency graph for $\{\sigma_x, x \in \mathbb{Z}^d\}$

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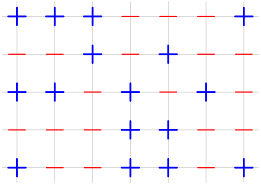

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Low temperature case proved by Malyshev and Minlos ('91).
Agian, proof based on cluster expansion...

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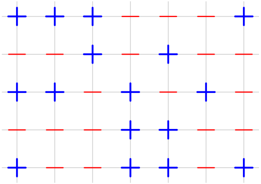

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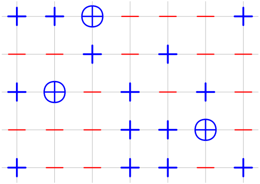
Question: does it hold near the critical point?
(At the critical point, the answer is NO, since already covariances do not decay exponentially)

Ising model: CLT for global patterns



Circled spins:
occurrence of the $+$ pattern 2 3 1

(notion inspired from patterns in permutations.)

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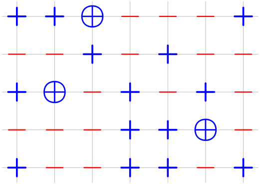

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$S_n^{\mathcal{P}} :=$ number of occurrences of $\mathcal{P}$ within $\Lambda_n = [-n, n]^d$.

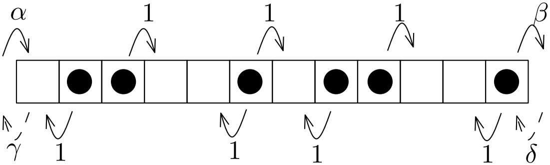
Theorem (Dousse, F., 2016)

Assume $\mathrm{Var}(S_n^{\mathcal{P}}) \geq cst|\Lambda_n|^{2|\mathcal{P}|-2+\eta}$ *for* $\eta > 0$. *Then we have*

$$\frac{S_n^{\mathcal{P}} - \mathbb{E}(S_n^{\mathcal{P}})}{\sqrt{\mathrm{Var}(S_n^{\mathcal{P}})}} \xrightarrow[n\to\infty]{d} \mathcal{N}(0,1).$$

The lower bound of the variance is always fulfilled for patterns of only positive spins (as in the example).

Symmetric simple exclusion process (SSEP)



$\tau = (\tau_1, \cdots, \tau_N)$ particle configuration with stationary distribution.

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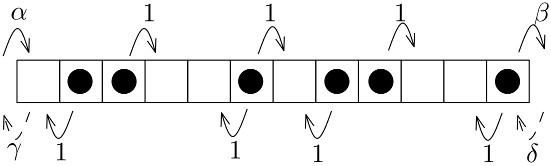

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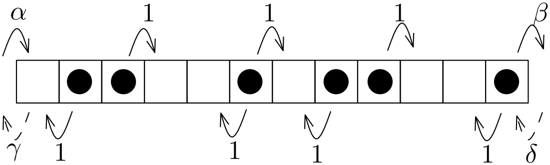

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Ingredients of the proof:

- joint moments of the τ_i given by matrix ansatz;
- in case of SSEP, this gives an induction formula for cumulants (Derrida, Lebowitz, Speer, 2006).

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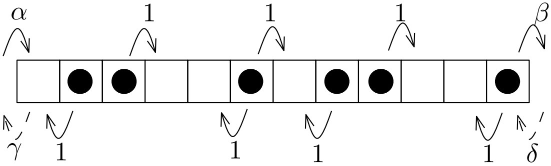

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Consequences:

- functional CLT for the particle distribution function $F_\tau(u) = \sum_{i=1}^{\lfloor uN \rfloor} \tau_i$;
- also, e.g., for the number $\sum_i \tau_i(1 - \tau_{i+1})$ of particles that can jump to their right (using stability by powers).

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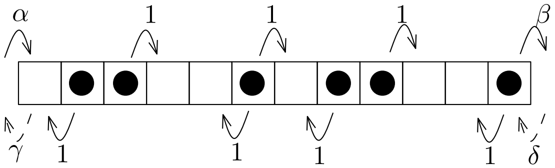

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The same is conjectured for ASEP in general.

Conclusion

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- There is a weighted variant which include linear statistics of Markov chain, magnetization in Ising model (except at low temperature);
- If you relax assumptions to only get a CLT and not a uniform control on cumulants, the weighted version applies to even more examples;
- We would really like to get uniform bounds on cumulants for these extra examples. . .