On Kerov polynomials for Jack characters

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Part 1: Background

Jack polynomials

- Family $J_\lambda^{(\alpha)}$ of symmetric functions depending on a parameter $\alpha > 0$.
- deformation of Schur functions:
$$J_\lambda^{(1)} = \frac{n!}{\dim(\lambda)} s_\lambda.$$
- widely studied since their introduction by Jack in 1970. [7]

Consider the expansion of Jack polynomials on the power sum basis:
$$J_\lambda^{(\alpha)} = \sum_{\mu \vdash |\lambda|} \theta_\mu^{(\alpha)}(\lambda)\, p_\mu.$$
Special case: $\theta_\mu^{(1)}(\lambda)$ is the normalized character value of the symmetric group.

Dual approach

We review here some work of Lassalle. [9]
Fix a partition μ. Define Jack characters as
$$\mathrm{Ch}_\mu^{(\alpha)}(\lambda) = \binom{|\lambda| - |\mu| + m_1(\mu)}{m_1(\mu)} z_\mu\, \theta^{(\alpha)}_{\mu, 1^{|\lambda|-|\mu|}}(\lambda).$$

- roughly $\theta_\mu^{(\alpha)}(\lambda)$ with a normalization factor ;
- $\mathrm{Ch}_\mu^{(\alpha)}$ is a function on all Young diagrams ;
- many results in the case $\alpha = 1$

Proposition (Lassalle)

$\mathrm{Ch}_\mu^{(\alpha)}$ is a (α-)shifted symmetric function in λ_1, $\lambda_2, \dots$ (*i.e.* symmetric in $\lambda_1 - 1/\alpha$, $\lambda_2 - 2/\alpha$, $\dots$)

Lassalle's conjecture

Interesting multiplicative basis of shifted symmetric functions:

free cumulants $(R_k^{(\alpha)}(\lambda))_{k \geq 2}$

Conjecture (Lassalle)

$\mathrm{Ch}_\mu^{(\alpha)}$ can be written as a polynomial (called Kerov polynomial) in $(R_k^{(\alpha)})_{k\geq 2}, \alpha, 1-\alpha$ with non-negative integer coefficients.

- The polynomiality in $(R_k^{(\alpha)})_{k\geq 2}$ is proved by Lassalle, but not in α!
- This conjecture is inspired by the case $\alpha = 1$, where the coefficients are shown to count some graphs on orientable surfaces. [3]

Part 2: Polynomiality

Our first main result is a partial answer to Lassalle conjecture:

Theorem (DF, 2013)

$\mathrm{Ch}_\mu^{(\alpha)}$ is a polynomial in $(R_k^{(\alpha)})_{k\geq 2}, \alpha$ with rational coefficients.

Three ingredients in the proof:

- use an algorithm given by Lassalle to compute the coefficients by induction on the size of μ (each step involves an overdetermined linear system (S));
- rewrite this algorithm with suitable normalizations and an auxiliary basis $(M_k^{(\alpha)})_{k\geq 2}$: we get a new system (S')
- extract a triangular subsystem from (S').

Corollary (Lapointe-Vinet theorem)

$\theta_\mu^{(\alpha)}(\lambda)$ is a polynomial in α with rational coefficients.

This has long been an open problem (until paper [8] from Lapointe and Vinet in 1995 ; they in fact prove that the coefficients are integers).

Part 3: Structure constants

When μ runs over all partitions, the family $(\mathrm{Ch}_\mu^{(\alpha)})_\mu$ is a basis of the shifted symmetric function algebra. Hence,
$$\mathrm{Ch}_\mu^{(\alpha)} \cdot \mathrm{Ch}_\nu^{(\alpha)} = \sum_\rho g^\rho_{\mu,\nu}(\alpha)\, \mathrm{Ch}_\rho^{(\alpha)},$$
for some scalars $g^\rho_{\mu,\nu}(\alpha)$, called structure constants.

Theorem (DF, 2013)

The structure constants $g^\rho_{\mu,\nu}(\alpha)$ are polynomials with rational coefficients in α.

Idea of the proof: write $\mathrm{Ch}_\mu^{(\alpha)}$ and $\mathrm{Ch}_\nu^{(\alpha)}$ as polynomial in free cumulants, multiply them and them write free cumulants in terms of $\mathrm{Ch}_\rho^{(\alpha)}$.

This result contains:

- by specialization $\alpha = 1, 2$, the polynomiality in n of structure constants of the symmetric group algebra $\mathbb{C}[S_n]$ (Farahat-Higman 1959, [4]) and Hecke algebra of (S_{2n}, H_n) (Aker-Can 2012/Tout 2013, [1, 10]).
- the polynomiality in $b = \alpha - 1$ in the b-conjecture of Goulden and Jackson (1995).

Part 4: Fluctuations of large diagrams.

Jack-Plancherel measure

- probability measure $\mathbb{P}_n^{(\alpha)}$ on Young diagrams λ with n boxes ;
- deformation of the well-known Plancherel measure (corresponding to $\alpha = 1$) ;
- defined using Jack polynomials. It can be characterized by:
$$\mathbb{E}_{\mathbb{P}_n^{(\alpha)}}(\theta_\mu^{(\alpha)}) = \begin{cases} 1 & \text{if } \mu = (1^n); \\ 0 & \text{else.} \end{cases}$$

Problem

Fix a partition μ. Study the asymptotic behaviour of the random variables $\mathrm{Ch}_\mu^{(\alpha)}\left(\lambda^{(n)}\right)$ where $\lambda^{(n)}$ is distributed *via* $\mathbb{P}_n^{(\alpha)}$.

Link with other parts? Structure constants appear when we compute moments
$$\mathbb{E}_{\mathbb{P}_n^{(\alpha)}}\left(\mathrm{Ch}_\mu^{(\alpha)}\left(\lambda^{(n)}\right)^k\right).$$

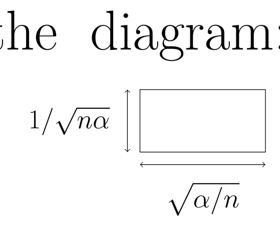
Results

Theorem (DF, 2013)

In probability,
$$\frac{1}{n^{|\mu|}} \mathrm{Ch}_\mu^{(\alpha)}\left(\lambda^{(n)}\right) \longrightarrow \begin{cases} 1 & \text{if } \mu = (1^k); \\ 0 & \text{else.} \end{cases}$$
Besides, the variables
$$\frac{1}{k n^{k-1}} \mathrm{Ch}_{(k)}^{(\alpha)}\left(\lambda^{(n)}\right)$$
converges in law towards independent standard normal variables.

The flucutation result uses multivariate Stein's method, as suggested in [5].

Consequence on the "shape" of the diagram: if we draw $\lambda^{(n)}$ with rectangular boxes, one has the same limit shape as for $\alpha = 1$.



Remark. Part 4 is based on the work of Ivanov, Kerov, Olshanski in the case $\alpha = 1$. [6]

Thank you

Thank you for your attention. Here is a list of references for more on the subject.

References

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