Large random Young diagrams and representation theory

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and Random Combinatorial Structures
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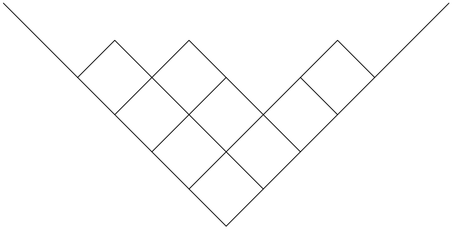
Teaser

Here is, in Russian representation, the Young diagram corresponding to $\lambda = 4, 2, 2, 1$:



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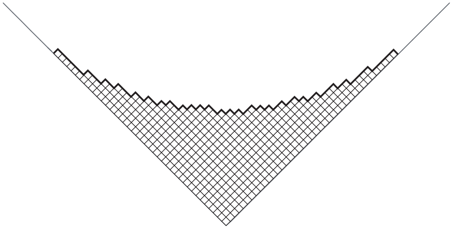

How does it look like when we choose randomly a large (renormalized) Young diagram?

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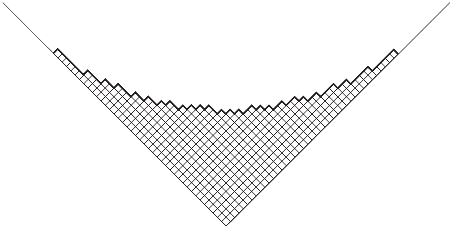


How does it look like when we choose randomly a large (renormalized) Young diagram?

For some measures, representation theory of symmetric groups and free cumulants allow us to find easily answers to this question!

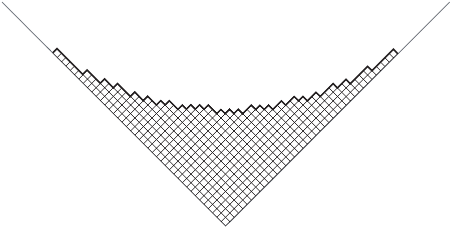
Outline of the talk

1. Limit law theorem for Plancherel's measure revisited

2. Generalizations to balanced and non-balanced random Young diagrams

Normalized border of a Young diagram

A Young diagram drawn with Russian convention



The Young diagram is determined by the continuous, piecewise affine function ω_λ in black.
Renormalization (area=1):

$$\omega_{\overline{\lambda}}(x) = (1/\sqrt{|\lambda|}) \cdot \omega_\lambda(\sqrt{|\lambda|}x).$$

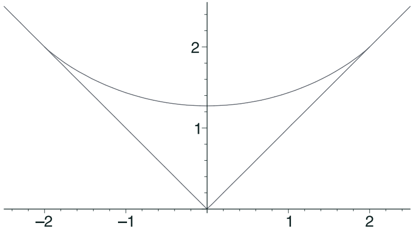
Existence of a limiting curve

Theorem (Logan and Shepp 77, Kerov and Vershik 77)

Let us take randomly (with Plancherel measure) a sequence of Young diagram λ_n of size n. Then, after renormalization, in probability, for the uniform convergence topology on continuous functions, one has:

$$\omega_{\overline{\lambda_n}} \longrightarrow \delta_\Omega,$$

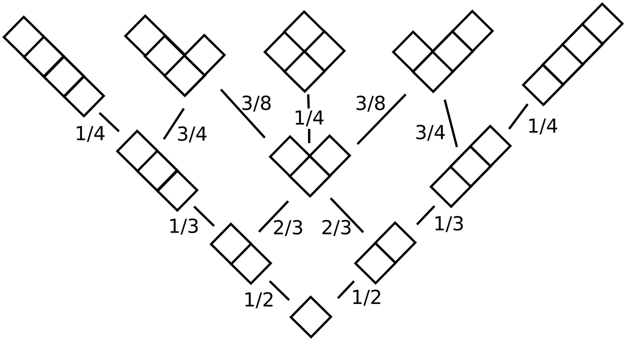
where Ω is an explicit function drawn here:



The Plancherel measure

$\mathcal{P}_n$: a measure on Young diagrams of size n.

1. can be defined by a Markov process:



The Plancherel measure

$\mathcal{P}_n$: a measure on Young diagrams of size n.

2. can be defined, using representation theory:

- Partitions of n index irreducible representations of $\mathfrak{S}_n$
- Therefore :

$$\mathbb{C}[\mathfrak{S}_n] \simeq \bigoplus_{\lambda \vdash n} V_\lambda^{\dim V_\lambda}$$

- In this context :

$$\mathcal{P}_n(\{\lambda\}) = \frac{(\dim V_\lambda)^2}{n!} = \frac{\dim(\text{isotypic component of type } \lambda)}{\dim \mathbb{C}[\mathfrak{S}_n]}$$

Normalized character values have simple expectations!

Fix $\sigma \in \mathfrak{S}_n$. Let us consider the random variable:

$$X_\sigma(\lambda) = \chi^\lambda(\sigma) = \mathrm{tr}\left(\rho_\lambda(\sigma)\right) = \frac{\mathrm{Tr}\left(\rho_\lambda(\sigma)\right)}{\dim V_\lambda}.$$

Let us compute its expectation:

$$\begin{aligned}\mathbb{E}_{\mathcal{P}_n}(X_\sigma) &= \frac{1}{n!}\sum_{\lambda\vdash n}(\dim V_\lambda)\cdot\mathrm{Tr}\left(\rho_\lambda(\sigma)\right)\\ &= \frac{1}{n!}\mathrm{Tr}_{\left(\bigoplus_{\lambda\vdash n} V_\lambda^{\dim V_\lambda}\right)}(\sigma) = \frac{1}{n!}\mathrm{Tr}_{\mathbb{C}[\mathfrak{S}_n]}(\sigma) = \mathrm{tr}_{\mathbb{C}[\mathfrak{S}_n]}(\sigma)\end{aligned}$$

Last espression is easy to evaluate:

$$\mathbb{E}_{\mathcal{P}_n}(X_\sigma) = \delta_{\sigma,\mathrm{Id}_n}$$

And now?

- Character values do not give directly informations on the shape of the diagram. ☹
- Is there some other random variables, linked to the shape of the diagram, which can be expressed in terms of normalized character values?
- Yes, thanks Kerov's and Olshanski's algebra of *polynomial functions on the set of Young diagrams*. ☺

Polynomial functions on the set of Young diagrams

Let $\mu \vdash k$ and $\sigma \in \mathfrak{S}_k$ of type μ. We define:

$$\Sigma_\mu(\lambda) = \begin{cases} n(n-1)\dots(n-k+1)\chi^\lambda(\sigma) & \text{if } \lambda \vdash n \geq k \\ 0 & \text{if } \lambda \vdash n < k \end{cases}$$

Consequence:

$$\mathbb{E}_{\mathcal{P}_n}(\Sigma_\mu) = \begin{cases} n(n-1)\dots(n-k+1) & \text{if } \mu = \mathbf{1}^k \text{ with } k \geq n \\ 0 & \text{else} \end{cases}$$

Theorem

The random variables Σ_μ span linearly an algebra denoted $\mathcal{P}ol$.

We will describe an other basis of this algebra.

Moments of transition measure

Let μ_λ be the measure defined by:

$$\int_{\mathbb{R}} \frac{d\mu_\lambda(x)}{z-x} = \frac{1}{z} \exp\left(\int_{\mathbb{R}} \frac{(\omega'(x) - \mathrm{sgn}(x))dx}{2(z-x)} \right)$$

Theorem (Kerov, Olshanski, 1994)

If $M_k(\mu_\lambda) = \int_R x^k d\mu_\lambda(x)$, one has:

$$\mathcal{P}ol = \mathbb{C}[\lambda \mapsto M_k(\mu_\lambda)_{k \geq 2}]$$

$\Rightarrow$ one has an expansion

$$\prod_j M_{k_j} = \sum_\mu c_\mu \Sigma_\mu.$$

Idea behind the following slides

Problem: we don't have good descriptions of the expansion of $\prod_j M_{k_j}$ in terms of Σ_μ.

But we interested in asymptotics of quantities :

$$M_k(\overline{\lambda}) = \frac{1}{n^{k/2}} M_k(\lambda).$$

We do not need to know the whole expansion.

gradation

We can define a gradation on $\mathcal{P}ol$ by:

$$\deg(M_k) = k$$

Theorem (Biane, 1998)

Σ_μ has degree $|\mu| + \ell(\mu)$ and

$$\Sigma_\mu = \prod_i R_{\mu_i+1} + \textit{smaller degree terms},$$

where $R_k(\lambda)$ is the k-th free cumulant of the measure $d\mu_\lambda$ defined by:

$$M_k = \sum_{\pi \in NCP(k)} \prod_{b \in \pi} R_{|b|} \qquad \textit{note that } \deg(R_k) = k.$$

Inverting Biane's theorem

Formula

$$\Sigma_\mu = \prod_i R_{\mu_i+1} + \text{smaller degree terms}$$

can be read as:
One has a triangular change of basis between (Σ_μ) and $(\prod_i R_{\mu_i+1})$.

Therefore,

$$\prod_i R_{\mu_i+1} = \Sigma_\mu + \text{smaller degree terms}$$

Remark: the degree of $X = \sum_\mu c_\mu \Sigma_\mu \in \mathcal{P}ol$ is $\max_{c_\mu \neq 0} \deg(\Sigma_\mu)$).

Therefore,

$$\mathbb{E}\left(\prod R_{\mu_j+1}(\overline{\lambda})\right) = \sqrt{n}^{-|\mu|-\ell(\mu)} \mathbb{E}\left(\prod R_{\mu_j+1}(\lambda)\right)$$
$$= \sqrt{n}^{-|\mu|-\ell(\mu)} \mathbb{E}(\Sigma_\mu) + o(1)$$

Limits of free cumulants

In particular,

$$\lim_{n\to\infty} \mathbb{E}_{\mathcal{P}_n}(R_k(\overline{\lambda})) = \begin{cases} 1 & \text{if } k=2 \\ 0 & \text{else.} \end{cases}$$

and

$$\lim_{n\to\infty} \mathrm{Var}_{\mathcal{P}_n}(R_k(\overline{\lambda})) = 0.$$

Therefore, in probability,

$$R_k(\overline{\lambda}) \to \delta_{k,2}$$

$\to \mu_\lambda$ converges to the semi-circle law.

Limiting curve

Lemma (technical, due to Kerov)

convergence of cumulants ⇒ uniform convergence of ω.

Moreover, one can compute ω from the cumulant sequence.

Can be generalized!

1. Take a (reducible) family of representations of $\mathfrak{S}_n$, whose characters are easy to compute. For instance,

$$V = (\mathbb{C}^r)^{\otimes n}, \text{ with } r \sim c \cdot n^\alpha$$

The normalized character is $\chi(\sigma) = r^{\#\text{ cycles of }\sigma - n}$

2. Consider the associated measures on Young diagram:

$$SW_n(\{\lambda\}) = \frac{\dim(\text{isotypic component of type }\lambda)}{\dim((\mathbb{C}^r)^{\otimes n})}$$

$$= \frac{\left|\left\{\begin{array}{c}\text{standard tableaux}\\ \text{de forme }\lambda\end{array}\right\}\right| \cdot \left|\left\{\begin{array}{c}\text{semi-standard tableaux}\\ \text{de forme }\lambda\text{ (entries} \leq N)\end{array}\right\}\right|}{r^n}$$

Can be generalized!

1. If $\alpha \geq 1/2$, one can use the same method as Plancherel measure because:
$$\mathbb{E}(\Sigma_\mu) = O(n^{(|\mu|+\ell(\mu))/2}).$$
2. if $\alpha > 1/2$, same limit curve than the Plancherel case.
if $\alpha = 1/2$, limit curve is the curve with free cumulants
$$0, 1, c, c^2, \dots$$
(result obtained by Biane, 2001: he also computed an explicit formula for these curves)
3. this works in general if $\chi(\rho_1\rho_2) \sim \chi(\rho_1)\chi(\rho_2)$ as soon as ρ_1 and ρ_2 has disjoint support (also a necessary condition).

Second order asymptotics

Recall: for the first-order asymptotics, one has used

$$\prod_i R_{\mu_i+1} = \Sigma_\mu + \text{smaller degree terms}$$

If we know explicitely the next term in the expansion, one can compute the fluctuations of R_k!

- in the case of Plancherel's measure, fluctuations are gaussian: one can deduce the fluctuations of ω_λ around the limit function Ω (Kerov).
- in more generality, P. Śniady has given sufficient conditions for the fluctuations of the R_k's to be gaussian.

Limits ?

- This methods works only for representations such that:

$$\mathbb{E}(\Sigma_\mu) = O(n^{(|\mu|+\ell(\mu))/2}).$$

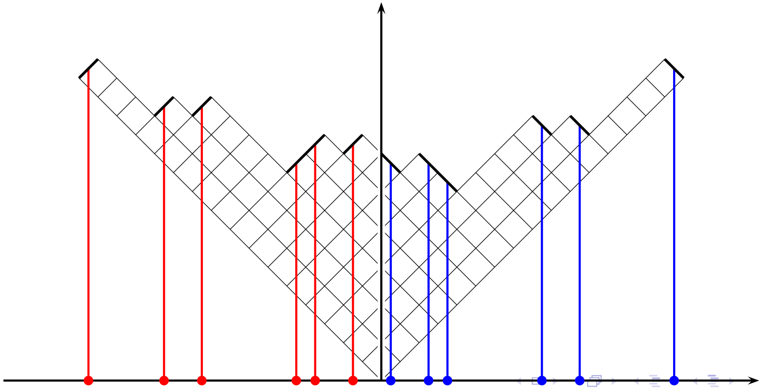
- If this is not satisfied (for instance, case $\alpha < 1/2$ a few slides ago), the diagrams have quite big (i.e. $\gg \sqrt{n}$) row(s) and/or column(s) (they are not *balanced*).
  $\Rightarrow \overline{\lambda}$ is perhaps not the good renormalization.

- But, still, one would like to describe asymptotically the shape of the diagram.

Power sums of Frobenius coordinates

In the non-balanced case, free cumulants should be replaced by power sums of (modified) Frobenius coordinates:

$$p_k(\lambda) = \sum_{i=1}^{d} (a_i)^k - (b_i)^k.$$



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This intuition comes from the following results:

Properties of the p_i's

$$\mathcal{P}ol = \mathbb{C}[p_1, p_2, \dots] \text{ (Kerov, Olshanski, 1994)}$$

If λ is not balanced,

$$\Sigma_\mu(\lambda) = \prod_i p_{\mu_i}(\lambda)(1 + o(1)).$$

Motivation for a new gradation

Let us look more precisely to the mesure SW_n in the case $\alpha < 1/2$. Expectation of characters:

$$\begin{aligned}\mathbb{E}(\Sigma_\mu) &= n(n-1)\dots(n-|\mu|+1)(c\ n^\alpha)^{\ell(\mu)-|\mu|} \\ &\sim c^{\ell(\mu)-|\mu|}\, n^{\alpha\ell(\mu)-\alpha|\mu|+|\mu|}\end{aligned}$$

As we need a result of kind

$$\mathbb{E}(\Sigma_\mu) = O(n^{\deg(\Sigma)}),$$

we will define a gradation such that:

$$\deg(\Sigma_\mu) = \alpha\ell(\mu) - \alpha|\mu| + |\mu|$$

New gradation

Definition of the gradation

Let us define:

$$\deg_\alpha(p_\mu) = \alpha\ell(\mu) - \alpha|\mu| + |\mu|.$$

One has:

- $$\Sigma_\mu = p_\mu + \text{ smaller degree terms.}$$
- If $X \in \mathcal{P}ol$, then $\mathbb{E}_{SW_n}(X) = O(n^{\deg_\alpha(X)})$

Convergence of power sums

Same ideas as before:

$$p_\mu = \Sigma_\mu + \text{ smaller degree terms.}$$

Therefore:

$$\mathbb{E}_{SW_n}(p_\mu) = (c^{\ell(\mu)-|\mu|} + o(1)) \cdot n^{\alpha\ell(\mu)-\alpha|\mu|+|\mu|}$$

i.e.

$$\lim_{n\to\infty} \mathbb{E}_{SW_n}\left(\frac{p_k}{n^{\alpha-\alpha k+k}}\right) = \frac{1}{c^{k-1}}$$

$$\lim_{n\to\infty} \mathrm{Var}_{SW_n}\left(\frac{p_k}{n^{\alpha-\alpha k+k}}\right) = 0$$

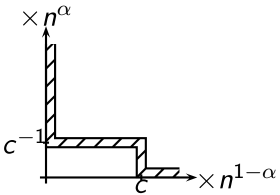
$\Rightarrow$ convergence in probability of *normalized* power sums towards those of measure $c\delta_{c^{-1}}$.

Result on the diagram

After a technical step, one can obtain:

Theorem (F., Méliot, 2010)

With the probability measure described before, $\forall \varepsilon, \eta > 0$, $\exists n_0$ s.t. $\forall n \geq n_0$, the border of the diagram λ_n is, after rescaling and with probability greater than $1 - \varepsilon$, contained in the hatched area of width η:



q-Plancherel measure: definition and motivation

We can also consider a q-deformation Plancherel measure defined by E. Strahov (2008).
Motivations:

- links with Hecke algebras and representations of $GL(n, \mathbb{F}_q)$.
- image by Robinson-Schensted of the distribution $q^{imaj(\sigma)}/[n]!$ on permutations.

Definition:

$$\mathbb{E}_{q\text{-}\mathcal{P}_n}(\chi_q^{\cdot}(T_\mu)) = 0,$$

where the χ_q^λ are the irreducible characters of the generic Hecke algebra.
Luckily, this can be translated in terms of usual characters:

$$\mathbb{E}_{q\text{-}\mathcal{P}_n}(\Sigma_\mu) = \frac{(1-q)^{|\mu|}}{\prod_i 1 - q^{\mu_i}} n^{\downarrow |\mu|}$$

!!

Asymptotics of q-Plancherel measure

Applying usual method with gradation $\deg_0(p_\mu) = |\mu|$, one obtains

Theorem (F., Méliot, 2010)

In probability, under q-Plancherel measure,

$$\forall k \geq 1, \ \frac{p_k(\lambda)}{|\lambda|^k} \longrightarrow_{M_{n,q}} \frac{(1-q)^k}{1-q^k}.$$

Moreover,

$$\forall i \geq 1, \ \frac{\lambda_i}{n} \longrightarrow_{M_{n,q}} (1-q)\, q^{i-1};$$
$$\forall i \geq 1, \ \frac{\lambda'_i}{n} \longrightarrow_{M_{n,q}} 0,$$

We also obtained the second-order asymptotics.

Conclusion

- Showing that some parameters of the diagrams converge is very simple!
- 1. implies immediately convergence of character.
  2. can be used to find a continuous limiting object with some extra works.
  3. not precise enough to study the first row, except if it has size $\Theta(n)$.
- Perspective: would be interesting to generalize it to other groups and objects...

Many thanks!

Thank you for listening!

Any questions?