Cumulants and triangles in Erdős-Rényi random graphs

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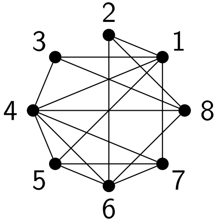
A problem in random graphs

Erdős-Rényi model of random graphs $G(n, p)$:

- G has n vertices labelled $1, \ldots, n$;
- each edge $\{i, j\}$ is taken independently with probability p;



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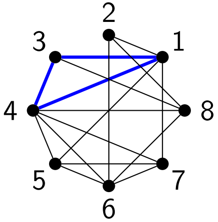

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Fix $p \in]0;1[$.
Describe asymptotically the fluctuations of the number T_n of triangles.

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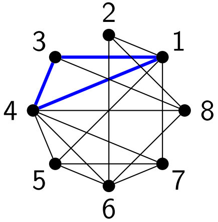

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Describe asymptotically the fluctuations of the number T_n of triangles.

Answer (Ruciński, 1988)

The fluctuations are asymptotically Gaussian.

Outline

1. Intro: cumulant method for number of triangles in $G(n, p)$
2. First extension: stronger conclusion
3. Second extension: weaker hypothesis
4. Ideas of proof

A good tool for that: mixed cumulants

- the r-th mixed cumulant κ_r of r random variables is a specific r-linear symmetric polynomial in joint moments. Examples:

$$\kappa_1(X) := \mathbb{E}(X), \quad \kappa_2(X,Y) := \mathrm{Cov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$$

$$\kappa_3(X,Y,Z) := \mathbb{E}(XYZ) - \mathbb{E}(XY)\mathbb{E}(Z) - \mathbb{E}(XZ)\mathbb{E}(Y) - \mathbb{E}(YZ)\mathbb{E}(X) + 2\mathbb{E}(X)\mathbb{E}(Y)\mathbb{E}(Z).$$

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- if the variables can be split in two mutually independent sets, then the cumulant vanishes.

- if, for each $r \neq 2$, the sequence $\kappa_r(X_n)$ converges towards 0 and if $\mathrm{Var}(X_n)$ has a limit, then X_n converges in distribution towards a Gaussian law.

Application to the number of triangles

$$T_n = \sum_{\Delta=\{i,j,k\}\subset[n]} B_\Delta, \text{ where } B_\Delta(G) = \begin{cases} 1 & \text{if } G \text{ contains the triangle } \Delta; \\ 0 & \text{otherwise.} \end{cases}$$

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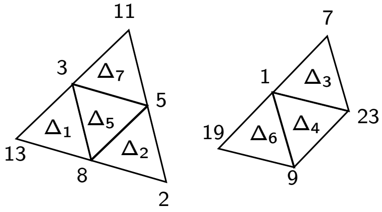

$\{\Delta_1, \Delta_2, \Delta_5, \Delta_7\}$ is independent from $\{\Delta_3, \Delta_4, \Delta_6\}$. Reminder: presence of different edges are independent events.

$$\kappa_\ell(B_{\Delta_1}, \dots, B_{\Delta_7}) = 0.$$

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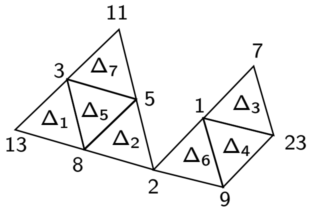

Example:

$\{\Delta_1,\Delta_2,\Delta_5,\Delta_7\}$ is independent from $\{\Delta_3,\Delta_4,\Delta_6\}$.
Triangles need to share an edge to be dependent!

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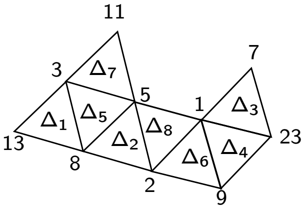

$\kappa_\ell(B_{\Delta_1},\dots,B_{\Delta_8}) \neq 0.$

This configuration contributes to the sum. Call it configuration of dependent triangles.

Lemma: Such a configuration has at most $\ell+2$ vertices (here $\ell = 8$).

Bound on the cumulant

$$\kappa_\ell(T_n) = \sum_{\Delta_1,\dots,\Delta_\ell} \kappa_\ell(B_{\Delta_1},\dots,B_{\Delta_\ell}).$$

Fact 1: number of non-zero terms is smaller than $C_\ell n^{\ell+2}$.

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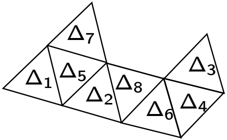
- the number of unlabelled configurations of dependent triangles does not depend on n (only on ℓ) ;



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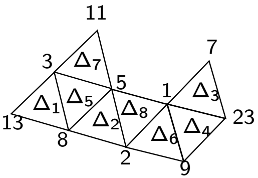

- each configuration can be labelled in at most $n^{\ell+2}$ ways.

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Fact 1: number of non-zero terms is smaller than $C_\ell n^{\ell+2}$.

Fact 2 (easy): each non-zero terms is bounded by C'_ℓ.

Conclusion:

$$|\kappa_\ell(T_n)| = O_\ell(n^{\ell+2})$$

The central limit theorem for triangles

Proposition (Leonov, Shirryaev, 1955)

If $X_1, \dots, X_\ell$ can be split into two sets of mutually independent variables, then

$$\kappa_\ell(X_1, \cdots, X_\ell) = 0$$

Corollary (Janson, 1988)

For each ℓ, there exists a constant C_ℓ such that

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Corollary (Ruciński, 1988)

$$\widetilde{T_n} := \frac{T_n - \mathbb{E}(T_n)}{\sqrt{\mathrm{Var}(T_n)}} \to \mathcal{N}(0,1)$$

Proof: $\mathrm{Var}(T_n) \approx n^4$ thus, $\kappa_\ell(\widetilde{T_n}) = \kappa_\ell(T_n)/Var(T_n)^{\ell/2} = O_\ell(n^{2-\ell})$.

Transition

First extension: stronger conclusion

Statement

Theorem (F., Méliot, Nighekbali, 2014)

Let $X_1, \dots, X_\ell$ be random variables with finite moments of order ℓ,

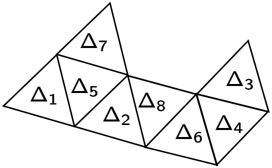
$$|\kappa_\ell(X_1, \cdots, X_\ell)| \leq 2^{\ell-1} ||X_1||_\ell \cdots ||X_\ell||_\ell \cdot \mathrm{ST}\left(G_{\mathrm{dep}}(X_1, \cdots, X_\ell)\right),$$

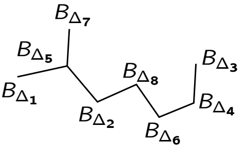
where $\mathrm{ST}\left(G_{\mathrm{dep}}(X_1, \cdots, X_\ell)\right)$ is the number of spanning trees of a dependency graph of $X_1, \cdots, X_\ell$.

A dependency graph for the list $(B_{\Delta_1}, \cdots, B_{\Delta_\ell})$:

$$B_{\Delta_i} \sim B_{\Delta_j} \Leftrightarrow \Delta_i \text{ and } \Delta_j \text{ share an edge}$$

Example:





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Corollary (FMN, 2014)

There exists an absolute constant C such that

$$|\kappa_\ell(T_n)| \leq (C\ell)^\ell n^{\ell+2}$$

Naive bound: $(C\ell)^{3\ell} n^{\ell+2}$

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Let $X_n = (T_n - \mathbb{E}(T_n))/n^{5/3}$. Then (uniformly on compacts of $\mathbb{C}$),

$$\mathbb{E}\left(e^{z X_n}\right) = \exp\left(n^{2/3} z^2/2\right) \exp(L_p z^3/6)(1 + o(1)).$$

(L_p is an explicit constant that depends only on p).

Mod-Gaussian convergence and consequences

Corollary (FMN,2014)

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This type of estimates for the Laplace transform is called mod-Gaussian convergence (Kowalski, Nikeghbali). It implies:

- a central limit theorem (here, we recover the result of Ruciński);
- description of the normality zone and asymmetry of deviations at the edge of this zone;
- speed of convergence in the central limit theorem (here of order $O(1/n)$); we recover a result of Krokowski, Reichenbachs and Thaele, 2015).

Discussion

Our result applies to sum of mostly independent variables (*i.e.* most of the variables are independent)

- Number of copies of a given subgraph;
- Number of arithmetic progression in a random subset of $\{1, \dots, n\}$;
- Number of descents/inversions in random permutations...

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The normality zone and speed of convergence results bound applies in the general context of mod-Gaussian convergence. Again, lots of examples:

- determinant of unitary random matrices,
- number of zeros of a complex analytic function with random coefficients,
- Curie-Weiss model in statistical physics...

Transition

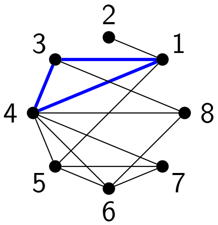
Second extension: weaker hypothesis

(work in progress)

Erdős-Rényi model $G(n, M)$

- G has n vertices labelled $1, \ldots, n$;
- The edge-set of G is taken uniformly among all possible edge-sets of cardinality M.

Example with $n = 8$ and $M = 14$



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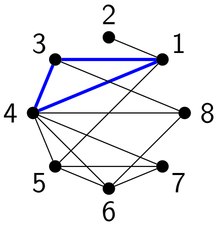

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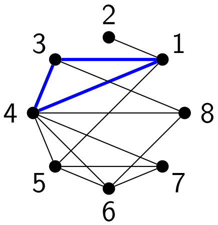


Example with $n = 8$ and $M = 14$

If $p = M/\binom{n}{2}$, each edge appears with probability p, but no independence any more!

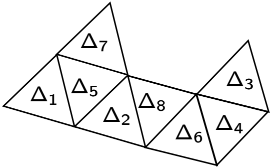
Question

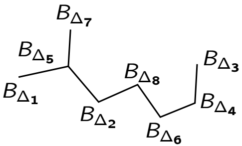
Let $M_n = \lfloor p\binom{n}{2} \rfloor$, with p fixed.
Describe asymptotically the fluctuations of the number T_n of triangles in $G(n, M_n)$.

Proposition (F., > 2015)

Let $\Delta_1, \dots, \Delta_\ell$ be triangles. Define $G_{\Delta_1,\dots,\Delta_\ell}$ as before and denote r its number of connected components. Then

$$\kappa_\ell\left(B_{\Delta_1}, \cdots, B_{\Delta_\ell}\right) \leq \frac{C_\ell}{M_n^{r-1}}.$$





This graph is not a dependency graph any more, but the more connected components it has, the smaller the cumulant is.

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Corollary

For each ℓ, there exists a constant C_ℓ such that

$$\kappa_\ell(T_n) \leq C_\ell \, n^{\ell+2}.$$

Corollary

$$\frac{T_n - \mathbb{E}(T_n)}{\sqrt{\mathrm{Var}(T_n)}} \to \mathcal{N}(0,1)$$

First proved by Janson in 1994 using a coupling with $G(n, p_n)$.

"Weak dependency graph": a general theory?

Other examples, where the order of magnitude of joint cumulants depends on the number of components of some underlying graph:

1. Patterns in random words with a fixed number of occurrences of each letter;
2. Images of distinct integers in a random permutation of size n are $1/n$-dependent;
3. Indicators of particles that can jump in the steady state of the symmetric simple exclusion process;
4. Entries in Haar-distributed orthogonal matrices;
5. Relations in random set partitions.

Yields various central limit theorems in all cases (work in progress!)

Transition

Ideas of proof

Moment-cumulant relation

Mixed cumulants can be expressed in terms of mixed moments:

$$\kappa(X_1, \dots, X_r) = \sum_{\pi} \mu(\pi) M_\pi,$$

where

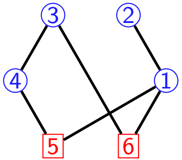
- π runs over set-partitions of $[\ell]$,
- $\mu(\pi) = \mu(\pi, \{[\ell]\})$ is the Möbius function of the set-partition poset (it is explicit but we will only use $\sum_{\text{interval}} \mu(\pi) = 0$),
- $M_\pi = \prod_{B \in \pi} \mathbb{E}\left[\prod_{i \in B} X_i\right]$.

Example:

$$\begin{aligned} M_{\{\{1,3\},\{2,4\}\}} &= \mathbb{E}(X_1 X_3)\mathbb{E}(X_2 X_4) \\ \kappa_3(X,Y,Z) &= \mathbb{E}(XYZ) - \mathbb{E}(XY)\mathbb{E}(Z) - \mathbb{E}(XZ)\mathbb{E}(Y) \\ &\quad - \mathbb{E}(YZ)\mathbb{E}(X) + 2\mathbb{E}(X)\mathbb{E}(Y)\mathbb{E}(Z). \end{aligned}$$

Using independence to simplify M_π

Example: take $\pi = \big\{\{1,2,3,4\},\{5,6\}\big\}$ and

$H := G_{\mathrm{dep}}(X_1,\dots,X_6) =$ 

$$\begin{aligned}\text{Then } M_\pi &:= \mathbb{E}(X_1X_2X_3X_4)\mathbb{E}(X_5X_6)\\ &= \mathbb{E}(X_1X_2)\mathbb{E}(X_3X_4)\mathbb{E}(X_5)\mathbb{E}(X_6)\\ &= M_{\big\{\{1,2\},\{3,4\},\{5\},\{6\}\big\}}.\end{aligned}$$

In general, $M_\pi = M_{\phi_H(\pi)}$, with obvious definition of $\phi_H(\pi)$:
"replace each part π_i of π by the connected components of $H[\pi_i]$".

Rewriting the summation

$$\kappa(X_1, \ldots, X_r) = \sum_{\pi} \mu(\pi) M_\pi = \sum_{\pi} \mu(\pi) M_{\phi_H(\pi)}$$

$$= \sum_{\pi'} M_{\pi'} \left(\sum_{\substack{\pi \text{ s.t.} \\ \phi_H(\pi)=\pi'}} \mu(\pi) \right)$$

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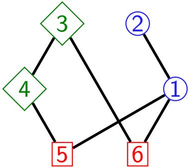
- $\phi_H(\pi) = \pi' \Rightarrow$ for all part π'_i of π', the induced graph $H[\pi'_i]$ is connected.
- if so, we have to compute

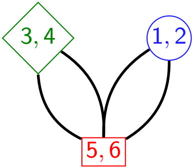
$$\alpha_H^{\pi'} := \sum_{\substack{\pi \text{ s.t.} \\ \phi_H(\pi)=\pi'}} \mu(\pi).$$

Bounding $\alpha_H^{\pi'}$

Consider the contracted graph H/π'. Example:



$$\pi' = \{\{1,2\},\ \{3,4\},\ \{5,6\}\} \quad \mapsto$$



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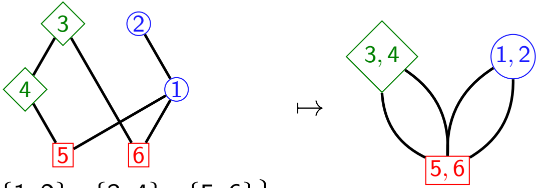

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It is a multigraph.

Lemma

$$\left|\alpha_H^{\pi'}\right| \leq \mathsf{ST}(H/\pi').$$

In the example: $\mathsf{ST}(H/\pi') = 4$.

Bounding everything

Reminder:

$$\kappa(X_1, \dots, X_\ell) = \sum_{\pi'} M_{\pi'} \alpha_H^{\pi'}$$

where the sum runs over set-partition π' such that the induced graphs $H[\pi'_i]$ are connected.

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We have the following inequalities

$$\begin{aligned} |M'_\pi| &\leq ||X_1||_\ell \cdots ||X_\ell||_\ell \quad \text{(Hölder inequality);} \\ \left|\alpha_H^{\pi'}\right| &\leq \mathrm{ST}(H/\pi'); \\ \mathbf{1}_{H[\pi'_i] \text{ connected}} &\leq \mathrm{ST}(H[\pi'_i]) \end{aligned}$$

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Thus

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A combinatorial identity

Lemma

$$2^{\ell-1}\,\mathsf{ST}(H) = \sum_{\pi'} \mathsf{ST}(H/\pi') \left(\prod_i \mathsf{ST}(H[\pi'_i]) \right),$$

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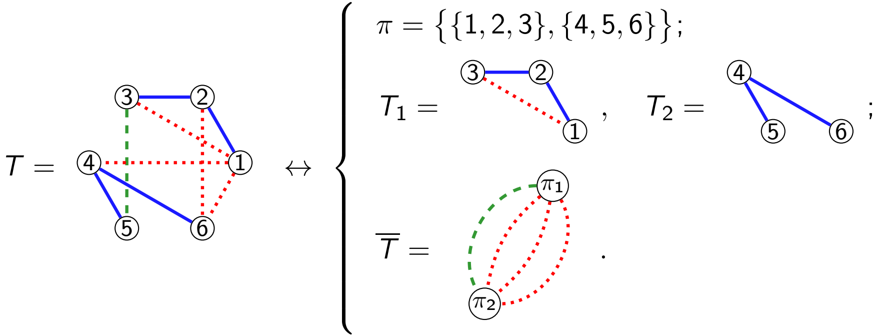

Progress report

We have "proved":

Theorem

Let $X_1, \ldots, X_\ell$ be random variables with finite moments of order ℓ,

$$|\kappa_\ell(X_1, \cdots, X_\ell)| \leq 2^{\ell-1} ||X_1||_\ell \cdots ||X_\ell||_\ell \cdot \mathrm{ST}\left(G_{\mathrm{dep}}(X_1, \cdots, X_\ell)\right).$$

Next step:

Corollary

There exists an absolute constant C such that

$$|\kappa_\ell(T_n)| \leq (C\ell)^\ell n^{\ell+2}.$$

A sharp bound on cumulants of T_n

Recall that $\kappa_\ell(T_n) = \sum_{\Delta_1,\dots,\Delta_\ell} \kappa_\ell(B_{\Delta_1},\dots,B_{\Delta_\ell})$.

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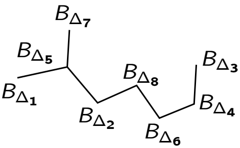

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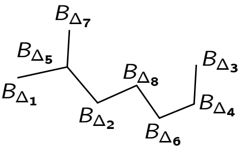

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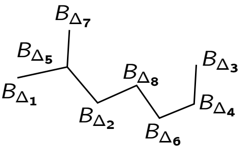

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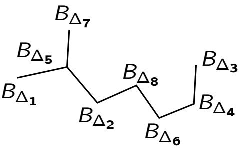

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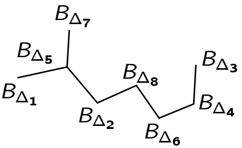

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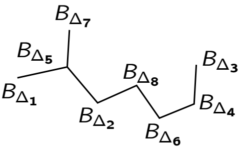

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Mod-Gaussian convergence

Let $X_n = (T_n - \mathbb{E}(T_n))/n^{5/3}$, then

$$\log \mathbb{E}(\exp(zX_n)) = \sum_{\ell \geq 2} \kappa_\ell(X_n) z^\ell/\ell!$$

$$= n^{2/3}\sigma^2 z^2/2 + L\, z^3/6 + \underbrace{\sum_{\ell \geq 4} n^{5/3}\, \kappa_\ell(T_n)\, z^\ell/\ell!}_{\text{call it } R}$$

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almost as if X_n would be the sum of $n^{2/3}$ independent standard Gaussian.
$\longrightarrow$ results on normality zone and speed of convergence follow by adaptation of standard techniques (Berry-Esseen lemma, change of probability measure).

A word on the proof of the second extension (1/2)

First consider edge-disjoint triangles $\Delta_1, \dots, \Delta_\ell$.

Then, for $A \subset \{1, \dots, \ell\}$,

$$M_A := \mathbb{E}\left(\prod_{i \in A} B_{\Delta_i}\right) = \frac{(M_n)_{3\ell}}{\binom{n}{2}_{3\ell}}$$

Notation: $(x)_k = x(x-1)(x-2)\dots(x-k+1)$.

We want to prove that

$$\kappa(B_{\Delta_1}, \dots, B_{\Delta_\ell}) = O(M_n^{-\ell+1})$$

This case is already difficult!

A word on the proof of the second extension (2/2)

Lemma

Define T_B by $M_A = \prod_{B \subset A}(1 + T_B)$, i.e. $T_B = -1 + \prod_{A \subset B} M_A^{(-1)^{|B|-|A|}}$.

$$\text{Then } T_B = O(M_n^{-|B|+1}).$$

Proof: elementary analysis.

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Fact: the sum [...] vanish unless the monomial is $O(M_n^{\ell-1})$. □

Open questions

- What about $G(n, p_n)$ with $p_n \to 0$ and $n\,p_n \to \infty$. One has:

$$\kappa_\ell(T_n) \le C_\ell\, n^3\, p_n^2\, \max(np_n^2, 1)^{\ell-1} \quad \text{(Mikhailov, 1991);}$$

$$\kappa_\ell(T_n) \le (C\ell)^\ell\, n^{\ell+2}\, p_n^3 \quad \text{(FMN, 2014).}$$

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- Future work: Stein's method/Lovász local lemma with weak dependency graphs?