Dual combinatorics of Jack polynomials

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What is this talk about?

- Symmetric functions:

$$x_1^3 + x_2^3 + x_3^3 + \dots$$

$$\sum_{i<j} x_i x_j$$

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- in particular Jack polynomials $J_\lambda^{(\alpha)}$.

$$J_{(2)}^{(\alpha)} = (\alpha+1) \cdot x_1^2 + 2 \cdot x_1 \cdot x_2 + (\alpha+1) \cdot x_2^2 \\ + 2 \cdot x_1 \cdot x_3 + 2 \cdot x_2 \cdot x_3 + (\alpha+1) \cdot x_3^2 + \dots$$

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- Symmetric functions.
- in particular Jack polynomials $J_\lambda^{(\alpha)}$.
- We present a new approach to the study of Jack polynomials (called dual), due to Michel Lassalle with a lot of open questions.
- Partial answers (for $\alpha = 1, 2$) involve combinatorics and representation theory.

Outline of the talk

1. Definitions and notations
2. Dual approach and Lassalle's conjectures
3. Solution to the $\alpha = 1$ case using Young symmetrizer
4. Overview of the $\alpha = 2$ case
5. Leads towards the general case

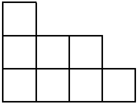
Partitions

Definition

A partition (of n) is a non-increasing list of integer (of sum n).
If λ is a partition of n, we denote $\lambda \vdash n$

Example : $(4,3,1) \vdash 8$.

Graphical representation as Young diagram :



Symmetric functions

Definition

A symmetric function is a symmetric *polynomial* in infinitely many variables $x_1, x_2, \dots$.

i.e.

- bounded degree ;
- when we set $x_{n+1} = x_{n+2} = \cdots = 0$, we have a symmetric polynomial in $x_1, \dots, x_n$.

Examples:

$$p_3 = x_1^3 + x_2^3 + x_3^3 + \dots, \quad e_2 = \sum_{i<j} x_i x_j$$

Swaping the indices of two variables does not change the polynomials.

Symmetric functions

Definition

A symmetric function is a symmetric *polynomial* in infinitely many variables $x_1, x_2, \dots$.

Let $\lambda = (\lambda_1, \dots, \lambda_r)$ be a partition. Set

$$m_\lambda(x_1, x_2, \dots) = x_1^{\lambda_1} \dots x_r^{\lambda_r} + \text{its images by swaping indices.}$$

Proposition

The family $(m_\lambda)_{\lambda \text{ partition}}$ is a linear basis of the symmetric function ring.

called monomial basis.

Symmetric functions

Definition

A symmetric function is a symmetric *polynomial* in infinitely many variables $x_1, x_2, \dots$.

Set $p_0 = 1,\ p_k = x_1^k + x_2^k + \dots$.   power sums

Proposition

The family $(p_i)_{i \geq 1}$ is an algebraic basis of the symmetric function ring. In other words, any symmetric function writes uniquely as a linear function of

$$\left(p_\lambda = \prod_i p_{\lambda_i} \right),$$

where λ runs over partitions.

Schur functions

Definition (Jacobi, 1841)

Let λ be a partition. Define

$$s_\lambda(x_1, \dots, x_n) = \frac{\det\left(x_i^{\lambda_j+n-j}\right)}{\det\left(x_i^{n-j}\right)}.$$

Then (s_λ) is a linear basis of symmetric function ring.

Example:

$$s_{(2,1)}(x_1, x_2, x_3) = x_1^2 \cdot x_2 + x_1 \cdot x_2^2 + x_1^2 \cdot x_3 + 2 \cdot x_1 \cdot x_2 \cdot x_3 \\ + x_2^2 \cdot x_3 + x_1 \cdot x_3^2 + x_2 \cdot x_3^2$$

Representation theory of symmetric group

- S_n : group of permutations of n.
- We are interested in its representation that is group morphisms $S_n \to \mathrm{GL}(V)$, V $\mathbb{C}$-vector space of finite dimension.

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  - it is enough to study the irreducible representations.
  - these irreducible representations ρ^λ are enumerated by the number of conjugacy classes in S_n, that is of partitions of n.

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- what the general theory says us:
  - it is enough to study the irreducible representations.
  - these irreducible representations ρ^λ are enumerated by the number of conjugacy classes in S_n, that is of partitions of n.
  - what is really important is to compute characters (=trace), that is a collections of numbers

    $$\chi^\lambda_\mu := \mathrm{tr}(\rho^\lambda(\pi)) \quad \text{(with } \pi \text{ of cycle type } \mu\text{)}$$

    indexed by two partitions.

Frobenius formula

Theorem (Frobenius, 1900)

Let λ be a partition of n, then

$$s_\lambda = \sum_{\mu \vdash n} \chi^\lambda_\mu \frac{p_\mu}{z_\mu},$$

where $z_\mu = \prod_{i \geq 1} i^{m_i} m_i!$ if μ has m_1 parts equal to 1,...

This result makes a link between two different theories: symmetric functions and representation theory of the symmetric group.

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Orthonormality of irreducible characters $\Rightarrow \langle s_\lambda, s_\rho \rangle = \delta_{\lambda,\rho}$

Proposition

The basis (s_λ) may be obtained from the monomial basis by Gram-Schmidt orthonormalization process. (use lexicographic order on partitions).

Jack polynomials

Consider the following deformation of Hall scalar product:

$$\langle p_\mu, p_\nu \rangle_\alpha = \alpha^{\ell(\mu)} z_\mu \delta_{\mu,\nu}$$

$\ell(\mu)$: length (number of parts) of the partition μ.

Definition

Jack polynomials $PQ_\lambda^{(\alpha)}$ are obtained from the monomial basis by Gram-Schmidt orthonormalization process (with respect to the deformed scalar product).

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Specialization: $J_\lambda^{(1)} = c_\lambda^{(1)} s_\lambda = \frac{n!}{\dim(V_\lambda)} s_\lambda$.
V_λ: irreducible representation of S_n associated to λ.

Jack "characters"

Main object in the talk

Let λ and μ be partitions of n. Define $\theta_\mu^{\lambda,(\alpha)}$ by

$$J_\lambda^{(\alpha)} = \sum_{\mu \vdash n} \theta_\mu^{\lambda,(\alpha)} \cdot p_\mu.$$

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Let λ and μ be partitions of n. Define $\theta_\mu^{\lambda,(\alpha)}$ by

$$J_\lambda^{(\alpha)} = \sum_{\mu \vdash n} \theta_\mu^{\lambda,(\alpha)} \cdot p_\mu.$$

Unfortunately, $\theta_\mu^{\lambda,(\alpha)}$ has no (known) representation-theoretical interpretation for general α.

But, it shares (conjecturally) a lot of properties with

$$\theta_\mu^{\lambda,(1)} = z_\mu n! \frac{\chi_\mu^\lambda}{\dim(\lambda)},$$

whence the name Jack characters.

A function on the set of all Young diagrams

Definition

Let μ be a partition of k without part equal to 1. Define

$$\mathrm{Ch}_\mu^{(\alpha)}(\lambda) = \begin{cases} z_\mu \theta_{\mu 1^{n-k}}^{\lambda,(\alpha)} & \text{if } n = |\lambda| \geq k; \\ 0 & \text{otherwise.} \end{cases}$$

$\mathrm{Ch}_\mu^{(\alpha)}$ is a function of all Young diagrams.

A function on the set of all Young diagrams

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$\mathrm{Ch}_\mu^{(\alpha)}$ is a function of all Young diagrams.

Specialization: if $|\mu| < |\lambda|$,

$$\mathrm{Ch}_\mu^{(1)}(\lambda) = \frac{|\lambda|!}{(|\lambda| - |\mu|)!} \cdot \frac{\chi^\lambda_{\mu 1^{n-k}}}{\dim(V_\lambda)}.$$

Introduced by S. Kerov, G. Olshanski in the case $\alpha = 1$, by M. Lassalle in the general case.

A function on the set of all Young diagrams

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Proposition (M. Lassalle)

For any r, the application

$$(\lambda_1, \dots, \lambda_r) \mapsto \mathrm{Ch}_\mu^{(\alpha)}\left((\lambda_1, \dots, \lambda_r)\right)$$

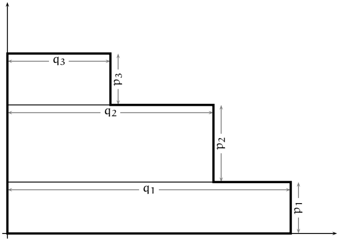
is a polynomial in λ_1, ..., λ_r. Besides, it is symmetric in $\lambda_1 - 1$, ..., $\lambda_r - r$.

In other words, $\mathrm{Ch}_\mu^{(\alpha)}$ is a shifted symmetric function.

Multirectangular coordinates (R. Stanley)

Consider two lists **p** and **q** of positive integers of the same size, with **q** non-decreasing.
We associate to them the partition

$$\lambda(\mathbf{p},\mathbf{q}) = \big(\underbrace{q_1,\dots,q_1}_{p_1 \text{ times}},\underbrace{q_2,\dots,q_2}_{p_2 \text{ times}},\dots\big).$$



Young diagram of $\lambda(\mathbf{p},\mathbf{q})$

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Conjecture (M. Lassalle)

Let μ be a partition of k. $(-1)^k \operatorname{Ch}_\mu^{(\alpha)}(\lambda(\mathbf{p},\mathbf{q}))$ is a polynomial in

$$p_1, p_2, \dots, -q_1, -q_2, \dots, \alpha - 1$$

with non-negative integer coefficients.

polynomiality in **p** and **q**: consequence of shifted symmetry
polynomiality in α: F., Dołęga 2012

Multirectangular coordinates (R. Stanley)

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with non-negative integer coefficients.

Hard, interesting still open part: non-negativity (and integrity).

Case $\alpha = 1$

Goal of the next few slides: sketch the proof of Lassalle's conjecture in the case $\alpha = 1$.

Theorem (F. 2007, conjectured by Stanley 2003)

Let μ be a partition of k. $(-1)^k \mathrm{Ch}_\mu^{(1)}(\lambda(\mathbf{p}, \mathbf{q}))$ is a polynomial in

$$p_1, p_2, \dots, -q_1, -q_2, \dots$$

with non-negative integer coefficients.

Reminder: if $|\mu| < |\lambda|$,

$$\mathrm{Ch}_\mu^{(1)}(\lambda) = \frac{|\lambda|!}{(|\lambda| - |\mu|)!} \cdot \frac{\chi^\lambda_{\mu 1^{n-k}}}{\dim(V_\lambda)}.$$

Hence, we need to know how to compute $\chi^\lambda_{\mu 1^{n-k}}$.
Next step: construction of irreducible representations of S_n.

Young's symmetrizer (1/3)

Let λ be a partition of n.

Choose a filling T_0 of λ.

Example:

$$\lambda = (2,2), \quad T_0 = \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array}.$$

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$$a_\lambda = \sum_{\substack{\sigma \in S_n \\ \sigma \in \mathrm{RS}(T_0)}} \sigma \in \mathbb{C}[S_n],$$

where $\mathrm{RS}(T_0)$ is the row stabilizer of T_0 ;

Example:

$$\lambda = (2,2), \quad T_0 = \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array}.$$

$$a_\lambda = \mathrm{id} + (1\ 3) + (2\ 4) + (1\ 3)(2\ 4)$$

Everything depends on T_0, although that is hidden in notations.

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$$b_\lambda = \sum_{\substack{\tau \in S_n \\ \tau \in \mathrm{CS}(T_0) \in \mathbb{C}[S_n]}} \varepsilon(\tau)\tau$$

$\mathrm{CS}(T_0)$ is the column stabilizer of T_0.

Example:

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$$a_\lambda = \mathrm{id} + (1\ 3) + (2\ 4) + (1\ 3)(2\ 4)$$

$$b_\lambda = \mathrm{id} - (1\ 2) - (3\ 4) + (1\ 2)(3\ 4)$$

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Young symmetrizer (2/3)

Consider

$$a_\lambda \cdot b_\lambda = \sum_{\substack{\sigma \in S_n \\ \sigma \in \mathrm{RS}(T_0)}} \sum_{\substack{\tau \in S_n \\ \tau \in \mathrm{CS}(T_0)}} \varepsilon(\tau)\sigma\tau$$

Lemma

Then $p_\lambda = \alpha_\lambda a_\lambda \cdot b_\lambda$ is a projector (*i.e.* $p_\lambda^2 = p_\lambda$) for a well-chosen constant α_λ.

Young symmetrizer (3/3)

Reminder: $p_\lambda = \alpha_\lambda a_\lambda \cdot b_\lambda$ is a projector.

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Set $V_\lambda = \mathbb{C}[S_n]p_\lambda$, subspace of the group algebra.

Then S_n acts by left multiplication on V_λ.

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Theorem (Young, 1901)

$(V_\lambda)_{\lambda \vdash n}$ forms a complete set of irreducible representations of S_n.

note: in fact, $\alpha_\lambda = \frac{\dim(V_\lambda)}{n!}$.

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Next step : compute the trace.

Reformulation

Our goal

Let μ be a partition of n and π a permutation of cycle-type μ. We want to compute the trace χ^λ_μ of

$$\rho^\lambda(\pi) : \begin{array}{rcl} \mathbb{C}[S_n]p_\lambda & \to & \mathbb{C}[S_n]p_\lambda \\ x & \mapsto & \pi \cdot x \end{array}$$

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Lemma

$$\mathrm{tr}(\rho^\lambda(\pi)) = \mathrm{tr}\left(\begin{array}{rcl} \mathbb{C}[S_n] & \to & \mathbb{C}[S_n] \\ x & \mapsto & \pi \cdot x \cdot p_\lambda \end{array} \right)$$

Proof: $\mathbb{C}[S_n] = \mathbb{C}[S_n]p_\lambda \oplus \mathbb{C}[S_n](1 - p_\lambda)$
and the application $(x \mapsto \pi x p_\lambda)$ is $\rho^\lambda(\pi)$ on $\mathbb{C}[S_n]p_\lambda$ and 0 on $\mathbb{C}[S_n](1 - p_\lambda)$

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Corollary

$$\chi^\lambda_\mu = \mathrm{tr}(\rho^\lambda(\pi)) = \alpha_\lambda \sum_{\substack{\sigma \in S_n \\ \sigma \in \mathrm{RS}(T_0)}} \sum_{\substack{\tau \in S_n \\ \tau \in \mathrm{CS}(T_0)}} \varepsilon(\tau)\, \mathrm{tr}(x \mapsto \pi \cdot x \cdot \sigma \cdot \tau)$$

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Corollary

$$\chi^\lambda_\mu = \mathrm{tr}(\rho^\lambda(\pi)) = \alpha_\lambda \sum_{\substack{\sigma \in S_n \\ \sigma \in \mathrm{RS}(T_0)}} \sum_{\substack{\tau \in S_n \\ \tau \in \mathrm{CS}(T_0)}} \varepsilon(\tau) \sum_{g \in S_n} \delta_{\pi \cdot g \cdot \sigma \cdot \tau,\, g}$$

First formula

$$n! \frac{\mathrm{tr}(\rho^\lambda(\pi))}{\dim(V_\lambda)} = \sum_{\substack{\sigma \in S_n \\ \sigma \in \mathrm{RS}(T_0)}} \sum_{\substack{\tau \in S_n \\ \tau \in \mathrm{CS}(T_0)}} \varepsilon(\tau) \sum_{g \in S_n} \delta_{\pi g \sigma \tau = g}$$

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... (some combinatorial manipulations on sums) ...

$$n! \frac{\chi^\lambda_\mu}{\dim(V_\lambda)} = \sum_{\substack{\sigma,\tau \in S_n \\ \sigma\tau = \pi}} \varepsilon(\tau) F_{\sigma,\tau}(\lambda),$$

where

$$F_{\sigma,\tau}(\lambda) = \left| \left\{ \begin{array}{c} \text{fillings } T \text{ of } \lambda \\ \text{such that } \sigma \in \mathrm{RS}(T),\ \tau \in \mathrm{CS}(T) \end{array} \right\} \right|$$

Example for $\sigma = (1,2) \in S_6, \tau = (1,3) \in S_6$: filling $T =$

5		
2	1	
4	3	6

Further simplifications

Reminder:
$$n!\frac{\chi^\lambda_\mu}{\dim(V_\lambda)} = \sum_{\substack{\sigma,\tau \in S_n \\ \sigma\tau = \pi}} \varepsilon(\tau) F_{\sigma,\tau}(\lambda).$$

We are interested in $\chi^\lambda_{\mu 1^{n-k}}$ $\Rightarrow$ we can choose $\pi \in S_k \subset S_n$.

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Observation:

- terms vanish except for σ, τ also in S_k ;
- for σ, τ in S_k,
$$F_{\sigma,\tau}(\lambda) = (n-k)! \widetilde{N}_{\sigma,\tau}(\lambda),$$
where $\widetilde{N}_{\sigma,\tau}(\lambda) = \left| \left\{ \begin{array}{c} \text{injective functions } f : \{1, \cdots, k\} \to \lambda \\ \text{such that } \sigma \in \mathrm{RS}(f),\ \tau \in \mathrm{CS}(f) \end{array} \right\} \right|$

Example for $\sigma = (1,2) \in S_3, \tau = (1,3) \in S_3$: filling $T =$

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	3	

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We obtain:
$$\frac{n!}{(n-k)!} \frac{\chi^\lambda_{\mu 1^{n-k}}}{\dim(V_\lambda)} = \sum_{\substack{\sigma,\tau \in S_k \\ \sigma\tau=\pi}} \varepsilon(\tau) \widetilde{N}_{\sigma,\tau}(\lambda),$$

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We have obtained:

$$\frac{n!}{(n-k)!} \frac{\chi^\lambda_{\mu 1^{n-k}}}{\dim(V_\lambda)} = \sum_{\substack{\sigma,\tau \in S_k \\ \sigma\tau = \pi}} \varepsilon(\tau) \widetilde{N}_{\sigma,\tau}(\lambda),$$

where $\widetilde{N}_{\sigma,\tau}(\lambda) = \left| \left\{ \begin{array}{c} \text{injective functions } f : \{1, \cdots, k\} \to \lambda \\ \text{such that } \sigma \in \mathrm{RS}(f),\ \tau \in \mathrm{CS}(f) \end{array} \right\} \right|$

Further simplifications

Reminder: $$n! \frac{\chi^\lambda_\mu}{\dim(V_\lambda)} = \sum_{\substack{\sigma,\tau \in S_n \\ \sigma\tau=\pi}} \varepsilon(\tau) F_{\sigma,\tau}(\lambda).$$

We are interested in $\chi^\lambda_{\mu 1^{n-k}}$ $\Rightarrow$ we can choose $\pi \in S_k \subset S_n$.

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One can forget injectivity condition : non-injective functions have a total 0-contribution.

End of our proof

Theorem (F., Śniady 2007, conjectured by Stanley 2006)

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Proof: the few previous slides!

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Theorem (F., Śniady 2007, conjectured by Stanley 2006)

$$\mathrm{Ch}^{(1)}_{\mu}(\lambda) = \sum_{\substack{\sigma,\tau \in S_k \\ \sigma\tau = \pi}} \varepsilon(\tau) N_{\sigma,\tau}(\lambda)$$

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Theorem (F., Śniady 2007, conjectured by Stanley 2006)

$$(-1)^k \operatorname{Ch}_\mu^{(1)}(\lambda) = \sum_{\substack{\sigma,\tau \in S_k \\ \sigma\tau=\pi}} (-1)^{|C(\tau)|} N_{\sigma,\tau}(\lambda)$$

$|C(\tau)|$: nombre de cycle de τ.

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Let σ, τ in S_k. Then $N_{\sigma,\tau}\big(\lambda(\mathbf{p},\mathbf{q})\big)$ is a polynomial in $\mathbf{p}$ and $\mathbf{q}$ with non-negative integer coefficients and degree $|C(\sigma)|$ in $\mathbf{p}$ and $|C(\tau)|$ in $\mathbf{q}$.

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Corollary

$(-1)^k \operatorname{Ch}_\mu^{(1)}\big(\lambda(\mathbf{p},\mathbf{q})\big)$ is a polynomial in $\mathbf{p}$ and $-\mathbf{q}$ with non-negative integer coefficients.

An example of $N_{\sigma,\tau}\big(\lambda(\mathbf{p},\mathbf{q})\big)$

Let $\sigma = (1\ 2)$ and $\tau = \mathrm{id}_2$.
$N_{\sigma,\tau}(\lambda)$ count the number of ordered choice of two boxes of λ in the same row.

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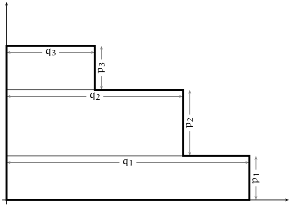

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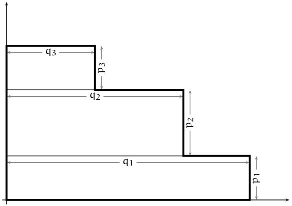

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Hence

$$N_{(1\ 2),\mathrm{id}_2}\big(\lambda(\mathbf{p},\mathbf{q})\big) = \sum_{i\geq 1} p_i q_i^2.$$

End of our proof (reminder)

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Pair of permutations and graphs embedded in surfaces

There is a (classical) bijection between

$$S_k \times S_k \quad \Longleftrightarrow \quad \left\{ \begin{array}{c} \text{bicolored graphs} \\ \text{embedded in orientable surfaces} \\ \text{with } k \text{ labelled edges.} \end{array} \right\}$$

$$\left(\begin{array}{c} \text{up to isomorphism} \\ \text{with a slight technical condition} \end{array} \right)$$

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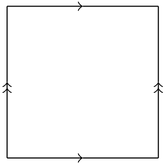

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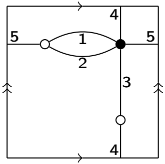

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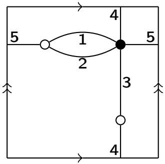
$$\begin{array}{c} \sigma = (1\ 5\ 2)(3\ 4) \\ \tau = (1\ 2\ 3\ 5\ 4) \end{array} \quad \longleftrightarrow$$



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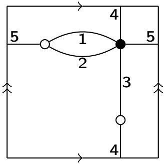

- cycles of the product $\leftrightarrow$ "faces" of the map;
- $N_{\sigma,\tau}$ depends only on the underlying graph (neither on the embedding nor on edge multiplicities).

Stanley's formula in terms of map

Theorem (F., Śniady 2007, conjectured by Stanley 2006)

$$(-1)^k \operatorname{Ch}_\mu^{(1)}(\lambda) = \sum_{\substack{M \text{ bipartite oriented map} \\ \text{of face-type } \mu}} (-1)^{|V_\bullet(M)|} N_{G(M)}(\lambda)$$

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It is classical to count maps via characters of the symmetric group using Frobenius counting formula (Stanley, Jackson, Vinsenti, Jones, Zagier, Goupil, Schaeffer, Poulhalon).

But both formulas do not seem to be linked!

Transition

We just proved Lassalle's conjecture for $\alpha = 1$.

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Theorem (F. Śniady, 2011)

Lassalle's conjecture holds also for $\alpha = 2$.

Next two slides:

- representation-theoretical interpretation of $\theta_\mu^{\lambda,(2)}$ (involves Gelfand pair) ;
- combinatorial formula for $\mathrm{Ch}_\mu^{(2)}$.

Definition of Gelfand pairs

Let G be a finite group and K a subgroup of G. We say that (G, K) is a Gelfand pair if

- The induced representation $\mathbf{1}_K^G$ is multiplicity free;
- or equivalently, the $\mathbb{C}[K \backslash G / K]$ is commutative

$\mathbb{C}[K \backslash G / K]$: subalgebra of $\mathbb{C}[G]$ formed by elements invariants by left and right multiplication by $k \in K$

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Theory of Gelfand pairs extends representation theory of finite groups (RTFG).

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$Z(\mathbb{C}[G])$	$\mathbb{C}[K \backslash G / K]$
representations	?
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Theorem (Stembridge, 1992)

$\theta_\mu^{\lambda,(2)}$ are the zonal spherical values of the Gelfand pair (S_{2n}, H_n) (H_n is the hyperoctahedral group).

Combinatorial formula for $\mathrm{Ch}_\mu^{(2)}$

Theorem (F., Śniady 2011)

$$(-1)^k 2^{\ell(\mu)}\, \mathrm{Ch}_\mu^{(2)}(\lambda) = \sum_{\substack{M \text{ bipartite non-oriented maps} \\ \text{of face-type } \mu}} (-2)^{|V_\bullet(M)|} N_{G(M)}(\lambda)$$

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There is a formula, analog to Frobenius counting formula, counting non-oriented maps using zonal spherical functions of (S_{2n}, H_n) (Goulden, Jackson, 1996). But, once again, it does not seem related to our formula!

A combinatorial solution to the general case ?

Conjecture (hope ?)

There exists a weight $w_M(\alpha - 1)$, polynomial with non-negative coefficients in $\alpha - 1$, such that

$$(-1)^k \,\mathrm{Ch}_\mu^{(\alpha)}(\lambda) = \sum_{\substack{M \text{ bipartite non-oriented map} \\ \text{of face-type } \mu}} w_M(\alpha - 1) N_{G(M)}(\lambda)$$

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Goulden and Jackson (1996) have a similar conjecture for an extension of Frobenius counting formula. But still open!

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A partial result (Dołęga, F., Śniady, 2013)

There exist a combinatorial weight $w_M(\alpha-1)$ such that, for any rectangular Young diagrams, the formula above holds.

But this specific weight does not work in general (fails for $\mu = (9)$ and λ non trivial superposition of 3 rectangles).

Conclusion and perspectives

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- In any case, Jack polynomials are well-studied objects and a new combinatorial description would be welcome.
- from a combinatorial point of view, the conjecture suggest an interpolation between oriented and non-oriented framework: puzzling!