Multi-parameter hook formula for labelled trees

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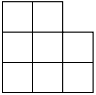
Frame-Robinson-Thrall formula (1954) for counting tableaux

Fix a Young diagram λ with n boxes.



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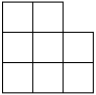
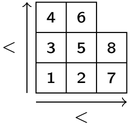


Then the number of standard Young tableaux



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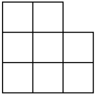

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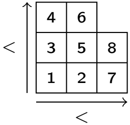
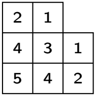


is given by

$$\frac{n!}{\prod_{\square \in \lambda} h_\square}.$$

$h_\square$: hook-length of the box $\square$, *i.e.* number of boxes at its right in the same row or above it in the same column.

In our example: the hook-lengths are



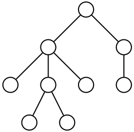
so there are

$8!/(5 * 4 * 4 * 3 * 2 * 2) = 42$ standard Young tableaux of shape λ.

Knuth formula for increasing trees (1973)

The same kind of formula holds for trees!

Fix a Tree T with n nodes.



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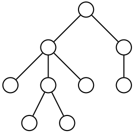
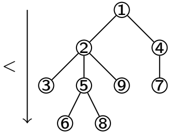


Then the number of increasing labellings of this tree



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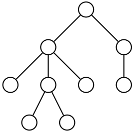

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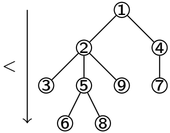

is given by

$$\frac{n!}{\prod_{\circ \in V(T)} h_\circ}.$$

$h_\circ$: hook-length of the vertex $\circ$, *i.e.* the number of vertices in the subtree of T rooted in $\circ$.

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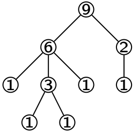

so there are

$9!/(9 * 6 * 3 * 2) = 1120$ increasing labellings of T.

Hook summation formulas

But these objects are in bijection with permutations.

- By Robinson-Schensted algorithm, pairs of standard Young tableaux of the same shape are in bijections with permutations, so

$$\sum_{\lambda \vdash n} \left(\frac{n!}{\prod_{\square \in \lambda} h_{\square}} \right)^2 = n!.$$

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$$\sum_{\lambda \vdash n} \left(\frac{n!}{\prod_{\square \in \lambda} h_\square} \right)^2 = n!.$$

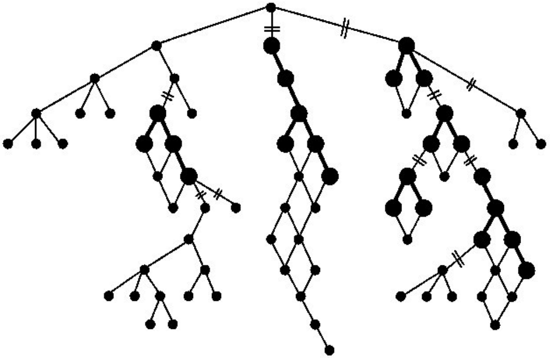
- By binary search tree algorithm, increasing labellings of binary trees are in bijection with permutations, so

$$\sum_{T \text{ binary tree}} \frac{n!}{\prod_{\circ \in V_T} h_\circ} = n!$$

These formulas are called hook summation formulas.

A large amount of work around these hook formulas

- formulas for other objects than trees or Young diagrams: in particular, d-complete posets that include both.



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- in summation formulas, one can replace $1/h_\square^2$ or $1/h_\circ$ by more involved expressions such that the sum remains simple.

$$\sum_{\substack{T \text{ binary} \\ \text{tree of size } n}} \prod_{v \in T} \left(x + \frac{1}{h_T(v)} \right) = \frac{1}{(n+1)!} \prod_{i=1}^{n-1} \left((n+1+i)x + n+1-i \right).$$

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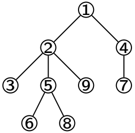
- in summation formulas, one can replace $1/h_\square^2$ or $1/h_\circ$ by more involved expressions such that the sum remains simple.
- interpretations in combinatorial Hopf algebra theory, in convex geometry, in commutative algebra.
- ...

Main result

A hook summation formula over labelled increasing tree with n nodes.

A labelled increasing tree T



Childs of a given vertex are not ordered. By convention, we draw them in increasing order from left to right.

⚠ in our formula, we sum over labelled trees.

Main result

A hook summation formula over labelled increasing tree with n nodes.

Theorem (FGL, 2013)

Let $(x_i)_{1\leq i\leq n}$ and $(y_{i,j})_{1\leq i\leq j\leq n}$ be formal parameters.

$$\sum_T \left[\prod_{i=2}^{n} x_{f_i(T)} \left(\sum_{j\in\mathfrak{h}_i(T)} y_{i,j}\right)\right] = x_1 y_{n,n} \prod_{i=2}^{n-1}\left(y_{i,i}\sum_{j=1}^{i} x_j + x_i \sum_{j=i+1}^{n} y_{i,j}\right).$$

$f_i(T)$: parent of i in T;
$\mathfrak{h}_i(T)$: vertex set of the subtree of T rooted in i.

Example :

$$\text{weight}\left(\begin{array}{c}①\\|\\②\\|\\③\end{array}\right) = x_1(y_{2,2}+y_{2,3})x_2 y_{3,3}$$

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- it contains the refined enumeration (by degrees) of Cayley trees. In fact, S. Carrell proved that it is equivalent to some refinement of Cayley formula due to V. Strehl.

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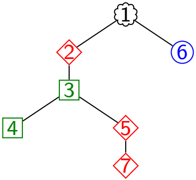
- As we shall see, it has a combinatorial flavor.

Combinatorial reformulation

Fix a set-partition of $\{2, \ldots, n\}$ (in the example $\pi = \{\{2,5,7\},\{3,4\},\{6\}\}$). One has to find a bijection between



increasing trees T such that, for any two elements in the same part, one is the ancestor of the other.

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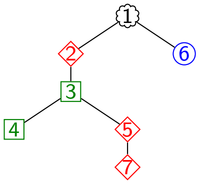

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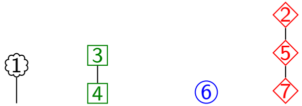

$a_\square = 1 \quad a_\circ = 3 \quad \emptyset$

a number for each part (except the one containing n) less or equal than the maximum of the part (called *anchor point*)

$$a_\square \leq 4, \ \ a_\circ \leq 6.$$

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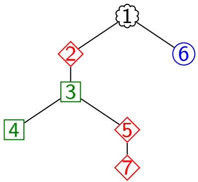

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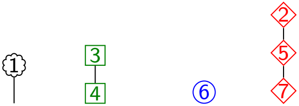

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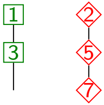
which respects the degree:

$$\deg_{\text{left}}(i) = \deg_{\text{right}}(i) + |a^{-1}(i)|.$$

Elementary splicing

Consider two chains, the first one ending by a free edge.



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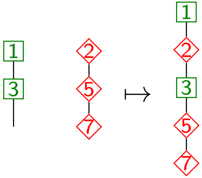

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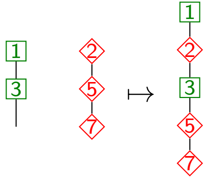

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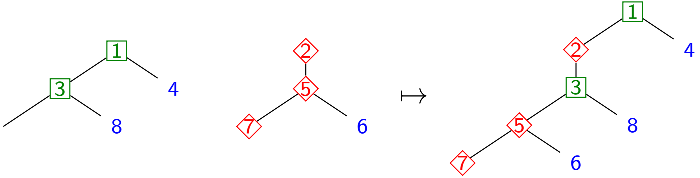

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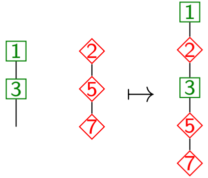

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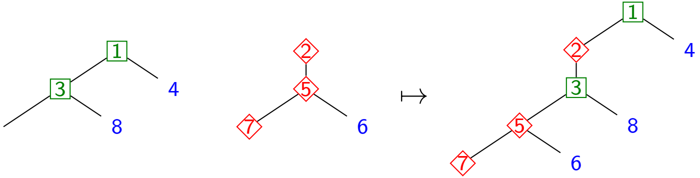
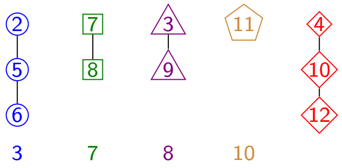


Works with additional vertices.

⚠ We must specify a chain on the second tree. We will always choose the one ending by the vertex with maximum label.

The bijection on an example



Start with the set of chains above with anchor points.

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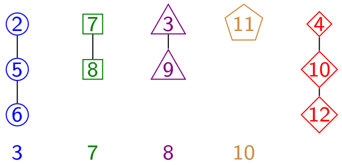

Start with the set of chains above with anchor points.

Step 0: we add a root labeled 1 with a free edge to the list.

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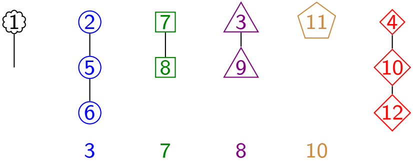

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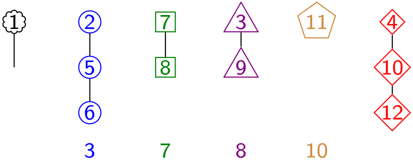

We will splice successively the chains together.

First step: we add a free edge to 3 and splice 2, 5, 6 with 3, 9 (*external splice*).

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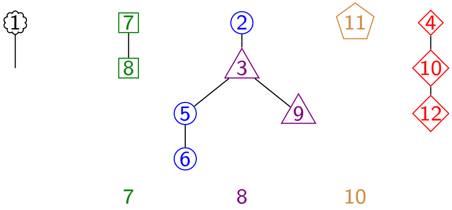

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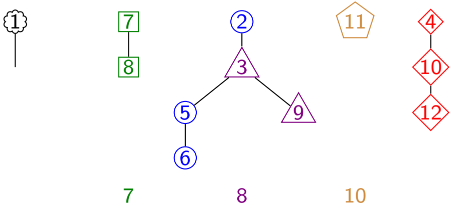

Second step: 7 is in the component we must splice. Thus, we splice 7, 8 on the free edge and add a free edge to 7 (*internal splice*).

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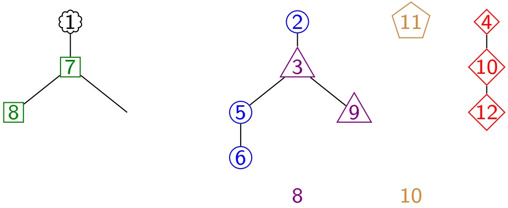

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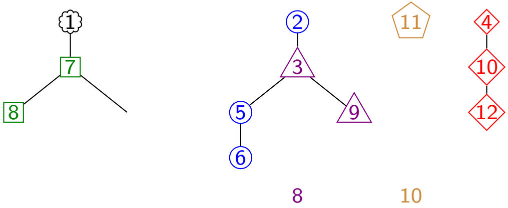
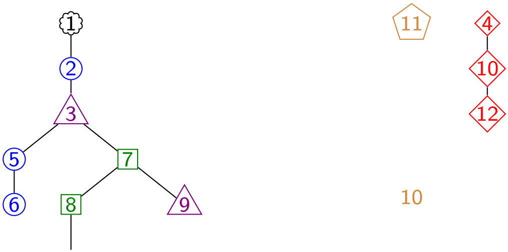


Third step: 8 is in the root component ⇒ again an internal splice. We splice the tree $2, 3, 5, 6, 9$ onto the free edge and add a free edge to 8.

The bijection on an example



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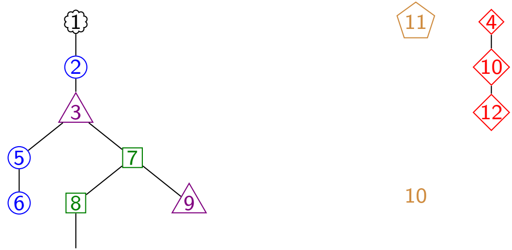
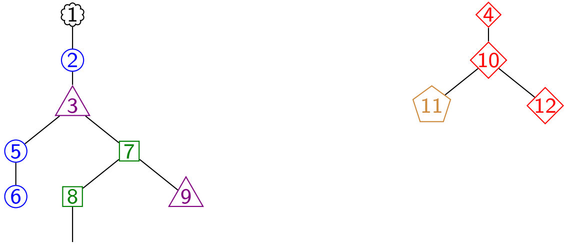


Fourth step: an external splice. We add a free edge to 10 and splice 11 onto it.

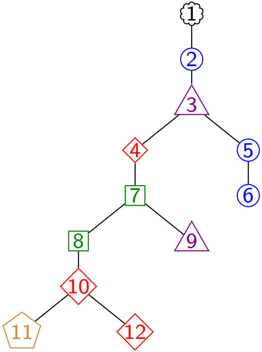
The bijection on an example



The bijection on an example

Last step: we splice the tree containing the maximum onto the free edge.

The bijection on an example



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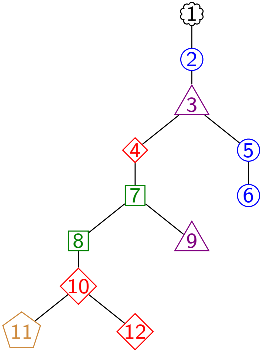

Here is the resulting partitioned tree.
The degree condition is fulfilled by construction.

Summary and conclusion

Construction by successive splicings:

- if the anchor point is in the component itself or in the root component, we splice onto the free edge and add an edge to the anchor point (internal splicing).
- if the anchor point is in another component, we add a free edge to the anchor point and splice the tree on this free edge (external splicing).

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The described procedure defines a bijection.

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Corollary (FGL, 2013)

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