Dual combinatorics of zonal polynomials

Valentin Féray (LaBRI, CNRS, Bordeaux)
Piotr Śniady (University of Wroclaw)

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  monomial symmetric functions, power sums, Schur functions.
- Another basis:
  zonal polynomials, analogue of Schur functions.
- Main result:
  a simple combinatorial formula for zonal polynomials in terms of power-sums.

Partitions

Definition

An integer partition λ of n (denoted $\lambda \vdash n$) is a non-increasing sequence of non-negative integers of sum n.

Example: $\lambda = (2,2,1) \vdash 5$.
length $\ell(\lambda)$: number of non-zeros entries.

Two different orders on partitions of n:

- lexicographic order $\leq_{\text{lex}}$;
- dominance order:

$$\lambda \leq_{\text{dom}} \mu \Leftrightarrow \forall\, i, \lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i.$$

Note: $\leq_{\text{lex}}$ is a total order refining $\leq_{\text{dom}}$.

Symmetric functions

Λ: ring of symmetric functions.

Augmented monomial symmetric functions:

$$\widetilde{M}_\lambda = \sum_{\substack{i_1,\dots,i_r \\ \text{pairwise distinct}}} x_{i_1}^{\lambda_1} \dots x_{i_r}^{\lambda_r}.$$

Example: for $j \geq k$,

$$\widetilde{M}_{(j,k)}(x_1, x_2, x_3) = x_1^j x_2^k + x_1^j x_3^k + x_2^j x_1^k + x_2^j x_3^k + x_3^j x_1^k + x_3^j x_2^k.$$

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Power sums: for $k \geq 1$, set

$$p_k(x_1,\dots,x_n) = x_1^k + x_2^k + \dots + x_n^k.$$

It is an *algebraic basis*, *i.e.* $p_\mu = \prod_i p_{\mu_i}$ is a linear basis.

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It is an *algebraic basis*, *i.e.* $p_\mu = \prod_i p_{\mu_i}$ is a linear basis.

Hall scalar product:

$$\langle p_\mu, p_\nu \rangle := \delta_{\mu,\nu} z_\mu,$$

where $z_\mu = \prod_i i^{m_i(\mu)} m_i(\mu)!$.

Schur functions

Another linear basis: $(h_\lambda s_\lambda)$ defined by

Remarks:

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- $h_\lambda = \frac{n!}{\dim(V_\lambda)}$.

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orthogonality $\langle s_\lambda, s_\mu \rangle = 0$ whenever $\lambda \neq \mu$

triangularity If $s_\lambda = \sum_\mu c^\lambda_\mu \widetilde{M}_\mu$, then $c^\lambda_\mu = 0$ for $\mu \not\leq_{\text{dom}} \lambda$.

normalization $[p_{1^n}] h_\lambda s_\lambda = 1$.

Unicity: Gram-Schmidt orthogonalization process with $\leq_{\text{lex}}$.

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Young symmetrizer formula

Fix a filling T of the Young diagram λ.

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$$h_\lambda s_\lambda = \sum_{\sigma} \sum_{\tau} \qquad ,$$

where σ (resp. τ) is a permutation preserving the rows (resp. columns) of T.

$\tau \backslash \sigma$	Id	(1 2)	(1 3)	(2 3)	(1 2 3)	(1 3 2)
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$$h_\lambda s_\lambda = \sum_{\sigma} \sum_{\tau} \varepsilon(\tau) p_{\text{cycle-type}(\sigma\tau^{-1})},$$

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Table of $\sigma\tau^{-1}$:

$\tau \backslash \sigma$	Id	(1 2)	(1 3)	(2 3)	(1 2 3)	(1 3 2)
Id , $\varepsilon = 1$	Id	(1 2)	(1 3)	(2 3)	(1 2 3)	(1 3 2)
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Id , $\varepsilon = 1$	p_{1^4}	p_{211}	p_{211}	p_{211}	p_{31}	p_{31}
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Finally,

$$h_{31} s_{31} = p_{1^4} + 2p_{211} - p_{22} - 2p_4$$

Definition of Jack polynomials

Deformed scalar product:

$$\langle p_\mu, p_\nu \rangle_\alpha = \delta_{\mu,\nu} z_\mu \alpha^{\ell(\mu)}.$$

$J_\lambda^{(\alpha)}$: one-parameter deformation of $(h_\lambda s_\lambda)$ defined by

orthogonality $\langle J_\lambda^{(\alpha)}, J_\mu^{(\alpha)} \rangle_\alpha = 0$ whenever $\lambda \neq \mu$.

triangularity If $J_\lambda^{(\alpha)} = \sum_\mu c_\mu^\lambda \widetilde{M}_\mu$, then $c_\mu^\lambda = 0$ for $\mu \not\leq_{\text{dom}} \lambda$.

normalization $[p_{1^n}] J_\lambda^{(\alpha)} = 1$.

Rk: $J_\lambda^{(1)} = h_\lambda s_\lambda$.
$\alpha = 2$: zonal polynomials (they have a representation-theoretical meaning)

An extension of Young symmetrizer formula?

Problem (Hanlon, 1988)

Find an α deformation of Young symmetrizer formula.
More precisely, find a statistics $f(\sigma, \tau)$ such that

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Our result: for $\alpha = 2$, it is easier to change the sum index and consider pairings instead of permutations!

Pairings

Definition

A pairing of $[2n] = \{1, 2, \dots, 2n\}$ is a partition of the set $[2n]$ into pairs.

Example: $S_0 = \big\{\{1,2\}, \{3,4\}, \dots, \{2n-1, 2n\}\big\}$.
Short notation: $12|34|\dots$.

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We consider

$S_1 = 13|24|56$;
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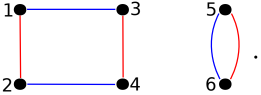
We associate the graph



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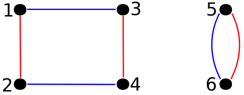

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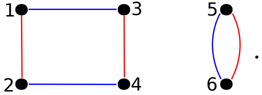

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Sign: $\varepsilon(S_1, S_2) := (-1)^{n - \#\text{cycles}}$.

The main theorem

Fix a filling T of the Young diagram $\mathbf{2}\lambda$.

Example : $\lambda = 21$, $T = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 \\ \cline{1-2}\end{array}$, $S(T) = 12|34|56$.

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$$J_\lambda^{(2)} = \sum_{S_1} \sum_{S_2} \qquad ,$$

- S_1 is a pairing preserving the rows of T.
- S_2 is a pairing associating elements of the $2i+1$-th column of T with elements of the $2i+2$-th column.

$S_2 \backslash S_1$	12|34|56	13|24|56	14|23|56
12|34|56,			
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Finally,

$$J_{21}^{(2)} = p_{1^3} + p_{21} - 2p_3$$

Outline of the proof

Set $Y_\lambda = \sum_{S_1} \sum_{S_2} \varepsilon(S(T), S_2) p_{\mathrm{type}(S_1, S_2)}$. One has to prove

1. Triangularity : if $Y_\lambda = \sum_\mu c_\mu^\lambda \widetilde{M}_\mu$, then $c_\mu^\lambda = 0$ whenever $\mu \not\leq_{\mathrm{lex}} \lambda$.
2. Orthogonality : $\langle Y_\lambda, Y_\mu \rangle = 0$ if $\lambda \neq \mu$.
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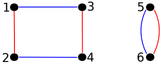
Rk: our original proof used representation theory.

Proof of triangularity (1/2)

2 pairings	$\mapsto$	graph	$\mapsto$	set-partition	$\mapsto$	partition
$S_1 = 13\|24\|56.$ $S_2 = 12\|34\|56;$	$\mapsto$		$\mapsto$	$\Pi_{S_1,S_2} =$ $1234\|56$	$\overset{\text{type}}{\mapsto}$	$(2,1)$

Recall that $Y_\lambda = \sum_{S_1} \sum_{S_2} \varepsilon(S(T), S_2) p_{\text{type}(\Pi_{S_1,S_2})}$.

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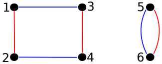
$S_1 = 13\vert 24\vert 56$. $S_2 = 12\vert 34\vert 56$;	$\mapsto$		$\mapsto$	$\Pi_{S_1,S_2} = 1234\vert 56$	$\overset{\text{type}}{\mapsto}$	$(2,1)$

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In terms of $\widetilde{M}$,

$$Y_\lambda = \sum_{S_1} \sum_{S_2} \varepsilon(S(T), S_2) \left(\sum_{\substack{\Pi_{S_1,S_2} \text{ finer} \\ \text{than } \Pi}} \widetilde{M}_{\text{type}(\Pi)} \right)$$

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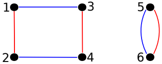

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$$= \sum_{\substack{\Pi \text{ even} \\ \text{set-partition}}} \widetilde{M}_{\text{type}(\Pi)} \left(\sum_{\substack{S_1, S_2 \\ \Pi_{S_1,S_2} \leq \Pi}} \varepsilon(S(T), S_2) \right)$$

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Idea. Find i and j (depending on Π but not on S_1, S_2!) such that, for any S_1 and S_2:

$$\varepsilon(S(T), S_2) + \varepsilon(S(T), S_2^{(i,j)}) = 0;$$

$$\Pi_{S_1,S_2} \leq \Pi \Leftrightarrow \Pi_{S_1,S_2^{(i,j)}} \leq \Pi;$$

$$S_2 \text{ fulfills the column condition} \Leftrightarrow S_2^{(i,j)} \text{ fulfills the column condition},$$

where $S_2^{(i,j)}$ is obtained from S_2 by exchanging i and j.

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Fact. it is enough to choose i and j such that:

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Fact. it is enough to choose i and j such that:

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Lemma: They always exist if $\mathrm{type}(\Pi) \not\leq_{\mathrm{dom}} \lambda$. □

Conclusion

Let $\mu \vdash k$ and $\lambda \vdash n$ with $k \leq n$.

With this formula, one can write

$$[p_{\mu 1^{n-k}}] J_\lambda^{(2)}$$

as a sum whose index set depends on k and not on n.

Conclusion

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In particular, we can prove some recent conjectures of M. Lassalle on $[p_{\mu 1^{n-k}}] J_\lambda^{(\alpha)}$ in the case $\alpha = 2$.

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Let $\mu \vdash k$ and $\lambda \vdash n$ with $k \leq n$.

With this formula, one can write

$$[p_{\mu 1^{n-k}}] J_\lambda^{(2)}$$

as a sum whose index set depends on k and not on n.

In particular, we can prove some recent conjectures of M. Lassalle on $[p_{\mu 1^{n-k}}] J_\lambda^{(\alpha)}$ in the case $\alpha = 2$.

Still work to do: is there an extension for general α?

Thanks for listening!