A combinatorial algebra of functions on Young diagrams

Valentin Féray

Laboratoire Bordelais de Recherche en Informatique
CNRS

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One word about combinatorics team in Bordeaux

Centers of interest:

- Bijective, enumerative and algebraic combinatorics of maps (Bonichon, Bousquet-Mélou, Cori, F., Marcus, Zvonkine).
- Combinatorics of alternating sign matrices and tableaux (Aval, Duchon, Guibert, Viennot).
- Symmetric functions and generalization (Aval, F.).
- Generating series and random sampling (Bousquet-Mélou, Duchon, Marckert).

Irreducible character values of symmetric group

- Irreducible representations of $S_n \simeq$ Young diagrams $\lambda \vdash n$.
- We are interested in normalized character values:

$$\chi^\lambda(\sigma) = \frac{\mathrm{tr}\left(\rho^\lambda(\sigma)\right)}{\dim(V_\lambda)}.$$

- We will look at it as a function

$$\lambda \mapsto \chi^\lambda(\sigma).$$

- Motivations:
  - Shape of large random Young diagrams;
  - Convergence rate of some process, complexity of algorithms.

Outline of the talk

1. Introduction
   - Shifted symmetric functions
   - Why do we need a bigger algebra?
2. Algebra of quasi-symmetric functions on Young diagrams
   - Functions indexed by graphs
   - Linear basis and relations
   - A combinatorial invariant
3. Application: combinatorics of Kerov's polynomials
4. To go further

Kerov's and Olshanski's approach

Let us define

$$\mathrm{Ch}_\mu : \begin{array}{ccc} \mathcal{Y} & \to & \mathbb{Q}; \\ \lambda & \mapsto & n(n-1)\dots(n-k+1)\chi^\lambda(\sigma), \end{array}$$

where $n = |\lambda|$, $k = |\mu|$
and σ is a permutation in S_n of cycle type $\mu 1^{n-k}$.

Examples:

$$\mathrm{Ch}_\mu(\lambda) = 0 \quad \text{as soon as } |\lambda| < |\mu|$$

$$\mathrm{Ch}_{1^k}(\lambda) = n(n-1)\dots(n-k+1) \quad \text{for any } \lambda \vdash n$$

$$\mathrm{Ch}_{(2)}(\lambda) = n(n-1)\chi^\lambda((1\ 2)) = \sum_i (\lambda_i)^2 - (\lambda_i')^2$$

$$\mathrm{Ch}_{\mu \cup 1}(\lambda) = (n - |\mu|)\,\mathrm{Ch}_\mu(\lambda) \quad \text{for any } \lambda \vdash n$$

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Proposition

The functions Ch_μ, when μ runs over all partitions, are linearly independent. Moreover, they span a subalgebra $\Lambda^\star$ of functions on Young diagrams.

Example: $\mathrm{Ch}_{(2)} \cdot \mathrm{Ch}_{(2)} = 4 \cdot \mathrm{Ch}_{(3)} + \mathrm{Ch}_{(2,2)} + 2\,\mathrm{Ch}_{(1,1)}$.

A formula for character values

Theorem (F. 2006, conjectured by Stanley)

Let $\mu \vdash k$.

$$\mathrm{Ch}_\mu = \sum_M \pm N_{G(M)},$$

where:

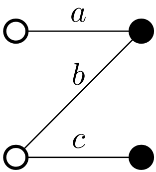
- *the sum runs over rooted bipartite maps with k edges and face-length $\mu_1, \mu_2, \dots$*
- *$G(M)$ is the underlying graph of M.*
- *N_G is a function on Young diagrams which will be defined later.*

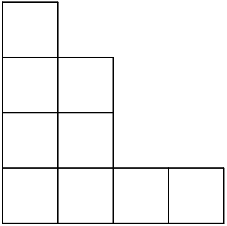
In general, $N_G \notin \Lambda^\star$.

$\longrightarrow$ We have to work in the bigger algebra $\mathcal{Q} := \mathrm{Vect}(N_G)$.

Definition of the N_G

Let G be a bipartite graph and λ a partition :





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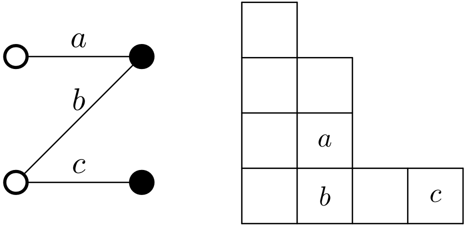

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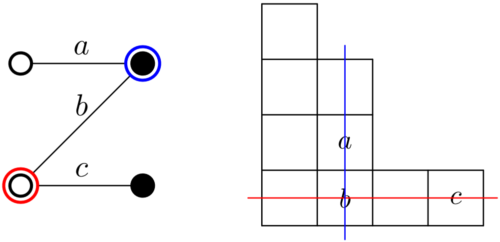

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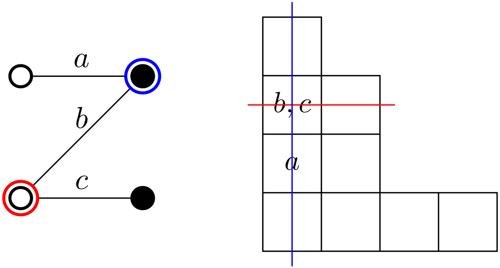

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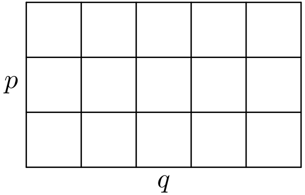
An interesting particular case: rectangular partition

$\lambda =$ 

$$N_G(\lambda) = p^{|V_\circ(G)|} \cdot q^{|V_\bullet(G)|}$$

Indeed, one has to choose independently:

- one row per white vertex ;
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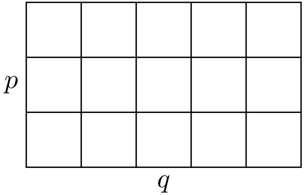
$\lambda =$ 

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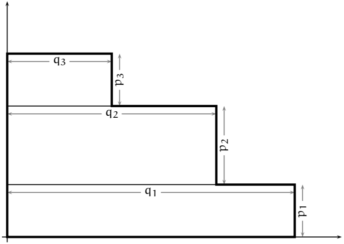
In this case,

$$\mathrm{Ch}_{mu}\Big(\underbrace{q,\dots,q}_{p \text{ times}}\Big) = \sum_M \pm p^{|V_\circ(M)|} \cdot q^{|V_\bullet(M)|} \qquad \text{(Stanley, 2003)}$$

Stanley's coordinates

$\lambda =$ 

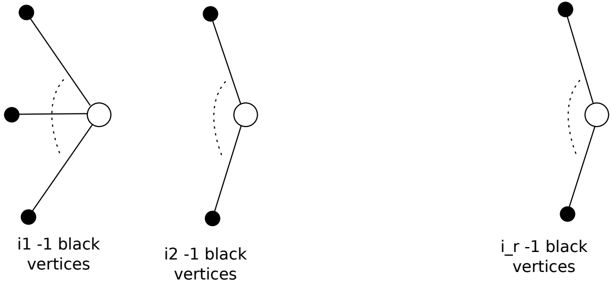
$$N_G(\lambda) = \sum_{\varphi : V_\circ(G) \to \mathbb{N}^\star} \prod_{\circ \in V_\circ} p_{\varphi(\circ)} \prod_{\bullet \in V_\bullet} q_{\psi(\bullet)},$$

where $\psi(\bullet) = \max_{\circ \text{ neighbourg of } \bullet} \varphi(\circ)$.

The graphs G_I

Definition

Let $I = (i_1, i_2, \dots, i_r)$ be a composition. Define G_I as the following bipartite graph:



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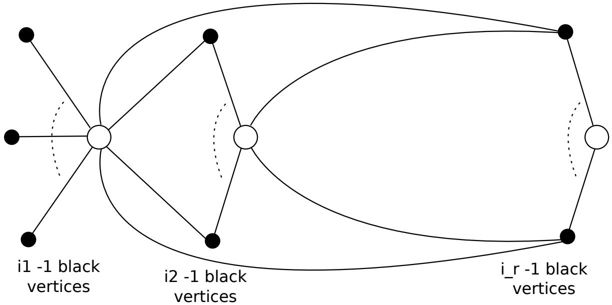
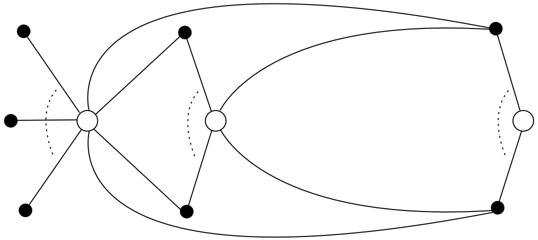


Proposition

The N_{G_I}'s are linearly independent when I runs over all compositions.

The N_{G_I}'s are linearly independent: proof

$G_I =$



Consider $N_{G_I}(p_1, p_2, \dots, p_r, q_1, q_2, \dots, q_r)$ (we truncate the alphabets)
As total degree in p is r, monomials without powers of p are:

$$M_J = p_1 q_1^{j_1 - 1} p_2 q_2^{j_2 - 1} \cdots p_r q_r^{j_r - 1},$$

where J is a composition of n (total number of vertices)

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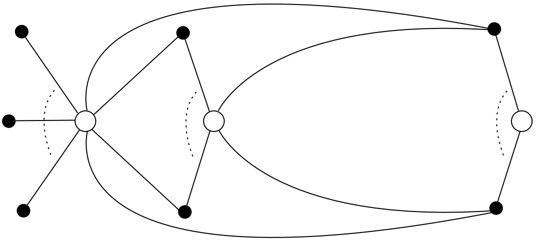


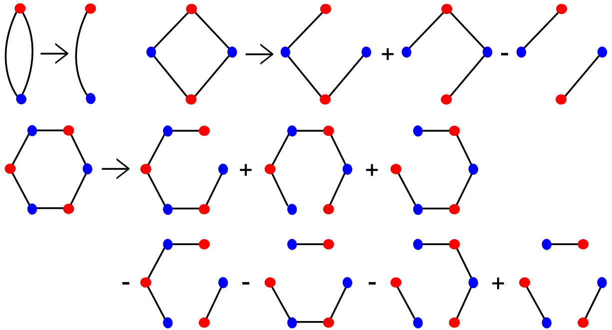
$$N_{G_I} = c_I M_I + \sum_{\substack{|J|=|I|=n,\ \ell(J)=\ell(I)=r \\ J \geq I}} c_{I,J} M_J$$

$+$ non-p-square-free terms.

$\geq$ stands for the right-dominance order. □

The *poinçonnage* relation

Select a cycle in a bipartite graph. Let us consider the *local* operation:



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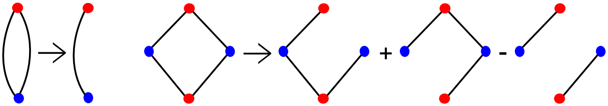
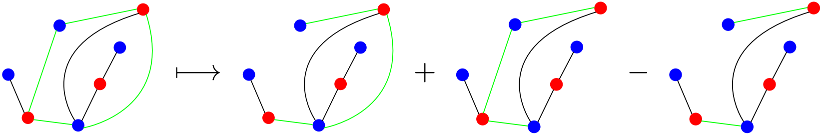


Example:



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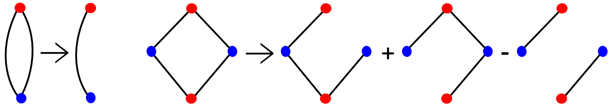

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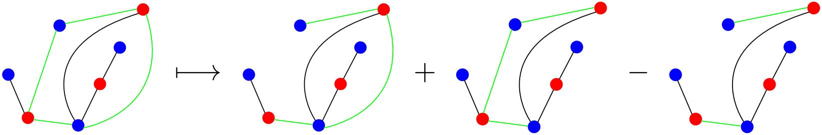

Proposition

N is invariant by this transformation.

Linear basis and relation of Vect(N_G)

Theorem

- *The N_{G_I} span the whole space $\mathcal{Q} = \mathrm{Vect}(N_G)$.*
- *All relations can be deduced from the poinçonnage relation.*

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Sketch of proof.

If $G \neq G_I$ for all I, one has, using *poinçonnage*:

$$N_G = \sum \pm N_{G'},$$

where the sum runs over graphs G' with strictly more edges. □

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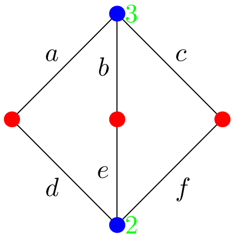
Remark:

- $\mathcal{Q}$ is isomorphic with quasi symmetric functions.
- Combinatorial description of the coproduct.

q-admissible graphs



Bipartite G endowed with $q : V_{\bullet} \to \mathbb{N}^{\star}$

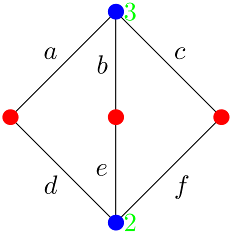
associated system $\longrightarrow$

$$(S_G^q) \begin{cases} x_a + x_b + x_c = 3 - 1 \\ x_d + x_e + x_f = 2 - 1 \\ x_a + x_d = 1 \\ x_b + x_e = 1 \\ x_c + x_f = 1 \end{cases}$$

Definition

G is said q-admissible
if the associated system has a solution with $x_i > 0$.

q-admissible graphs



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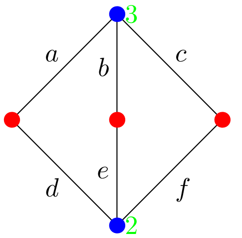
Proposition

G is q-admissible

$\Leftrightarrow \forall A \subset V_\bullet$ *non trivial*, $|\text{Neighbours}(A)| > \sum_{v \in A}(q_v - 1)$

q-admissible graphs



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Theorem (Dołęga, F., Śniady, 2009)

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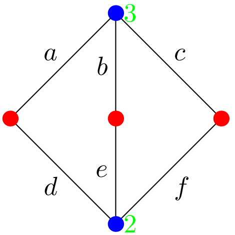

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Non trivial proof, using Euler's characteristic

Consequence in $\mathcal{Q}$

Let π be a partition. Define

$$F_\pi(G) = (-1)^{\# c.c.} \left| \left\{ q : V_\bullet(G) \to \mathbb{N}^\star \text{ s.t. } \begin{array}{l} \mathrm{Im}(q) = \{\pi_i\} \text{ (as multisets)} \\ G \ q\text{-admissible} \end{array} \right\} \right|$$

Then, if $X = \sum c_G N_G \in \mathcal{Q}$,

$$F_\pi(X) = \sum c_G F_\pi(G)$$

is well-defined.

Free cumulants

Definition (Free cumulants)

$$R_k = \sum_T \pm N_T,$$

where the sum runs over bipartite rooted planar tree with k vertices.

- can be defined more directly using the shape of the diagram.
- $R_2, R_3, \dots$ form an algebraic basis of $\Lambda^\star$.

Therefore $\mathrm{Ch}_\mu = K_\mu(R_2, R_3, \dots)$,
where K_μ is a polynomial (Kerov's polynomial).

Question (Kerov, 2000)

Combinatorics of K_μ?

Coefficients of Kerov's polynomials

If τ is a partition, denote $R_\tau = \prod_i R_{\tau_i}$.

Easy to check that $F_\pi(R_\tau) = (-1)^{\ell(\pi)} \delta_{\tau,\pi}$.

Therefore

$$\begin{aligned} F_\pi(\mathrm{Ch}_\mu) &= (-1)^\pi [R_\pi] K_\mu \\ &= \sum \pm F_\pi(G), \end{aligned}$$

where the sum runs over bipartite rooted maps whose faces have length $\mu_1, \mu_2, \dots$

Theorem (Dołęga, F., Śniady, 2009)

$[R_\pi] K_\mu$ counts signed maps of face-type μ with conditions on the number of neighbours of subsets of black vertices.

Extension to Jack polynomials

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- They belong to $\Lambda^\star$.
- Combinatorics like in $\alpha = 1$ case?

Maps on locally oriented surfaces

Case $\alpha = 2$ (zonal polynomials):

Theorem (F., Śniady 2010)

Let $\mu \vdash k$.

$$\mathrm{Ch}^{(2)}_\mu = \sum_M \pm N_{G(M)},$$

where the sum runs over rooted bipartite maps on ***locally oriented surfaces*** *with k edges and face-length $\mu_1, \mu_2, \dots$*

$\Longrightarrow$ combinatorial description in terms of the R_ℓ's.

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$\Longrightarrow$ combinatorial description in terms of the R_ℓ's.

Conjecture for general $\alpha = 1 + \beta$:
Maps are counted with a weight depending on β (like in Matching-Jack's conjecture).

End of the talk

Thanks for listening

Do you have any questions?