Mixed augmented variational formulation (MAVF) for lower hybrid full-wave calculations

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Abstract. In the continuation of the works led in cylindrical geometry [2], a full toroidal description for an arbitrary poloidal cross-section of the plasma has been developed. For simulation purpose a mixed augmented variational formulation (MAVF), which is particularly well suited for solving Maxwell equations, is considered [4]. The discretization of the MAVF is carried out using Taylor-Hood P2-iso-P1 finite elements. This formulation provide a natural implementation for parallel processing, a particularly important aspect when simulations for plasmas of large size must be considered. Details on the specific application of the MAVF to the LH problem are presented, as well as the structure of the corresponding matrices. A first application to a realistic small tokamak configuration is considered.

Keywords: Full wave, Lower hybrid, Finite element

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INTRODUCTION

The propagation and the absorption of the lower hybrid electromagnetic wave is a powerful method to generate current drive by Landau wave-particle resonance in tokamaks [5]. However, since its wavelength $\lambda$ at the LH frequency is very small a compared to the machine size $R$, a conventional full wave description represents a considerable numerical effort [1]. Therefore, the problem is addressed by an appropriate mathematical finite element technique, which incorporates naturally parallel processing capabilities. It is based on a mixed augmented variational (weak) formulation taking account of the divergence constraint and essential boundary conditions, which provides an original and efficient scheme to describe in a global manner both propagation and absorption of electromagnetic waves in plasmas. With such a description, usual limitations of the conventional ray tracing related to the approximation $\lambda << \phi_B << R$, where $\phi_B$ is the size of the beam transverse to the rf power flow direction, may be overcome. Since conditions are corresponding to $\lambda << \phi_B \sim R$, the code under development may be considered as a WKB full wave, dielectric properties being local [3].

ELECTROMAGNETIC WAVES: MODEL PROBLEM

A time-harmonic case is considered so that time-dependence is of the form $\exp(-i \omega t)$, for an excited wave frequency $\omega > 0$. Then, Maxwell equations becomes a second order
partial differential equation on the the electric field \( E \).

\[
\begin{align*}
\text{curl curl} E - k^2 K_r E &= \overrightarrow{0} \quad \text{in } \Omega, \quad (1) \\
\text{div}(K_r E) &= 0 \quad \text{in } \Omega, \quad (2)
\end{align*}
\]

where

\[ K_r = \varepsilon_r + \frac{i}{\varepsilon_0 \omega} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \gamma_e \end{pmatrix} \]

Here, \( k := \omega/c > 0 \) where \( c \) is the speed of light. The relative dielectric permittivity tensor is denoted \( \varepsilon_r, \gamma_e \) is the linear absorption and \( \omega_L, \omega_c \) the lower hybrid frequency. Cold plasma approximation of \( \varepsilon_r \) is used, which turns out to be valid in the hybrid regime \( \omega_i \ll \omega \ll \omega_c \), see [6], [7]. Let \( \Gamma \) be the boundary of the domain \( \Omega \) which represents the toroidal plasma volume. If the edge of the antenna is denoted \( \Gamma_a \), then the boundary condition is:

\[ \text{curl} E \times n = i \omega \mu_0 J_s \quad \text{on } \Gamma_a \quad (3) \]

where \( J_s \) is a current source equivalent to the electric field excited at the antenna. On the other part of the boundary \( \Gamma_c = \Gamma \setminus \Gamma_a \),

\[ E \times n = \overrightarrow{0} \quad (4) \]

and the solution satisfies \( (K_r E) \cdot n = 0 \) on the whole boundary \( \Gamma \).

**Variational formulation**

The variational scheme presented in this paper is based upon the use of square integrable trial functions with square integrable curl and div in \( \Omega \). This function space is denoted \( X(\Omega) \).

In order to take account of the divergence condition (2) as a constraint, Lagrangian multiplier \( p \) is introduce belonging to \( L^2(\Omega) \), the set of square integrable functions defined in \( \Omega \). Using the same idea, the boundary condition (4) is considered as an additional constraint and two auxiliary Lagrangians multipliers \( \lambda \) and \( \lambda_n \) square integrable functions in \( \Gamma_c \) are also introduced. In addition two penalization terms are added, one for each constraint, and corresponding penalization parameters \( \alpha \) and \( \beta \) must be introduced. It results the following mixed augmented variational formulation (MAVF), see [7], [8] :

Find \((E, p, \lambda, \lambda_n)\) such that

\[
\begin{align*}
\alpha, \beta (E, F) + b(F, p) + c(v, \lambda) &= L_{\alpha, \beta}(F) \quad (5) \\
b(E, q) &= 0 \quad (6) \\
c(u, \mu) + d(\mu, \lambda_n) &= (g, \mu \times n)_{\Gamma_c}, \quad (7) \\
d(\lambda, \mu) &= 0, \quad (8)
\end{align*}
\]
where

\[
\begin{align*}
A_{\alpha,\beta}(E, F) & := (\text{curl}E, \text{curl}F)_{\Omega} - k^2(K, E, F)_{\Omega} + \\
& \quad + \alpha(\text{div}(K, E), \text{div}(K, F))_{\Omega} + \beta(E \times n, F \times n)_{\Gamma_c},
\end{align*}
\]

\[
b(F, p) := (\text{div}(K, F), p)_{\Omega},
\]

\[
c(E, \mu) := (E \times n, \mu \times n)_{\Gamma_c},
\]

\[
L_{\alpha,\beta}(F) := i\omega \mu_0 (J_s, F)_{\Gamma_a} + \beta(g, F \times n)_{\Gamma_c},
\]

\[
d(\lambda, \mu) := (\lambda \cdot n, \mu)_{\Gamma_c},
\]

with parameters \(\alpha, \beta \in \mathbb{C}\). Here \((\cdot, \cdot)_{\Omega}\) and \((\cdot, \cdot)_{\Gamma}\) denote the standard inner products in \(L^2(\Omega)\) and \(L^2(\Gamma)\), respectively. The coercivity of the sesquilinear form \(a_{\alpha,\beta}\) has been examined in [9].

The domain \(\Omega\) is described by usual toroidal coordinates \((R, Z, \phi)\), which are related to the cartesian ones \((x, y, z)\) by \(x = R \cos \phi, y = R \sin \pi, z = -Z\). The problem is reduced to a two dimensional one by assuming that all function \(f(R, Z, \phi)\) may be developed as Fourier series in \(\phi\) where the coefficients \(f_{\nu}(R, Z)\) are defined on a meridian of \(\Omega\), \(\bar{\Omega} := \{(R, Z) \in \mathbb{R}^2 | (R, Z, \pi/2) \in \Omega\}\). Therefore, a 2-D problem is obtained for each triple of Fourier coefficient \((E_{\nu}, p_{\nu}, \lambda_{\nu})\), \(\nu \in \mathbb{Z}\) [10]. This problem is of the form (5)-(8), with \(\Omega\) replaced by \(\bar{\Omega}\).

**Finite Elements**

In \(\bar{\Omega}\) we consider a triangular mesh \(\mathcal{T}_{2h}\) of diameter \(2h\). The number of \(\mathcal{T}_{2h}\) nodes is denoted \(N_{2h}\). The Lagrangian multiplier \(p_{\nu}\) is approximated by piecewise linear and continuous finite element on a mesh \(\mathcal{T}_h\). Then we look for a Lagrangian multiplier \(p_{\nu}^{2h}\) in a finite dimensional space \(W^{2h}\) included in \(L^2(\Omega)\). Let \(\mathcal{T}_h\) be triangular mesh of \(\Omega\) of diameter \(h\) obtained by standard subdivision of \(\mathcal{T}_{2h}\). The number of \(\mathcal{T}_h\) nodes is denoted \(N_h\). The vector field \(E_{\nu}\) is approximated by piecewise linear and continuous finite elements on a mesh \(\mathcal{T}_h\). Then the computed solution belongs to \(V^h\) a finite dimensional subspace of \(X(\Omega)\) with \(\text{dim}V^h = 3N_h\). This approximation scheme are known as the Taylor-Hood \(P_2\)-iso-\(P_1\) finite element method to discretize the MAVF (5)-(8).

Similary \(\lambda_{\nu}\) and \(\lambda_{\nu,n}\) shall also be discretized with piecewise linear and continuous finite elements on \(\Gamma_c\). We denote the corresponding finite dimensional vector space by \(U^h\) and \(U^h\).

From (5)-(8) we obtain a system of linear equations for the coefficients, which is of the form

\[
\begin{pmatrix}
A_{\alpha,\beta} & B^H & C^H & 0 \\
B & 0 & 0 & 0 \\
C & 0 & 0 & X^H \\
0 & 0 & X & 0
\end{pmatrix}
\begin{pmatrix}
E^h_v \\
p_{\nu}^{2h} \\
\lambda^h_v \\
\lambda_{\nu,n}^h
\end{pmatrix}
= \begin{pmatrix}
L_{\alpha,\beta} & 0 \\
0 & g \\
0 & 0
\end{pmatrix},
\]

where \(E^h_v, p_{\nu}^{2h}, \lambda^h_v\) and \(\lambda_{\nu,n}^h\) are a approximation of the nodal values of \(E_{\nu}, p_{\nu}, \lambda_{\nu}\) and \(\lambda_{\nu,n}\). The linear system (9) can be written in the general form \(Ax = b\). We note that
the stiffness matrix $A$ is complex-valued and not-Hermitian. Therefore, the number of admissible solvers is rather limited. For this special sparse linear system, we have used the iterative methods GMRES (Generalized minimal residual methods, see [12]).

There were test cases where the prescribed residual was not reached after the prescribed maximal number of iterations. The more Work are underway to better improve the convergence of the method.

**Practical values**

- $n_e = 50 \times 10^{19} m^{-3}$: Electron density.
- $T_e = 1 kev$: Electron temperature.
- $m_e = 0.91 \times 10^{-30} kg$: Electron mass.
- $q_e = 1.6 \times 10^{-19} c$: Electron charge.
- $n_i = n_e$: Ion density of specie.
- $T_i = T_e$: Ion temperature.
- $Z_i = 1$: Ion charge.
- $m_i = 1.67 \times 10^{-27} kg$: Ion mass.
- $a = 0.12 m$: Torus minor radius.
- $R_0 = 0.38 m$: Torus major radius.
- $\varepsilon_0 = 8.85 \times 10^{-12} F m^{-1}$: Permittivity of free space.
- $\mu_0 = 4 \times 10^{-7} \pi H m^{-1}$: Permeability of free space.
- $c = 3 \times 10^8$: Speed of light in free space.

- $f_{LH} = 3.7 \times 10^9 Hz$: Lower hybrid frequency.
- $B = 0.5 T$: Intensity of the magnetic field.
- $\omega_{LH} = 2\pi f_{LH}$: Temporal frequency.
- $\omega_{pe} = 56.4 \sqrt{n_e}$: Electron plasma frequency.
- $\Omega_{ce} = 0.176 \times 10^{12} B$: Electron cyclotron frequency.
- $\omega_{pi} = 1.6 \times 10^{-19} \sqrt{\frac{n_i}{m_i \varepsilon_0}}$: Ion plasma frequency.
- $\Omega_{ci} = 1.6 \times 10^{-19} \frac{Z_i m_i}{m_i} B$: Ion cyclotron frequency.
- $v_{the} = c \sqrt{\frac{T_e}{511}}$: Thermal speed.
- $k_|| = \omega \frac{n_e}{\varepsilon_0}$: A logarithm of a plasma parameter.
- $\ln(\Lambda(r)) = 15.2 - \frac{1}{4} \ln(\frac{n_e}{10^{27}}) + \ln(T_e)$: Collision frequency.
\[
\begin{align*}
S(r) &= 1 - \frac{\alpha(r)}{\omega} \left[ \frac{\omega^2_{pe}(r)}{\alpha(r)^2 - \omega^2_{ce}(r)} + \sum_{ions} \frac{\omega^2_{pi}(r)}{\alpha(r)^2 - \omega^2_{ci}(r)} \right], \\
D(r) &= \frac{1}{\omega} \left[ - \frac{\omega_{ce}(r) \omega^2_{pe}(r)}{\alpha(r)^2 - \omega^2_{ce}(r)} + \sum_{ions} \frac{\omega_{ci}(r) \omega^2_{pi}(r)}{\alpha(r)^2 - \omega^2_{ci}(r)} \right], \\
P(r) &= 1 - \frac{1}{\omega \alpha(r)} \left[ \omega^2_{pe}(r) + \sum_{ions} \omega^2_{pi}(r) \right], \\
\gamma_c(r) &= \frac{\sqrt{\pi} \omega^2 n_e q_e^2}{\sqrt{2 m_e v_{the}^3(r) k^3(r)}} \exp \left( - \frac{\omega^2}{2 v_{the}^2(r) k^2(r)} \right),
\end{align*}
\]

and \( \mathbf{J}_e = (0, 0, 1)^t \) on the antenna \( \Gamma_a \).
Numerical results

We now give some numerical examples. In all test the parameters $\alpha$ and $\beta$ are set equal to 1.

- Case 1: In this case we suppose that $\omega = 2\omega_{LH}$.

**FIGURE 1.** The solution $E$ for $k = 50$ and $\nu = 200$.

**FIGURE 2.** Power density $P_0$ for $k = 50$ and $\nu = 200$. 
FIGURE 3. The solution $E$ for $k = \frac{\omega}{c}$ and $\nu = 450$.

FIGURE 4. Power density $P_0$ for $k = \frac{\omega}{c}$ and $\nu = 450$.

- Case 2: In this case we suppose that $\omega = \omega_{LH}$. 
FIGURE 5. The solution $E$ for $k = \frac{\omega}{c}$ and $\nu = 450$.

FIGURE 6. Power density $P_0$ for $k = \frac{\omega}{c}$ and $\nu = 450$.

**Code development status**

Actually the code is written for a single domain (MatLab). Nevertheless, realistic simulations can be carried out for plasma of very small size, in order to validate the concept. Once this phase achieved, the code should be rewritten in C/FORTRAN for parallel processing implementation, thus studying influence of numerical scheme on the solution of the Maxwell equation, knowing that the physical problem is correctly
described. This next step will allow us to consider fully realistic tokamak regimes, and in particular ITER size plasmas. While the academic and preliminary numerical tests were successfully carried out, physical benchmarking is under way. Especially, the detailed implementation of the antenna at the border of the plasma volume is performed. Ultimately, from the wave field pattern, a full quasilinear absorption may be considered, by using full wave results in a 3-D kinetic solver like LUKE code for self-consistent current-drive calculations [3].

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