GEOMETRIC THETA-LIFTING FOR UNITARY GROUPS

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Abstract. In this note we define the geometric theta-lifting functors in the global nonramified setting. They are expected to provide new cases of the geometric Langlands functoriality.

1. INTRODUCTION

1.1. In this note we define the geometric theta-lifting functors in the global nonramified setting. They are expected to provide new cases of the geometric Langlands functoriality. At level of functions the theta-lifting for unitary groups was studied by many authors ([1, 3, 4, 13, 15]).

The definition uses a splitting of the metaplectic extension over the unitary groups. The splitting existing in the litterature at the level of functions ([6, 4, 14, 15]) are not entirely satisfactory. Namely, given a local field $F$ and its degree two extension $E$, for the corresponding unitary group the splitting in loc.cit. is defined up to a multiplication by a character of $E^*/F^*$ with values in $C^1$ (the group of complex numbers of absolute value one). We give a more canonical geometric construction of this splitting.

1.2. Notations. We follow the conventions of [10]. So, we work over an algebraically closed field $k$ of characteristic $p \neq 2$. We fix a prime $\ell \neq p$, write $\bar{Q}_{\ell}$ for the algebraic closure of $Q_{\ell}$. All our stacks are defined over $k$. For an algebraic stack locally of finite type $S$ we have the derived categories $D(S), D^-(S), D^+(S)$ of complexes of $\bar{Q}_{\ell}$-sheaves on $S$ as in loc.cit.

Let $X$ be a smooth projective connected curve. Write $\Omega$ for the canonical line bundle on $X$. Let $\pi : Y \to X$ be an étale degree 2 cover with $Y$ connected. Write $\sigma$ for the nontrivial automorphism of $Y$ over $X$, $\pi_* \mathcal{O} = \mathcal{O} \oplus \mathcal{E}$, where $\mathcal{E}$ is the sheaf of $\sigma$-antiinvariants, so $\mathcal{E}^2 \cong \mathcal{O}$. Once and for all we pick a line bundles $\mathcal{E}', \Omega^1_\mathbb{Z}$ on $X$ with isomorphisms $\mathcal{E}'^2 \cong \mathcal{E}, (\Omega^1_\mathbb{Z})^2 \cong \Omega$.

For an algebraic group $G$ write $\text{Bun}_G$ for the stack of $G$-torsors on $X$. For $n \geq 1$ write $\text{Bun}_n$ (resp., $\text{Bun}_{n,Y}$) for the stack of $G$-torsors on $X$ (resp., on $Y$).

2. MODULI OF UNITARY BUNDLES AND GEOMETRIC THETA-LIFTING

2.1. Pick $\epsilon = \pm 1$. For $n \geq 1$ an $\epsilon$-hermitian vector bundle on $Y$ (with respect to $\pi$) is a datum of $V \in \text{Bun}_{n,Y}$ with a nondegenerate form $\phi : V \otimes \sigma^*V \to \mathcal{O}_Y$ such that the
2.1.1. For an $\epsilon$-hermitian vector bundle $V$ on $Y$ viewing $\phi$ as an isomorphism $\phi : \sigma^* V \to V^*$, one gets $(\sigma^* \phi)^* = \epsilon \phi$. Consider the isomorphism

$$
(\sigma^* \phi, \phi) : V \oplus \sigma^* V \to \sigma^* V^* \oplus V^*
$$

on $Y$. Let $M = \pi_* C$. Let $\sigma$ act on $V \oplus \sigma^* V$ such that the natural isomorphism $V \oplus \sigma^* V \cong \pi^* M$ is $\sigma$-invariant. Then (1) is $\sigma$-equivariant, so descends to an isomorphism $\bar{\phi} : M \to M^*$ such that $\bar{\phi}^* = \epsilon \bar{\phi}$. We also view the latter as a map $\bar{\phi} : M \otimes M \to \mathcal{O}_X$. So, $\phi$ is symmetric for $\epsilon = 1$ (resp., anti-symmetric for $\epsilon = -1$).

Consider the isomorphism

$$
(-\sigma^* \phi, \phi) : V \oplus \sigma^* V \to \sigma^* V^* \oplus V^*
$$

As above, there is a unique isomorphism $\phi' : M \to M^* \otimes \mathcal{E}$ such that $\pi^* (\phi') = (-\sigma^* \phi, \phi)$. Besides, $\phi'$ is symmetric for $\epsilon = -1$ (resp., antisymmetric for $\epsilon = 1$).

Let $\text{Bun}_\mathcal{O}_{2n}$ be the stack classifying $M \in \text{Bun}_{2n}$ with a nondegenerate symmetric form $\text{Sym}^2 M \to \mathcal{O}_X$. Let $q_n : \text{Bun}_U \to \text{Bun}_\mathcal{O}_{2n}$ be the map sending $(V, \phi)$ to $(M = \pi_* V, \bar{\phi})$. Let $q_n : \text{Bun}_U \to \text{Bun}_\mathcal{O}_{2n}$ be the map sending $(V, \phi)$ to $(M \otimes \mathcal{E}', \phi')$. Here we have viewed $\phi'$ as a map $\text{Sym}^2 (M \otimes \mathcal{E}') \to \mathcal{O}_X$.

The stack $\text{Bun}_\text{Sp}_{2n}$ classifies $M \in \text{Bun}_{2n}$ with a symplectic form $\wedge^2 M \to \mathcal{O}_X$. Let $p_n : \text{Bun}_U \to \text{Bun}_\text{Sp}_{2n}$ be the map sending $(V, \phi)$ to $(M \otimes \mathcal{E}', \phi')$. Here $M = \pi_* V$. Let also $p_n : \text{Bun}_U \to \text{Bun}_\text{Sp}_{2n}$ be the map sending $(V, \phi)$ to $(M, \bar{\phi})$.

2.1.2. We have a canonical identification $\pi^* \mathcal{E} \cong \mathcal{O}$ such that the descent data for $\pi^* \mathcal{E}$ are given by the action of $\sigma$ on $\mathcal{O}$ as $-1$. This gives an isomorphism $\delta : \text{Bun}_U \to \text{Bun}_{U^-}$ sending $(V, \phi)$ to $(V \otimes \pi^* \mathcal{E}', \phi)$. The diagram commutes

$$
\begin{array}{ccc}
\text{Bun}_U & \xrightarrow{\phi_n} & \text{Bun}_{\mathcal{O}_{2n}} \\
\downarrow q_n & & \downarrow p_n \\
\text{Bun}_{U^-} & \xrightarrow{\phi_n} & \text{Bun}_{\text{Sp}_{2n}}
\end{array}
$$

2.1.3. The stack $\text{Bun}_{U_1}$ is a group stack described in ([8], Appendix A). We have the norm map $N : \text{Bun}_{1,Y} \to \text{Bun}_1$ sending $\mathcal{L}$ to $\mathcal{L} \otimes \det(\pi_* \mathcal{L})$, this is a homomorphism of group stacks. The stack $\text{Bun}_{U_1}$ classifies $V \in \text{Bun}_{1,Y}$ together with a trivialization $N(V) \to \mathcal{O}_X$. The stack $\text{Bun}_{U_1}$ has connected components $\text{Bun}_{U_1}^a$ indexed by $a \in \mathbb{Z}/2\mathbb{Z}$, $\text{Bun}_{U_1}^0$ being the connected component of unity. The stack $\text{Bun}_{U_1}^1$ classifies $V \in \text{Bun}_{1,Y}$ with an isomorphism $N(V) \cong \mathcal{E}$. 

The group stack $\text{Bun}_{U_1}$ acts on $\text{Bun}_{U_n}$ sending $(\mathcal{L}, \phi_{\mathcal{L}}) \in \text{Bun}_{U_1}, (V, \phi) \in \text{Bun}_{U_n}$ to $(V \otimes \mathcal{L}, \phi \otimes \phi_{\mathcal{L}}) \in \text{Bun}_{U_n}$. Let $\rho_n : \text{Bun}_{U_n} \to \text{Bun}_{U_1}$ be the map sending $(V, \phi)$ to $(\det V, \det \phi)$.

2.1.4. For $n > 1$ let $(V, \phi) \in \text{Bun}_{U_n}$. Then $\det \phi : \sigma^* \det V \overset{\sim}{\to} \det V^*$ can be seen as a trivialization $\xi : N(\det V) \overset{\sim}{\to} \mathcal{O}$. Let $\text{Bun}_{SU_n}$ be the stack classifying $V \in \text{Bun}_{U_n}$ with an isomorphism $\det V \overset{\sim}{\to} \mathcal{O}_Y$ such that the induced isomorphism $N(\det V) \overset{\sim}{\to} N(\mathcal{O}_Y) \overset{\sim}{\to} \mathcal{O}_X$ is $\xi$. So, $\text{Bun}_{SU_n}$ is the fibre of the map $\rho_n : \text{Bun}_{U_n} \to \text{Bun}_{U_1}$ over the unit $\mathcal{O}_Y \in \text{Bun}_{U_1}$ of this group stack.

By ([5], Theorem 2) the stack $\text{Bun}_{SU_n}$ is connected. By ([5], Theorem 3), if $n > 1$ then $\text{Pic}(\text{Bun}_{SU_n}) \overset{\sim}{\to} \mathbb{Z}$. The line bundle on $\text{Bun}_{SU_n}$ sending $V \in \text{Bun}_{U_n}$ to $\det \Gamma(Y, V)$ is twice a generator of $\text{Pic}(\text{Bun}_{SU_n})$ by ([12], Remark 3.6(2)).

2.2. Twist by $\Omega$. As in [8] write $\text{Bun}_{G_n}$ for the stack classifying $M \in \text{Bun}_{2n}$ with a symplectic form $\wedge^2 M \to \Omega_X$. Write $\text{Bun}_{O_{2n}, \Omega}$ for the stack classifying $M \in \text{Bun}_{2n}$ with a nondegenerate symmetric bilinear form $\text{Sym}^2 M \to \Omega$.

Write $\text{Bun}_{U_n, s}$ for the stack classifying $V \in \text{Bun}_{n, Y}$ together with an isomorphism

$$\phi : \sigma^* V \overset{\sim}{\to} V^* \otimes \pi^* \Omega$$

such that $\phi$ is skew-hermitian. In other words, $(\sigma^* \phi)^* = -\phi$. Here $s$ stands for 'skew-hermitian'. Write $\Omega_Y$ for the canonical line bundle on $Y$, one has $\pi^* \Omega \overset{\sim}{\to} \Omega_Y$.

As in Section 2.1.1, for $(V, \phi) \in \text{Bun}_{U_n, s}$ let $M = \pi_* V$. There is a unique symplectic form $\phi : M \overset{\sim}{\to} M^* \otimes \Omega$ such that

$$\pi^* \phi = (\sigma^* \phi, \phi) : V \otimes \sigma^* V \to (\sigma^* V^* \oplus V^*) \otimes \pi^* \Omega$$

There is also a unique symmetric bilinear form $\phi' : M \to M^* \otimes \Omega \otimes \mathcal{E}$ such that

$$\pi^* \phi' = (-\sigma^* \phi, \phi) : V \otimes \sigma^* V \to (\sigma^* V^* \oplus V^*) \otimes \pi^* (\Omega \otimes \mathcal{E})$$

This defines a diagram

$$\text{Bun}_{O_{2n}, \Omega} \overset{q_{n,s}}{\leftarrow} \text{Bun}_{U_n, s} \overset{p_{n,s}}{\to} \text{Bun}_{G_n},$$

where $p_{n,s}$ sends $(V, \phi)$ to $(M = \pi_* V, \phi)$, and $q_{n,s}$ sends $(V, \phi)$ to $(M \otimes \mathcal{E}', \phi')$.

2.3. Square root. Denote by $A_n$ the line bundle on $\text{Bun}_{G_n}$ whose fibre at $M$ is $\det \Gamma(X, M)$. Write $\widetilde{\text{Bun}}_{G_n}$ for the gerbe of square roots of $A_n$ over $\text{Bun}_{G_n}$.

Our immediate purpose is to construct a distinguished square root of the line bundle $p_{n,s}^* A_n$. Our construction will depend only on the choices of $\Omega^2_1$, $\mathcal{E}'$ that we made in Section 1.2, and also on a choice of $i \in k$ with $i^2 = -1$.

2.3.1. For $W \in \text{Bun}_n$ write for brevity $d(W) = \det \Gamma(X, W)$, we view it as a $\mathbb{Z}/2\mathbb{Z}$-graded line.

For $A_i \in \text{Bun}_1$ set

$$K(A_1, A_2) = \frac{d(A_1 \otimes A_2) \otimes d(\mathcal{O})}{d(A_1) \otimes d(A_2)}$$

We view this line as a $\mathbb{Z}/2\mathbb{Z}$-graded. Then $K$ is bilinear up to a canonical isomorphism by ([9], Section 4.2.1-4.2.2).
One has a canonical Pfaffian line bundle $Pf$ on $\text{Bun}_{\mathbb{D}^{2n},\Omega}$ defined in ([2], Section 4.2.1). Pick $i \in k$ with $i^2 = -1$. This choice yields a canonical $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism $Pf(W)^2 \overset{\sim}{\to} d(W)$ for $W \in \text{Bun}_{\mathbb{D}^{2n},\Omega}$ as in loc.cit. An alternative construction of $Pf$ is given in [7].

Given $(V, \phi) \in \text{Bun}_{U_n,s}$ let $M = \phi_* V$. The symplectic form $\tilde{\phi} : \wedge^2 M \to \Omega$ induces an isomorphism $\det M \overset{\sim}{\to} \Omega^n$. By ([8], Lemma 1) one has canonically for $(V, \phi) \in \text{Bun}_{U_n,s}$ and $M = \pi_* V$

\begin{equation}
(3) \quad d(M \otimes \mathcal{E}') \overset{\sim}{\to} d(M) \otimes K(\Omega^n, \mathcal{E}') \otimes \frac{d(\mathcal{E}')^{2n}}{d(\Omega)^{2n}}
\end{equation}

Our choice of $\Omega^1$, bilinearity of $K$, and (3) yield an isomorphism

\begin{equation}
(4) \quad Pf(M \otimes \mathcal{E}')^2 \overset{\sim}{\to} d(M) \otimes \left( K(\Omega^1, \mathcal{E}') \otimes \frac{dX(e)}{X(\Omega^1, \mathcal{E}') \otimes dX(\mathcal{E}')} \right)^n
\end{equation}

Denote by $L_n$ the line bundle on $\text{Bun}_{U_n,s}$ whose fibre at $(V, \phi)$ is

\[ Pf(M \otimes \mathcal{E}) \otimes \left( \frac{dX(\Omega)}{K_X(\Omega^1, \mathcal{E}) \otimes dX(\mathcal{E})} \right)^n \]

Then (4) yields the desired isomorphism over $\text{Bun}_{U_n,s}$

\[ L_n^2 \overset{\sim}{\to} \mathfrak{p}_{n,s}^* \mathfrak{A}_n \]

Let $\tilde{\mathfrak{p}}_{n,s} : \text{Bun}_{U_n,s} \to \tilde{\text{Bun}}_{G_n}$ be the map sending $(V, \psi)$ to $\mathfrak{p}_{n,s}(V, \phi) = (M, \tilde{\phi})$ and a line $L_n(V, \phi)$ equipped with the above isomorphism $L_n(V, \phi)^2 \overset{\sim}{\to} d(M)$.

**2.4. Dual pair $U_n, U_m$.** Let $n, m \geq 1$. Write

\[ \tau : \text{Bun}_{U_n} \times \text{Bun}_{U_m,s} \to \text{Bun}_{U_{nm},s} \]

for the map sending $(V_1, \phi_1) \in \text{Bun}_{U_n}$, $(V_2, \phi_2) \in \text{Bun}_{U_m,s}$ to $V_1 \otimes V_2$ with the induced isomorphism

\[ \phi_1 \otimes \phi_2 : \sigma^*(V_1 \otimes V_2) \overset{\sim}{\to} (V_1 \otimes V_2)^* \otimes \pi^* \Omega \]

The groups $U_n, U_m$ form a dual pair in $G_{nm}$ essentially via the map $\mathfrak{p}_{nm,s} \tau$. Let $\text{Aut}$ be the theta-sheaf on $\text{Bun}_{G_n}$ given in ([11], Definition 1). Define the theta-lifting functors

\[ F_s : \text{D}^-(\text{Bun}_{U_n})! \to \text{D}^-(\text{Bun}_{U_m,s}), \quad F : \text{D}^-(\text{Bun}_{U_{nm},s})! \to \text{D}^-\left(\text{Bun}_{U_n}\right) \]

following the framework of the geometric Langlands functoriality proposed in ([10], Section 2) for the kernel

\[ M = \tau^* \tilde{\mathfrak{p}}_{nm,s}^* \text{Aut}[\dim \text{Bun}_{U_n} \times \text{Bun}_{U_m,s} - \dim \text{Bun}_{G_{nm}}] \]

That is, for $K \in \text{D}^-(\text{Bun}_{U_n})!$ and $K' \in \text{D}^-(\text{Bun}_{U_{nm},s})!$ we let

\[ F_s(K) = (q_s)_!(q^* K \otimes M)[\dim \text{Bun}_{U_n}] \quad \text{and} \quad F(K') = q'_!(q^* K' \otimes M)[\dim \text{Bun}_{U_{nm},s}] \]

for the diagram of projections

\[ \text{Bun}_{U_n} \overset{q_s}{\leftarrow} \text{Bun}_{U_n} \times \text{Bun}_{U_m,s} \overset{q_s}{\rightarrow} \text{Bun}_{U_{nm},s} \]
References


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