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ISOMORPHISMS BETWEEN MODULI SPACES OF $SL(2)$-BUNDLES WITH CONNECTIONS ON $\mathbb{P}^1 \setminus \{x_1, \dots, x_4\}$.

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Okamoto found in [Ok1] that Painlevé equations and in particular P_{VI} have unexpectedly large groups of symmetries. One knows from [Fu] that solutions to P_{VI} correspond to isomonodromic deformations of a certain kind of linear differential equations. This kind of differential equations corresponds to a certain kind of $SL(2)$-bundles with connections on $\mathbb{P}^1 \setminus \{x_1, \dots, x_4\}$. Moduli spaces of these bundles form a family parametrized by the cross-ratio of $x_1, \dots, x_4 \in \mathbb{P}^1$, and P_{VI} can be considered as a connection on this family.

Our aim is to find all isomorphisms between these moduli spaces and to give a geometric description of these isomorphisms.

In this work the basic field is $\mathbb{C}$,i.e., 'space' means '$\mathbb{C}$-space', '$\mathbb{P}^1$' means '$\mathbb{P}^1_{\mathbb{C}}$' and so on.

1.

Let C be the moduli space of $(X, x_1, \dots, x_4)$, where X is a smooth projective curve of genus 0, $x_1, \dots, x_4 \in X$, $x_i \neq x_j$ for $i \neq j$. Obviously $C \simeq \mathbb{P}^1 \setminus \{0, 1, \infty\}$.

The group $\mathbb{S}_4$ acts on C permuting x_i and the kernel of this action is Klein's four-group Kl.

Let $(\lambda_1, \dots, \lambda_4) \in \mathbb{C}^4$ be such that $2\lambda_i \notin \mathbb{Z}$ and

$$\sum_{i=1}^{4} \epsilon_i \lambda_i \notin \mathbb{Z} \qquad (1)$$

for any $\epsilon_i \in \mu_2 := \{1, -1\}$. Denote by Λ the set of all such $(\lambda_1, \dots, \lambda_4)$. Let $\theta = (X, x_1, \dots, x_4; \lambda_1, \dots, \lambda_4) \in \Theta := C \times \Lambda$.

Definition. A θ-bundle is a triple (L, ∇, φ) such that L is a rank 2 vector bundle on X, $\nabla : L \to L \otimes \Omega_X(x_1 + \dots + x_4)$ is a connection, $\varphi : \Lambda^2 L \widetilde{\to} O_X$ is a horizontal isomorphism, and the residue R_i of ∇ at the point x_i has eigenvalues $\{\lambda_i, -\lambda_i\}$.

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θ-bundles form an algebraic stack $\mathcal{M}_\theta$. We denote by M_θ the coarse moduli space corresponding to $\mathcal{M}_\theta$ (see [LM] for the definitions). We denote by $M \to \Theta$ the family of all M_θ.

Remark. (1) implies that if (L, ∇, φ) is a θ-bundle then (L, ∇) is irreducible. In particular this shows that $\mathcal{M}_\theta$ is a μ_2-gerbe over M_θ.

One can check that $\operatorname{Pic}\mathcal{M}_\theta$ is the free abelian group with generators δ, $\xi_1, \dots, \xi_4$ (see [AL]). Here δ (resp. ξ_i) is the class of the line bundle on $\mathcal{M}_\theta$ whose fiber over (L, ∇, φ) is $\det R\Gamma(X, L)$ (resp. $l_i := \ker(R_i - \lambda_i) \subset L_{x_i}$). $\operatorname{Pic} M_\theta \subset \operatorname{Pic}\mathcal{M}_\theta$ is the subgroup of index 2 such that $\delta \in \operatorname{Pic} M_\theta$, $\xi_i \notin \operatorname{Pic} M_\theta$. We identify $\operatorname{Pic} M_\theta$ for all $\theta \in \Theta$ and write simply Pic instead of $\operatorname{Pic} M_\theta$.

Define $\deg : \operatorname{Pic} \to \mathbb{Z}$ by $\deg(a\delta + \sum_{i=1}^4 a_i\xi_i) := -a$. Set $\operatorname{Pic}^0 := \ker(\deg)$. Let $\langle\cdot,\cdot\rangle$ be the bilinear form on Pic^0 such that $\langle \sum_{i=1}^4 a_i\xi_i, \sum_{i=1}^4 b_i\xi_i\rangle := -\frac{1}{2}\sum_{i=1}^4 a_ib_i$. Denote by G the group of automorphisms of Pic preserving deg and $\langle\cdot,\cdot\rangle$.

Theorem 1. *If $\theta_1 \in \Theta$, $g \in G$ there exist unique $\theta_2 \in \Theta$ and $f_g : M_{\theta_1} \widetilde{\to} M_{\theta_2}$ such that $(f_g)_* = g \in \operatorname{Aut}(\operatorname{Pic})$. Any isomorphism $f : M_{\theta_1} \widetilde{\to} M_{\theta_2}$ $(\theta_1, \theta_2 \in \Theta)$ equals f_g for some $g \in G$.*

Remark. It follows from Theorem 1 that $f_{gh} = f_g \circ f_h$.

Set $V := \operatorname{Pic} \otimes_{\mathbb{Z}} \mathbb{C}$, $V_0 := \operatorname{Pic}^0 \otimes_{\mathbb{Z}} \mathbb{C} \subset V$. Then $R := \{v \in \operatorname{Pic}^0 M | \langle v, v\rangle = -2\}$ is a D_4 root system. Since $\mathbb{S}_4$ acts on R permuting ξ_i we have a map $\mathbb{S}_4 \to \operatorname{Aut}(R)$. One can show that this map induces an isomorphism $\mathbb{S}_4/Kl \widetilde{\to} \operatorname{Aut}(R)/W(R)$, where $W(R)$ is the Weyl group of R. The composition $G \to \operatorname{Aut}(R) \to \operatorname{Aut}(R)/W(R) = \mathbb{S}_4/Kl$ gives us an action of G on C. We denote by $\iota : \Lambda \to V$ the embedding $(\lambda_1, \dots, \lambda_4) \mapsto -\delta - 2\sum_{i=1}^4 \lambda_i\xi_i$. One can easily check (see Remark *ii* at the end of this section) that $\iota(\Lambda)$ is stable under the action of G, so ι defines an action of G on Λ. Hence G acts on $\Theta = C \times \Lambda$.

Theorem 2. *Suppose $\theta_1, \theta_2 \in \Theta$; $g \in G$; $f_g : M_{\theta_1} \widetilde{\to} M_{\theta_2}$. Then $\theta_2 = g\theta_1$.*

Denote by P_{VI} the (algebraic) connection on $M \to \Theta$ along C whose (analytic) integral curves correspond to isomonodromic deformations of θ-bundles.

Theorem 3. *P_{VI} is the unique algebraic connection on $M \to \Theta$ along C.*

It is well known that M_θ is symplectic. In Section 4 we construct a concrete symplectic structure ω.

Theorem 4. *Suppose $g \in G$. Then :*

i) The morphisms $f_g : M_\theta \to M_{g\theta}$ form a family $f_g : M \to M$.

ii) The maps f_g preserve ω and P_{VI}.

We will sketch proofs of Theorems 1-4 in Sections 6, 7.

Remarks.

i) $\operatorname{Pic}^0 \subset V_0$ is the weight lattice of R.

ii) Let us give an explicit description of $\iota(\Lambda)$ in terms of R. Denote by Q the root lattice of R. Then $\iota(\Lambda)$ is the set of $\gamma \in V$ such that $\deg\gamma = 1$, $\langle\gamma + \delta, q\rangle \notin \mathbb{Z}$ for any $q \in Q$. Since Pic^0 is the lattice dual to Q, $\iota(\Lambda)$ is the set of $\gamma \in V$ such that $\deg\gamma = 1$, $\langle\gamma + p, q\rangle \notin \mathbb{Z}$ for any $p \in \mathrm{Pic}$, $q \in Q$, $\deg p = -1$. So $\iota(\Lambda)$ is stable under the action of $g \in G$.

iii) There is an obvious isomorphism between G and the semidirect product of $\mathrm{Aut}(R)$ and Pic^0. Here $\mathrm{Aut}(R)$ is identified with the stabilizer of $\delta \in \mathrm{Pic}$ in G, and $p \in \mathrm{Pic}^0$ is identified with $g \in G$ defined by $g(\gamma) := \gamma + \deg(\gamma)p$, $\gamma \in \mathrm{Pic}$.

2.

In this section we give some examples of isomorphisms $f : M_\theta \widetilde{\to} M_{\theta'}$. Suppose $\theta = (X, x_1, \ldots, x_4; \lambda_1, \ldots, \lambda_4) \in \Theta$, (L, ∇, φ) is a θ-bundle.

Let $\sigma \in \mathbb{S}_4$, $\theta' = (X, x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(4)}; \lambda_{\sigma^{-1}(1)}, \ldots, \lambda_{\sigma^{-1}(4)})$. Clearly (L, ∇, φ) is also a θ'-bundle. This gives us $f_\sigma : M_\theta \widetilde{\to} M_{\theta'}$. One can easily compute $(f_\sigma)_* \in \mathrm{Aut}(V)$. The result is: $(f_\sigma)_*\delta = \delta$, $(f_\sigma)_*\xi_i = \xi_{\sigma(i)}$.

Let $\epsilon = (\epsilon_1, \ldots, \epsilon_4) \in (\mu_2)^4$, $\theta' = (X, x_1, \ldots, x_4; \epsilon_1\lambda_1, \ldots, \epsilon_4\lambda_4)$. The notions of θ-bundle and θ'-bundle are equivalent, so we get $f_\epsilon : M_\theta \widetilde{\to} M_{\theta'}$. Clearly $(f_\epsilon)_*\delta = \delta$, $(f_\epsilon)_*\xi_i = \epsilon_i\xi_i$.

Let $l = \sum_{i=1}^4 a_i\xi_i \in \mathrm{Pic}^0$, $\theta' = (X, x_1, \ldots, x_4; \lambda_1 + \frac{a_1}{2}, \ldots, \lambda_4 + \frac{a_4}{2})$. Consider bundles L' such that $L(-N(x_1 + \cdots + x_4)) \subset L' \subset L(N(x_1 + \cdots + x_4))$ for $N \gg 0$ and the connection ∇' induced on L' by ∇ has poles of first order at x_i. There is a unique bundle L' such that the residue of ∇' at x_i has eigenvalues $(\lambda_i, -a_i - \lambda_i)$. Clearly φ induces a horizontal isomorphism $\varphi' : \Lambda^2 L' \widetilde{\to} O(\sum_{i=1}^4 a_i x_i)$.

There exists a triple (γ, d, ψ) such that γ is a line bundle on X, $d : \gamma \to \gamma \otimes \Omega(x_1 + \cdots + x_4)$ is a connection, $\mathrm{res}_{x_i} d = \frac{a_i}{2}$, $\psi : \gamma^{\otimes 2} \to O(-\sum_{i=1}^4 a_i x_i)$ is a horizontal isomorphism. (γ, d, ψ) is unique up to an isomorphism. Obviously $(L' \otimes \gamma, \nabla' \otimes d, \varphi' \otimes \psi)$ is a θ'-bundle. This gives $f_l : M_\theta \widetilde{\to} M_{\theta'}$. It is easy to check that $(f_l)_*\delta = \delta + l$, $(f_l)_*\xi_i = \xi_i$.

In Section 8 we give a nontrivial example of $f : M_\theta \widetilde{\to} M_{\theta'}$.

3.

Now we give a geometric description of M_θ which goes back to Okamoto [Ok2]. Suppose $\theta = (X, x_1, \ldots, x_4; \lambda_1, \ldots, \lambda_4) \in \Theta$. We denote by $\overline{K}_\theta$ the Hirzebruch surface $\mathbb{P}(O_X \oplus \Omega_X(x_1 + \cdots + x_4))$. Let $s_\infty = \mathbb{P}(O_X) \subset \overline{K}_\theta$ be the infinite section, so $K_\theta = \overline{K}_\theta \setminus s_\infty$ is the total space of the bundle $\Omega_X(x_1 + \cdots + x_4)$. Let $b_i \subset K_\theta$ be the fiber over x_i, $\mathrm{res}_i : b_i \widetilde{\to} \mathbb{A}^1$ the canonical isomorphism. Let $c_i^\pm = (\mathrm{res}_i)^{-1}(\lambda_i^\pm)$, where $\lambda_i^\pm = \pm\lambda_i$ for $i \neq 1$, $\lambda_1^+ = \lambda_1, \lambda_1^- = 1 - \lambda_1$. Blowing up $c_i^\pm \in \overline{K}_\theta$ we obtain a variety $\overline{M}_\theta$. Denote by b_i', s_∞' the proper preimages of b_i, s_∞. We denote by $\widetilde{M}_\theta$ the complement to b_i', s_∞' in $\overline{M}_\theta$. Denote by $b_i^\pm \subset \overline{M}_\theta$ the preimages of $c_i^\pm$.

Proposition 1. *There is an isomorphism* $f : \widetilde{M}_\theta \widetilde{\to} M_\theta$ *such that* $f^*(\delta) \simeq O(-b_1^-)$, $f^*(\xi_i^{\otimes 2}) \simeq O(b_i^- - b_i^+)$.

Remark. Theorem 1 implies that f is uniquely determined by $f^*(\delta), f^*(\xi_i) \in \mathrm{Pic}\,\widetilde{M}_\theta$.

Let us sketch a construction of f. Let (L,∇,φ) be a θ-bundle. Consider $L' := \{s \in L | s(x_1) \in l_1\}$, where $l_1 := \ker(R_1 - \lambda_1)$. Then $\nabla' := \nabla|_{L'}$ has a pole of order 1 at x_1. Since (L',∇') is irreducible $L' \simeq O_X \oplus O_X(-1)$. Fix $s \in H^0(X,L')$, $s \neq 0$. Define $j : O_X \oplus (\Omega_X(x_1+\cdots+x_4))^{-1} \to L'$ by $(f,\tau) \mapsto fs + \tau\nabla s \in L'$. Then $\det j$ has a unique simple zero $x \in X$. Denote by l the kernel of $j_x : (O_X \oplus (\Omega_X(x_1+\cdots+x_4))^{-1})_x \to L'_x$. l defines a point of $\mathbb{P}(O_X \oplus \Omega_X(x_1+\cdots+x_4)) \setminus \mathbb{P}(O_X) = K_\theta$. We have constructed a morphism $M_\theta \to K_\theta$. It induces an isomorphism $f : \widetilde{M}_\theta \to M_\theta$. ☐

One can easily check the following formulas:

$$(2) \qquad (s'_\infty, s'_\infty) = -2 \qquad (b'_i, b'_j) = \begin{cases} -2, & i = j \\ 0, & i \neq j \end{cases} \qquad (s'_\infty, b'_i) = 1$$

$$(i,j = 1,\ldots,4)$$

It follows from (2) that $\overline{M}_\theta$ is the least smooth compactification of $\widetilde{M}_\theta$. Clearly we can identify $\operatorname{Pic}\overline{M}_\theta$ for all $\theta \in \Theta$. So we write simply $\overline{\operatorname{Pic}}$ instead of $\operatorname{Pic}\overline{M}_\theta$. The kernel of the natural map $\overline{\operatorname{Pic}} \to \operatorname{Pic}$ is the free abelian group $\operatorname{Pic}_\infty$ with basis s'_∞, b'_i (so any class $\alpha \in \operatorname{Pic}_\infty$ contains a unique divisor C such that $\operatorname{supp} C \subset \overline{M}_\theta \setminus M_\theta$). The restriction of the intersection form on $\overline{\operatorname{Pic}}$ to $\operatorname{Pic}_\infty$ is non-positive. Its kernel is generated by $D := 2s'_\infty + \sum_{i=1}^4 b'_i$.

Proposition 2.
i) $\Omega^2_{\overline{M}_\theta} \simeq O_{\overline{M}_\theta}(-D)$.
ii) If $\gamma \in \operatorname{Pic}, \overline{\gamma} \in \overline{\operatorname{Pic}}$ *are such that* $\overline{\gamma}|_{M_\theta} = \gamma$ *then* $(\overline{\gamma}, D) = \deg\gamma$.
iii) For $\gamma_1, \gamma_2 \in \operatorname{Pic}^0$ *there exist* $\overline{\gamma}_j \in \overline{\operatorname{Pic}} \otimes_{\mathbb{Z}} \mathbb{C}$ *such that* $(\overline{\gamma}_j, s'_\infty) = (\overline{\gamma}_j, b'_i) = 0$ *and* $\overline{\gamma}_j|_{M_\theta} = \gamma_j$ $(j = 1,2; i = 1,\ldots,4)$*; in this situation* $(\overline{\gamma}_1, \overline{\gamma}_2) = \langle\gamma_1,\gamma_2\rangle$. ☐

Remark. Denote by $Q \subset V_0$ the lattice dual to Pic^0 (so Q is the root lattice of R). Then $\gamma \in Q$ iff there is a $\overline{\gamma} \in \overline{\operatorname{Pic}}$ such that $(\overline{\gamma}, s'_\infty) = (\overline{\gamma}, b'_i) = 0$ and $\overline{\gamma}|_{M_\theta} = \gamma$.

Lemma 1. $H^0(M_\theta, O_{M_\theta}) = \mathbb{C}$

Proof. Riemann-Hilbert correspondence yields an analytic isomorphism $(M_\theta)_{an} \simeq (W_\theta)_{an}$, where W_θ is an affine variety (W_θ is the moduli space of two-dimensional representations of $\pi_1(X \setminus \{x_1,\ldots,x_4\})$ with fixed conjugacy classes of local monodromies). Hence M_θ contains no projective curves. Let $f \in H^0(M_\theta, O_{M_\theta})$. The divisor of f on $\overline{M}_\theta$ can be represented as $(f) = C_\infty + \overline{C}$, where $\operatorname{supp} C_\infty \subset \overline{M}_\theta \setminus M_\theta$ and $\overline{C}$ is the closure of the divisor of f on M_θ. Since $((f),D) = (C_\infty, D) = 0$ we have $(\overline{C}, D) = 0$. But $\overline{C} \geq 0$ and M_θ contains no projective curves so $\overline{C} = 0$. So $C_\infty \sim 0$. Since $\operatorname{supp} C_\infty \subset \overline{M}_\theta \setminus M_\theta$ this implies $C_\infty = 0$. ☐

4.

The description of M_θ can be reformulated in the following way.

Let π' be the composition $M_\theta \to K_\theta \to X$. The fiber of π' over the point $x_i \in X$ has two connected components $b_i^{\pm}$.

Glueing together two copies of X outside $x_1, \dots, x_4$, we obtain a scheme N'. We have the natural morphism $\pi_N : N' \to X$. Set $\{x_i^+, x_i^-\} := (\pi_N)^{-1}(x_i)$. There exists a unique morphism $\pi : M_\theta \to N'$ such that $\pi' = \pi_N \circ \pi$ and $\pi^{-1}(x_i^{\pm}) = b_i^{\pm}$. π defines a structure of an affine bundle on M_θ (i.e., $\pi : M_\theta \to N'$ is a torsor over some vector bundle on N'). So $\pi^* : \operatorname{Pic} N' \widetilde{\to} \operatorname{Pic}$.

Then Proposition 1 implies :

Proposition 3. *Set* $\beta := (\pi^*)^{-1}(-\delta - 2\sum_{i=1}^4 \lambda_i \xi_i) \in \operatorname{Pic} N' \otimes_{\mathbb{Z}} \mathbb{C}$. *Then*

i) The vector bundle associated with $\pi : M_\theta \to N'$ *is* $\Omega_{N'}$.

ii) $\alpha \in H^1(N', \Omega^1_{N'})$ *corresponding to the* $\Omega_{N'}$*-torsor* $\pi : M_\theta \to N'$ *is the image of* β. □

Since M_θ is an $\Omega^1_{N'}$-torsor there is a natural exact sequence $0 \to \pi^*\Omega^1_{N'} \to T_{M_\theta} \to \pi^* T_{N'} \to 0$, where $T_{N'}$ is the tangent bundle. This exact sequence yields $\omega \in H^0(M_\theta, \Omega^2_{M_\theta})$ defined by $\omega(s_1 \wedge s_2) = \langle s_1, \overline{s}_2 \rangle$ for $s_1 \in \pi^*\Omega^1_{N'}$, $s_2 \in T_{M_\theta}$. Here $\overline{s}_2 \in \pi^* T_{N'}$ is the image of $s_2 \in T_{M_\theta}$, $\langle \cdot, \cdot \rangle$ is the natural pairing between $\pi^*\Omega_{N'}$ and $\pi^* T_{N'}$. $d\omega = 0$ because $\dim M_\theta = 2$. By the results of [BK] we get:

Corollary 1. *The image of* ω *in* $H^2_{DR}(M_\theta, \mathbb{C})$ *coincides with the image of* $-\delta - 2\sum_{i=1}^4 \lambda_i \xi_i \in \operatorname{Pic} \otimes_{\mathbb{Z}} \mathbb{C}$. □

Remarks.

i) Consider any variety obtained by the same way as $\widetilde{M}_\theta$, but for any points $c_i^{\pm} \in b_i \subset K_\theta$, $c_i^+ \neq c_i^-$ $(i = 1, \dots, 4)$. By the same arguments we get a symplectic structure on this variety and can compute the corresponding de Rham cohomology class. This class is the image of $\sum_{i=1}^4 (\lambda_i^+ [b_i^+] + \lambda_i^- [b_i^-]) \in \operatorname{Pic} \otimes_{\mathbb{Z}} \mathbb{C}$ in $H^2_{DR}(\widetilde{M}_\theta, \mathbb{C})$. Here $b_i^{\pm} \subset \widetilde{M}_\theta$ is the preimage of $c_i^{\pm} \in K_\theta$, $\lambda_i^{\pm} := \operatorname{res}_i(c_i^{\pm})$, $\operatorname{res}_i : b_i \to \mathbb{A}^1$ is the canonic isomorphism, $[b_i^{\pm}] \in \operatorname{Pic}$ is the class of the divisor $b_i^{\pm}$.

ii) It is well known that the analog of M_θ for any curve X and any points $x_1, \dots, x_n \in X$ has a natural symplectic structure. This structure depends only on a choice of an invariant scalar product on $sl(2)$. We conjecture that Corollary 1 is true in this situation for a suitable choice of this product.

5.

Denote by $\operatorname{Exc}_\theta$ the set of all exceptional curves of the first kind on $\overline{M}_\theta$.

Proposition 4. *The map* $C \mapsto [C]$ *is a bijection*

$$\operatorname{Exc}_\theta \widetilde{\to} \operatorname{Pic}^1 := \{\gamma \in \operatorname{Pic} \mid \deg(\gamma) = 1\}.$$

Proof. Suppose $C \in \mathrm{Exc}_\theta$. By adjunction formula $(C, D) = 1$, i.e., $\deg(O_{M_\theta}(C)) = 1$. So one has the mapping $\phi : \mathrm{Exc}_\theta \to \mathrm{Pic}^1$ defined by $\phi(C) := [C] \in \mathrm{Pic}$. Proposition 1 shows that $-\delta \in \mathrm{Im}\,\phi$. Using the isomorphisms f_l from Section 2, $l \in \mathrm{Pic}^0$, one sees that ϕ is surjective. Clearly the map $\mathrm{Exc}_\theta \to \mathrm{Pic}\,\overline{M}_\theta : C \mapsto O_{\overline{M}_\theta}(C)$ is injective, and its image is contained in $\mathrm{Ex} := \{\gamma \in \mathrm{Pic}\,\overline{M}_\theta | (\gamma^2) = -1,\ (\gamma, s'_\infty) \geq 0,\ (\gamma, b'_i) \geq 0,\ (\gamma, D) = 1\}$. It is easy to prove that the map $\mathrm{Ex} \to \mathrm{Pic}^1 : \gamma \mapsto \gamma|_{M_\theta}$ is injective. This completes the proof. □

In fact we have proved that both maps $\mathrm{Exc}_\theta \to \mathrm{Ex}$ and $\mathrm{Ex} \to \mathrm{Pic}^1$ are bijective. Denote by C_α the image of $\alpha \in \mathrm{Pic}^1$ in $\mathrm{Ex} \subset \overline{\mathrm{Pic}}$. One can check the following formulas:

$$(s'_\infty, C_\alpha) = 0 \qquad (b'_i, C_\alpha) = \begin{cases} 1, & \alpha \in P_i \\ 0, & \alpha \notin P_i \end{cases}$$

$$(C_\alpha, C_\beta) = -1 - \left[\frac{\langle \alpha - \beta, \alpha - \beta \rangle}{2}\right] \tag{3}$$

$$(\alpha, \beta \in \mathrm{Pic}^1; i = 1, \ldots, 4)$$

Here $P_i := (-\delta + \xi_i + \xi_1) + Q \in \mathrm{Pic}^1 / Q$. Obviously s'_∞, $b'_1, \ldots, b'_4$, and C_α generate $\overline{\mathrm{Pic}}$.

Denote by $\overline{G}$ the group of all automorphisms of $\overline{\mathrm{Pic}}$ preserving s'_∞, the intersection form, and the set $\{b'_1, \ldots, b'_4\}$. The restriction map $\overline{\mathrm{Pic}} \to \mathrm{Pic}$ induces a homomorphism $p : \overline{G} \to \mathrm{Aut}(\mathrm{Pic})$.

Lemma 2. *p is a bijection $\overline{G} \widetilde{\to} G$.*

Proof. Clearly $\overline{G}$ preserves D, so Proposition 2 implies $p(\overline{G}) \subset G$. Now we construct the inverse map.

Suppose $g \in G$. Denote by Γ the free abelian group with basis $s'_\infty, b'_i, C_\alpha$, ($i = 1, \ldots, 4$, $\alpha \in \mathrm{Pic}^1$). Denote by B the symmetric bilinear form on Γ defined by (2) and (3). Since the intersection form on $\overline{\mathrm{Pic}}$ is non-degenerate $\overline{\mathrm{Pic}} = \Gamma / \ker B$. We define $\widetilde{g} : \Gamma \to \Gamma$ on the generators by $\widetilde{g}(s'_\infty) = s'_\infty$, $\widetilde{g}(C_\alpha) = C_{g(\alpha)}$, $\widetilde{g}(b'_i) = b'_j$ iff $g(P_i) = P_j$ and extend it to Γ by linearity. Since $\widetilde{g}$ preserves B, $\widetilde{g}$ induces $\overline{g} : \overline{\mathrm{Pic}} \to \overline{\mathrm{Pic}}$. Clearly $\overline{g} \in \overline{G}$. □

6. Proof of Theorems 1 and 2

Fix $\theta_1 = (X^{(1)}, x_1^{(1)}, \ldots, x_4^{(1)}; \lambda_1^{(1)}, \ldots, \lambda_4^{(1)}) \in \Theta$, $g \in G$.

Step 1. Suppose $\theta_2 = (X^{(2)}, x_1^{(2)}, \ldots, x_4^{(2)}; \lambda_1^{(2)}, \ldots, \lambda_4^{(2)}) \in \Theta$, $f : M_{\theta_1} \widetilde{\to} M_{\theta_2}$ are such that $f_* = g \in \mathrm{Aut}\,\mathrm{Pic}$. Let us prove that θ_2 and f are uniquely determined by θ_1 and g.

Since $\overline{M}_{\theta_r}$ is the least smooth compactification of M_{θ_r} $(r = 1, 2)$ one can extend f to $\overline{f} : \overline{M}_{\theta_1} \widetilde{\to} \overline{M}_{\theta_2}$. Clearly $\overline{g} := (\overline{f})_* \in \overline{G}$ is the image of g via the isomorphism $G \widetilde{\to} \overline{G}$ constructed in Lemma 2. Hence $\overline{f}(s_\infty^{(1)}) = s_\infty^{(2)}$, $\overline{f}(b_{\sigma(i)}^{(1)}) = b_i^{(2)}$, where $s_\infty^{(r)}$, $b_i^{(r)}$ denote the curves $s'_\infty, b'_i \subset \overline{M}_{\theta_r}$, $r = 1, 2$, and $\sigma \in \mathbb{S}_4$ is the permutation such that $g^{-1}(P_i) = P_{\sigma(i)}$.

Let $E_i^\pm \in \mathrm{Exc}_{\theta_1}$ correspond to $g^{-1}[b_i^\pm] \in \mathrm{Pic}^1$. Let $\overline{K}_g$ be the variety obtained by blowing down $E_i^\pm \subset \overline{M}_{\theta_1}$. Clearly the composition $\overline{M}_{\theta_1} \widetilde{\to} \overline{M}_{\theta_2} \to \overline{K}_{\theta_2}$ induces an isomorphism $\overline{f}_K : \overline{K}_g \widetilde{\to} \overline{K}_{\theta_2}$. Let $s_{(\infty)}, b_{(i)} \subset \overline{K}_g$, and $c_{(i)}^\pm \in b_{(i)}$ be the images of $s_\infty^{(1)}$, $b_{\sigma(i)}^{(1)}$, and $E_i^\pm \subset \overline{M}_{\theta_1}$ respectively. Then $\overline{f}_K$ has the following properties: $\overline{f}_K(s_{(\infty)}) \subset \overline{K}_{\theta_2}$ is the infinite section, $\overline{f}_K(b_{(i)}) \subset \overline{K}_{\theta_2}$ is the fiber over $x_i^{(2)} \in X^{(2)}$, and $\overline{f}_K(c_{(i)})^\pm = (\mathrm{res}_i)^{-1}(\lambda_i^{(2)\pm})$. Here $\lambda_i^{(2)\pm} = \pm\lambda_i^{(2)}$ for $i \neq 1$, $\lambda_1^{(2)+} = \lambda_1^{(2)}$, $\lambda_1^{(2)-} = 1 - \lambda_1^{(2)}$.

Clearly $\theta_2 \in \Theta$ and $\overline{f}_K : \overline{K}_g \widetilde{\to} \overline{K}_{\theta_2}$ with the above properties are uniquely determined by $\overline{K}_g$, $s_{(\infty)} \subset \overline{K}_g$, $b_{(i)} \subset \overline{K}_g$, and $c_{(i)}^+, c_{(i)}^- \in b_{(i)}$ $(i = 1, \ldots, 4)$.

Remark. The map $\overline{M}_{\theta_r} \to \overline{K}_{\theta_r} \to X^{(r)}$ induces an isomorphism $(s_\infty^{(r)}, s_\infty^{(r)} \cap b_1^{(r)}, \ldots, s_\infty^{(r)} \cap b_4^{(r)}) \widetilde{\to} (X^{(r)}, x_1^{(r)}, \ldots, x_4^{(r)})$, $r = 1, 2$. So $(X^{(2)}, x_1^{(2)}, \ldots, x_4^{(2)}) \simeq (X^{(1)}, x_{\sigma(1)}^{(1)}, \ldots, x_{\sigma(4)}^{(1)})$.

Step 2. Let us construct $f_g : M_{\theta_1} \widetilde{\to} M_{\theta_2}$. We keep the notation of Step 1.

$\overline{K}_g$ is a smooth rational projective surface. It is easy to check that $[b_{(1)}] = [b_{(2)}] = [b_{(3)}] = [b_{(4)}] \in \mathrm{Pic}\,\overline{K}_g$. $[b_{(1)}]$ and $[s_{(\infty)}]$ form a basis in $\mathrm{Pic}\,\overline{K}_g$. One can prove that $(s_{(\infty)}, b_{(i)}) = 1$, $(s_{(\infty)}, s_{(\infty)}) = -2$, $(b_{(i)}, b_{(i)}) = 0$. Combining this fact with the remark from Step 1 we can find an isomorphism $\overline{K}_g \widetilde{\to} \overline{K}_2 := \mathbb{P}(O_{X^{(2)}} \oplus \Omega_{X^{(2)}}(x_1^{(2)} + \cdots + x_4^{(2)}))$ such that $s_{(\infty)}$ corresponds to the infinite section and $b_{(i)}$ corresponds to the fiber over $x_i^{(2)}$. Here $X^{(2)} := X^{(1)}$, $x_i^{(2)} := x_{\sigma(i)}^{(1)}$. Then $c_{(i)}^\pm$ corresponds to $(\mathrm{res}_i)^{-1}(\lambda_{(i)}^\pm)$ for some $\lambda_{(i)}^\pm \in \mathbb{C}$, $\lambda_{(i)}^+ \neq \lambda_{(i)}^-$. By Remark i from Section 4 the map $M_{\theta_1} \to \overline{K}_2$ yields a symplectic structure on M_{θ_1} such that the corresponding de Rham cohomology class is the image of $v_2 := \sum_{i=1}^4 (\lambda_{(i)}^+[E_i^+] + \lambda_{(i)}^-[E_i^-]) \in \mathrm{Pic} \otimes_{\mathbb{Z}} \mathbb{C}$. By Lemma 1 two symplectic structures on M_{θ_1} should coincide up to $a \in \mathbb{C}^*$. So $av_2 = v_1 := -\delta - 2\sum_{i=1}^4 \lambda_i^{(1)}\xi_i$. Using $\deg : \mathrm{Pic} \otimes_{\mathbb{Z}} \mathbb{C} \to \mathbb{C}$ we obtain that $\sum_{i=1}^4 (\lambda_{(i)}^+ + \lambda_{(i)}^-) \neq 0$.

Therefore replacing $\overline{K}_g \widetilde{\to} \overline{K}_2$ by its composition with a suitable automorphism of $\overline{K}_2$ over $X^{(2)}$ we can come to the situation where $\lambda_{(i)}^+ + \lambda_{(i)}^-$ equals 0 for $i \neq 1$ and 1 for $i = 1$. Then $a = 1$, $v_1 = v_2$.

Set $\lambda_i^{(2)} := \lambda_{(i)}^+$. Then $v_2 = g^{-1}(-\delta - 2\sum_{i=1}^4 \lambda_i^{(2)}\xi_i)$. Therefore

$$-\delta - 2\sum_{i=1}^{4} \lambda_i^{(2)}\xi_i = gv_2 = gv_1 = g(-\delta - 2\sum_{i=1}^{4} \lambda_i^{(1)}\xi_i) \tag{4}$$

So $(\lambda_1^{(2)},\dots,\lambda_4^{(2)})$ is obtained from $(\lambda_1^{(1)},\dots,\lambda_4^{(1)})$ by the action of $g\in G$. Hence $(\lambda_1^{(2)},\dots,\lambda_4^{(2)})\in\Lambda$ and $\theta_2:=(X^{(2)},x_1^{(2)},\dots,x_4^{(2)};\lambda_1^{(2)},\dots,\lambda_4^{(2)})\in\Theta$.

The composition $\overline{M}_{\theta_1}\to\overline{K}_g\widetilde{\to}\overline{K}_2=\overline{K}_{\theta_2}$ lifts to an isomorphism $\overline{f}_g:\overline{M}_{\theta_1}\widetilde{\to}\overline{M}_{\theta_2}$. $\overline{f}_g$ induces $f_g:M_{\theta_1}\widetilde{\to}M_{\theta_2}$. By the construction $\overline{f}_g(E_i^{\pm})$ is the closure of $b_i^{\pm}$. Since $[b_i^{\pm}]$ generate Pic we have $(f_g)_*=g$.

Step 3. Let us prove Theorem 2. We have already proved that $(\lambda_1^{(2)},\dots,\lambda_4^{(2)})$ is obtained from $(\lambda_1^{(1)},\dots,\lambda_4^{(1)})$ by the action of $g\in G$. Now we prove that $(X^{(2)},x_1^{(2)},\dots,x_4^{(2)})\simeq(X^{(1)},x_{\sigma(1)}^{(1)},\dots,x_{\sigma(4)}^{(1)})$ is obtained from $(X^{(1)},x_1^{(1)},\dots,x_4^{(1)})$ by the action of $g\in G$. Consider two particular cases.

Case 1. Suppose $g|_{\mathrm{Pic}^0}\in W(R)$. Then g induces the identity automorphism of Pic^0/Q. So the action of g on Pic^1/Q is a translation and $\sigma\in Kl$. Hence $(X^{(2)},x_1^{(2)},\dots,x_4^{(2)})\simeq(X^{(1)},x_1^{(1)},\dots,x_4^{(1)})$.

Case 2. Suppose g is defined by $g(\xi_i)=\xi_{\sigma'(i)}$, $g(\delta)=\delta$, where $\sigma'\in\mathbb{S}_4$ is such that $\sigma'(1)=1$. Then $f_g=f_{\sigma'}$ (see Section 2). By definition $\sigma=(\sigma')^{-1}$ and $(X^{(2)},x_1^{(2)},\dots,x_4^{(2)})$ is obtained from $(X^{(1)},x_1^{(1)},\dots,x_4^{(1)})$ by the action of $\sigma'\in\mathbb{S}_4$.

Any g can be represented as $g_1f_{\sigma'}$, where $g_1|_{\mathrm{Pic}^0}\in W(R)$, $\sigma'\in\mathbb{S}_4$, $\sigma'(1)=1$, so Theorem 2 follows from these particular cases.

Step 4. Let us prove the last statement of Theorem 1. Take any $f:M_{\theta_1}\widetilde{\to}M_{\theta_2}$ $(\theta_1,\theta_2\in\Theta)$. Extend f to $\overline{f}:\overline{M}_{\theta_1}\widetilde{\to}\overline{M}_{\theta_2}$. Set $\overline{g}:=(\overline{f})_*\in\mathrm{Aut}(\overline{\mathrm{Pic}})$. Clearly $\overline{g}\in\overline{G}$. Hence $g:=f_*\in G\subset\mathrm{Aut}\,\mathrm{Pic}$ and $f=f_g$. □

Remark. Consider the isomorphisms $f_\sigma(\sigma\in\mathbb{S}_4), f_\epsilon(\epsilon\in(\mu_2)^4), f_l(l\in\mathrm{Pic}^0)$, and τ constucted in Section 2 and Section 8. One can easily check that $(f_\sigma)_*,(f_\epsilon)_*,(f_l)_*$, and τ_* generate G, so any isomorphism $f:M_{\theta_1}\widetilde{\to}M_{\theta_2}$, $\theta_1,\theta_2\in\Theta$ can be represented as a composition of the isomorphisms $f_\sigma,f_\epsilon,f_l,\tau$. That gives us some geometric description of f. Besides, Theorem 2 is obvious for $f_\sigma,f_\epsilon,f_l,\tau$, so it can be proved for any $f:M_{\theta_1}\widetilde{\to}M_{\theta_2}$ using this decomposition. This is another proof of Theorem 2.

7. Proof of Theorems 3 and 4

Proof of Theorem 3. Suppose there are two connections on $M\to\Theta$ along C. For any fixed $\theta\in\Theta$ two such connections differ by a vector field on M_θ. So it suffices to prove the following lemma.

Lemma 3. $H^0(M_\theta,T_{M_\theta})=0$.

Proof. Since M_θ is symplectic $T_{M_\theta}\simeq\Omega^1_{M_\theta}$. Suppose $\eta\in H^0(M_\theta,\Omega^1_{M_\theta})$. Then by Lemma 1 $d\eta\in H^0(M_\theta,\Omega^2_{M_\theta})=\mathbb{C}\omega$. But the image of ω in $H^2_{DR}(M_\theta,\mathbb{C})$ does not vanish, so $d\eta=0$. It follows from Proposition 3 that M_θ can be covered with open subsets isomorphic to $\mathbb{A}^2$, so locally η lies in the image of $d:O\to\Omega^1$. But $\ker(d)=\mathbb{C}$ and $H^1_{Zar}(M_\theta,\mathbb{C})=0$, so $\eta=df$, $f\in H^0(M_\theta,O)$. Lemma 1 shows that $\eta=0$. □

Proof of Theorem 4. All the above constructions are still valid for families of M_θ and so the first part of Theorem 4 is obvious. Suppose $g \in G$. (4) implies that f_g preserves ω. The fact that $f_g : M \widetilde{\to} M$ preserves P_{VI} follows from Theorem 3. □

8.

Now we give another geometric description of M_θ, $\theta = (X, x_1, \dots, x_4; \lambda_1, \dots, \lambda_4) \in \Theta$.

Let $\Delta \subset X^2$ be the diagonal. Set $\beta_i := (x_i, x_i) \in \Delta$. We denote by $\widetilde{K}_\theta$ the variety obtained by blowing up $\beta_1, \dots, \beta_4 \in X^2$. Let $\widetilde{b}_i \subset \widetilde{K}_\theta$ be the preimage of β_i, $r_i : \mathbb{P}^1 \widetilde{\to} \widetilde{b}_i$ the isomorphism such that $r_i(0), r_i(\infty), r_i(1)$ lie on the proper preimages of $\{x_i\} \times X$, $X \times \{x_i\}$, Δ respectively. Set $u := \sum_{i=1}^4 \lambda_i$, $v_i := \lambda_i^+ - \lambda_i^-$, $\mu_i := r_i(\frac{v_i}{u - v_i}) \in \widetilde{b}_i$.

Proposition 5. *There is a unique map $\overline{M}_\theta \to \widetilde{K}_\theta$ which is the blow-up at μ_1, ..., μ_4 such that*:

i) s'_∞, b'_i are the proper preimages of $\Delta, \widetilde{b}_i$ respectively,

ii) b_i^- is the preimage of μ_i,

iii) the morphism $\overline{M}_\theta \to X$ from Section 3 equals the composition $\overline{M}_\theta \to \widetilde{K}_\theta \to X \times X \to X$, where $X \times X \to X$ is the first projection.

Proof. Blow down the curves b_i^- and b'_i on $\overline{M}_\theta$, $i = 1, \dots, 4$. Denote by $\widetilde{P}$ the obtained variety. Proposition 1 implies that $\widetilde{P}$ is the natural compactification of the Ω_X-torsor whose sheaf of sections is $\{s \in \Omega_X(x_1 + \cdots + x_4) | \operatorname{res}_{x_i} s = \lambda_i\}$. It follows from (1) that this torsor is not trivial ,i.e., $\widetilde{P} \simeq \mathbb{P}(\mathcal{E})$ for a non-trivial extension $0 \to \Omega_X \to \mathcal{E} \to O_X \to 0$. Thus $\mathcal{E} \simeq (O_X(-1))^2$ and $\widetilde{P} \simeq X^2$. There is a unique isomorphism $\widetilde{P} \widetilde{\to} X^2$ such that the natural projection $\widetilde{P} \to X$ and the first projection $X^2 \to X$ are identified and $\Delta \subset X^2$ is the image of $s'_\infty \subset \overline{M}_\theta$. To complete the proof one can check the formula for μ_i by direct calculation. □

Corollary 2. *Set $\theta' := (X, x_1, \dots, x_4; \lambda'_1, \dots, \lambda'_4)$, $\lambda'_j := \lambda_j - \frac{1}{2}\sum_{i=1}^4 \lambda_i$. The map $X^2 \widetilde{\to} X^2$ defined by $(x, y) \mapsto (y, x)$ induces an isomorphism $\overline{\tau} : \overline{M}_\theta \widetilde{\to} \overline{M}_{\theta'}$ such that $\tau := \overline{\tau}|_{M_\theta}$ is an isomorphism $M_\theta \widetilde{\to} M_{\theta'}$.* □

One can easily check that $\tau_*(\delta) = \delta, \tau_*(\xi_i) = \xi_i - \frac{1}{2}\sum_{i=1}^4 \xi_i$ (so $\tau_*|_{V_0}$ is the reflection corresponding to $\sum_{i=1}^4 \xi_i \in R$).

Remark. Denote by N the coarse moduli space of indecomposable $SL(2)$-bundles on X with quasiparabolic structure at $x_1, \dots, x_4$. For a θ-bundle (L, ∇, φ) we set $l_i := \ker(R_i - \lambda_i) \subset L_{x_i}$. $(L, \varphi, l_1, \dots, l_4)$ is an indecomposable quasiparabolic bundle on X. This yields a morphism $M_\theta \to N$. One can show that there exists an isomorphism $N \widetilde{\to} N'$ such that the diagram

$$\begin{array}{ccc} M_\theta & \overset{\tau}{\widetilde{\to}} & M_{\theta'} \\ \downarrow & & \downarrow \\ N & \widetilde{\to} & N' \end{array}$$

commutes. Here θ' and τ were defined in Corollary 2 and the morphism $M_{\theta'} \to N'$ is analogous to the morphism $M_\theta \to N'$ constructed in Section 4.

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