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LOCAL GEOMETRIZED RANKIN-SELBERG METHOD FOR GL(n)

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Abstract
Following G. Laumon [12], to a nonramified ℓ-adic local system E of rank n on a curve X one associates a complex of ℓ-adic sheaves ${}_n\mathscr{K}_E$ on the moduli stack of rank n vector bundles on X with a section, which is cuspidal and satisfies the Hecke property for E. This is a geometric counterpart of the well-known construction due to J. Shalika [19] and I. Piatetski-Shapiro [18]. We express the cohomology of the tensor product ${}_n\mathscr{K}_{E_1} \otimes {}_n\mathscr{K}_{E_2}$ in terms of cohomology of the symmetric powers of X. This may be considered as a geometric interpretation of the local part of the classical Rankin-Selberg method for $\mathrm{GL}(n)$ *in the framework of the geometric Langlands program.*

0. Introduction

This is the first in a series of two papers, where we propose a geometric version of the classical Rankin-Selberg method for computation of the scalar product of two cuspidal automorphic forms on $\mathrm{GL}(n)$ over a function field. This geometrization fits in the framework of the geometric Langlands program initiated by V. Drinfeld, A. Beilinson, and Laumon.

Let X be a smooth, projective, geometrically connected curve over $\mathbb{F}_q$. Let ℓ be a prime invertible in $\mathbb{F}_q$. According to the Langlands correspondence for $\mathrm{GL}(n)$ over function fields (proved by L. Lafforgue), to any smooth geometrically irreducible $\bar{\mathbb{Q}}_\ell$-sheaf E of rank n on X is associated a (unique up to a multiple) cuspidal automorphic form $\varphi_E : \mathrm{Bun}_n(\mathbb{F}_q) \to \bar{\mathbb{Q}}_\ell$, which is a Hecke eigenvector with respect to E. The function φ_E is defined on the set $\mathrm{Bun}_n(\mathbb{F}_q)$ of isomorphism classes of rank n vector bundles on X.

The classical method of Rankin and Selberg for $\mathrm{GL}(n)$ may be divided into two parts: local and global. The global result calculates for any integer d the scalar product

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of two (appropriately normalized) automorphic forms

$$\sum_{L\in \mathrm{Bun}_n^d(\mathbb{F}_q)} \frac{1}{\#\operatorname{Aut} L}\ \varphi_{E_1^*}(L)\varphi_{E_2}(L), \tag{1}$$

where $\mathrm{Bun}_n^d(\mathbb{F}_q)$ is the set of isomorphism classes of vector bundles L on X of rank n and degree d, and $\#\operatorname{Aut} L$ stands for the number of elements in $\operatorname{Aut} L$. More precisely, this scalar product vanishes if and only if E_1 and E_2 are nonisomorphic. In the case $E_1 \xrightarrow{\sim} E_2 \xrightarrow{\sim} E$, the answer is expressed in terms of the action of the geometric Frobenius endomorphism on $\mathrm{H}^1(X \otimes \bar{\mathbb{F}}_q,\ \operatorname{End} E)$.

The computation of (1) is based on the equality of formal series

$$\sum_{d\geq 0}\ \sum_{(\Omega^{n-1}\hookrightarrow L)\in {}_n\mathscr{M}_d(\mathbb{F}_q)} \frac{1}{\#\operatorname{Aut}(\Omega^{n-1}\hookrightarrow L)}\varphi_{E_1^*}(L)\varphi_{E_2}(L)t^d = L(E_1^*\otimes E_2,\ q^{-1}t). \tag{2}$$

Here ${}_n\mathscr{M}_d(\mathbb{F}_q)$ is the set of isomorphism classes of pairs $(\Omega^{n-1}\hookrightarrow L)$, where L is a vector bundle on X of rank n and degree $d+n(n-1)(g-1)$, and Ω is the canonical invertible sheaf on X (Ω^{n-1} is embedded in L as a subsheaf; i.e., the quotient is allowed to have torsion). We have denoted by $L(E_1^*\otimes E_2, t)$ the L-function attached to the local system $E_1^*\otimes E_2$ on X.

Recall that the existence of the automorphic form φ_E is a descent problem (cf. [12]). Using an explicit construction due to Shalika [19] and Piatetski-Shapiro [18], one associates to a smooth $\bar{\mathbb{Q}}_\ell$-sheaf E of rank n on X a function $\tilde{\varphi}_E : {}_n\mathscr{M}_d(\mathbb{F}_q) \to \bar{\mathbb{Q}}_\ell$, which is cuspidal and satisfies the Hecke property with respect to E. The Langlands conjecture predicts that when E is geometrically irreducible, $\tilde{\varphi}_E$ is constant along the fibres of the projection ${}_n\mathscr{M}_d(\mathbb{F}_q) \to \mathrm{Bun}_n^{d+n(n-1)(g-1)}(\mathbb{F}_q)$; that is, $\tilde{\varphi}_E$ is the pullback of a function φ_E on $\mathrm{Bun}_n(\mathbb{F}_q)$. So, (2) is a statement independent of the Langlands conjecture. In fact, (2) is of local nature: it is true for any local systems E_1 and E_2 of rank n on X after replacing φ_E by $\tilde{\varphi}_E$.

The main result of this paper is a strengthened geometric version of the equality

$$\sum_{(\Omega^{n-1}\hookrightarrow L)\in {}_n\mathscr{M}_d(\mathbb{F}_q)} \frac{1}{\#\operatorname{Aut}(\Omega^{n-1}\hookrightarrow L)}\tilde{\varphi}_{E_1^*}(L)\tilde{\varphi}_{E_2}(L) = q^{-d}\sum_{D\in X^{(d)}(\mathbb{F}_q)} \operatorname{tr}\bigl(\mathrm{Fr},\ (E_1^*\otimes E_2)^{(d)}_D\bigr) \tag{3}$$

of coefficients in (2) for each $d\geq 0$. Here $X^{(d)}$ is the dth symmetric power of X, $(E_1^*\otimes E_2)^{(d)}$ is a constructable $\bar{\mathbb{Q}}_\ell$-sheaf on $X^{(d)}$ (cf. Section 1), and Fr is the geometric Frobenius endomorphism.

Let ${}_n\mathscr{M}_d$ denote the moduli stack of pairs $(\Omega^{n-1}\overset{s}{\hookrightarrow} L)$, where L is a vector bundle of rank n and degree $d+n(n-1)(g-1)$ on X, and s is an inclusion of

$\mathscr{O}_X$-modules. Following Drinfeld [3] ($n = 2$) and P. Deligne ($n = 1$), Laumon [12] has defined a complex of $\bar{\mathbb{Q}}_\ell$-sheaves ${}_n\mathscr{K}_E^d$ on ${}_n\mathscr{M}_d$, which is a geometric counterpart of $\tilde{\varphi}_E$.* The geometric Langlands conjecture predicts that when E is a smooth geometrically irreducible $\bar{\mathbb{Q}}_\ell$-sheaf of rank n on X, ${}_n\mathscr{K}_E^d$ descends with respect to the projection ${}_n\mathscr{M}_d \to \mathrm{Bun}_n$, where Bun_n is the moduli stack of rank n vector bundles on X.

We establish for any smooth $\bar{\mathbb{Q}}_\ell$-sheaves E_1, E_2 of rank n on X and any $d \geq 0$ a canonical isomorphism

$$\mathrm{R}\Gamma_c({}_n\mathscr{M}_d\,,\, {}_n\mathscr{K}_{E_1^*}^d \otimes {}_n\mathscr{K}_{E_2}^d) \widetilde{\to} \mathrm{R}\Gamma\big(X^{(d)},\, (E_1^* \otimes E_2)^{(d)}\big)(d)[2d],$$

which is a geometric version of (3). In fact, a more general statement is proved.

0.1. Conventions and notation

0.1.1.

Fix an algebraically closed ground field k of characteristic $p > 0$, a prime $\ell \neq p$, and an algebraic closure $\bar{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$. All the schemes and stacks we use are defined over k. Throughout the paper, X denotes a fixed smooth projective connected curve of genus $g \geq 1$ (over k).

We work with algebraic stacks in smooth topology and with (perverse) $\bar{\mathbb{Q}}_\ell$-sheaves on them. If $\mathscr{X}$ is an algebraic stack locally of finite type, then the notion of a (perverse) $\bar{\mathbb{Q}}_\ell$-sheaf on $\mathscr{X}$ localizes in smooth topology and, hence, makes perfect sense. However, the corresponding derived category is problematic. We adopt the point of view that an appropriate formalism exists. (It is partially established in [13].) Let $f : \mathscr{X} \to \mathscr{Y}$ be a morphism of algebraic stacks. The functors $f^*, f_*, f_!$ are understood in the derived category sense.

We say that f is a *generalized affine fibration of rank* m in the following cases: first, if locally in smooth topology on $\mathscr{Y}$ there exists a homomorphism $L \to L'$ of locally free coherent sheaves on $\mathscr{Y}$ and an L'-torsor $\mathscr{Y}' \to \mathscr{Y}$ such that f is identified with $\mathscr{Y}'/L \to \mathscr{Y}$, the quotient being taken in stack sense, and $\mathrm{rk}\, L' - \mathrm{rk}\, L = m$; second, if the map f can be written as the composition of generalized affine fibrations of first type of ranks $m_1, \dots, m_k$ with $\sum m_i = m$. We essentially use the fact that for a generalized affine fibration f of rank m one has $f_! \bar{\mathbb{Q}}_\ell \widetilde{\to} \bar{\mathbb{Q}}_\ell(-m)[-2m]$.

We fix a nontrivial additive character $\psi : \mathbb{F}_p \to \bar{\mathbb{Q}}_\ell^*$ and denote by $\mathscr{L}_\psi$ the Artin-Schreier sheaf on $\mathbb{A}_k^1$ associated to ψ (see [2, Section 1.7]). Fix also a square root of p in $\bar{\mathbb{Q}}_\ell$, and use it to define the sheaf $\bar{\mathbb{Q}}_\ell(1/2)$ over $\mathrm{Spec}\, \mathbb{F}_p$ and, hence, over $\mathrm{Spec}\, k$.

*We normalize ${}_n\mathscr{K}_E^d$ as in Remark 1 (cf. Section 2.1). This also gives a normalization of $\tilde{\varphi}_E$ as the function "trace of Frobenius" of ${}_n\mathscr{K}_E^d$.

0.1.2.

When we say that a stack $\mathscr{Y}$ classifies *something*, it should always be clear what an S-family of *something* is for any k-scheme S; that is, what the groupoid $\operatorname{Hom}(S, \mathscr{Y})$ and what the functors $\operatorname{Hom}(S_2, \mathscr{Y}) \to \operatorname{Hom}(S_1, \mathscr{Y})$ are for each morphism $S_1 \to S_2$.

For example, if $\mathscr{Y}$ is the stack that classifies pairs $M_1 \hookrightarrow M_2$ with M_1 (resp., M_2) being a coherent sheaf on X of generic rank i_1 and of degree d_1 (resp., of generic rank i_2 and of degree d_2), then $\operatorname{Hom}(S, \mathscr{Y})$ is the groupoid whose objects are inclusions $M_1 \hookrightarrow M_2$ of coherent sheaves on $S \times X$ which are S-flat and such that the quotient M_2/M_1 is also S-flat, and for any point $s \in S$ the conditions on the generic rank and on the degree of $M_i \mid_{s \times X}$ $(i = 1, 2)$ hold. Morphisms from an object $M_1 \hookrightarrow M_2$ to an object $M'_1 \hookrightarrow M'_2$ are by definition the isomorphisms $M_1 \widetilde{\to} M'_1$ and $M_2 \widetilde{\to} M'_2$, making the natural diagram commutative.

We denote by Sh_i the moduli stack of coherent sheaves on X of generic rank i. This is an algebraic stack locally of finite type. Its connected components are numbered by $d \in \mathbb{Z}$; the component Sh_i^d classifies coherent sheaves of rank i and of degree d on X. The stack Sh_0^d is, in fact, of finite type.

By $\operatorname{Pic} X \subset \operatorname{Sh}_1$ we denote the open substack classifying invertible $\mathscr{O}_X$-modules. This is the Picard stack of X. Its connected component $\operatorname{Pic}^d X$ classifies line bundles of degree d on X.

Denote by ${}^{\leq n}\operatorname{Sh}_0^d \subset \operatorname{Sh}_0^d$ the open substack given by the following property: For a scheme S an object F of $\operatorname{Hom}(S, \operatorname{Sh}_0^d)$ lies in $\operatorname{Hom}(S, {}^{\leq n}\operatorname{Sh}_0^d)$ if the geometric fibre of F at any point of $X \times S$ is of dimension at most n. We write $X^{(d)}$ for the dth symmetric power of X. The morphism norm is denoted by $\operatorname{div} : \operatorname{Sh}_0^d \to X^{(d)}$ (cf. [10, Section 6]). It sends the $\mathscr{O}_X$-module $\mathscr{O}_{D_1+\dots+D_s} \oplus \mathscr{O}_{D_2+\dots+D_s} \oplus \cdots \oplus \mathscr{O}_{D_s}$ to $D_1 + 2D_2 + \cdots + sD_s$ if $D_1, \dots, D_s$ are effective divisors on X.

0.1.3.

Fix the maximal torus of diagonal matrices in $\operatorname{GL}(n)$ and the Borel subgroup of upper-triangular matrices. Then the set of weights of $\operatorname{GL}(n)$ is identified with $\mathbb{Z}^n$. The fundamental weights are given by $\omega_i = (1, \dots, 1, 0, \dots, 0) \in \mathbb{Z}^n$, where 1 occurs i times $(i = 1, \dots, n)$.

Define the following semigroups $\Lambda_n^+ \subset \Lambda_n \subset \Lambda_n^p$, consisting of weights. Let $\Lambda_n = \mathbb{Z}_+^n$ and $\Lambda_n^p = \{\lambda \in \mathbb{Z}^n \mid \lambda_1 + \cdots + \lambda_i \geq 0 \text{ for all } i\}$. The superscript p should designate that Λ_n^p contains the $\mathbb{Z}_+$-span of positive roots. Set also $\Lambda_n^+ = \{\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0) \mid \lambda_i \in \mathbb{Z}\}$. Similarly, we let $\Lambda_n^- = \{\lambda = (0 \leq \lambda_1 \leq \cdots \leq \lambda_n) \mid \lambda_i \in \mathbb{Z}\}$.

For $d \geq 0$ we also introduce $\Lambda_{n,d} \subset \Lambda_n$, $\Lambda_{n,d}^+ \subset \Lambda_n^+$, and so on, where the subscript d means that we impose the condition $\sum \lambda_i = d$. The half sum of positive roots is denoted by ρ.

For a weight λ of GL(n) we introduce the schemes X_+^λ, X_-^λ, X^λ, and X_p^λ, which should be thought of as the moduli schemes of Λ_n^+(resp., of Λ_n^-, Λ_n, Λ_n^p)-valued divisors on X of degree λ. The precise definition is as follows.

Set $X_p^\lambda = \prod_{i=1}^n X^{(\lambda_1+\cdots+\lambda_i)}$. A point of X_p^λ is a collection of (not necessarily effective) divisors $(D_1,\dots,D_n)$ on X with $D_1+\cdots+D_i \in X^{(\lambda_1+\cdots+\lambda_i)}$. Let $X^\lambda \hookrightarrow X_p^\lambda$ be the closed subscheme given by $D_i \geq 0$ for all i. Let X_+^λ (resp., X_-^λ) be the closed subscheme of X^λ given by $D_1 \geq \cdots \geq D_n$ (resp, $D_1 \leq \cdots \leq D_n$).

Given a closed point (D_i) of X_i^λ with $D_i = \sum_x d_{i,x}x$, we associate to it a divisor on X with values in Λ_n^p. The value of this divisor at x is the weight $(d_{1,x},\dots,d_{n,x})$. In the same way, a closed point of X^λ (resp., X_+^λ, X_-^λ) can be viewed as a Λ_n (resp., Λ_n^+, Λ_n^-)-valued divisor on X.

0.1.4.
For $\lambda \in \Lambda_{n,d}^+$, define the polynomial functor V^λ of a $\bar{\mathbb{Q}}_\ell$-vector space V as follows. Let $\lambda = (\lambda_1,\dots,\lambda_{n'},0,\dots,0)$ with $\lambda_{n'} > 0$. Denote by U^λ the irreducible representation of S_d (over $\mathbb{Q}$) associated to λ. So, for example, if $\lambda = (d,0,\dots,0)$, then $U^\lambda \tilde{\to} \mathbb{Q}$ is trivial, and if $\lambda = (1,\dots,1)$, then U^λ is the signature representation. Set

$$V^\lambda = (V^{\otimes d} \otimes_{\mathbb{Q}} U^\lambda)^{S_d},$$

where it is understood that S_d acts by permutations on $V^{\otimes d}$ and diagonally on the tensor product. If $m = \dim V < n'$, then $V^\lambda = 0$; otherwise, V^λ is the irreducible representation of GL(V) of the highest weight $(\lambda_1,\dots,\lambda_{n'},0,\dots,0) \in \Lambda_{m,d}^+$.

1. Laumon's perverse sheaf $\mathscr{L}_E^d$

Let E be a smooth $\bar{\mathbb{Q}}_\ell$-sheaf on X. Recall the definition of Laumon's perverse sheaf $\mathscr{L}_E^d$ on Sh_0^d associated to E (see [12]). Denote by $\mathrm{sym} : X^d \to X^{(d)}$ the natural map, and consider the smooth $\bar{\mathbb{Q}}_\ell$-sheaf $E^{\boxtimes d}$ on X^d. Notice that $\mathrm{sym}_!(E^{\boxtimes d})[d]$ is a perverse sheaf. Set

$$E^{(d)} = \left(\mathrm{sym}_!(E^{\boxtimes d})\right)^{S_d}.$$

Since $E^{(d)}$ is a direct summand of $\mathrm{sym}_!(E^{\boxtimes d})$, $E^{(d)}[d]$ is also a perverse sheaf.

Denote by $\mathscr{F}l^{1,\dots,1}$ (1 occurs d times) the stack of complete flags $(F_1 \subset \cdots \subset F_d)$, where F_i is a coherent torsion sheaf on X of length i. The morphism $\mathfrak{p} : \mathscr{F}l^{1,\dots,1} \to \mathrm{Sh}_0^d$ that sends $(F_1 \subset \cdots \subset F_d)$ to F_d is representable and proper. The morphism $\mathfrak{q} : \mathscr{F}l^{1,\dots,1} \to \mathrm{Sh}_0^1 \times \cdots \times \mathrm{Sh}_0^1$ that sends $(F_1 \subset \cdots \subset F_d)$ to $(F_1, F_2/F_1, \dots, F_d/F_{d-1})$ is a generalized affine fibration. This, in particular, implies that $\mathscr{F}l^{1,\dots,1}$ is smooth.

Springer's sheaf Spr_E^d on Sh_0^d is defined as

$$\mathrm{Spr}_E^d = \mathfrak{p}_!\mathfrak{q}^*(\mathrm{div}^{\times d})^*(E^{\boxtimes d}).$$

Since $\mathfrak{p}$ is small, Spr^d_E is a perverse sheaf that coincides with the Goresky-MacPherson extension of its restriction to any nonempty open substack of Sh^d_0. It also carries a natural S_d-action (cf. [12, Theorem 3.3.1]). Set

$$\mathscr{L}^d_E = \mathrm{Hom}_{S_d}(\mathrm{triv}, \mathrm{Spr}^d_E),$$

where triv denotes the trivial representation of the symmetric group S_d. Again, $\mathscr{L}^d_E$ is a direct summand of Spr^d_E, so $\mathscr{L}^d_E$ is perverse and coincides with the Goresky-MacPherson extension of its restriction to any nonempty open substack of Sh^d_0. We have a smooth morphism $X^{(d)} \to \mathrm{Sh}^d_0$ that sends a divisor D to $\mathscr{O}_D$, and the pullback of $\mathscr{L}^d_E$ under this map is identified with $E^{(d)}$.

2. Main results

2.1

Fix $n > 0$, $d \geq 0$. Let Ω be the canonical invertible sheaf on X. Denote by ${}_n\mathscr{Q}_d$ the stack that classifies collections

$$\big(0 = L_0 \subset L_1 \subset \cdots \subset L_n \subset L, \ \ (s_i)\big), \tag{4}$$

where $L_n \subset L$ is a modification of rank n vector bundles on X with $\deg(L/L_n) = d$, (L_i) is a complete flag of subbundles on L_n, and $s_i : \Omega^{n-i} \widetilde{\to} L_i/L_{i-1}$ is an isomorphism ($i = 1, \dots, n$). We have a map $\mu : {}_n\mathscr{Q}_d \to \mathbb{A}^1_k$ that at the level of k-points sends the above collection to the sum of $n-1$ classes in

$$k \widetilde{\to} \mathrm{Ext}^1(\Omega^{n-i-1}, \Omega^{n-i}) \widetilde{\to} \mathrm{Ext}^1(L_{i+1}/L_i, L_i/L_{i-1})$$

which correspond to the succesive extensions $0 \to L_i/L_{i-1} \to L_{i+1}/L_{i-1} \to L_{i+1}/L_i \to 0$.

Let $\beta : {}_n\mathscr{Q}_d \to {}^{\leq n}\mathrm{Sh}^d_0$ be the map that sends (4) to L/L_n. It is of finite type and smooth of relative dimension $b = b(n,d) = nd + (1-g)\sum_{i=1}^{n-1} i^2$. Therefore, ${}_n\mathscr{Q}_d$ is smooth and of finite type. So, if E is a smooth $\bar{\mathbb{Q}}_\ell$-sheaf on X, then on ${}_n\mathscr{Q}_d$ we have a perverse $\bar{\mathbb{Q}}_\ell$-sheaf

$${}_n\mathscr{F}^d_{E,\psi} = \beta^*\mathscr{L}^d_E \otimes \mu^*\mathscr{L}_\psi[b]\Big(\frac{b}{2}\Big).$$

Let $\pi_0 : {}_n\mathscr{Q}_d \to X^{(d)}$ be the map that sends (4) to the divisor $D \in X^{(d)}$ for which the inclusion of invertible sheaves $\wedge^n L_n \hookrightarrow \wedge^n L$ induces an isomorphism $\wedge^n L_n(D) \widetilde{\to} \wedge^n L$. We also have a map $X^{(d)} \to \mathrm{Pic}^d X$ that sends a divisor D to $\mathscr{O}_X(D)$.

Let ${}_n\mathscr{M}_d$ be the stack classifying pairs $(\Omega^{n-1} \hookrightarrow L)$, where L is an n-bundle on X with

$$\deg L - \deg(\Omega^{(n-1)+(n-2)+\cdots+(n-n)}) = d.$$

The forgetful map $\zeta : {}_n\mathcal{Q}_d \to {}_n\mathcal{M}_d$ is representable, and the following diagram commutes:

$$\begin{array}{ccc} {}_n\mathcal{Q}_d & \stackrel{\pi_0}{\to} & X^{(d)} \\ \downarrow \zeta & & \downarrow \\ {}_n\mathcal{M}_d & \stackrel{\theta}{\to} & \mathrm{Pic}^d X \end{array}$$

where θ is the map that sends $(\Omega^{n-1} \hookrightarrow L)$ to $\det L \otimes \Omega^{(1-n)+(2-n)+\cdots+(n-n)}$. Denote by

$$\pi : {}_n\mathcal{Q}_d \times_{{}_n\mathcal{M}_d} {}_n\mathcal{Q}_d \to X^{(d)} \times_{\mathrm{Pic}^d X} X^{(d)}$$

the morphism $\pi_0 \times \pi_0$. Since $X^{(d)} \to \mathrm{Pic}^d X$ is representable and separated, the diagonal map $i : X^{(d)} \to X^{(d)} \times_{\mathrm{Pic}^d X} X^{(d)}$ is a closed immersion. Our main result is the next theorem.

MAIN LOCAL THEOREM

For any smooth $\bar{\mathbb{Q}}_\ell$-sheaves E, E' on X of ranks m, m', respectively, with $\min\{m, m'\} \leq n$*, there exists a canonical isomorphism*

$$\pi_!({}_n\mathcal{F}^d_{E,\psi} \boxtimes {}_n\mathcal{F}^d_{E',\psi^{-1}}) \xrightarrow{\sim} i_*(E \otimes E')^{(d)}(d)[2d]$$

in the derived category on $X^{(d)} \times_{\mathrm{Pic}^d X} X^{(d)}$.

Remark 1

(i) The stack ${}_n\mathcal{Q}_d \times_{{}_n\mathcal{M}_d} {}_n\mathcal{Q}_d$ is of finite type, though ${}_n\mathcal{M}_d$ is not, so that π is of finite type but not representable.

(ii) Define the complex ${}_n\mathcal{K}^d_E$ on ${}_n\mathcal{M}_d$ as ${}_n\mathcal{K}^d_E = \zeta_!({}_n\mathcal{F}^d_{E,\psi})$. The geometric Langlands conjecture claims that if E is a smooth irreducible $\bar{\mathbb{Q}}_\ell$-sheaf of rank n on X, then for each $d \geq 0$ the complex ${}_n\mathcal{K}^d_E$ descends with respect to the projection ${}_n\mathcal{M}_d \to \mathrm{Bun}_n$.

2.2

Actually, we prove a more general statement. Recall that for $\bar{\mathbb{Q}}_\ell$-vector spaces E, E' of dimensions m, m', respectively, we have

$$\mathrm{Sym}^d(E \otimes E') = \bigoplus_{\lambda \in \Lambda^+_{r,d}} E^\lambda \otimes (E')^\lambda,$$

where $r = \min\{m, m'\}$. To formulate the version of the Main Local Theorem that we actually prove, we globalize the above equality as follows.

For $\lambda \in \Lambda^+_{n,d}$ and a smooth $\bar{\mathbb{Q}}_\ell$-sheaf E on X, we define a constructable $\bar{\mathbb{Q}}_\ell$-sheaf E^λ_+ on X^λ_+ (cf. Section 3.1) which is a global analog of the corresponding polynomial functor. The fibre of E_+^λ at $D = \sum_x \lambda_x x$ is the tensor product over closed points of X,

$$\bigotimes_{x\in X} (E_x)^{\lambda_x},$$

where E_x denotes the fibre of E at x. For example, for $\lambda = (d, 0, \dots, 0)$ we have $X_+^\lambda = X^{(d)}$ and $E_+^\lambda = E^{(d)}$. Another example is that for $\lambda = \omega_i$ we obtain $X_+^\lambda = X$ and $E_+^\lambda = \wedge^i E$.

Denote by $\pi^\lambda : X_+^\lambda \to X^{(d)}$ the map that sends $(D_1 \geq \cdots \geq D_n \geq 0) \in X_+^\lambda$ to $\sum D_i$.

LEMMA 1
For any smooth $\bar{\mathbb{Q}}_\ell$-sheaves E, E' on X of ranks m, m', respectively, there is a canonical filtration

$$0 = {}^{\leq 0}(E\otimes E')^{(d)} \subset {}^{\leq 1}(E\otimes E')^{(d)} \subset \cdots$$

on $(E\otimes E')^{(d)}$ by constructable subsheaves with the following property. First, if $\min\{m, m'\} \leq n$, then ${}^{\leq n}(E\otimes E')^{(d)} = (E\otimes E')^{(d)}$. Second, there is a canonical refinement of this filtration such that

$$\mathrm{gr}\ {}^{\leq n}(E\otimes E')^{(d)} \widetilde{\to} \bigoplus_{\lambda\in\Lambda^+_{n,d}} \pi^\lambda_*(E^\lambda_+ \otimes E'^\lambda_+)$$

for each n.

MAIN LOCAL THEOREM$_n$
For any smooth $\bar{\mathbb{Q}}_\ell$-sheaves E, E' on X there exists a canonical isomorphism

$$\pi_!({}_n\mathscr{F}^d_{E,\psi} \boxtimes {}_n\mathscr{F}^d_{E',\psi^{-1}}) \widetilde{\to} i_*\, {}^{\leq n}(E\otimes E')^{(d)}(d)[2d]$$

in the derived category on $X^{(d)} \times_{\mathrm{Pic}^d X} X^{(d)}$.

2.3
The proof consists of the following steps. Let us denote by ${}_n\mathscr{X}_d$ the stack classifying collections $(L, (t_i))$, where L is a vector bundle on X of rank n,

$$t_i : \Omega^{(n-1)+(n-2)+\cdots+(n-i)} \hookrightarrow \wedge^i L$$

is an inclusion of $\mathscr{O}_X$-modules ($i = 1, \dots, n$), and

$$\deg L - \deg(\Omega^{(n-1)+(n-2)+\cdots+(n-n)}) = d.$$

Given an object of ${}_n\mathscr{Q}_d$, we get the morphisms

$$t_i : \Omega^{(n-1)+\cdots+(n-i)} \widetilde{\to} \wedge^i L_i \hookrightarrow \wedge^i L.$$

This defines a map $\varphi : {}_n\mathcal{Q}_d \to {}_n\mathcal{X}_d$. Notice that $\zeta : {}_n\mathcal{Q}_d \to {}_n\mathcal{M}_d$ factors as ${}_n\mathcal{Q}_d \stackrel{\varphi}{\to} {}_n\mathcal{X}_d \to {}_n\mathcal{M}_d$, where the second arrow is the forgetful map. Since ${}_n\mathcal{X}_d \to {}_n\mathcal{M}_d$ is representable and separated, the natural map ${}_n\mathcal{Q}_d \times_{{}_n\mathcal{X}_d} {}_n\mathcal{Q}_d \to {}_n\mathcal{Q}_d \times_{{}_n\mathcal{M}_d} {}_n\mathcal{Q}_d$ is a closed immersion. Let

$$\pi' : {}_n\mathcal{Q}_d \times_{{}_n\mathcal{X}_d} {}_n\mathcal{Q}_d \to X^{(d)} \times_{\mathrm{Pic}^d X} X^{(d)}$$

be the restriction of π to ${}_n\mathcal{Q}_d \times_{{}_n\mathcal{X}_d} {}_n\mathcal{Q}_d$. The first step is to establish the following result.

THEOREM A
For any smooth $\bar{\mathbb{Q}}_\ell$-sheaves E, E' on X, the natural map

$$\pi_!({}_n\mathcal{F}^d_{E,\psi} \boxtimes {}_n\mathcal{F}^d_{E',\psi^{-1}}) \to \pi'_!({}_n\mathcal{F}^d_{E,\psi} \boxtimes {}_n\mathcal{F}^d_{E',\psi^{-1}})$$

is an isomorphism.

Our proof of Theorem A is based on Proposition 1, which is a corollary of the geometric Casselman-Shalika formula for GL(n) (cf. [16], [7], [17]). We present it in Section 3, written independently of the rest of the paper.

The second step is as follows. Let $\phi : {}_n\mathcal{X}_d \to X^{(d)}$ be the map that sends $(L, (t_i))$ to the divisor $D \in X^{(d)}$ such that t_n induces an isomorphism

$$\Omega^{(n-1)+\cdots+(n-n)}(D) \stackrel{\sim}{\to} \wedge^n L,$$

so that $\phi \circ \varphi = \pi_0$. We write $f : {}_n\mathcal{Q}_d \times_{{}_n\mathcal{X}_d} {}_n\mathcal{Q}_d \to X^{(d)}$ for the composition ${}_n\mathcal{Q}_d \times_{{}_n\mathcal{X}_d} {}_n\mathcal{Q}_d \to {}_n\mathcal{X}_d \stackrel{\phi}{\to} X^{(d)}$, where the first map is the natural projection. The morphism f is of finite type but not representable. Since the diagram

$$\begin{array}{ccc} {}_n\mathcal{Q}_d \times_{{}_n\mathcal{X}_d} {}_n\mathcal{Q}_d & \hookrightarrow & {}_n\mathcal{Q}_d \times_{{}_n\mathcal{M}_d} {}_n\mathcal{Q}_d \\ \downarrow f & & \downarrow \pi \\ X^{(d)} & \stackrel{i}{\to} & X^{(d)} \times_{\mathrm{Pic}^d X} X^{(d)} \end{array}$$

commutes, the Main Local Theorem is just a combination of Theorem A with the following result.

THEOREM B
For any smooth $\bar{\mathbb{Q}}_\ell$-sheaves E, E' on X, there is a canonical isomorphism

$$f_!({}_n\mathcal{F}^d_{E,\psi} \boxtimes {}_n\mathcal{F}^d_{E',\psi^{-1}}) \stackrel{\sim}{\to} {}^{\leq n}(E \otimes E')^{(d)}(d)[2d]. \quad (5)$$

We present two different proofs of Theorem B. In the first proof, which occupies Sections 6.1–6.5, we derive Theorem B from the following result.

THEOREM C
Let ${}^{\le n}\operatorname{div} : {}^{\le n}\operatorname{Sh}_0^d \to X^{(d)}$ *denote the restriction of* $\operatorname{div} : \operatorname{Sh}_0^d \to X^{(d)}$. *For any smooth* $\bar{\mathbb{Q}}_\ell$*-sheaves* E, E' *on* X, *the complex* $({}^{\le n}\operatorname{div})_!(\mathscr{L}_E^d \otimes \mathscr{L}_{E'}^d)$ *is placed in degrees less than or equal to* $-2d$, *and for the highest cohomology sheaf of this complex we have canonically*

$$\mathrm{R}^{-2d}({}^{\le n}\operatorname{div})_!(\mathscr{L}_E^d \otimes \mathscr{L}_{E'}^d)(-d) \widetilde{\to} {}^{\le n}(E \otimes E')^{(d)}.$$

In Section 6.6 we present an alternative proof of Theorem B. The idea of this proof was communicated to the author by D. Gaitsgory. This proof requires the additional assumption $\min\{\operatorname{rk} E, \operatorname{rk} E'\} \le n$. The reader interested in the proof of the Main Local Theorem under this assumption may skip Sections 6.1–6.5.

3. Around the geometric Casselman-Shalika formula for $\mathrm{GL}(n)$

3.1
The purpose of Section 3 is to present Proposition 1, which is a corollary of the geometric Casselman-Shalika formulae for $\mathrm{GL}(n)$ (cf. [16], [7], [17]). To formulate it we introduce some notation.

Fix $\lambda \in \Lambda_{n,d}^-$. Recall that X_-^λ is the scheme of collections $(D_1, \dots, D_n)$, where D_i is an effective divisor on X of degree λ_i with $D_1 \le \dots \le D_n$. Let

$$i_\lambda : X_-^\lambda \to {}^{\le n}\operatorname{Sh}_0^d$$

be the map that sends $(D_1, \dots, D_n)$ to

$$\Omega^{n-1}(D_1)/\Omega^{n-1} \oplus \Omega^{n-2}(D_2)/\Omega^{n-2} \oplus \dots \oplus \mathscr{O}(D_n)/\mathscr{O}.$$

According to [12, Theorem 3.3.8], if E is a smooth $\bar{\mathbb{Q}}_\ell$-sheaf on X, then the complex $i_\lambda^*\mathscr{L}_E^d$ is placed in degrees less than or equal to $2a(\lambda)$ with respect to the usual t-structure, where

$$a(\lambda) \stackrel{\text{def}}{=} \big\langle \lambda, (n-1, n-2, \dots, 0)\big\rangle.$$

Moreover, if $m \in \mathbb{N}$ is such that $\lambda = (0, \dots, 0, \lambda_{n-m+1}, \dots, \lambda_n)$ with $\lambda_{n-m+1} > 0$, then the $2a(\lambda)$ th cohomology sheaf of $i_\lambda^*\mathscr{L}_E^d$ vanishes if and only if $\operatorname{rk} E < m$.

For a weight $\lambda = (\lambda_1, \dots, \lambda_n)$, set $\lambda^t = (\lambda_n, \dots, \lambda_1)$. Also, denote by $t : X_-^\lambda \widetilde{\to} X_+^{\lambda^t}$ the isomorphism that sends $(D_1, \dots, D_n)$ to $(D_n, \dots, D_1)$.

Definition 1
For any smooth $\bar{\mathbb{Q}}_\ell$-sheaf E on X, define the sheaf E_-^λ on X_-^λ by

$$E_-^\lambda = \mathscr{H}^{2a(\lambda)}(i_\lambda^*\mathscr{L}_E^d)\big(a(\lambda)\big).$$

Also, define the sheaf $E_+^{\lambda^t}$ on $X_+^{\lambda^t}$ by $E_+^{\lambda^t} = t_* E_-^{\lambda}$.

Let $S^\lambda \to X_-^\lambda$ be the vector bundle whose fibre over $(D_1, \dots, D_n)$ is the vector space of collections $(\sigma_1, \dots, \sigma_{n-1})$, where

$$\sigma_i \in \mathrm{Hom}\big(\Omega^{n-i-1}, \Omega^{n-i}(D_i)/\Omega^{n-i}\big).$$

By $\mu_S : S^\lambda \to \mathbb{A}^1$ we denote the map that at the level of k-points sends $(\sigma_1, \dots, \sigma_{n-1})$ to the sum of $n-1$ classes in $k \xrightarrow{\sim} \mathrm{Ext}^1(\Omega^{n-i-1}, \Omega^{n-i})$ corresponding to the pullbacks of

$$0 \to \Omega^{n-i} \to \Omega^{n-i}(D_i) \to \Omega^{n-i}(D_i)/\Omega^{n-i} \to 0$$

with respect to $\sigma_i : \Omega^{n-i-1} \to \Omega^{n-i}(D_i)/\Omega^{n-i}$.

Let $\mathscr{W}^\lambda$ be the following stack of collections: $(D_1, \dots, D_n) \in X_-^\lambda$ and a flag $(F^1 \subset \cdots \subset F^n)$ of coherent torsion sheaves on X with trivializations

$$F^i/F^{i-1} \xrightarrow{\sim} \Omega^{n-i}(D_i)/\Omega^{n-i}$$

for $i = 1, \dots, n$. The projection $\tau : \mathscr{W}^\lambda \to X_-^\lambda$ is a generalized affine fibration of rank zero.

Let $\kappa : \mathscr{W}^\lambda \to S^\lambda$ be the morphism over X_-^λ defined as follows. Given an S-point of $\mathscr{W}^\lambda$, consider for $i = 1, \dots, n-1$ the exact sequence

$$0 \to \Omega^{n-i-1} \to \Omega^{n-i-1}(D_{i+1}) \to \Omega^{n-i-1}(D_{i+1})/\Omega^{n-i-1} \to 0.$$

(Here Ω should be understood as the sheaf of relative differentials $\Omega_{S\times X/S}$.) It induces a map

$$\mathscr{H}om\big(\Omega^{n-i-1}, \Omega^{n-i}(D_i)/\Omega^{n-i}\big) \to \mathscr{E}xt^1\big(\Omega^{n-i-1}(D_{i+1})/\Omega^{n-i-1}, \Omega^{n-i}(D_i)/\Omega^{n-i}\big) \quad (6)$$

which is an isomorphism of $\mathscr{O}_{S\times X}$-modules, because $D_{i+1} \geq D_i$. The map κ sends this point of $\mathscr{W}^\lambda$ to $(\sigma_1, \dots, \sigma_{n-1})$, where σ_i is the global section of $\mathscr{H}om(\Omega^{n-i-1}, \Omega^{n-i}(D_i)/\Omega^{n-i})$, whose image under (6) corresponds to the extension

$$0 \to F^i/F^{i-1} \to F^{i+1}/F^{i-1} \to F^{i+1}/F^i \to 0.$$

Denote also by $\beta_{\mathscr{W}} : \mathscr{W}^\lambda \to {}^{\leq n}\mathrm{Sh}_0^d$ the morphism that sends $(F^1 \subset \cdots \subset F^n)$ to F^n.

PROPOSITION 1

For any smooth $\bar{\mathbb{Q}}_\ell$-sheaf E on X, there is a canonical isomorphism

$$\tau_!\,(\beta_{\mathscr{W}}^* \mathscr{L}_E^d \otimes \kappa^* \mu_S^* \mathscr{L}_\psi) \xrightarrow{\sim} E_-^\lambda. \quad (7)$$

The proof is given in Section 3.3.

3.2. Local lemma

For the convenience of the reader, we begin with a local counterpart of Proposition 1, working completely in a local setting.

Let $\mathscr{O}$ be a complete local k-algebra with residue field k, which is regular of dimension one (that is, choosing a generator ω of the maximal ideal $\mathfrak{m} \subset \mathscr{O}$, one identifies $\mathscr{O}$ with the ring $k[[\omega]]$ of formal power series of one variable). Denote by K the field of fractions of $\mathscr{O}$. Let Ω be the completed module of relative differentials of $\mathscr{O}$ over k (so, Ω is a free $\mathscr{O}$-module generated by $d\omega$). For $i \geq 0$ we write Ω^i for the ith tensor power of Ω (over $\mathscr{O}$). For an integer m, denote by $\Omega^i(m) \subset \Omega^i \otimes_{\mathscr{O}} K$ the $\mathscr{O}$-submodule generated by $\omega^{-m} d\omega^{\otimes i}$.

Recall that we have fixed $\lambda \in \Lambda^-_{n,d}$. Consider the stack $\mathscr{W}^\lambda_{\mathrm{loc}}$ classifying collections: a flag of torsion sheaves $(F^1 \subset \cdots \subset F^n)$ over $\operatorname{Spf} \mathscr{O}$ with trivializations

$$F^i/F^{i-1} \xrightarrow{\sim} \Omega^{n-i}(\lambda_i)/\Omega^{n-i}$$

for $i = 1, \ldots, n$. (The subscript "loc" stands for local counterparts of certain stacks or morphisms.) Clearly, $\mathscr{W}^\lambda_{\mathrm{loc}} \to \operatorname{Spec} k$ is a generalized affine fibration of rank zero. We also have the scheme S^λ_{loc} whose set of k-points is the set of $(\sigma_1, \ldots, \sigma_{n-1})$ with

$$\sigma_i \in \operatorname{Hom}\big(\Omega^{n-i-1}, \Omega^{n-i}(\lambda_i)/\Omega^{n-i}\big).$$

Besides, we have a map $(\mu_S)_{\mathrm{loc}} : S^\lambda_{\mathrm{loc}} \to \mathbb{A}^1$ that at the level of k-points sends $(\sigma_1, \ldots, \sigma_{n-1})$ to $\sum \operatorname{res} \sigma_i$. One also defines a morphism $\kappa_{\mathrm{loc}} : \mathscr{W}^\lambda_{\mathrm{loc}} \to S^\lambda_{\mathrm{loc}}$ in the same way as κ.

Let ${}^{\leq n}\operatorname{Sh}^d_0(\mathscr{O})$ be the stack classifying coherent torsion sheaves F on $\operatorname{Spf} \mathscr{O}$ of length d for which $\dim(F \otimes_{\mathscr{O}} k) \leq n$. It is stratified by locally closed substacks $\operatorname{Sh}^\nu(\mathscr{O})$ indexed by $\nu \in \Lambda^+_{n,d}$. The stratum $\operatorname{Sh}^\nu(\mathscr{O})$ classifies sheaves isomorphic to

$$\mathscr{O}/\mathfrak{m}^{\nu_1} \oplus \cdots \oplus \mathscr{O}/\mathfrak{m}^{\nu_n}.$$

Let $\mathscr{B}_\nu$ be the intersection cohomology sheaf associated to the constant sheaf on the stratum $\operatorname{Sh}^\nu(\mathscr{O})$. Let

$$\beta_{\mathscr{W},\mathrm{loc}} : \mathscr{W}^\lambda_{\mathrm{loc}} \to {}^{\leq n}\operatorname{Sh}^d_0(\mathscr{O})$$

be the map that sends $(F^1 \subset \cdots \subset F^n)$ to F^n.

A local version of Proposition 1 can be stated as follows.

LEMMA 2

For any $\nu \in \Lambda^+_{n,d}$, we have canonically

$$\mathrm{R}\Gamma_c\big(\mathscr{W}^\lambda_{\mathrm{loc}},\ \ \beta^*_{\mathscr{W},\mathrm{loc}}\mathscr{B}_\nu \otimes \kappa^*_{\mathrm{loc}}(\mu_S)^*_{\mathrm{loc}}\mathscr{L}_\psi\big) \xrightarrow{\sim} \begin{cases} 0 & \text{if } \nu^t \neq \lambda, \\ \bar{\mathbb{Q}}_\ell[-d](a(\lambda)) & \text{if } \nu^t = \lambda. \end{cases}$$

Proof

Denote by $\check{\mathscr{W}}^{\lambda}_{\mathrm{loc}}$ the stack of collections: a flag of torsion sheaves $(\check{F}^1 \subset \cdots \subset \check{F}^n)$ on Spf $\mathscr{O}$ with trivializations

$$\check{F}^i/\check{F}^{i-1} \widetilde{\to} \Omega^{n-i}/\Omega^{n-i}(-\lambda_{n-i+1})$$

for $i = 1, \ldots, n$. We have an isomorphism $\mathscr{W}^{\lambda}_{\mathrm{loc}} \widetilde{\to} \check{\mathscr{W}}^{\lambda}_{\mathrm{loc}}$ that sends $(F^1 \subset \cdots \subset F^n)$ to the flag $(\check{F}^1 \subset \cdots \subset \check{F}^n)$ with

$$\check{F}^i = \mathscr{E}xt^1(F^n/F^{n-i}, \Omega^{n-1})$$

for $i = 1, \ldots, n$. This duality allows us to switch between dominant and antidominant weights of GL(n).

Put $\check{L}_i = \Omega^{n-1} \oplus \cdots \oplus \Omega^{n-i}$ for $i = 1, \ldots, n$. Denote by $\mathscr{G}r^{d,+}(\check{L}_n)$ the moduli scheme of $\mathscr{O}$-sublattices $\check{\mathscr{R}} \subset \check{L}_n$ such that

$$\dim(\check{L}_n/\check{\mathscr{R}}) = d.$$

Choosing a trivialization $\check{L}_n \widetilde{\to} \mathscr{O}^n$, one identifies this scheme with the connected component $\mathscr{G}r^{d,+} = \mathscr{G}r^{d,+}(\mathscr{O}^n)$ of the positive part of the affine grassmanian for GL(n).

We have a locally closed subscheme $\check{S} \hookrightarrow \mathscr{G}r^{d,+}(\check{L}_n)$ whose set of k-points consists of $\check{\mathscr{R}}$ with the following property. If $\check{\mathscr{R}}_i = \check{\mathscr{R}} \cap \check{L}_i$, then the image of the inclusion $\check{\mathscr{R}}_i/\check{\mathscr{R}}_{i-1} \hookrightarrow \Omega^{n-i}$ is $\Omega^{n-i}(-\lambda_{n-i+1})$ for $i = 1, \ldots, n$.

We have a map $\check{\eta}_{\mathrm{loc}} : \check{S} \to \check{\mathscr{W}}^{\lambda}_{\mathrm{loc}}$ given by $\check{F}^i = \check{L}_i/\check{\mathscr{R}}_i$ for $i = 1, \ldots, n$. One checks that $\check{\eta}_{\mathrm{loc}}$ is an affine fibration of rank $a(\lambda^t)$. We also have a smooth and surjective map

$$\mathscr{G}r^{d,+}(\check{L}_n) \to {}^{\leq n}\,\mathrm{Sh}^d_0(\mathscr{O})$$

that sends $\check{\mathscr{R}} \subset \check{L}_n$ to $\check{L}_n/\check{\mathscr{R}}$, and we denote by $\mathscr{G}r^{\nu}(\check{L}_n)$ the preimage of the stratum $\mathrm{Sh}^{\nu}(\mathscr{O})$ under this map. The Goresky-MacPherson extension of $\bar{\mathbb{Q}}_\ell[2\langle\nu, \rho\rangle](\langle\nu, \rho\rangle)$ from $\mathscr{G}r^{\nu}(\check{L}_n)$ to its closure is a perverse sheaf denoted by $\mathscr{A}_\nu$ (cf. [6]). So, our assertion is nothing but the geometric Casselman-Shalika formulae (cf. [16], [7], [17]):

$$\mathrm{R}\Gamma_c\big(\check{S},\ \mathscr{A}_\nu \otimes \check{\eta}^*_{\mathrm{loc}}\kappa^*_{\mathrm{loc}}(\mu_S)^*_{\mathrm{loc}}\mathscr{L}_\psi\big) \widetilde{\to} \begin{cases} 0 & \text{if } \nu^t \neq \lambda, \\ \bar{\mathbb{Q}}_\ell[-2\langle\nu, \rho\rangle](-\langle\nu, \rho\rangle) & \text{if } \nu^t = \lambda. \end{cases} \qquad \square$$

We need Lemma 2 in a slightly different form. Put $L_i = \Omega^{n-1}(\lambda_1) \oplus \cdots \oplus \Omega^{n-i}(\lambda_i)$ for $i = 1, \ldots, n$. Let $\mathscr{G}r^{d,+}(L_n)$ be the moduli scheme of $\mathscr{O}$-sublattices $\mathscr{R} \subset L_n$ such that $\dim(L_n/\mathscr{R}) = d$. As in the proof of Lemma 2, on $\mathscr{G}r^{d,+}(L_n)$ we get a stratification by locally closed subschemes $\mathscr{G}r^{\nu}(L_n)$ indexed by $\nu \in \Lambda^+_{n,d}$, and the perverse sheaves $\mathscr{A}_\nu$.

By $S \subset \mathcal{G}r^{d,+}(L_n)$ we denote the locally closed subscheme whose set of k-points consists of sublattices $\mathcal{R}$ with the following property. Let $\mathcal{R}_i = \mathcal{R} \cap L_i$. The condition is that the image of the inclusion

$$\mathcal{R}_i/\mathcal{R}_{i-1} \hookrightarrow L_i/L_{i-1} \widetilde{\to} \Omega^{n-i}(\lambda_i)$$

is the sublattice $\Omega^{n-i} \subset \Omega^{n-i}(\lambda_i)$ for $i = 1, \dots, n$. We also have a map $\eta_{\mathrm{loc}} : S \to \mathcal{W}^{\lambda}_{\mathrm{loc}}$ given by $F^i = L_i/\mathcal{R}_i$ for $i = 1, \dots, n$. This is an affine fibration of rank $a(\lambda)$.

Let $i : \operatorname{Spec} k \to S$ be the distinguished point that corresponds to $\mathcal{R} = \Omega^{n-1} \oplus \dots \oplus \mathcal{O}$. Put $\mu_{\mathrm{loc}} = (\mu_S)_{\mathrm{loc}} \circ \kappa_{\mathrm{loc}} \circ \eta_{\mathrm{loc}}$. We use Lemma 2 under the following form.

LEMMA 3
For any $\nu \in \Lambda^+_{n,d}$ we have canonically

$$\mathrm{R}\Gamma_c(S, \ \mathcal{A}_\nu \otimes \mu^*_{\mathrm{loc}}\mathcal{L}_\psi) \widetilde{\to} \begin{cases} 0 & \text{if } \nu^t \neq \lambda, \\ \bar{\mathbb{Q}}_\ell[-2\langle\lambda, \rho\rangle](-\langle\lambda, \rho\rangle) & \text{if } \nu^t = \lambda. \end{cases}$$

Moreover, this isomorphism is obtained by applying the functor $\mathrm{R}\Gamma_c$ *to the composition of the canonical maps*

$$\mathcal{A}_\nu \otimes \mu^*_{\mathrm{loc}}\mathcal{L}_\psi \to i_*i^*(\mathcal{A}_\nu \otimes \mu^*_{\mathrm{loc}}\mathcal{L}_\psi) \to \tau_{\geq 2\langle\lambda,\rho\rangle}\big(i_*i^*(\mathcal{A}_\nu \otimes \mu^*_{\mathrm{loc}}\mathcal{L}_\psi)\big).$$

Proof
As is easy to see, if $\nu^t \neq \lambda$, then the fibre $i^*(\mathcal{A}_\nu \otimes \mu^*_{\mathrm{loc}}\mathcal{L}_\psi)$ is placed in (usual) degrees strictly less then $2\langle\lambda, \rho\rangle$. For $\nu^t = \lambda$ this fibre equals

$$\bar{\mathbb{Q}}_\ell\big[-2\langle\lambda, \rho\rangle\big]\big(-\langle\lambda, \rho\rangle\big).$$

Since the closure of $\mathcal{G}r^{\lambda^t}(L_n)$ in $\mathcal{G}r^{d,+}(L_n)$ is the union of strata $\mathcal{G}r^{\nu}(L_n)$ with $\nu \leq \lambda^t$, our assertion follows from Lemma 2 combined with the geometric statement due to B. C. Ngo (cf. [17, Lemma 5.2]):

(i) $S \cap \mathcal{G}r^{\nu}(L_n) = \emptyset$ for $\nu < \lambda^t$,

(ii) $S \cap \mathcal{G}r^{\nu}(L_n)$ is the point $\operatorname{Spec} k \overset{i}{\hookrightarrow} S$ for $\nu = \lambda^t$. □

Remark 2
The shift in degree and the twist are calculated using the following two formulae. For any weight ν of $\mathrm{GL}(n)$ we have $a(\nu) - a(\nu^t) = 2\langle\nu, \rho\rangle$ and $2a(\nu) - 2\langle\nu, \rho\rangle = d(n-1)$, where $d = \sum \nu_i$.

3.3. Proof of Proposition 1
We return to our notation in the global case (as in Section 3.1).

Step 1. Denote by $\widetilde{\mathscr{W}}^\lambda$ the scheme of collections: $(D_1, \dots, D_n) \in X^\lambda_-$, a diagram

$$\begin{array}{ccc} L_1 & \subset \cdots \subset & L_n \\ \cup & & \cup \\ \mathscr{R}_1 & \subset \cdots \subset & \mathscr{R}_n \end{array} \qquad (8)$$

where $L_i = \Omega^{n-1}(D_1) \oplus \cdots \oplus \Omega^{n-i}(D_i)$ for $i = 1, \dots, n$, and $(\mathscr{R}_i)$ is a complete flag of vector subbundles on an n-bundle $\mathscr{R}_n$ such that the natural map

$$\mathscr{R}_i/\mathscr{R}_{i-1} \hookrightarrow L_i/L_{i-1} \widetilde{\to} \Omega^{n-i}(D_i)$$

induces an isomorphism $\mathscr{R}_i/\mathscr{R}_{i-1} \widetilde{\to} \Omega^{n-i}$. We have a map $\eta : \widetilde{\mathscr{W}}^\lambda \to \mathscr{W}^\lambda$ over X^λ_- given by

$$F^i = L_i/\mathscr{R}_i$$

for $i = 1, \dots, n$. This is an affine fibration of rank $a(\lambda)$, so that

$$\eta_! \bar{\mathbb{Q}}_\ell \widetilde{\to} \bar{\mathbb{Q}}_\ell\big(-a(\lambda)\big)\big[-2a(\lambda)\big].$$

Put $\tilde{\tau} = \tau \circ \eta$. We replace the functor $\tau_!(\cdot)$ by

$$\tilde{\tau}_! \eta^*(\cdot)\big(a(\lambda)\big)\big[2a(\lambda)\big].$$

The advantage is that $\tilde{\tau}$ is representable whereas τ is not.

The morphism $\tilde{\tau} : \widetilde{\mathscr{W}}^\lambda \to X^\lambda_-$ admits a canonical section $\xi : X^\lambda_- \to \widetilde{\mathscr{W}}^\lambda$ defined by

$$\mathscr{R}_i = \Omega^{n-1} \oplus \cdots \oplus \Omega^{n-i}$$

for $i = 1, \dots, n$. Notice that ξ is a closed immersion. The following diagram commutes:

$$\begin{array}{ccccc} \mathscr{W}^\lambda & \overset{\eta}{\leftarrow} & \widetilde{\mathscr{W}}^\lambda & \overset{\tilde{\tau}}{\to} & X^\lambda_- \\ \downarrow \beta_{\mathscr{W}} & & \uparrow \xi & \nearrow \text{id} & \\ {}_{\le n}\mathrm{Sh}^d_0 & \overset{i_\lambda}{\leftarrow} & X^\lambda_- & & \end{array}$$

Besides, the composition

$$X^\lambda_- \overset{\xi}{\to} \widetilde{\mathscr{W}}^\lambda \overset{\eta}{\to} \mathscr{W}^\lambda \overset{\kappa}{\to} S^\lambda \overset{\mu_S}{\to} \mathbb{A}^1$$

is the zero map. Now applying the functor $\tilde{\tau}_!$ to the canonical morphism

$$\eta^*(\beta^*_{\mathscr{W}} \mathscr{L}^d_E \otimes \kappa^* \mu^*_S \mathscr{L}_\psi) \to \xi_* \xi^* \eta^*(\beta^*_{\mathscr{W}} \mathscr{L}^d_E \otimes \kappa^* \mu^*_S \mathscr{L}_\psi),$$

we get a map

$$\tau_! \left(\beta^*_{\mathscr{W}} \mathscr{L}^d_E \otimes \kappa^* \mu^*_S \mathscr{L}_\psi\right) \to i^*_\lambda \mathscr{L}^d_E\big(a(\lambda)\big)\big[2a(\lambda)\big]. \qquad (9)$$

Define (7) as the composition of (9) with the canonical map

$$i_\lambda^* \mathscr{L}_E^d\big(a(\lambda)\big)\big[2a(\lambda)\big] \to \mathscr{H}^{2a(\lambda)}(i_\lambda^* \mathscr{L}_E^d)\big(a(\lambda)\big).$$

Now we check that (7) is an isomorphism fibre by fibre.

Step 2. Fix a k-point $(D_1, \dots, D_n)$ of X_-^λ. Let $D_i = \sum_x \lambda_{i,x}\, x$. So, the corresponding Λ_n^--valued divisor on X associates to $x \in X$ the antidominant weight

$$\lambda_x = (\lambda_{1,x}, \dots, \lambda_{n,x}) \in \Lambda_{n,d_x}^-$$

with $d_x = \sum_i \lambda_{i,x}$.

For $i = 1, \dots, n$, put $L_i = \Omega^{n-1}(D_1) \oplus \cdots \oplus \Omega^{n-i}(D_i)$. For every closed point $x \in X$, let

$$\big((L_1)_x \subset \cdots \subset (L_n)_x\big)$$

be the restriction of the flag $(L_1 \subset \cdots \subset L_n)$ to $\operatorname{Spec} \hat{\mathscr{O}}_{X,x}$.

Let $\mathscr{G}r^{d_x,+}((L_n)_x)$ denote the moduli scheme of sublattices $\mathscr{R} \subset (L_n)_x$ such that

$$\dim\big((L_n)_x/\mathscr{R}\big) = d_x.$$

By $S_x \subset \mathscr{G}r^{d_x,+}((L_n)_x)$ we denote the locally closed subscheme whose set of k-points consists of sublattices $\mathscr{R} \subset (L_n)_x$ with the following property. Let $\mathscr{R}_i = \mathscr{R} \cap (L_i)_x$. The condition is that the image of the natural inclusion

$$\mathscr{R}_i/\mathscr{R}_{i-1} \hookrightarrow (L_i)_x/(L_{i-1})_x \xrightarrow{\sim} \big(\Omega^{n-i}(\lambda_{i,x}\, x)\big)_x$$

is the sublattice $\Omega_x^{n-i} \subset (\Omega^{n-i}(\lambda_{i,x}\, x))_x$ for $i = 1, \dots, n$.

For every $x \in X$, one defines the stack $\mathscr{W}_x^\lambda$ and the scheme S_x^λ with morphisms

$$S_x \xrightarrow{\eta_x} \mathscr{W}_x^\lambda \xrightarrow{\kappa_x} S_x^\lambda \xrightarrow{(\mu_S)_x} \mathbb{A}^1,$$

which are local counterparts of the corresponding stacks and morphisms for the weight λ_x. So, after the base change $(D_1, \dots, D_n) : \operatorname{Spec} k \to X_-^\lambda$, the diagram

$$\widetilde{\mathscr{W}}^\lambda \xrightarrow{\eta} \mathscr{W}^\lambda \xrightarrow{\kappa} S^\lambda$$

becomes

$$\prod_{x\in X} S_x \to \prod_{x \in X} \mathscr{W}_x^\lambda \to \prod_{x\in X} S_x^\lambda,$$

the morphisms being the product of morphisms η_x and κ_x, respectively. The restriction of $\mu_S : S^\lambda \to \mathbb{A}^1$ to $\prod_{x\in X} S_x^\lambda$ is the sum of morphisms $(\mu_S)_x$.

For $\nu \in \Lambda_{n,d_x}^+$, the perverse sheaf $\mathscr{A}_\nu$ considered as a sheaf on $\mathscr{G}r^{d_x,+}((L_n)_x)$ is denoted $\mathscr{A}_{\nu,x}$ (cf. Section 3.2).

From [6, Proposition 3.1 and Lemma 4.2] it follows that the restriction of $\eta^*\beta^*_{\mathscr{W}}\mathscr{L}^d_E$ to $\prod_{x\in X} S_x$ is identified with $\boxtimes_{x\in X} F_x$, where F_x is the restriction of

$$\bigoplus_{\nu} \mathscr{A}_{\nu,x}[-d_x(n-1)]\Big(\frac{-d_x(n-1)}{2}\Big) \otimes E_x^{\nu}$$

under the inclusion $S_x \hookrightarrow \mathscr{G}r^{d_x,+}((L_n)_x)$ (the sum being taken over $\nu \in \Lambda^+_{n,d_x}$).

Now combining Lemma 3 with [12, Theorem 3.3.8]), we get the desired assertion. This concludes the proof of Proposition 1. □

4. Geometric Whittaker models for GL(n)

4.1. The stack ${}_n\mathscr{Y}_d$

Consider the stack ${}_n\mathscr{X}_d$ defined in Section 2.3. We impose Plücker's relations on a point $(L, (t_i))$ of ${}_n\mathscr{X}_d$, which means that generically (t_i) come from a complete flag of vector subbundles of L. Our definition is justified by the following simple observation.

Let V be a vector space of dimension n (over any field). For $n \geq k > i \geq 1$, let $\alpha_{k,i} : \wedge^k V \otimes \wedge^i V \to \wedge^{k+1} V \otimes \wedge^{i-1} V$ be the contraction map that sends $u \otimes (v_1 \wedge v_2 \wedge \cdots \wedge v_i)$ to

$$\sum_{j=1}^{i} (-1)^j (u \wedge v_j) \otimes (v_1 \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_i).$$

LEMMA 4

Given nonzero elements $t_i \in \wedge^i V$ for $1 \leq i \leq n$, the following are equivalent:

(1) *there exists a complete flag of vector subspaces $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$ such that $t_i \in \wedge^i V_i \subset \wedge^i V$;*

(2) *for $n \geq k > i \geq 1$, we have $\alpha_{k,i}(t_k \otimes t_i) = 0$.*

Proof

The statement is obvious in characteristic zero. Let us give an argument that holds in any characteristic. Write $e_1 \wedge \cdots \wedge e_i$ for the image of $e_1 \otimes \cdots \otimes e_i$ under $V^{\otimes i} \to \wedge^i V$.

We construct by induction on k the elements $e_1, \ldots, e_k \in V$ such that $t_i = e_1 \wedge \cdots \wedge e_i$ for $i = 1, \ldots, k$. Let $e_1 = t_1$, and assume that $e_1, \ldots, e_{k-1}$ are already constructed.

To construct e_k, we show by induction on i that $t_k = e_1 \wedge \cdots \wedge e_i \wedge \omega_{k-i}$ for some $\omega_{k-i} \in \wedge^{k-i} V$, and we define e_k as ω_1.

First, since $\alpha_{k,1}(t_k \otimes t_1) = -t_k \wedge e_1 = 0$, we get $t_k = e_1 \wedge \omega_{k-1}$ for some $\omega_{k-1} \in \wedge^{k-1} V$. Now assume that $t_k = e_1 \wedge \cdots \wedge e_{i-1} \wedge \omega_{k-i+1}$ for some $\omega_{k-i+1} \in \wedge^{k-i+1} V$ with $i < k$. Then

$$\alpha_{k,i}(t_k \otimes t_i) = \alpha_{k,i}\big(t_k \otimes (e_1 \wedge \cdots \wedge e_i)\big) = (-1)^i (t_k \wedge e_i) \otimes (e_1 \wedge \cdots \wedge e_{i-1}) = 0.$$

It follows that $t_k \wedge e_i = 0$. So, there exists $\omega_{k-i} \in \wedge^{k-i} V$ such that $t_k = e_1 \wedge \cdots \wedge e_i \wedge \omega_{k-i}$. We are done. □

Now we define the closed substack ${}_n\mathscr{Y}_d \hookrightarrow {}_n\mathscr{X}_d$ by the conditions $\alpha_{k,i}(t_k \otimes t_i) = 0$ for $n \geq k > i \geq 1$, where

$$\alpha_{k,i} : \wedge^k L \otimes \wedge^i L \to \wedge^{k+1} L \otimes \wedge^{i-1} L$$

are the contraction maps defined as above. Then the map φ factors through ${}_n\mathscr{Q}_d \to {}_n\mathscr{Y}_d \hookrightarrow {}_n\mathscr{X}_d$.

We stratify ${}_n\mathscr{Y}_d$ by locally closed substacks $\mathscr{V}_p^\lambda \subset {}_n\mathscr{Y}_d$ numbered by $\lambda \in \Lambda_{n,d}^p$. The stratum $\mathscr{V}_p^\lambda$ is defined by the following condition: the degree of the divisor of zeros of $t_i : \Omega^{(n-1)+\cdots+(n-i)} \hookrightarrow \wedge^i L$ equals $\lambda_1 + \cdots + \lambda_i$ for $i = 1, \ldots, n$. Recall that a point of X_p^λ is a collection of divisors $(D_1, \ldots, D_n)$ on X with $\deg(D_i) = \lambda_i$ and $D_1 + \cdots + D_i \geq 0$ for all i. So, the stack $\mathscr{V}_p^\lambda$ classifies collections

$$\big(0 = L'_0 \subset L'_1 \subset \cdots \subset L'_n = L,\ (s_i),\ (D_i) \in X_p^\lambda\big), \tag{10}$$

where (L'_i) is a complete flag of subbundles on a rank n vector bundle L and

$$s_i : \Omega^{(n-1)+\cdots+(n-i)}(D_1 + \cdots + D_i) \xrightarrow{\sim} \wedge^i L'_i$$

is an isomorphism.

Define the closed substack $\mathscr{V}^\lambda \hookrightarrow \mathscr{V}_p^\lambda$ as $\mathscr{V}_p^\lambda \times_{X_p^\lambda} X^\lambda$. So if $\lambda \notin \Lambda_n$, then $\mathscr{V}^\lambda$ is empty. Notice that the projection $\mathscr{V}_p^\lambda \times_{{}_n\mathscr{Y}_d} {}_n\mathscr{Q}_d \to \mathscr{V}_p^\lambda$ factors through $\mathscr{V}^\lambda \hookrightarrow \mathscr{V}_p^\lambda$, and for $\lambda \in \Lambda_{n,d}$ the corresponding morphism $\mathscr{V}_p^\lambda \times_{{}_n\mathscr{Y}_d} {}_n\mathscr{Q}_d = \mathscr{V}^\lambda \times_{{}_n\mathscr{Y}_d} {}_n\mathscr{Q}_d \to \mathscr{V}^\lambda$ is an affine fibration of rank $a(\lambda)$.

4.2. The sheaves ${}_n\mathscr{P}^d_{E,\psi}$

Definition 2
For any smooth $\bar{\mathbb{Q}}_\ell$-sheaf E on X, put ${}_n\mathscr{P}^d_{E,\psi} = \varphi_!({}_n\mathscr{F}^d_{E,\psi})$.

Clearly, the restriction of ${}_n\mathscr{P}^d_{E,\psi}$ to a stratum $\mathscr{V}_p^\lambda$ of ${}_n\mathscr{Y}_d$ vanishes outside the closed substack $\mathscr{V}^\lambda$ of $\mathscr{V}_p^\lambda$. For $\lambda \in \Lambda_{n,d}$, denote by ${}_n\mathscr{P}^\lambda_{E,\psi}$ the restriction of ${}_n\mathscr{P}^d_{E,\psi}$ to $\mathscr{V}^\lambda$.

Define the closed substack $\mathscr{V}^\lambda_- \hookrightarrow \mathscr{V}^\lambda$ as $\mathscr{V}^\lambda \times_{X^\lambda} X^\lambda_-$. Recall that the subscheme X^λ_- of X^λ is given by the condition $0 \le \mathrm{D}_1 \le \cdots \le D_n$, where $(D_i) \in X^\lambda$. Let $\mu_\lambda : \mathscr{V}^\lambda_- \to \mathbb{A}^1$ be the map that at the level of k-points sends (10) to the sum of $n-1$ classes in

$$k \tilde{\to} \operatorname{Ext}^1\big(\Omega^{n-i-1}(D_i), \Omega^{n-i}(D_i)\big)$$

corresponding to the pullbacks of the successive extensions

$$\begin{array}{ccccccc} 0 \to & L'_i/L'_{i-1} & \to L'_{i+1}/L'_{i-1} \to & L'_{i+1}/L'_i & \to 0 \\ & \downarrow \wr & & \downarrow \wr & \\ & \Omega^{n-i}(D_i) & & \Omega^{n-i-1}(D_{i+1}) & \end{array}$$

with respect to the inclusion $\Omega^{n-i-1}(D_i) \hookrightarrow \Omega^{n-i-1}(D_{i+1})$.

PROPOSITION 2

Let E be a smooth $\bar{\mathbb{Q}}_\ell$-sheaf on X of rank m and $\lambda \in \Lambda_{n,d}$. Then

(1) ${}_n\mathscr{P}^\lambda_{E,\psi}$ *vanishes unless*

$$\lambda_1 = \cdots = \lambda_{n-m} = 0; \qquad (*)$$

(2) *under condition* $(*)$ *the complex* ${}_n\mathscr{P}^\lambda_{E,\psi}$ *is supported at* $\mathscr{V}^\lambda_- \hookrightarrow \mathscr{V}^\lambda$*, and its restriction to* $\mathscr{V}^\lambda_-$ *is isomorphic to the tensor product of*

$$\mu_\lambda^* \mathscr{L}_\psi\Big(\frac{b-2a(\lambda)}{2}\Big)\big[b-2a(\lambda)\big]$$

with the inverse image of E^λ_- *under* $\mathscr{V}^\lambda_- \to X^\lambda_-$.

Remark 3

(i) The sheaf ${}_n\mathscr{P}^d_{E,\psi}$ was also considered by E. Frenkel, Gaitsgory, and K. Vilonen [8, Section 4.3]. They show that for any smooth $\bar{\mathbb{Q}}_\ell$-sheaf E on X, ${}_n\mathscr{P}^d_{E,\psi}$ is a perverse sheaf and the Goresky-MacPherson extension of its restriction to any nonempty open substack of ${}_n\mathscr{Y}_d$. Besides, the Verdier dual of ${}_n\mathscr{P}^d_{E,\psi}$ is canonically isomorphic to ${}_n\mathscr{P}^d_{E^*,\psi^{-1}}$ (see [8, Sections 4.6 and 4.7]). We notice that the stratification of ${}_n\mathscr{Y}_d$ used in [8, Section 4.10] is different from ours, so that our Proposition 2 is a strengthened version of [8, Proposition 4.12].

According to [7], [8], in the case $\operatorname{rk} E = n$ the perverse sheaf ${}_n\mathscr{P}^d_{E,\psi}$ can be thought of as a geometric counterpart of the Whittaker function canonically attached to E.

(ii) For $m > 0$, let ${}^m_n\mathscr{Y}_d \subset {}_n\mathscr{Y}_d$ denote the open substack given by the following conditions: the image of t_i is a *line subbundle* in $\wedge^i L$ for $i = 1, \dots, n-m$. In particular, ${}^m_n\mathscr{Y}_d = {}_n\mathscr{Y}_d$ for $m \ge n$. Then Proposition 2(1) claims that ${}_n\mathscr{P}^d_{E,\psi}$ is the extension by zero of its restriction to ${}^m_n\mathscr{Y}_d \subset {}_n\mathscr{Y}_d$.

(iii) The relation between the sheaves ${}_n\mathscr{P}^d_{E,\psi}$ for different n is as follows. Let ${}^m_n\mathscr{Q}_d$ be the preimage of ${}^m_n\mathscr{Y}_d$ under $\varphi : {}_n\mathscr{Q}_d \to {}_n\mathscr{Y}_d$. So, ${}^m_n\mathscr{Q}_d$ is the open substack of ${}_n\mathscr{Q}_d$ parametrizing collections (4) such that L/L_{n-m} is locally free.

Denote by ${}^m_n\mathscr{E}xt$ the stack of collections $(L_1 \subset \cdots \subset L_{n-m+1} \subset L, (s_i))$, where L/L_{n-m} is a vector bundle on X of rank m, and $s_i : \Omega^{n-i} \xrightarrow{\sim} L_i/L_{i-1}$ is an isomorphism ($i = 1, \dots, n-m+1$). Let $\mu^m_n : {}^m_n\mathscr{E}xt \to \mathbb{A}^1$ be the composition ${}^m_n\mathscr{E}xt \to {}_{n-m+1}\mathscr{Q}_0 \xrightarrow{\mu} \mathbb{A}^1$, where the first arrow is the map that forgets L.

Let ${}_m\mathscr{M}$ be the stack of pairs $(\Omega^{m-1} \hookrightarrow L)$, where L is a vector bundle on X of rank m. Taking the quotient by L_{n-m}, we get a map ${}^m_n\mathscr{E}xt \to {}_m\mathscr{M}$, which is a generalized affine fibration.

For $1 \le m \le n$ there is a commutative diagram

$$\begin{array}{ccc} {}^m_n\mathscr{Q}_d & \xrightarrow{\sim} & {}_m\mathscr{Q}_d \times_{{}_m\mathscr{M}} {}^m_n\mathscr{E}xt \\ \downarrow & & \downarrow \varphi\times\mathrm{id} \\ {}^m_n\mathscr{Y}_d & \xrightarrow{\sim} & {}_m\mathscr{Y}_d \times_{{}_m\mathscr{M}} {}^m_n\mathscr{E}xt \end{array}$$

where the left vertical arrow is the restriction of φ. So, the restriction of ${}_n\mathscr{P}^d_{E,\psi}$ to ${}^m_n\mathscr{Y}_d$ is isomorphic to

$$ {}_m\mathscr{P}^d_{E,\psi} \boxtimes (\mu^m_n)^*\mathscr{L}_\psi\big[b(m,d) - b(n,d)\big]\Big(\frac{b(m,d) - b(n,d)}{2}\Big). $$

4.3. The support of ${}_n\mathscr{P}^\lambda_{E,\psi}$

In this section we prove the following lemma.

LEMMA 5

The complex ${}_n\mathscr{P}^\lambda_{E,\psi}$ *vanishes outside the closed substack* $\mathscr{V}^\lambda_- \hookrightarrow \mathscr{V}^\lambda$.

This may be derived from the geometric Casselman-Shalika formulae, but we give a direct proof. We start with the following sublemma. Given $\lambda \in \Lambda^-_n$ and $\nu \in \Lambda_n$, denote by $\mathscr{U}^\nu_\lambda$ the stack of collections: $(D_1, \dots, D_n) \in X^\lambda_-$, $(D'_1, \dots, D'_n) \in X^\nu$ and a diagram

$$\begin{array}{ccc} L'_1 & \subset \cdots \subset & L'_n \\ \cup & & \cup \\ L_1 & \subset \cdots \subset & L_n \end{array} \tag{11}$$

where (L_i) (resp., (L'_i)) is a complete flag of vector subbundles on a rank n vector bundle L_n (resp., L'_n) on X with trivializations

$$ L'_i/L'_{i-1} \xrightarrow{\sim} \Omega^{n-i}(D_i + D'_i) $$

such that the image of the inclusion $L_i/L_{i-1} \hookrightarrow L'_i/L'_{i-1} \tilde{\to} \Omega^{n-i}(D_i + D'_i)$ equals $\Omega^{n-i}(D_i)$ for $i = 1, \dots, n$. Let

$$\varphi^\nu_\lambda : \mathscr{U}^\nu_\lambda \to (X^\lambda_- \times X^\nu) \times_{X^{\lambda+\nu}} \mathscr{V}^{\lambda+\nu}$$

be the map that forgets the flag (L_i). Here $X^\lambda_- \times X^\nu \to X^{\lambda+\nu}$ denotes the summation of divisors. The map φ^ν_λ is an affine fibration of rank $a(\nu)$.

Let $X_{\lambda,\nu} \hookrightarrow X^\lambda_- \times X^\nu$ be the closed subscheme defined by

$$D_i \geq D_{i-1} + D'_{i-1}$$

for $i = 2, \dots, n$. The composition $X_{\lambda,\nu} \hookrightarrow X^\lambda_- \times X^\nu \to X^{\lambda+\nu}$ factors through $X^{\lambda+\nu}_- \hookrightarrow X^{\lambda+\nu}$.

We also have a map $\mathscr{U}^\nu_\lambda \to \mathscr{V}^\lambda_-$ that forgets (L'_i) and (D'_i). By abuse of notation, the composition of this map with $\mu_\lambda : \mathscr{V}^\lambda_- \to \mathbb{A}^1$ is also denoted μ_λ.

SUBLEMMA 1

The complex $(\varphi^\nu_\lambda)_! \mu^*_\lambda \mathscr{L}_\psi$ is supported at the closed substack $X_{\lambda,\nu} \times_{X^{\lambda+\nu}_-} \mathscr{V}^{\lambda+\nu}_-$ of $(X^\lambda_- \times X^\nu) \times_{X^{\lambda+\nu}} \mathscr{V}^{\lambda+\nu}$ and is isomorphic to the inverse image of

$$\mu^*_{\lambda+\nu} \mathscr{L}_\psi \big[-2a(\nu)\big]\big(-a(\nu)\big)$$

from $\mathscr{V}^{\lambda+\nu}_-$.

Proof

Let us decompose φ^ν_λ into two affine fibrations $\mathscr{U}^\nu_\lambda \to \tilde{\mathscr{U}}^\nu_\lambda \xrightarrow{\tilde{\varphi}^\nu_\lambda} (X^\lambda_- \times X^\nu) \times_{X^{\lambda+\nu}} \mathscr{V}^{\lambda+\nu}$ defined as follows. Let $\tilde{\mathscr{U}}^\nu_\lambda$ be the stack of collections:

- $(D_i) \in X^\lambda_-$, $(D'_i) \in X^\nu$;
- a complete flag of vector bundles $(L'_1 \subset \cdots \subset L'_n)$ on X with trivializations

$$L'_i/L'_{i-1} \tilde{\to} \Omega^{n-i}(D_i + D'_i) \tag{12}$$

  for $i = 1, \dots, n$;
- for $i = 1, \dots, n-1$ diagrams

$$\begin{array}{ccccccc} 0 \to & L'_i/L'_{i-1} & \to & L'_{i+1}/L'_{i-1} & \to & L'_{i+1}/L'_i & \to 0 \\ & \cup & & \cup & & \cup & \\ 0 \to & \Omega^{n-i}(D_i) & \to & F_i & \to & \Omega^{n-i-1}(D_{i+1}) & \to 0 \end{array}$$

  where each row is an exact sequence of $\mathscr{O}_X$-modules, and both left and right vertical arrows are compatible with (12).

Now define the morphism $\mathscr{U}_\lambda^\nu \to \tilde{\mathscr{U}}_\lambda^\nu$ by $F_i = L_{i+1}/L_{i-1}$ for $i = 1, \dots, n-1$, where L_i are from diagram (11). One checks that this is an affine fibration of rank

$$(n-2)\nu_1 + (n-3)\nu_2 + \cdots + \nu_{n-2}.$$

Define also $\tilde{\varphi}_\lambda^\nu$ as the map that forgets all F_i. This is an affine fibration of rank $\nu_1 + \cdots + \nu_{n-1}$.

Clearly, $\mu_\lambda : \mathscr{U}_\lambda^\nu \to \mathbb{A}^1$ is constant along the fibres of $\mathscr{U}_\lambda^\nu \to \tilde{\mathscr{U}}_\lambda^\nu$. So, it suffices to prove the sublemma in the case $n = 2$.

In this case a fibre of φ_λ^ν is the affine space of maps $\xi : L_2/L_1 \to L_2'/L_1$ such that the diagram commutes:

$$\begin{array}{ccccc} 0 \to L_1'/L_1 \to & L_2'/L_1 & \to & L_2'/L_1' \to 0 \\ & \uparrow \xi & \nearrow i & \\ & L_2/L_1 & & \end{array}$$

where i is the canonical inclusion compatible with trivializations. On this affine space we have a free and transitive action of $\mathrm{Hom}(L_2/L_1,\ L_1'/L_1)$. The restriction of $\mu_\lambda^* \mathscr{L}_\psi$ to this affine space is a sheaf that changes under the action of $\mathrm{Hom}(L_2/L_1, L_1'/L_1)$ by a local system, say, $\tilde{\mu}^* \mathscr{L}_\psi$, where

$$\tilde{\mu} : \mathrm{Hom}(L_2/L_1, L_1'/L_1) \to k$$

is the following linear functional. It associates to $s \in \mathrm{Hom}(L_2/L_1,\ L_1'/L_1)$ the class of the pullback of

$$0 \to L_1 \to L_1' \to L_1'/L_1 \to 0 \tag{13}$$

under the composition $\mathscr{O}(D_1) \hookrightarrow \mathscr{O}(D_2) \widetilde{\to} L_2/L_1 \xrightarrow{s} L_1'/L_1$. The sequence (13) is just

$$0 \to \Omega(D_1) \to \Omega(D_1 + D_1') \to \Omega(D_1 + D_1')/\Omega(D_1) \to 0.$$

So, $\tilde{\mu} = 0$ if and only if $D_2 \geq D_1 + D_1'$. Besides, under this condition the pullback of

$$0 \to L_1' \to L_2' \to L_2'/L_1' \to 0$$

under $\mathscr{O}(D_1 + D_1') \hookrightarrow \mathscr{O}(D_2 + D_2') \widetilde{\to} L_2'/L_1'$ is identified (after tensoring by $\mathscr{O}(-D_1')$) with the pullback of

$$0 \to L_1 \to L_2 \to L_2/L_1 \to 0$$

under $\mathscr{O}(\mathrm{D}_1) \hookrightarrow \mathscr{O}(D_2) \widetilde{\to} L_2/L_1$. Our assertion follows. □

For $m \geq 0$ and $\nu \in \Lambda_{m,d}$, denote by $\mathscr{F}l^\nu$ the stack of flags $(F^1 \subset \cdots \subset F^m)$, where F^i is a coherent torsion sheaf on X with $\deg(F^i/F^{i-1}) = \nu_i$ for $i = 1, \dots, m$. Let $\mathrm{div}^\nu : \mathscr{F}l^\nu \to X^\nu$ denote the composition

$$\mathscr{F}l^\nu \to \mathrm{Sh}_0^{\nu_1} \times \cdots \times \mathrm{Sh}_0^{\nu_m} \xrightarrow{\mathrm{div} \times \cdots \times \mathrm{div}} X^\nu.$$

Set also ${}_n\mathscr{Q}^\nu = {}_n\mathscr{Q}_d \times_{\mathrm{Sh}_0^d} \mathscr{F}l^\nu$, where $\mathscr{F}l^\nu \to \mathrm{Sh}_0^d$ sends $(F^1 \subset \cdots \subset F^m)$ to F^m.

Denote by ${}_n^m J_d$ the set of $(n \times m)$-matrices $e = (e_i^j)$ $(1 \le i \le n,\ 1 \le j \le m)$ with $e_i^j \in \mathbb{Z}_+$, $\sum_{i,j} e_i^j = d$. We have a map $h : {}_n^m J_d \to \Lambda_{n,d} \times \Lambda_{m,d}$ that sends e to (λ, ν), where $\lambda_i = \sum_j e_i^j$ and $\nu_j = \sum_i e_i^j$. For $e \in {}_n^m J_d$, put $Y^e = \prod_{i,j} X^{(e_i^j)}$. So, Y^e classifies matrices of effective divisors (D_i^j) on X such that $\deg(D_i^j) = e_i^j$.

For every $\lambda \in \Lambda_{n,d}$, the stack $\mathscr{V}^\lambda \times_{{}_n\mathscr{Y}_d} {}_n\mathscr{Q}^\nu$ is stratified by locally closed substacks $\mathscr{Q}^e \hookrightarrow \mathscr{V}^\lambda \times_{{}_n\mathscr{Y}_d} {}_n\mathscr{Q}^\nu$ indexed by $e \in {}_n^m J_d$ such that $h(e) = (\lambda, \nu)$. The stratum $\mathscr{Q}^e$ is the stack classifying the following collections:

- a diagram

$$\begin{array}{ccccccc} L_1^m & \subset & L_2^m & \subset & \cdots & \subset & L_n^m \\ \cup & & \cup & & & & \cup \\ L_1^{m-1} & \subset & L_2^{m-1} & \subset & \cdots & \subset & L_n^{m-1} \\ \cup & & \cup & & & & \cup \\ \vdots & & \vdots & & & & \vdots \\ \cup & & \cup & & & & \cup \\ L_1^0 & \subset & L_2^0 & \subset & \cdots & \subset & L_n^0 \end{array} \tag{14}$$

  where L_i^j is a vector bundle of rank i on X, and all the maps are inclusions of $\mathscr{O}_X$-modules;
- a matrix $(D_i^j) \in Y^e$;
- isomorphisms $L_i^m / L_{i-1}^m \xrightarrow{\sim} \Omega^{n-i}(D_i^1 + \cdots + D_i^m)$ such that the image of the inclusion

$$L_i^j / L_{i-1}^j \hookrightarrow L_i^m / L_{i-1}^m \xrightarrow{\sim} \Omega^{n-i}(D_i^1 + \cdots + D_i^m)$$

  equals $\Omega^{n-i}(D_i^1 + \cdots + D_i^j)$ $(i = 1, \dots, n;\ j = 0, \dots, m)$.

We have a natural map

$$\varphi^e : \mathscr{Q}^e \to Y^e \times_{X^\lambda} \mathscr{V}^\lambda$$

that forgets all the rows in (14) except the top one. (Here $Y^e \to X^\lambda$ sends (D_i^j) to (D_i) with $D_i = \sum_j D_i^j$.) The morphism φ^e is an affine fibration of rank $a(\lambda)$.

Denote by $Y_-^e \hookrightarrow Y^e$ the closed subscheme given by the following conditions:

(1′) for $i \le n - j$, we have $D_i^j = 0$;

(2′) for $1 \le j \le m-1$ and $2 \le i \le n$, we have $D_i^1 + \cdots + D_i^j \ge D_{i-1}^1 + \cdots + D_{i-1}^{j+1}$.

The composition $\mathscr{Q}^e \hookrightarrow {}_n\mathscr{Q}^\nu \to {}_n\mathscr{Q}_d \xrightarrow{\mu} \mathbb{A}^1$ is denoted by μ_e.

SUBLEMMA 2

The complex $(\varphi^e)_!\mu_e^*\mathcal{L}_\psi$ is supported at $Y_-^e \times_{X_-^\lambda} \mathcal{V}_-^\lambda \hookrightarrow Y^e \times_{X^\lambda} \mathcal{V}^\lambda$ and is isomorphic to the inverse image of

$$\mu_\lambda^*\mathcal{L}_\psi\big(-a(\lambda)\big)\big[-2a(\lambda)\big]$$

from $\mathcal{V}_-^\lambda$.

Proof

Apply Sublemma 1 m times forgetting successively the rows in diagram (14) starting from the lowest one and moving up. □

Proof of Lemma 5

Since $\mathcal{L}_E^d$ is a direct summand of the Springer sheaf Spr_E^d (cf. Section 1), it suffices to show that the restriction of

$$\varphi_!(\beta^*\,\mathrm{Spr}_E^d \otimes \mu^*\mathcal{L}_\psi)$$

to $\mathcal{V}^\lambda$ vanishes outside $\mathcal{V}_-^\lambda$. Put $\nu = (1,\dots,1) \in \Lambda_{d,d}$. The composition ${}_n\mathcal{Q}^\nu \to {}_n\mathcal{Q}_d \xrightarrow{\mu} \mathbb{A}^1$ is also denoted by μ. By the projection formulae, we have to consider the direct image with respect to the projection

$$\mathcal{V}^\lambda \times_{{}_n\mathcal{Y}_d} {}_n\mathcal{Q}^\nu \to \mathcal{V}^\lambda \tag{15}$$

of $\mathrm{pr}_2^*\,\mu^*\mathcal{L}_\psi$ tensored by some local system that comes from X^ν. The stack $\mathcal{V}^\lambda \times_{{}_n\mathcal{Y}_d} {}_n\mathcal{Q}^\nu$ is stratified by locally closed substacks $\mathcal{Q}^e$ indexed by $e \in {}_n^dJ_d$ such that $h(e) = (\lambda,\nu)$. The restriction of (15) to $\mathcal{Q}^e$ can be decomposed as

$$\mathcal{Q}^e \xrightarrow{\varphi^e} Y^e \times_{X^\lambda} \mathcal{V}^\lambda \to X^\nu \times_{X^{(d)}} \mathcal{V}^\lambda \to \mathcal{V}^\lambda.$$

So, our assertion follows from Sublemma 2 because the composition $Y_-^e \hookrightarrow Y^e \to X^\lambda$ factors through $X_-^\lambda \hookrightarrow X^\lambda$. □

Remark 4

Using Sublemma 2, one may also check that for any $\lambda \in \Lambda_{n,d}^-$ and any smooth $\bar{\mathbb{Q}}_\ell$-sheaf E on X, $E_-^\lambda\,[\lambda_n]$ is a perverse sheaf on X_-^λ.

4.4. Proof of Proposition $\hat{E}$2

Recall that $\mathcal{V}_-^\lambda \times_{{}_n\mathcal{Y}_d} {}_n\mathcal{Q}_d$ is the stack classifying the following collections: $(D_1, \dots, D_n) \in X_-^\lambda$ and a diagram

$$\begin{array}{ccc} L_1' & \subset \cdots \subset & L_n' \\ \cup & & \cup \\ L_1 & \subset \cdots \subset & L_n \end{array} \tag{16}$$

where (L_i') (resp., (L_i)) is a complete flag of vector subbundles on a rank n vector bundle L_n' (resp., L_n) on X with trivializations

$$L_i'/L_{i-1}' \tilde{\to} \Omega^{n-i}(D_i)$$

such that the image of the natural inclusion $L_i/L_{i-1} \hookrightarrow L_i'/L_{i-1}' \tilde{\to} \Omega^{n-i}(D_i)$ equals Ω^{n-i} for $i = 1, \dots, n$. Denote by

$$\eta : \mathscr{V}_-^\lambda \times_{{}_n\mathscr{Y}_d} {}_n\mathscr{Q}_d \to \mathscr{V}_-^\lambda \times_{X_-^\lambda} \mathscr{W}^\lambda$$

the morphism over $\mathscr{V}_-^\lambda$, whose composition with the projection $\mathscr{V}_-^\lambda \times_{X_-^\lambda} \mathscr{W}^\lambda \to \mathscr{W}^\lambda$ sends (16) to the flag $(L_1'/L_1 \subset L_2'/L_2 \subset \cdots \subset L_n'/L_n)$. One checks that η is a (representable) affine fibration of rank $a(\lambda)$. Further, the composition

$$\mathscr{V}_-^\lambda \times_{{}_n\mathscr{Y}_d} {}_n\mathscr{Q}_d \xrightarrow{\eta} \mathscr{V}_-^\lambda \times_{X_-^\lambda} \mathscr{W}^\lambda \xrightarrow{\mathrm{id} \times \kappa} \mathscr{V}_-^\lambda \times_{X_-^\lambda} S^\lambda \xrightarrow{\mu_\lambda \times \mu_S} \mathbb{A}^1 \times \mathbb{A}^1 \xrightarrow{\mathrm{sum}} \mathbb{A}^1$$

coincides with the restriction of $\mu : {}_n\mathscr{Q}_d \to \mathbb{A}^1$ to the substack $\mathscr{V}_-^\lambda \times_{{}_n\mathscr{Y}_d} {}_n\mathscr{Q}_d \hookrightarrow {}_n\mathscr{Q}_d$. So, our assertion follows from Proposition 1. □

5. Proof of Theorem A

Recall the map $\phi : {}_n\mathscr{X}_d \to X^{(d)}$ (cf. Section 2.3). By abuse of notation, its restriction to ${}_n\mathscr{Y}_d \hookrightarrow {}_n\mathscr{X}_d$ is also denoted ϕ. We let

$$\tilde{\pi} : {}_n\mathscr{Y}_d \times_{{}_n\mathscr{M}_d} {}_n\mathscr{Y}_d \to X^{(d)} \times_{\mathrm{Pic}^d X} X^{(d)}$$

be the morphism $\phi \times \phi$. By $\tilde{\pi}'$ we denote the restriction of $\tilde{\pi}$ to the diagonal ${}_n\mathscr{Y}_d \hookrightarrow {}_n\mathscr{Y}_d \times_{{}_n\mathscr{M}_d} {}_n\mathscr{Y}_d$. Clearly, Theorem A is equivalent to the fact that the natural map

$$\tilde{\pi}_!({}_n\mathscr{P}^d_{E,\psi} \boxtimes {}_n\mathscr{P}^d_{E',\psi^{-1}}) \to \tilde{\pi}'_!({}_n\mathscr{P}^d_{E,\psi} \otimes {}_n\mathscr{P}^d_{E',\psi^{-1}})$$

is an isomorphism. For $\lambda, \nu \in \Lambda_{n,d}$, we denote by $\tilde{\pi}^{\lambda,\nu}$ the restriction of $\tilde{\pi}$ to the substack

$$\mathscr{V}^\lambda \times_{{}_n\mathscr{M}_d} \mathscr{V}^\nu \hookrightarrow {}_n\mathscr{Y}_d \times_{{}_n\mathscr{M}_d} {}_n\mathscr{Y}_d.$$

In the case $\lambda = \nu$ we write $(\tilde{\pi}^{\lambda,\lambda})'$ for the restriction of $\tilde{\pi}^{\lambda,\lambda}$ to the diagonal $\mathscr{V}^\lambda \hookrightarrow \mathscr{V}^\lambda \times_{{}_n\mathscr{M}_d} \mathscr{V}^\lambda$.

Using the stratification of ${}_n\mathscr{Y}_d \times_{{}_n\mathscr{M}_d} {}_n\mathscr{Y}_d$ induced by both stratifications of the first and the second multiple (cf. Section 4.1), Theorem A is reduced to the following statement.

PROPOSITION 3

For any $\lambda, \nu \in \Lambda_{n,d}$, the direct image $(\tilde{\pi}^{\lambda,\nu})_!({}_n\mathscr{P}^\lambda_{E,\psi} \boxtimes {}_n\mathscr{P}^\nu_{E',\psi^{-1}})$ vanishes unless $\lambda = \nu$. Under the condition $\lambda = \nu$ the natural map

$$(\tilde{\pi}^{\lambda,\lambda})_!({}_n\mathscr{P}^\lambda_{E,\psi} \boxtimes {}_n\mathscr{P}^\lambda_{E',\psi^{-1}}) \to (\tilde{\pi}^{\lambda,\lambda})'_!({}_n\mathscr{P}^\lambda_{E,\psi} \otimes {}_n\mathscr{P}^\lambda_{E',\psi^{-1}})$$

is an isomorphism.

Proof

Put $\mathscr{V}_1 = \mathscr{V}_-^{\lambda} \times_{{}_n\mathscr{M}_d} \mathscr{V}_-^{\nu}$. The restriction of $\tilde{\pi}^{\lambda,\nu}$ to $\mathscr{V}_1$ can be decomposed as

$$\mathscr{V}_1 \stackrel{{}^1\pi}{\to} X_-^{\lambda} \times_{\mathrm{Pic}^d X} X_-^{\nu} \to X^{(d)} \times_{\mathrm{Pic}^d X} X^{(d)},$$

where ${}^1\pi$ is the product of two projections $\mathscr{V}_-^{\lambda} \to X_-^{\lambda}$ and $\mathscr{V}_-^{\nu} \to X_-^{\nu}$. In the case $\lambda = \nu$ we denote by diag : $\mathscr{V}_-^{\lambda} \hookrightarrow \mathscr{V}_1$ the diagonal map.

By Proposition 2, our assertion is reduced to the following lemma.

LEMMA 6

For any $\lambda, \nu \in \Lambda_{n,d}^-$, the direct image ${}^1\pi_!(\mu_\lambda^*\mathscr{L}_\psi \boxtimes \mu_\nu^*\mathscr{L}_{\psi^{-1}})$ vanishes unless $\lambda = \nu$. Under the condition $\lambda = \nu$ the natural map

$${}^1\pi_!(\mu_\lambda^*\mathscr{L}_\psi \boxtimes \mu_\lambda^*\mathscr{L}_{\psi^{-1}}) \to {}^1\pi_!(\mathrm{diag})_*(\mathrm{diag})^*(\mu_\lambda^*\mathscr{L}_\psi \boxtimes \mu_\lambda^*\mathscr{L}_{\psi^{-1}})$$

is an isomorphism.

We need the next straightforward sublemma. Given a divisor D and a coherent sheaf M on X with a section $\mathscr{O}(D) \stackrel{s}{\to} M$, denote by $\mathscr{E}xt_{M,D}$ the stack classifying extensions of $\mathscr{O}_X$-modules $0 \to \Omega(D) \to ? \to M \to 0$, and by $\mu_s : \mathscr{E}xt_{M,D} \to \mathbb{A}^1$ the map that sends this extension to the class of its pullback under s.

SUBLEMMA 3

If $s \neq 0$, then $\mathrm{R}\Gamma_c(\mathscr{E}xt_{M,D},\ \mu_s^*\mathscr{L}_\psi) = 0$.

Proof of Lemma 6

The stack $\mathscr{V}_1$ classifies the following collections: $(D_1, \dots, D_n) \in X_-^{\lambda}$, $(D_1', \dots, D_n') \in X_-^{\nu}$, two flags $(L_1 \subset \cdots \subset L_n = L)$ and $(L_1' \subset \cdots \subset L_n' = L)$ of subbundles on a rank n vector bundle L on X with trivializations

$$s_i : \Omega^{n-i}(D_i) \tilde{\to} L_i/L_{i-1} \qquad \text{and} \qquad s_i' : \Omega^{n-i}(D_i') \tilde{\to} L_i'/L_{i-1}'$$

for $i = 1, \dots, n$ such that (L_1, s_1) and (L_1', s_1') coincide (in particular, we have $D_1 = D_1'$).

Let $\mathscr{V}_i \hookrightarrow \mathscr{V}_1$ be the closed substack defined by the condition that the flags

$$\big(L_1 \subset \cdots \subset L_i,\ \ (s_j)_{j=1,\dots,i}\big) \qquad \text{and} \qquad \big(L_1' \subset \cdots \subset L_i',\ \ (s_j')_{j=1,\dots,i}\big)$$

coincide. Let ${}^i\mathscr{L}_\psi$ be the restriction of $\mu_\lambda^*\mathscr{L}_\psi \boxtimes \mu_\nu^*\mathscr{L}_{\psi^{-1}}$ under $\mathscr{V}_i \hookrightarrow \mathscr{V}_1$. Also, let

$${}^i\pi : \mathscr{V}_i \to X_-^{\lambda} \times_{\mathrm{Pic}^d X} X_-^{\nu}$$

be the restriction of ${}^1\pi$ to $\mathscr{V}_i$. Arguing by induction, we show that for every $i = 1, \dots, n-1$, the natural map

$$({}^i\pi)_!({}^i\mathscr{L}_\psi) \to ({}^{i+1}\pi)_!({}^{i+1}\mathscr{L}_\psi)$$

is an isomorphism.

To do so, denote by $\mathscr{N}_i$ the stack of the following collections: $(D_1, \dots, D_n) \in X^\lambda_-$, $(D'_1, \dots, D'_n) \in X^\nu_-$, two flags $(M_{i+1} \subset \dots \subset M_n = M)$ and $(M'_{i+1} \subset \dots \subset M'_n = M)$ of subbundles on a rank $n-i$ vector bundle M on X with trivializations

$$s_j : \Omega^{n-j}(D_j) \xrightarrow{\sim} M_j/M_{j-1} \qquad \text{and} \qquad s'_j : \Omega^{n-j}(D'_j) \xrightarrow{\sim} M'_j/M'_{j-1}$$

for $j = i+1, \dots, n$. Let also ${}'\mathscr{N}_i \hookrightarrow \mathscr{N}_i$ be the closed substack defined by the condition that

$$(M_{i+1}, s_{i+1}) \qquad \text{and} \qquad (M'_{i+1}, s'_{i+1})$$

coincide. Taking the quotient by $L_i = L'_i$, we get a morphism $\gamma : \mathscr{V}_i \to \mathscr{N}_i$, which is a generalized affine fibration. Further, we have a commutative diagram

$$\begin{array}{ccccc} \mathscr{V}_{i+1} & \hookrightarrow & \mathscr{V}_i & & \\ \downarrow & & \downarrow \gamma & \searrow {}^i\pi & \\ {}'\mathscr{N}_i & \hookrightarrow & \mathscr{N}_i & \to & X^\lambda_- \times_{\mathrm{Pic}^d X} X^\nu_- \end{array}$$

where the square is Cartesian. Applying Sublemma 3 for the section $s_{i+1} - s'_{i+1} : \Omega^{n-i-1}(D_i) \to M$, one checks that the complex $\gamma_!({}^i\mathscr{L}_\psi)$ is supported at ${}'\mathscr{N}_i$, and our assertion follows. □

Proposition 3 also follows. □

This concludes the proof of Theorem A. □

6. Proof of Theorems B and C

6.1. Plan of the proof

Proposition 2 admits the following corollary.

COROLLARY 1

For any smooth $\bar{\mathbb{Q}}_\ell$-sheaves E, E' on X, the complex

$$_n\mathscr{S}^d_{E,E'} \overset{\mathrm{def}}{=} f_!({}_n\mathscr{F}^d_{E,\psi} \boxtimes {}_n\mathscr{F}^d_{E',\psi^{-1}})(-d)[-2d]$$

is a sheaf on $X^{(d)}$ placed in degree zero. It has a canonical filtration by constructable subsheaves such that

$$\operatorname{gr}\ {}_n\mathscr{S}^d_{E,E'} = \bigoplus_{\lambda\in\Lambda^+_{n,d}} \pi^\lambda_*(E^\lambda_+ \otimes E'^\lambda_+).$$

For each $r \le n$ there is a canonical inclusion ${}_r\mathscr{S}^d_{E,E'} \subset {}_n\mathscr{S}^d_{E,E'}$ compatible with filtrations.

Proof

By the projection formulae, ${}_n\mathscr{S}^d_{E,E'} \widetilde{\to} \phi_!({}_n\mathscr{P}^d_{E,\psi} \otimes {}_n\mathscr{P}^d_{E',\psi^{-1}})(-d)[-2d]$. Calculate this direct image with respect to the stratification of ${}_n\mathscr{Y}_d$ by locally closed substacks $\mathscr{V}^\lambda_p$ indexed by $\lambda \in \Lambda^p_{n,d}$ (cf. Section 4.1). Since the natural map $\mathscr{V}^{\lambda}_{-} \to X^\lambda_-$ is a generalized affine fibration of rank $b - d - 2a(\lambda)$, our first assertion follows from Proposition 2.

Recall the open substacks ${}^r_n\mathscr{Y}_d \subset {}_n\mathscr{Y}_d$ for $r \le n$ introduced in Section 4.2, Remark 3(iii). Let ${}^{\le r}\phi$ be the restriction of ϕ to ${}^r_n\mathscr{Y}_d$. By the same remark, ${}^{\le r}\phi_!({}_n\mathscr{P}^d_{E,\psi} \otimes {}_n\mathscr{P}^d_{E',\psi^{-1}})(-d)[-2d] \widetilde{\to} {}_r\mathscr{S}^d_{E,E'}$. Our second assertion follows. □

This reduces our proof of Theorem B to the following steps. For $\nu \in \Lambda_{m,d}$, $\nu' \in \Lambda_{m',d}$ and $c \overset{\text{def}}{=} (\nu, \nu')$, set $V^c = X^\nu \times_{X^{(d)}} X^{\nu'}$. Recall our notation ${}_n\mathscr{Q}^\nu = {}_n\mathscr{Q}_d \times_{\mathrm{Sh}^d_0} \mathscr{F}l^\nu$ (cf. Section 4.3). Let

$$f^c : {}_n\mathscr{Q}^\nu \times_{{}_n\mathscr{Y}_d} {}_n\mathscr{Q}^{\nu'} \to V^c$$

denote the composition ${}_n\mathscr{Q}^\nu \times_{{}_n\mathscr{Y}_d} {}_n\mathscr{Q}^{\nu'} \to \mathscr{F}l^\nu \times_{X^{(d)}} \mathscr{F}l^{\nu'} \overset{\operatorname{div}^\nu \times \operatorname{div}^{\nu'}}{\to} V^c$. The morphism f^c is of finite type. Let also ${}^n f^c$ be the restriction of f^c to the closed substack

$${}_n\mathscr{Q}^\nu \times_{{}_n\mathscr{Q}_d} {}_n\mathscr{Q}^{\nu'} \subset {}_n\mathscr{Q}^\nu \times_{{}_n\mathscr{Y}_d} {}_n\mathscr{Q}^{\nu'}.$$

The first step is as follows.

PROPOSITION 4

The morphism f^c is of relative dimension less than or equal to $b - d$, and the natural map of the highest cohomology sheaves

$$\mathrm{R}^{2(b-d)}(f^c)_!(\mu^*\mathscr{L}_\psi \boxtimes \mu^*\mathscr{L}_{\psi^{-1}}) \to \mathrm{R}^{2(b-d)}({}^n f^c)_!\bar{\mathbb{Q}}_\ell$$

is an isomorphism.

Further, set $W^c = \mathscr{F}l^\nu \times_{\mathrm{Sh}^d_0} \mathscr{F}l^{\nu'}$. Let $\operatorname{div}^c : W^c \to V^c$ be the map $\operatorname{div}^\nu \times \operatorname{div}^{\nu'}$. Set ${}^{\le n}W^c = W^c \times_{\mathrm{Sh}^d_0} {}^{\le n}\mathrm{Sh}^d_0$. Let also ${}^{\le n}\operatorname{div}^c$ denote the restriction of div^c to ${}^{\le n}W^c \subset W^c$.

The morphism ${}^n f^c$ is decomposed as ${}_n\mathscr{Q}^{\nu} \times_{{}_n\mathscr{Q}_d} {}_n\mathscr{Q}^{\nu'} \xrightarrow{\beta^c} {}^{\le n}W^c \xrightarrow{{}^{\le n}\mathrm{div}^c} V^c$, where β^c is the natural projection. The second step is the following lemma.

LEMMA 7

(i) *β^c is smooth and surjective with connected fibres of dimension b;*

(ii) div^c *is of relative dimension less than or equal to $-d$, so that*

$$\mathrm{R}^{2(b-d)}({}^n f^c)_!\bar{\mathbb{Q}}_\ell(b-d) \xrightarrow{\sim} \mathrm{R}^{-2d}({}^{\le n}\mathrm{div}^c)_!\bar{\mathbb{Q}}_\ell(-d).$$

Now assume $c = (\nu, \nu)$ with $\nu = (1, \dots, 1) \in \Lambda_{d,d}$. Let ${}_n\mathscr{S}^c_{E,E'}$ denote the direct image under $V^c \to X^{(d)}$ of the sheaf

$$\mathrm{R}^{2(b-d)}(f^c)_!(\mu^*\mathscr{L}_\psi \boxtimes \mu^*\mathscr{L}_{\psi^{-1}})(b-d)$$

tensored by the local system $(E^{\boxtimes d}) \boxtimes (E'^{\boxtimes d})$ on V^c. The group $S_d \times S_d$ acts naturally on ${}_n\mathscr{S}^c_{E,E'}$. By Corollary 1, ${}_n\mathscr{S}^d_{E,E'}$ is the sheaf of $(S_d \times S_d)$-invariants of ${}_n\mathscr{S}^c_{E,E'}$.

Combining Lemma 7 and Proposition 4, we learn that the complex $({}^{\le n}\mathrm{div})_!(\mathrm{Spr}^d_E \otimes \mathrm{Spr}^d_{E'})$ is placed in degrees less than or equal to $-2d$, and there is a canonical $(S_d \times S_d)$-equivariant isomorphism

$${}_n\mathscr{S}^c_{E,E'} \xrightarrow{\sim} \mathrm{R}^{-2d}({}^{\le n}\mathrm{div})_!(\mathrm{Spr}^d_E \otimes \mathrm{Spr}^d_{E'})(-d).$$

Thus, Theorem B is reduced to Theorem C.

6.2. The stack ${}_i\tilde{\mathscr{Z}}_d$

For $0 \le i \le n$, denote by ${}_i\tilde{\mathscr{Q}}_d$ the stack classifying collections

$$\big(0 = L_0 \subset L_1 \subset \cdots \subset L_i \subset F,\ \ (s_j)\big), \tag{17}$$

where $F \in \mathrm{Sh}_i$, (L_j) is a complete flag of vector subbundles on a rank i vector bundle L_i, $\deg(F/L_i) = d$, and $s_j : \Omega^{i-j} \xrightarrow{\sim} L_j/L_{j-1}$ is an isomorphism ($j = 1, \dots, i$).

We have the open substack ${}_i\mathscr{Q}_d \subset {}_i\tilde{\mathscr{Q}}_d$ given by the following condition: F is locally free. We also have a map ${}_i\tilde{\mathscr{Q}}_d \to \mathrm{Sh}_i$ that sends (17) to F. Define a substack

$${}_i\tilde{\mathscr{Z}}_d \hookrightarrow {}_i\tilde{\mathscr{Q}}_d \times_{\mathrm{Sh}_i} {}_i\tilde{\mathscr{Q}}_d$$

as follows. If S is a scheme, then an object

$$\big(F, (L_j, s_j), (L'_j, s'_j)\big) \tag{18}$$

of $\mathrm{Hom}(S, {}_i\tilde{\mathscr{Q}}_d \times_{\mathrm{Sh}_i} {}_i\tilde{\mathscr{Q}}_d)$ lies in $\mathrm{Hom}(S, {}_i\tilde{\mathscr{Z}}_d)$ if the collections (L_j, s_j) and (L'_j, s'_j) coincide outside a closed subscheme of $S \times X$ finite over S.

LEMMA 8
The map ${}_i\tilde{\mathscr{Z}}_d \hookrightarrow {}_i\tilde{\mathscr{Q}}_d \times_{\mathrm{Sh}_i} {}_i\tilde{\mathscr{Q}}_d$ *is a closed immersion. In particular, the stack* ${}_i\tilde{\mathscr{Z}}_d$ *is algebraic.*

Proof
An object (18) of $\mathrm{Hom}(S, {}_i\tilde{\mathscr{Q}}_d \times_{\mathrm{Sh}_i} {}_i\tilde{\mathscr{Q}}_d)$ gives rise to a pair of sections

$$t_j : \Omega^{(i-1)+\cdots+(i-j)} \widetilde{\to} \wedge^j L_j \to \wedge^j F$$

and

$$t'_j : \Omega^{(i-1)+\cdots+(i-j)} \widetilde{\to} \wedge^j L'_j \to \wedge^j F.$$

Clearly, (18) lies in $\mathrm{Hom}(S, {}_i\tilde{\mathscr{Z}}_d)$ if and only if the support of $t_j - t'_j$ is a closed subscheme of $S \times X$ finite over S (for all $j = 1, \ldots, i$).

Since F is S-flat, F (as well as its exterior powers) is locally free outside some closed subscheme of $S \times X$ finite over S. So, our assertion is a consequence of the following sublemma, communicated to the author by Drinfeld. □

SUBLEMMA 4
(1) *Let F be any coherent sheaf on $S \times X$, which is locally free outside a closed subscheme of $S \times X$ finite over S. Let s be a global section of F. Consider the following subfunctor Z of S (on the category of S-schemes): a morphism $S' \to S$ belongs to $Z(S')$ if the pullback of s to $S' \times X$ vanishes outside a closed subscheme of $S' \times X$ finite over S'. Then the subfunctor Z is closed.*
(2) *Suppose in addition that F is locally free. Then $S' \to S$ belongs to $Z(S')$ if and only if the pullback of s to $S' \times X$ vanishes.*

Proof
If $r : S \times X \to S \times \mathbb{P}^1$ is a finite morphism over S, then the functor Z does not change if we replace (X, F, s) by $(\mathbb{P}^1, r_*F, r_*s)$. After localizing with respect to S, we may assume that S is affine and that there is r as above with r_*F locally free over $S \times \mathbb{A}^1$. (Recall that if S is Noetherian, then any finite morphism $S \times X \to S \times \mathbb{P}^1$ over S is flat (cf. [9, expose IV, Section 5.9])). So, we are reduced to the case $X = \mathbb{P}^1$ with F locally free over $S \times \mathbb{A}^1$.

If $S = \mathrm{Spec}\, R$, then we have the projective $R[t]$-module $M = \mathrm{H}^0(S \times \mathbb{A}^1, F)$ and its element s. Represent M as a direct summand of a free $R[t]$-module M'. Clearly, M' is also a free R-module. If $s_i \in R$ are the coordinates of $s \in M'$ (over R), then Z is the closed subscheme of S defined by the equations $s_i = 0$. □

We have an open substack ${}_i\mathscr{Z}_d \subset {}_i\tilde{\mathscr{Z}}_d$ given by the condition that F is locally free. In particular, by Sublemma 4(2), ${}_n\mathscr{Z}_d = {}_n\mathscr{Q}_d \times_{{}_n\mathscr{Y}_d} {}_n\mathscr{Q}_d$. Besides, if (18) is a point of ${}_n\tilde{\mathscr{Z}}_d$, then the sections

$$\Omega^{(n-1)+\cdots+(n-n)} \tilde{\to} \det L_n \hookrightarrow \det F$$

and

$$\Omega^{(n-1)+\cdots+(n-n)} \tilde{\to} \det L'_n \hookrightarrow \det F$$

coincide, where $\det : \mathrm{Sh}_n \to \mathrm{Pic}\, X$ denotes the determinant map (cf. [11]). This yields a map $\tilde{f} : {}_n\tilde{\mathscr{Z}}_d \to X^{(d)}$ whose restriction to ${}_n\mathscr{Z}_d$ coincides with f. We prove in Section 6.3 that $\tilde{f}$ is of relative dimension less than or equal to $b-d$ but not of finite type. (The stack ${}_n\tilde{\mathscr{Z}}_d$ even has infinitely many irreducible components.)

For $k = 0, \dots, i$ we have a closed substack ${}^k_i\tilde{\mathscr{Z}}_d \hookrightarrow {}_i\tilde{\mathscr{Z}}_d$ given by the following condition: for a point (18) of ${}_i\tilde{\mathscr{Z}}_d$ the flags

$$\big(0 = L_0 \subset \cdots \subset L_k,\ \ (s_j)_{j=1,\dots,k}\big)$$

and

$$\big(0 = L'_0 \subset \cdots \subset L'_k,\ \ (s'_j)_{j=1,\dots,k}\big)$$

coincide. Notice that taking the quotient by $L_k = L'_k$, one gets a map ${}^k_i\tilde{\mathscr{Z}}_d \to {}_{i-k}\tilde{\mathscr{Z}}_d$, which is a generalized affine fibration. This observation is a key point in the proof of Proposition 4.

6.3. *Dimensions counting*

Proof of Lemma 7

(i) The map β^c is obtained by base change from the map $\beta : {}_n\mathscr{Q}_d \to {}^{\leq n}\mathrm{Sh}^d_0$, which is surjective and extends to a generalized affine fibration ${}_n\tilde{\mathscr{Q}}_d \to \mathrm{Sh}^d_0$ that sends (17) to F/L_n.

(ii) We stratify W^c by locally closed substacks $\mathscr{U}^e \hookrightarrow W^c$ indexed by $e \in {}^{m'}_{m}J_d$ with $h(e) = (\nu, \nu')$. A point

$$\big(F^1 \subset \cdots \subset F^m = F,\ \ (F^1)' \subset \cdots \subset (F^{m'})' = F\big) \tag{19}$$

of W^c lies in $\mathscr{U}^e$ if

$$\deg(F_i^j) = \sum_{k\leq i,\ \ l\leq j} e_k^l \quad \text{for } 1 \leq i \leq m,\ 1 \leq j \leq m',$$

where $F_i^j = F^i \cap (F^j)'$. If (19) is a point of $\mathscr{U}^e$, then for $1 \leq i \leq m,\ 1 \leq j \leq m'$, define $\tilde{F}_i^j \in \mathrm{Sh}_0$ from the co-Cartesian square

$$\begin{array}{ccc} F_{i-1}^j & \to & \tilde{F}_i^j \\ \uparrow & & \uparrow \\ F_{i-1}^{j-1} & \to & F_i^{j-1} \end{array}$$

and put $G_i^j = F_i^j/\tilde{F}_i^j$. Set also

$$\mathscr{W}^e = \prod_{i,j} \mathrm{Sh}_0^{e_i^j}.$$

The map $\mathscr{U}^e \to \mathscr{W}^e$ that sends (19) to the collection (G_i^j) is a generalized affine fibration of rank zero. We have a map $\mathscr{W}^e \to Y^e$ that sends (G_i^j) to the collection $(\operatorname{div} G_i^j)$, and we define $\operatorname{div}^e : \mathscr{U}^e \to Y^e$ as the composition $\mathscr{U}^e \to \mathscr{W}^e \to Y^e$. Since for any $i \geq 0$ the morphism $\operatorname{div} : \mathrm{Sh}_0^i \to X^{(i)}$ is of relative dimension less than or equal to $-i$, our assertion follows. □

Define the stack ${}_i\tilde{\mathscr{Z}}^c$ by the Cartesian square

$$\begin{array}{ccc} {}_i\tilde{\mathscr{Z}}^c & \to & {}_i\tilde{\mathscr{Z}}_d \\ \downarrow & & \downarrow \\ \mathscr{F}l^{\nu} \times_{X^{(d)}} \mathscr{F}l^{\nu'} & \to & \mathrm{Sh}_0^d \times_{X^{(d)}} \mathrm{Sh}_0^d \end{array}$$

where the right vertical arrow sends (18) to $(F/L_i\ , F/L_i')$. Let ${}_i\mathscr{Z}^c \subset {}_i\tilde{\mathscr{Z}}^c$ denote the preimage of ${}_i\mathscr{Z}_d$ under ${}_i\tilde{\mathscr{Z}}^c \to {}_i\tilde{\mathscr{Z}}_d$. In particular, we have ${}_n\mathscr{Z}^c = {}_n\mathscr{Q}^{\nu} \times_{{}_n\mathscr{Y}_d} {}_n\mathscr{Q}^{\nu'}$. Let

$$\tilde{f}^c : {}_n\tilde{\mathscr{Z}}^c \to V^c$$

denote the composition ${}_n\tilde{\mathscr{Z}}^c \to \mathscr{F}l^{\nu} \times_{X^{(d)}} \mathscr{F}l^{\nu'} \overset{\operatorname{div}^{\nu} \times \operatorname{div}^{\nu'}}{\to} V^c$. The restriction of $\tilde{f}^c$ to ${}_n\mathscr{Z}^c$ coincides with f^c. Notice that $\tilde{f}^c$ is locally of finite type but not of finite type in general.

LEMMA 9
The map $\tilde{f}^c$ is of relative dimension less than or equal to $b - d$.

Proof

Step 1. The stack ${}_n\mathscr{Q}^{\nu} \times_{{}_n\mathscr{Y}_d} {}_n\mathscr{Q}^{\nu'}$ is stratified by locally closed substacks $\mathscr{Q}^e \times_{{}_n\mathscr{Y}_d} \mathscr{Q}^{e'}$ indexed by pairs $e \in {}_n^m J_d$, $e' \in {}_n^{m'} J_d$ such that there exists $\lambda \in \Lambda_{n,d}$ with $h(e) = (\lambda, \nu)$, $h(e') = (\lambda, \nu')$. (cf. Section 4.3).

The restriction of $f^c : {}_n\mathscr{Z}^c \to V^c$ to a stratum $\mathscr{Q}^e \times_{{}_n\mathscr{Y}_d} \mathscr{Q}^{e'}$ is written as the composition

$$\mathscr{Q}^e \times_{{}_n\mathscr{Y}_d} \mathscr{Q}^{e'} \overset{\varphi^e \times \varphi^{e'}}{\to} (Y^e \times_{X^{\lambda}} Y^{e'}) \times_{X^{\lambda}} \mathscr{V}^{\lambda} \to Y^e \times_{X^{\lambda}} Y^{e'} \to V^c.$$

Since $\varphi^e : \mathscr{Q}^e \to Y^e \times_{X^{\lambda}} \mathscr{V}^{\lambda}$ is an affine fibration of rank $a(\lambda)$, and $\mathscr{V}^{\lambda} \to X^{\lambda}$ is a generalized affine fibration of rank $b - d - 2a(\lambda)$, it follows that f^c is of relative dimension less than or equal to $b - d$.

Step 2. Stratify Sh_n by fixing the degree of the maximal torsion subsheaf of $F \in \mathrm{Sh}_n$. Consider the induced stratification of ${}_n\tilde{\mathscr{Z}}^c$. A stratum ${}_n\tilde{\mathscr{Z}}^c_k \subset {}_n\tilde{\mathscr{Z}}^c$ classifies the following data: a point (18) of ${}_n\tilde{\mathscr{Z}}_d$, two flags of subsheaves

$$(L_n \subset L_n^1 \subset \cdots \subset L_n^m = F) \tag{20}$$

and

$$\big(L'_n \subset (L_n^1)' \subset \cdots \subset (L_n^{m'})' = F\big), \tag{21}$$

and an exact sequence $0 \to F_0 \to F \to M \to 0$ of $\mathscr{O}_X$-modules, where $F_0 \in \mathrm{Sh}_0^k$ and M is a vector bundle on X of rank i.

The preimages of flags (20) and (21) in F_0 give rise to a point of $\mathscr{F}l^\lambda \times_{\mathrm{Sh}_0^k} \mathscr{F}l^{\lambda'}$ for some $\lambda \in \Lambda_{m,k}$, $\lambda' \in \Lambda_{m',k}$. This yields a stratification of ${}_n\tilde{\mathscr{Z}}^c_k$ by locally closed substacks ${}_n\tilde{\mathscr{Z}}^c_{\lambda,\lambda'} \hookrightarrow {}_n\tilde{\mathscr{Z}}^c_k$ indexed by pairs $\lambda \in \Lambda_{m,k}$, $\lambda' \in \Lambda_{m',k}$.

For an object of ${}_n\tilde{\mathscr{Z}}^c_{\lambda,\lambda'}$ the vector bundle M together with the images of the corresponding flags on F defines a point of ${}_n\mathscr{Q}^{\nu-\lambda} \times_{{}_n\mathscr{Y}_{d-k}} {}_n\mathscr{Q}^{\nu'-\lambda'}$. The natural forgetful map

$${}_n\tilde{\mathscr{Z}}^c_{\lambda,\lambda'} \to (\mathscr{F}l^\lambda \times_{\mathrm{Sh}_0^k} \mathscr{F}l^{\lambda'}) \times ({}_n\mathscr{Q}^{\nu-\lambda} \times_{{}_n\mathscr{Y}_{d-k}} {}_n\mathscr{Q}^{\nu'-\lambda'})$$

is a generalized affine fibration of rank nk. Recall that $b = b(n,d)$ depends on n and d (cf. Section 2.2). By Step 1,

$${}_n\mathscr{Q}^{\nu-\lambda} \times_{{}_n\mathscr{Y}_{d-k}} {}_n\mathscr{Q}^{\nu'-\lambda'} \to X^{\nu-\lambda} \times_{X^{(d-k)}} X^{\nu'-\lambda'}$$

is of relative dimension less than or equal to $b(n,d-k)-(d-k) = b(n,d)-d-nk+k$. By Lemma 7(ii),

$$\mathscr{F}l^\lambda \times_{\mathrm{Sh}_0^k} \mathscr{F}l^{\lambda'} \to X^\lambda \times_{X^{(k)}} X^{\lambda'}$$

is of relative dimension less than or equal to $-k$. Our assertion follows. □

6.4. Proof of Proposition 4

Let ${}^k_i\mathscr{Z}_d$ be the preimage of ${}_i\mathscr{Z}_d$ under ${}^k_i\tilde{\mathscr{Z}}_d \hookrightarrow {}_i\tilde{\mathscr{Z}}_d$. For $k = 0, \dots, i$, define the stacks ${}^k_i\mathscr{Z}^c \subset {}^k_i\tilde{\mathscr{Z}}^c$ by the Cartesian squares

$$\begin{array}{ccccc} {}^k_i\mathscr{Z}^c & \hookrightarrow & {}^k_i\tilde{\mathscr{Z}}^c & \hookrightarrow & {}_i\tilde{\mathscr{Z}}^c \\ \downarrow & & \downarrow & & \downarrow \\ {}^k_i\mathscr{Z}_d & \hookrightarrow & {}^k_i\tilde{\mathscr{Z}}_d & \hookrightarrow & {}_i\tilde{\mathscr{Z}}_d \end{array}$$

Denote by ${}^i\mathscr{L}_\psi$ the restriction of $\mu^*\mathscr{L}_\psi \boxtimes \mu^*\mathscr{L}_{\psi^{-1}}$ under the composition

$${}^i_n\mathscr{Z}^c \hookrightarrow {}_n\mathscr{Z}^c \to {}_n\mathscr{Z}_d \tilde{\to} {}_n\mathscr{Q}_d \times_{{}_n\mathscr{Y}_d} {}_n\mathscr{Q}_d.$$

Let also ${}^if^c$ be the restriction of f^c to ${}^i_n\mathscr{Z}^c \hookrightarrow {}_n\mathscr{Z}^c$. Arguing by induction, we show that the natural map

$$\mathrm{R}^{2(b-d)}({}^if^c)_!({}^i\mathscr{L}_\psi) \to \mathrm{R}^{2(b-d)}({}^{i+1}f^c)_!({}^{i+1}\mathscr{L}_\psi)$$

is an isomorphism for $i = 1, \dots, n-1$.

The map $\mu : {}_n\mathscr{Q}_d \to \mathbb{A}^1$ extends naturally to a morphism ${}_n\tilde{\mathscr{Q}}_d \to \mathbb{A}^1$ defined in the same way; it is also denoted by μ. This allows us to extend ${}^i\mathscr{L}_\psi$ to a local system ${}^i\tilde{\mathscr{L}}_\psi$ on ${}^i_n\tilde{\mathscr{Z}}^c$, where ${}^i\tilde{\mathscr{L}}_\psi$ is defined as the restriction of $\mu^*\mathscr{L}_\psi \boxtimes \mu^*\mathscr{L}_{\psi^{-1}}$ under the composition

$${}^i_n\tilde{\mathscr{Z}}^c \hookrightarrow {}_n\tilde{\mathscr{Z}}^c \to {}_n\tilde{\mathscr{Z}}_d \hookrightarrow {}_n\tilde{\mathscr{Q}}_d \times_{\mathrm{Sh}_i} {}_n\tilde{\mathscr{Q}}_d.$$

We have the diagram

$$\begin{array}{ccccccc}
{}^{i+1}_n\mathscr{Z}^c & \hookrightarrow & & {}^i_n\mathscr{Z}^c & \overset{\delta}{\hookrightarrow} & {}^i_n\tilde{\mathscr{Z}}^c \\
\downarrow & & \searrow\alpha & \downarrow\beta & \swarrow\gamma & \\
{}^1_{n-i}\tilde{\mathscr{Z}}^c & \hookrightarrow & & {}_{n-i}\tilde{\mathscr{Z}}^c & &
\end{array}$$

in which the square is Cartesian, and γ is a generalized affine fibration of rank $b(n,d) - b(n-i,d)$. Since δ is an open immersion,

$$\mathrm{R}^{\mathrm{top}}\beta_!({}^i\mathscr{L}_\psi) \to \mathrm{R}^{\mathrm{top}}\gamma_!({}^i\tilde{\mathscr{L}}_\psi)$$

is an isomorphism over the image of (the smooth map) β. Applying Sublemma 3 for $M = F/L_i$, $D = 0$, and the section $s_{i+1} - s'_{i+1} : \Omega^{n-i-1} \to M$, we learn that $\mathrm{R}^{\mathrm{top}}\gamma_!({}^i\tilde{\mathscr{L}}_\psi)$ is supported at ${}^1_{n-i}\tilde{\mathscr{Z}}^c$. Therefore,

$$\mathrm{R}^{\mathrm{top}}\beta_!({}^i\mathscr{L}_\psi) \to \mathrm{R}^{\mathrm{top}}\alpha_!({}^{i+1}\mathscr{L}_\psi)$$

is an isomorphism. So, ${}^if^c$ is decomposed as ${}^i_n\mathscr{Z}^c \xrightarrow{\beta} {}_{n-i}\tilde{\mathscr{Z}}^c \to V^c$, where, by Lemma 9, the second map is of relative dimension less than or equal to $b(n-i,d)-d$. Though ${}_{n-i}\tilde{\mathscr{Z}}^c \to V^c$ is not of finite type in general, its restriction to the image of β is a morphism of finite type.

This concludes the proof of Proposition 4. □

6.5. Proof of Theorem C

Recall that we have the map $h : {}^{m'}_m J_d \to \Lambda_{m,d} \times \Lambda_{m',d}$, and for $e \in {}^{m'}_m J_d$ we write $Y^e = \prod_{i,j} X^{(e_i^j)}$ (cf. Section 4.3). Let

$$\mathrm{norm} : \bigsqcup_{e \in h^{-1}(c)} Y^e \to V^c$$

be the map that sends a matrix $(D_i^j) \in Y^e$ to the collection $((D_i),(D'_j))$, where $D_i = \sum_j D_i^j$ and $D'_j = \sum_i D_i^j$.

LEMMA 10

(i) *The scheme V^c is of pure dimension d, and its irreducible components are numbered by the set $h^{-1}(c)$. Namely, to $e \in h^{-1}(c)$ there corresponds the component* $\mathrm{norm}(Y^e)$.

(ii) *The map* norm *is the normalization of V^c. (More precisely, it is a finite morphism, an isomorphism over an open dense subscheme of V^c, and the scheme $\bigsqcup_{e\in h^{-1}(c)} Y^e$ is smooth.) So,*

$$\mathrm{norm}_* \bar{\mathbb{Q}}_\ell[d] \xrightarrow{\sim} \mathrm{IC},$$

where IC *is the intersection cohomology sheaf on V^c.*

Proof

Stratify V^c by locally closed subschemes ${}_eV^c \subset V^c$ indexed by $e \in h^{-1}(c)$. First, define ${}_eV^c$ as the open subscheme of Y^e given by the following condition:

$$\text{if } i > k, l > j, \text{ then } D_i^j \cap D_k^l = \emptyset.$$

Then the composition ${}_eV^c \hookrightarrow Y^e \overset{\mathrm{norm}}{\to} V^c$ is a locally closed immersion. As a subscheme of V^c, ${}_eV^c$ is given by the condition that

$$\text{for all } i, j, \text{ we have } \deg\Big(\Big(\sum_{k\le i} D_k\Big) \cap \Big(\sum_{l\le j} D_l'\Big)\Big) = \sum_{k\le i,\ l\le j} e_k^l.$$

For any $e \in h^{-1}(c)$, the scheme ${}_eV^c$ is smooth, nonempty, and irreducible of dimension d. This concludes the proof. □

LEMMA 11

There is a canonical isomorphism $\mathrm{R}^{-2d}(\mathrm{div}^c)_!\bar{\mathbb{Q}}_\ell(-d) \xrightarrow{\sim} \mathrm{norm}_* \bar{\mathbb{Q}}_\ell$.

We need the following straightforward sublemma.

SUBLEMMA 5

(i) *Let $r : Y \to Y'$ be a separated morphism of schemes of finite type. Assume that the fibres of r are of dimension less than or equal to d. Let F be a smooth $\bar{\mathbb{Q}}_\ell$-sheaf on Y, let $U \subset Y$ be an open subscheme, and let r_U be the restriction of r to U. Then the natural map* $\mathrm{R}^{2d}(r_U)_!F \to \mathrm{R}^{2d}r_!F$ *is injective.*

(ii) *Let $(U^j)_{j\in J}$ be a stratification of Y by locally closed subschemes which comes from a filtration of Y by closed subschemes. Let r^j be the restriction of r to U^j. Then* $\mathrm{R}^{2d}r_!F$ *admits a filtration by subsheaves with successive quotients being* $\mathrm{R}^{2d}r_!^jF \quad (j \in J)$.

Proof of Lemma 11

Recall that W^c is stratified by locally closed substacks $\mathscr{U}^e \hookrightarrow W^c$ indexed by $e \in h^{-1}(c)$ and that we have the maps $\operatorname{div}^e : \mathscr{U}^e \to Y^e$ (cf. the proof of Lemma 7). The diagram commutes:

$$\begin{array}{ccc} \mathscr{U}^e & \hookrightarrow & W^c \\ \downarrow \operatorname{div}^e & & \downarrow \operatorname{div}^c \\ Y^e & \overset{\text{norm}}{\to} & V^c \end{array}$$

We have $\mathrm{R}^{-2d}(\operatorname{div}^e)_!\bar{\mathbb{Q}}_\ell(-d) \widetilde{\to} \bar{\mathbb{Q}}_\ell$ canonically. Indeed, by Künneth formulae, this is reduced to the fact that for any $i \geq 0$ the fibres of $\operatorname{div} : \mathrm{Sh}_0^i \to X^{(i)}$ are connected of dimension $-i$.

By Sublemma 5(ii), on $\mathrm{R}^{-2d}(\operatorname{div}^c)_!\bar{\mathbb{Q}}_\ell(-d)$ there is a filtration parametrized by the set $h^{-1}(c)$ with successive quotients being $(\operatorname{norm}^e)_*\bar{\mathbb{Q}}_\ell$. We claim that any filtration with these successive quotients degenerates canonically into a direct sum. Indeed,

(i) the different successive quotients are supported on different irreducible components of V^c, so our filtration degenerates into a direct sum over some open dense subscheme of V^c;

(ii) the sheaf $(\operatorname{norm}^e)_*\bar{\mathbb{Q}}_\ell[d]$ is perverse and the Goresky-MacPherson extension of its restriction to any open dense subscheme of V^c;

(iii) the property "perverse and the Goresky-MacPherson extension of its restriction to a given open subscheme of V^c" is preserved for extensions.

□

Finally, assume $c = (\nu, \nu)$ with $\nu = (1, \dots, 1) \in \Lambda_{d,d}$. Then the set $h^{-1}(c)$ is in natural bijection with S_d, and the map norm becomes

$$\bigsqcup_{\sigma \in S_d} X^\nu_\sigma \to V^c,$$

where $X^\nu_\sigma = X^\nu$, and the map norm sends a point $(x_1, \dots, x_d) \in X^\nu_\sigma$ to $((x_1, \dots, x_d), (x_{\sigma 1}, \dots, x_{\sigma d}))$. The action of $S_d \times S_d$ on V^c lifts naturally to an action on

$$(E^{\boxtimes d} \boxtimes E'^{\boxtimes d}) \otimes \operatorname{norm}_* \bar{\mathbb{Q}}_\ell,$$

and it is easy to see that $\operatorname{pr}_!((E^{\boxtimes d} \boxtimes E'^{\boxtimes d}) \otimes \operatorname{norm}_* \bar{\mathbb{Q}}_\ell)^{S_d \times S_d} \widetilde{\to} (E \otimes E')^{(d)}$ canonically, where $\operatorname{pr} : V^c \to X^{(d)}$ denotes the projection. On the other hand, by Lemma 11,

$$\mathrm{R}^{-2d} \operatorname{div}_!(\mathrm{Spr}^d_E \otimes \mathrm{Spr}^d_{E'})(-d) \widetilde{\to} \operatorname{pr}_!\big((E^{\boxtimes d} \boxtimes E'^{\boxtimes d}) \otimes \operatorname{norm}_* \bar{\mathbb{Q}}_\ell\big).$$

One checks that this isomorphism is $(S_d \times S_d)$-equivariant. Taking the invariants, one gets

$$\mathrm{R}^{-2d} \operatorname{div}_!(\mathscr{L}^d_E \otimes \mathscr{L}^d_{E'})(-d) \widetilde{\to} (E \otimes E')^{(d)}.$$

By Sublemma 5(i), the natural map $\mathrm{R}^{-2d}({}^{\leq n}\mathrm{div}^c)_!\bar{\mathbb{Q}}_\ell \to \mathrm{R}^{-2d}(\mathrm{div}^c)_!\bar{\mathbb{Q}}_\ell$ is an inclusion. It follows that

$$\mathrm{R}^{-2d}({}^{\leq n}\mathrm{div})_!(\mathrm{Spr}^d_E \otimes \mathrm{Spr}^d_{E'})(-d) \to \mathrm{R}^{-2d}\,\mathrm{div}_!(\mathrm{Spr}^d_E \otimes \mathrm{Spr}^d_{E'})(-d) \tag{22}$$

is an inclusion. Taking the $S_d \times S_d$-invariants in (22), one gets an inclusion ${}_n\mathscr{S}^d_{E,E'} \subset (E \otimes E')^{(d)}$, whose image is denoted ${}^{\leq n}(E \otimes E')^{(d)}$. Since n and d were arbitrary, Lemma 1 follows now from Corollary 1, and the proof of Theorem C is complete. □

So, Theorem B and the Main Local Theorem are also proved. □

6.6. *Second proof of Theorem B*

In this section we present an alternative proof of Theorem B under the additional assumption $\min\{\mathrm{rk}\, E, \mathrm{rk}\, E'\} \leq n$. The idea of this proof was suggested to the author by Gaitsgory.

Let ${}_n\,\mathrm{Mod}_d$ denote the stack classifying modifications $(L \subset L')$ of rank n vector bundles on X with $\deg(L'/L) = d$. Let $q : {}_n\,\mathrm{Mod}_d \to \mathrm{Sh}^d_0$ be the map that sends $(L \subset L')$ to L'/L, and let $\mathrm{supp} : {}_n\,\mathrm{Mod}_d \to X^{(d)}$ denote $\mathrm{div} \circ q$. For $d' \geq 0$, let $\mathfrak{p}_{\mathscr{Y}} : {}_n\mathscr{Y}_d \times_{\mathrm{Bun}_n} {}_n\,\mathrm{Mod}_{d'} \to {}_n\mathscr{Y}_{d+d'}$ be the map that sends $((t_i), L \subset L')$ to $((t'_i), L')$, where t'_i is the composition

$$\Omega^{(n-1)+\cdots+(n-i)} \stackrel{t_i}{\hookrightarrow} \wedge^i L \hookrightarrow \wedge^i L'.$$

The map $\mathfrak{p}_{\mathscr{Y}}$ is representable and proper. Let $\mathfrak{q}_{\mathscr{Y}} : {}_n\mathscr{Y}_d \times_{\mathrm{Bun}_n} {}_n\,\mathrm{Mod}_{d'} \to {}_n\mathscr{Y}_d$ denote the projection. The map $\mathfrak{q}_{\mathscr{Y}}$ is smooth of relative dimension nd'.

The key ingredient is the Hecke property of Whittaker sheaves (see [8, Proposition 7.5]). It admits the following immediate corollary. (The argument given in [8, Proposition 7.5] for $\mathrm{rk}\, E = n$ holds, in fact, for $\mathrm{rk}\, E \leq n$.)

PROPOSITION 5

For any smooth $\bar{\mathbb{Q}}_\ell$-sheaf E on X and any $d \geq 0$, there is a natural map

$$(\mathfrak{q}_{\mathscr{Y}} \times \mathrm{supp})_!\mathfrak{p}^*_{\mathscr{Y}}({}_n\mathscr{P}^{d+1}_{E,\psi}) \to {}_n\mathscr{P}^d_{E,\psi} \boxtimes E\Big(\frac{2-n}{2}\Big)[2-n],$$

which is an isomorphism if $\mathrm{rk}\, E \leq n$.

Let ${}_n\widetilde{\mathrm{Mod}}_d$ be the stack of flags $(L_0 \subset \cdots \subset L_d)$, where $(L_i \subset L_{i+1}) \in {}_n\,\mathrm{Mod}_1$ for all i. Let $\widetilde{\mathrm{supp}} : {}_n\widetilde{\mathrm{Mod}}_d \to X^d$ be the map that sends $(L_0 \subset \cdots \subset L_d)$ to $(\mathrm{div}(L_1/L_0), \ldots, \mathrm{div}(L_d/L_{d-1}))$. Let $p : {}_n\mathscr{Y}_d \times_{\mathrm{Bun}_n} {}_n\widetilde{\mathrm{Mod}}_{d'} \to {}_n\mathscr{Y}_d \times_{\mathrm{Bun}_n} {}_n\,\mathrm{Mod}_{d'}$ be the projection, and let $\tilde{\mathfrak{p}}_{\mathscr{Y}} : {}_n\mathscr{Y}_d \times_{\mathrm{Bun}_n} {}_n\widetilde{\mathrm{Mod}}_{d'} \to {}_n\mathscr{Y}_{d+d'}$ be the composition $\mathfrak{p}_{\mathscr{Y}} \circ p$.

COROLLARY 2
For any smooth $\bar{\mathbb{Q}}_\ell$-sheaf E on X and any $d, d' \geq 0$, there is a natural map

$$(\mathfrak{q}_{\mathscr{Y}} \times \widetilde{\operatorname{supp}})_! \, \tilde{\mathfrak{p}}_{\mathscr{Y}}^* ({}_n\mathscr{P}_{E,\psi}^{d+d'}) \to {}_n\mathscr{P}_{E,\psi}^d \boxtimes E^{\boxtimes d'}\Big(\frac{2d' - nd'}{2}\Big)[2d' - nd'], \qquad (23)$$

which is an isomorphism if $\operatorname{rk} E \leq n$.

Proof
The map (23) is defined as follows. Let $\mathfrak{p}_{\mathscr{Q}} : {}_n\mathscr{Q}_d \times_{\operatorname{Bun}_n} {}_n\operatorname{Mod}_{d'} \to {}_n\mathscr{Q}_{d+d'}$ be the map that sends $(L_1 \subset \cdots \subset L_n \subset L \subset L')$ to $(L_1 \subset \cdots \subset L_n \subset L')$. Let $\tilde{\mathfrak{p}}_{\mathscr{Q}} : {}_n\mathscr{Q}_d \times_{\operatorname{Bun}_n} {}_n\widetilde{\operatorname{Mod}}_{d'} \to {}_n\mathscr{Q}_{d+d'}$ denote the composition

$${}_n\mathscr{Q}_d \times_{\operatorname{Bun}_n} {}_n\widetilde{\operatorname{Mod}}_{d'} \to {}_n\mathscr{Q}_d \times_{\operatorname{Bun}_n} {}_n\operatorname{Mod}_{d'} \stackrel{\mathfrak{p}_{\mathscr{Q}}}{\to} {}_n\mathscr{Q}_{d+d'},$$

where the first arrow is the projection. Consider the commutative diagram

$$\begin{array}{ccc} {}_n\mathscr{Q}_d \times_{\operatorname{Bun}_n} {}_n\widetilde{\operatorname{Mod}}_{d'} & \stackrel{\tilde{\mathfrak{p}}_{\mathscr{Q}}}{\to} & {}_n\mathscr{Q}_{d+d'} \\ \downarrow \varphi\times\mathrm{id} & & \downarrow \varphi \\ {}_n\mathscr{Y}_d \times_{\operatorname{Bun}_n} {}_n\widetilde{\operatorname{Mod}}_{d'} & \stackrel{\tilde{\mathfrak{p}}_{\mathscr{Y}}}{\to} & {}_n\mathscr{Y}_{d+d'} \end{array}$$

Since ${}_n\mathscr{F}_{E,\psi}^{d+d'}$ is a direct summand of

$$(\tilde{\mathfrak{p}}_{\mathscr{Q}})_! ({}_n\mathscr{F}_{E,\psi}^d \boxtimes \widetilde{\operatorname{supp}}^* E^{\boxtimes d'})[nd']\Big(\frac{nd'}{2}\Big),$$

it follows that ${}_n\mathscr{P}_{E,\psi}^{d+d'}$ is a direct summand of

$$(\tilde{\mathfrak{p}}_{\mathscr{Y}})_! ({}_n\mathscr{P}_{E,\psi}^d \boxtimes \widetilde{\operatorname{supp}}^* E^{\boxtimes d'})[nd'](nd'/2).$$

This yields a morphism

$$\tilde{\mathfrak{p}}_{\mathscr{Y}}^* ({}_n\mathscr{P}_{E,\psi}^{d+d'}) \to {}_n\mathscr{P}_{E,\psi}^d \boxtimes \widetilde{\operatorname{supp}}^* E^{\boxtimes d'}[nd']\Big(\frac{nd'}{2}\Big).$$

Since $\mathfrak{q}_{\mathscr{Y}} \times \widetilde{\operatorname{supp}}$ is smooth of relative dimension $d'(n-1)$, the desired map is obtained from the last one by adjointness. To show that (23) is an isomorphism under the condition $\operatorname{rk} E \leq n$, apply d' times Proposition 5. □

Denote by ${}^{\mathrm{rss}}X^{d'} \subset X^{d'}$ the open subscheme that parametrizes pairwise different points $(x_1, \ldots, x_{d'}) \in X^{d'}$ (here “rss” stands for “regular semisimple”). Let ${}_n^{\mathrm{rss}}\widetilde{\operatorname{Mod}}_{d'}$ be the preimage of ${}^{\mathrm{rss}}X^{d'}$ under $\widetilde{\operatorname{supp}}$. The symmetric group $S_{d'}$ acts on ${}_n^{\mathrm{rss}}\widetilde{\operatorname{Mod}}_{d'}$, and the restriction of $\tilde{\mathfrak{p}}_{\mathscr{Y}}$ to ${}_n\mathscr{Y}_d \times_{\mathrm{Bun}_n} {}^{\mathrm{rss}}_n\widetilde{\mathrm{Mod}}_{d'}$ is $S_{d'}$-invariant. So, the action of $S_{d'}$ on ${}_n\mathscr{Y}_d \times_{\mathrm{Bun}_n} {}^{\mathrm{rss}}_n\widetilde{\mathrm{Mod}}_{d'}$ lifts to an action on

$$\tilde{\mathfrak{p}}^*_{\mathscr{Y}}({}_n\mathscr{P}^{d+d'}_{E,\psi}).$$

Since the restriction ${}^{\mathrm{rss}}_n\widetilde{\mathrm{Mod}}_{d'} \to {}^{\mathrm{rss}}X^{d'}$ of $\widetilde{\mathrm{supp}}$ is $S_{d'}$-equivariant, $S_{d'}$ acts on the complex

$$(\mathfrak{q}_{\mathscr{Y}} \times \widetilde{\mathrm{supp}})_!\tilde{\mathfrak{p}}^*_{\mathscr{Y}}({}_n\mathscr{P}^{d+d'}_{E,\psi})$$

restricted to ${}_n\mathscr{Y}_d \times {}^{\mathrm{rss}}X^{d'}$. On the other hand, $S_{d'}$ acts on $E^{\boxtimes d'}$ and, hence, on the right-hand side of (23). Using the explicit description of map (23), one easily proves the following lemma.

LEMMA 12
For any smooth $\bar{\mathbb{Q}}_\ell$-sheaf E on X, the map (23) restricted to ${}_n\mathscr{Y}_d \times {}^{\mathrm{rss}}X^{d'}$ is $S_{d'}$-equivariant.

Recall that for any smooth $\bar{\mathbb{Q}}_\ell$-sheaf E on X, the Verdier dual of ${}_n\mathscr{P}^d_{E,\psi}$ is canonically isomorphic to ${}_n\mathscr{P}^d_{E^*,\psi^{-1}}$ (see [8, Lemma 4.7]). So, to prove Theorem B we must establish a canonical isomorphism

$$\phi_*\,\mathrm{R}\mathscr{H}om({}_n\mathscr{P}^d_{E,\psi},\ {}_n\mathscr{P}^d_{E',\psi}) \tilde{\to} \mathscr{H}om(E, E')^{(d)},$$

where $\phi : {}_n\mathscr{Y}_d \to X^{(d)}$ is the map defined in Section 2.3. The statement of Theorem B being symmetric with respect to interchanging E and E', we assume $\mathrm{rk}\, E \leq n$.

For any smooth $\bar{\mathbb{Q}}_\ell$-sheaf E on X, set ${}_n\tilde{\mathscr{P}}^d_{E,\psi} = \varphi_!(\beta^*\,\mathrm{Spr}^d_E \otimes \mu^*\mathscr{L}_\psi)[b](b/2)$. In other words, ${}_n\tilde{\mathscr{P}}^d_{E,\psi}$ is a complex on ${}_n\mathscr{Y}_d$ obtained by replacing in the definition of ${}_n\mathscr{P}^d_{E,\psi}$ Laumon's sheaf $\mathscr{L}^d_E$ by Springer's sheaf Spr^d_E. Theorem B follows now from the next statement.

PROPOSITION 6
Let E, E' be smooth $\bar{\mathbb{Q}}_\ell$-sheaves on X with $\mathrm{rk}\, E \leq n$. Then there exists a canonical S_d-equivariant isomorphism

$$\phi_*\,\mathrm{R}\mathscr{H}om({}_n\mathscr{P}^d_{E,\psi},\ {}_n\tilde{\mathscr{P}}^d_{E',\psi}) \tilde{\to} \mathrm{sym}_*\big(\mathscr{H}om(E, E')^{\boxtimes d}\big).$$

Proof
The idea is that Proposition 6 is a tautological consequence of Corollary 2 obtained by applying the formalism of six functors. The equivariance property follows from Lemma 12. The precise argument is as follows.

Consider the commutative diagram

$$\begin{array}{ccccc} & & {}_n\mathcal{Y}_0 \times_{\mathrm{Bun}_n} {}_n\mathrm{Mod}_d & \overset{\mathfrak{p}_{\mathcal{Y}}}{\to} & {}_n\mathcal{Y}_d \\ & \swarrow \mathfrak{q}_{\mathcal{Y}} & \downarrow \mathfrak{q}_{\mathcal{Y}} \times \mathrm{supp} & & \downarrow \phi \\ {}_n\mathcal{Y}_0 & \leftarrow & {}_n\mathcal{Y}_0 \times X^{(d)} & \to & X^{(d)} \end{array}$$

Set $\Psi^0 = {}_n\mathcal{P}^0_{E,\psi}$ for brevity. (It does not depend on E, though it does depend on ψ.) By definition,

$${}_n\tilde{\mathcal{P}}^d_{E',\psi} \widetilde{\to} \mathfrak{p}_{\mathcal{Y}*}(\mathfrak{q}^*_{\mathcal{Y}}\Psi^0 \otimes q^* \operatorname{Spr}^d_{E'})[nd]\Big(\frac{nd}{2}\Big).$$

LEMMA 13

We have $\mathfrak{q}^*_{\mathcal{Y}}\Psi^0 \otimes q^* \operatorname{Spr}^d_{E'} \widetilde{\to} \mathrm{R}\mathcal{H}om(q^* \operatorname{Spr}^d_{E'^*}, \mathfrak{q}^*_{\mathcal{Y}}\Psi^0)$ *canonically and* S_d*-equivariantly.*

Proof

Using the fact that both $\mathfrak{q}_{\mathcal{Y}}$ and $\mathfrak{q}_{\mathcal{Y}} \circ p$ are smooth of relative dimension nd, we get

$$\begin{aligned} \mathfrak{q}^*_{\mathcal{Y}}\Psi^0 \otimes q^* \operatorname{Spr}^d_{E'} &\widetilde{\to} p_!(\widetilde{\mathrm{supp}}^* E'^{\boxtimes d} \otimes p^*\mathfrak{q}^*_{\mathcal{Y}}\Psi^0) \\ &\widetilde{\to} p_* \mathrm{R}\mathcal{H}om\big(\widetilde{\mathrm{supp}}^*(E'^*)^{\boxtimes d},\ p^*\mathfrak{q}^*_{\mathcal{Y}}\Psi^0\big) \\ &\widetilde{\to} p_* \mathrm{R}\mathcal{H}om\big(\widetilde{\mathrm{supp}}^*(E'^*)^{\boxtimes d},\ p^!\mathfrak{q}^!_{\mathcal{Y}}\Psi^0[-2nd](-nd)\big) \\ &\widetilde{\to} \mathrm{R}\mathcal{H}om\big(p_!\widetilde{\mathrm{supp}}^*(E'^*)^{\boxtimes d},\ \mathfrak{q}^*_{\mathcal{Y}}\Psi^0\big). \end{aligned}$$

□

Using Lemma 13, we get

$$\begin{aligned} &\mathrm{R}\mathcal{H}om({}_n\mathcal{P}^d_{E,\psi}, {}_n\tilde{\mathcal{P}}^d_{E',\psi}) \\ &\quad \widetilde{\to} \mathfrak{p}_{\mathcal{Y}*}\mathrm{R}\mathcal{H}om\big(\mathfrak{p}^*_{\mathcal{Y}}({}_n\mathcal{P}^d_{E,\psi}),\ \mathfrak{q}^*_{\mathcal{Y}}\Psi^0 \otimes q^* \operatorname{Spr}^d_{E'}\big)[nd]\Big(\frac{nd}{2}\Big) \\ &\quad \widetilde{\to} \mathfrak{p}_{\mathcal{Y}*}\mathrm{R}\mathcal{H}om\big(\mathfrak{p}^*_{\mathcal{Y}}({}_n\mathcal{P}^d_{E,\psi}) \otimes q^* \operatorname{Spr}^d_{E'^*},\ \mathfrak{q}^!_{\mathcal{Y}}\Psi^0\big)[-nd]\Big(\frac{-nd}{2}\Big). \end{aligned}$$

Let $j : {}_n\mathcal{Q}_0 \hookrightarrow {}_n\mathcal{Y}_0$ denote the natural open immersion. Since ${}_n\mathcal{Q}_0 \to \operatorname{Spec} k$ is a generalized affine fibration, we have $\mathrm{R}\Gamma({}_n\mathcal{Q}_0, \bar{\mathbb{Q}}_\ell) \widetilde{\to} \bar{\mathbb{Q}}_\ell$. So, our assertion is reduced to the following lemma.

LEMMA 14

There is a canonical S_d*-equivariant isomorphism over* ${}_n\mathcal{Y}_0 \times X^{(d)}$*:*

$$\begin{aligned} &(\mathfrak{q}_{\mathcal{Y}} \times \mathrm{supp})_* \mathrm{R}\mathcal{H}om\big(\mathfrak{p}^*_{\mathcal{Y}}({}_n\mathcal{P}^d_{E,\psi}) \otimes q^* \operatorname{Spr}^d_{E'^*},\ \mathfrak{q}^!_{\mathcal{Y}}\Psi^0\big) \\ &\qquad \widetilde{\to} (j \times \mathrm{id})_*\big(\bar{\mathbb{Q}}_\ell \boxtimes \mathrm{sym}_* \mathcal{H}om(E, E')^{\boxtimes d}\big)[nd]\Big(\frac{nd}{2}\Big). \end{aligned}$$

Proof
Let $\mathrm{pr}_{\mathscr{Y}} : {}_n\mathscr{Y}_0 \times X^d \to {}_n\mathscr{Y}_0$ denote the projection. Using the commutative diagram

$$\begin{array}{ccc} {}_n\mathscr{Y}_0 \times_{\mathrm{Bun}_n} {}_n\mathrm{Mod}_d & \overset{p}{\leftarrow} & {}_n\mathscr{Y}_0 \times_{\mathrm{Bun}_n} {}_n\widetilde{\mathrm{Mod}}_d \\ \downarrow \mathfrak{q}_{\mathscr{Y}} \times \mathrm{supp} & & \downarrow \mathfrak{q}_{\mathscr{Y}} \times \widetilde{\mathrm{supp}} \\ {}_n\mathscr{Y}_0 \times X^{(d)} & \overset{\mathrm{id} \times \mathrm{sym}}{\leftarrow} & {}_n\mathscr{Y}_0 \times X^d \end{array}$$

we obtain

$$\begin{aligned}
&(\mathfrak{q}_{\mathscr{Y}} \times \mathrm{supp})_* \mathrm{R}\mathscr{H}om\big(\mathfrak{p}^*_{\mathscr{Y}}({}_n\mathscr{P}^d_{E,\psi}) \otimes q^* \mathrm{Spr}^d_{E'^*},\ \mathfrak{q}^!_{\mathscr{Y}} \Psi^0\big) \\
&\quad \widetilde{\to} (\mathfrak{q}_{\mathscr{Y}} \times \mathrm{supp})_* \mathrm{R}\mathscr{H}om\big(p_!\big(\tilde{\mathfrak{p}}^*_{\mathscr{Y}}({}_n\mathscr{P}^d_{E,\psi}) \otimes \widetilde{\mathrm{supp}}^*(E'^*)^{\boxtimes d}\big),\ \mathfrak{q}^!_{\mathscr{Y}} \Psi^0\big) \\
&\quad \widetilde{\to} (\mathfrak{q}_{\mathscr{Y}} \times \mathrm{supp})_* p_* \mathrm{R}\mathscr{H}om\big(\tilde{\mathfrak{p}}^*_{\mathscr{Y}}({}_n\mathscr{P}^d_{E,\psi}) \otimes \widetilde{\mathrm{supp}}^*(E'^*)^{\boxtimes d},\ p^! \mathfrak{q}^!_{\mathscr{Y}} \Psi^0\big) \\
&\quad \widetilde{\to} (\mathrm{id} \times \mathrm{sym})_* (\mathfrak{q}_{\mathscr{Y}} \times \widetilde{\mathrm{supp}})_* \mathrm{R}\mathscr{H}om\big(\tilde{\mathfrak{p}}^*_{\mathscr{Y}}({}_n\mathscr{P}^d_{E,\psi}) \otimes \widetilde{\mathrm{supp}}^*(E'^*)^{\boxtimes d}, \\
&\qquad\qquad\qquad\qquad (\mathfrak{q}_{\mathscr{Y}} \times \widetilde{\mathrm{supp}})^! \mathrm{pr}^!_{\mathscr{Y}} \Psi^0\big) \\
&\quad \widetilde{\to} (\mathrm{id} \times \mathrm{sym})_* \mathrm{R}\mathscr{H}om\big((\mathfrak{q}_{\mathscr{Y}} \times \widetilde{\mathrm{supp}})_!\big(\tilde{\mathfrak{p}}^*_{\mathscr{Y}}({}_n\mathscr{P}^d_{E,\psi}) \otimes \widetilde{\mathrm{supp}}^*(E'^*)^{\boxtimes d}\big), \mathrm{pr}^!_{\mathscr{Y}} \Psi^0\big) \\
&\quad \widetilde{\to} (\mathrm{id} \times \mathrm{sym})_* \mathrm{R}\mathscr{H}om\big(\Psi^0 \boxtimes (E \otimes E'^*)^{\boxtimes d},\ \mathrm{pr}^!_{\mathscr{Y}} \Psi^0\big)\ [nd - 2d]\Big(\frac{nd-2d}{2}\Big),
\end{aligned}$$

where the last isomorphism comes from Corollary 2. Since $\mathrm{pr}_{\mathscr{Y}}$ is smooth of relative dimension d, our assertion follows. □

This concludes the proof of Proposition 6. □

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References

[1] A. BRAVERMAN AND D. GAITSGORY, *Geometric Eisenstein series*, preprint, arXiv:math.AG/9912097

[2] P. DELIGNE, “Applications de la formule des traces aux sommes trigonométriques” in *Cohomologie étale*, Séminaire de Géométrie Algébrique du Bois-Marie (SGA 4 1/2), Lecture Notes in Math. **569**, Springer, Berlin, 1977, 168–232. MR 57:3132 453

[3] V. DRINFELD, *Two-dimensional ℓ-adic representations of the fundamental group of a curve over a finite field and automorphic forms on* GL(2), Amer. J. Math. **105** (1983), 85–114. MR 84i:12011 453

[4] B. FEIGIN, M. FINKELBERG, A. KUZNETSOV, and I. MIRKOVIĆ, *Semiinfinite flags, II: Local and global intersection cohomology of quasimaps' spaces*, preprint, arXiv:alg-geom/9711009

[5] M. FINKELBERG and I. MIRKOVIĆ, *Semiinfinite flags, I: Case of global curve P^1*, preprint, arXiv:alg-geom/9707010

[6] E. FRENKEL, D. GAITSGORY, D. KAZHDAN, and K. VILONEN, *Geometric realization of Whittaker functions and the Langlands conjecture*, J. Amer. Math. Soc. **11** (1998), 451–484. MR 99f:11148 463, 467

[7] E. FRENKEL, D. GAITSGORY, and K. VILONEN, *Whittaker patterns in the geometry of moduli spaces of bundles on curves*, Ann. of Math. (2) **153** (2001), 699–748. MR CMP 1 836 286 459, 460, 463, 469

[8] ———, *On the geometric Langlands conjecture*, preprint, arXiv:math.AG/0012255 469, 487, 489

[9] A. GROTHENDIECK, *Revêtements étales et groupe fondamental*, Séminaire de Géométrie Algébrique du Bois-Marie 1960/61 (SGA 1), Lecture Notes in Math. **224**, Springer, Berlin, 1971. MR 50:7129 480

[10] ———, *Techniques de construction et théorèmes d'existence en géométrie algébrique, IV: Les schémas de Hilbert*, Sem. Bourbaki **221** (1960/61), Soc. Math. France, Paris, 1995, 249–276. MR CMP 1 611 822 454

[11] F. KNUDSEN and D. MUMFORD, *The projectivity of the moduli space of stable curves, I: Preliminaries on "det" and "Div"*, Math. Scand. **39** (1976), 19–55. MR 55:10465 481

[12] G. LAUMON, *Correspondance de Langlands géométrique pour les corps de fonctions*, Duke Math. J. **54** (1987), 309–359. MR 88g:11086 451, 452, 453, 455, 456, 460, 467

[13] G. LAUMON and L. MORET-BAILLY, *Champs algébriques*, Ergeb. Math. Grenzgeb. (3) **39**, Springer, Berlin, 2000. MR 2001f:14006 453

[14] S. LYSENKO, *Local geometrized Rankin-Selberg method for* GL(n) *and its application*, C. R. Acad. Sci. Paris Sér. I Math. **329** (1999), 1065–1070. MR 2001h:11081

[15] ———, *Orthogonality relations between the automorphic sheaves attached to 2-dimensional irreducible local systems on a curve*, Ph.D. thesis, Université Paris-Sud, 1999, http://math.u-psud.fr/~biblio/the/1999/2111/the_1999_2111.html

[16] B. C. NGÔ, *Preuve d'une conjecture de Frenkel-Gaitsgory-Kazhdan-Vilonen pour les groupes linéaires généraux*, Israel J. Math. **120** (2000), 259–270. MR CMP 1 815 378 459, 460, 463

[17] B. C. NGÔ and P. POLO, *Résolutions de Demazure affines et formule de Casselman-Shalika géométrique*, J. Algebraic Geom. **10** (2001), 515–547. MR CMP 1 832 331 459, 460, 463, 464

[18] I. I. PIATETSKI-SHAPIRO, "Euler subgroups" in *Lie Groups and Their Representations (Budapest, 1971)*, ed. I. M. Gelfand, Halstead, New York, 1975, 597–620. MR 53:10720 451, 452

[19] J. A. SHALIKA, *The multiplicity one theorem for* GL(n), Ann. of Math. (2) **100** (1974), 171–193. MR 50:545

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