Izvestiya: Mathematics **60**:5 963–984

Izvestiya RAN: Ser. Mat. **60**:5 89–110
UDC 517.5

On the functional equation $f(p(z)) = g(q(z))$, where p and q are "generalized" polynomials and f and g are meromorphic functions

S. A. Lysenko

Abstract. We find all the solutions of the equation $f(p(z)) = g(q(z))$, where p and q are polynomials and f and g are transcendental meromorphic functions in $\mathbb{C}$. In fact, a more general algebraic problem is solved.

Introduction

0.1. Motivation. This paper (like [3]) arose as an extension of the following result due to Flatto [1].

Let $p, q \in \mathbb{C}[z]$ be polynomials of the same degree. Let

$$f \circ p = g \circ q, \tag{1}$$

where f and g are non-constant entire functions in $\mathbb{C}$. Then one of the following two properties holds:

(i) $p(z) = \lambda q(z) + a$, where $\lambda,\ a \in \mathbb{C}$;

(ii) $p(z) = r(z)^2 + a,\ \ q(z) = br(z)^2 + cr(z) + d$, where $r(z)$ is a polynomial and $a, b, c, d \in \mathbb{C}$ with $b \neq 0$.

Flatto raised the question ([4], Question 5) of whether an analogue of his theorem exists in case $\deg p \neq \deg q$. The same question arises in the case when f and g are not entire, but meromorphic functions (in $\mathbb{C}$ or only in a deleted neighbourhood of infinity). Partial results in connection with Flatto's problem were obtained in [11], [2], [12] and [3] (a survey of most of these results can be found in [2]). The aim of this paper is to describe all the pairs p, q for which there exist non-constant meromorphic functions f, g that satisfy equation (1) and such that there do not exist rational f, g with this property (in fact a more general problem is solved; see § 0.2 and § 1).

Equation (1) is also related to the following interesting problem: describe the equivalence relations R on $\mathbb{C}$ such that

1) R as a subset of $\mathbb{C}^2$ is the union of a sequence of algebraic curves, and

---

The author was partially supported by grant INTAS No. 94-4720

AMS 1991 *Mathematics Subject Classification.* Primary 30F99, 30D60, 12F20, 20F38, 39B32. Secondary 12E10, 14H55, 30C10, 32J05, 58F23.

2) there exists a non-constant meromorphic function in $\mathbb{C}$ that is constant on the equivalence classes of R.

Such an equivalence relation can be considered as a generalization of the discrete group of biholomorphic automorphisms of $\mathbb{C}$. (A discrete group Γ defines an equivalence relation $z \sim w$ if $z = \gamma(w)$ for some $\gamma \in \Gamma$; this equivalence relation satisfies properties 1) and 2).)

Assume that the pair p, q gives a solution of equation (1) (with meromorphic f, g). Then it corresponds to an equivalence relation $R_{p,q}$.

Let

$$R_p = \{(z,u) \in \mathbb{C}^2 \mid p(z) = p(u)\}, \qquad R_q = \{(z,u) \in \mathbb{C}^2 \mid q(z) = q(u)\}.$$

We define $R_{p,q}$ as the equivalence relation generated by R_p and R_q. The properties 1) and 2) hold for it. In some sense $R_{p,q}$ arises from the action of the group Γ generated by $z \mapsto h_p(z) := p^{-1}(p(z))$ and $z \mapsto h_q(z) := q^{-1}(q(z))$. We notice that h_p and h_q are holomorphic only in a neighbourhood of infinity, so that Γ is the group of germs of holomorphic maps $(\overline{\mathbb{C}}, \infty) \to (\overline{\mathbb{C}}, \infty)$.

0.2. Statement of the problem. Let (X, ∞_X) and (Y, ∞_Y) be compact Riemann surfaces with marked points $\infty_X \in X$ and $\infty_Y \in Y$. We shall frequently write ∞ instead of ∞_X, ∞_Y. Let $p : (X, \infty) \to (Y, \infty)$ be a holomorphic map. We shall call it a *generalized polynomial* if $p^{-1}(\{\infty\}) = \{\infty\}$.

We shall study equation (1) in the case when $p : (X, \infty_X) \to (Y, \infty_Y)$ and $q : (X, \infty_X) \to (Z, \infty_Z)$ are generalized polynomials and f and g are meromorphic functions in deleted neighbourhoods of ∞_Y and ∞_Z, respectively. A meromorphic function on a compact Riemann surface W will be called a *rational function on W* (this agrees with the terminology used in algebraic geometry.) The problem is to find all the pairs p, q for which there exist non-constant f, g satisfying (1) and for which there do not exist rational f, g with this property. In fact a more general algebraic problem, formulated in § 1, will be solved.

0.3. Main result. There are several standard solutions of equation (1).

Example 1. Let $p(z) = z^n$, $q(z) = (z+1)^m$, where n, m, $\operatorname{lcm}(n,m) \in \{2,3,4,6\}$. Then there exist non-constant functions f, g, meromorphic in $\mathbb{C}$ and satisfying (1), and there do not exist rational f, g with this property.

Remark 1. Every diagram of generalized polynomials

$$\begin{tikzcd} (X,\infty) \arrow[dr, "p"'] & & (Z,\infty) \arrow[dl, "q"] \\ & (Y,\infty) & \end{tikzcd}$$

such that $\gcd(\deg p, \deg q) = 1$ can be completed to a diagram of generalized polynomials

$$\begin{tikzcd} (W,\infty) \arrow[r, "p_1"'] \arrow[d, "q_1"] & (Z,\infty) \arrow[d, "q"] \\ (X,\infty) \arrow[r, "p"'] & (Y,\infty) \end{tikzcd}$$

in which $\deg p_1 = \deg p$, $\deg q_1 = \deg q$, and this is unique up to a canonical isomorphism. In fact $W = W_0$, where W_0 is the normalization (desingularization) of the analytic curve $X \times_Y Z = \{(z,x) \in Z \times X \mid q(z) = p(x)\}$. (If $\deg p$ and $\deg q$ are relatively prime, then W_0 has only one point over ∞_Y, and thus W_0 is connected. Hence $W_0 \to X$ and $W_0 \to Y$ are generalized polynomials.) We notice that if X, Y, Z are of genus 0, then, as a rule, W is of non-zero genus.

Example 2. Let $\widetilde{p}$, $\widetilde{q}$ be the pair of polynomials from Example 1, $\deg \widetilde{p} = n$, $\deg \widetilde{q} = m$. Let $h\colon (Y,\infty) \to (\mathbb{CP}^1,\infty)$, $r\colon (Z,\infty) \to (\mathbb{CP}^1,\infty)$ be generalized polynomials, $\deg h = \alpha$, $\deg r = \beta$. Let $\gcd(\alpha,n) = \gcd(\beta,m) = \gcd(\alpha,\beta) = 1$. Using Remark 1, we obtain a commutative diagram of generalized polynomials

$$\begin{tikzcd} W \arrow[r, "h_2"] \arrow[d, "r_2"] & Z_1 \arrow[r, "q_1"] \arrow[d, "r_1"] & Z \arrow[d, "r"] \\ Y_1 \arrow[r, "h_1"] \arrow[d, "p_1"] & \mathbb{CP}^1 \arrow[r, "\tilde{q}"] \arrow[d, "\tilde{p}"] & \mathbb{CP}^1 \\ Y \arrow[r, "h"] & \mathbb{CP}^1 & \end{tikzcd}$$

in which $\deg r_1 = \deg r_2 = \beta$, $\deg h_1 = \deg h_2 = \alpha$, $\deg p_1 = n$, $\deg q_1 = m$.

Let $s : (X,\infty) \to (W,\infty)$ be an arbitrary generalized polynomial. We set $p = p_1 \circ r_2 \circ s$, $q = q_1 \circ h_2 \circ s$. Then the pair p, q gives a solution of our problem.

All solutions are obtained in this way. (For example, in case $\deg p = \deg q$ this means that h, h_1, h_2 and r, r_1, r_2 are isomorphisms, so that in fact $p = \widetilde{p} \circ s$ and $q = \widetilde{q} \circ s$.)

0.4. The structure of the paper. In § 1 we replace the original problem by a more general algebraic problem, which is what we actually solve, and we formulate the corresponding results. We define irreducible pairs of generalized polynomials and divide the results into two parts. In § 3 we reduce the problem to the study of irreducible pairs. In § 4 we proceed to the proper study of irreducible pairs and also formulate the fundamental group-theoretic lemma that plays a central role in the proof of the main result. This lemma is proved in § 5.

0.5. Conventions. All Riemann surfaces are assumed to be connected. We recall that the following three notions are equivalent: a compact Riemann surface, a smooth connected projective algebraic curve over $\mathbb{C}$, a finitely generated field over $\mathbb{C}$

of transcendence degree 1. We identify a point of a curve and the corresponding valuation of the field of rational functions on this curve.

0.6. Thanks. I express my deep appreciation to V. G. Drinfel'd for useful advice and his constant interest in this work.

§ 1. Statement of results

We denote by $\mathfrak{I}$ the group of (all) germs of conformal maps $(\mathbb{CP}^1, \infty) \to (\mathbb{CP}^1, \infty)$.

Definition. Let Γ be a subgroup of $\mathfrak{I}$. We will say that Γ is *discrete* if there exists a non-constant function F, meromorphic in a deleted neighbourhood of infinity in $\mathbb{C}$, such that $F(g(z)) = F(z)$ for all $g \in \Gamma$.

A necessary condition for a group Γ to be discrete was obtained in [3] using the results of local holomorphic dynamics [13]. This condition will be a fundamental method for proving our theorems. For $g \in \mathbb{C}((\frac{1}{z}))$ we write $\operatorname{ord}_\infty g = n$ if $g = \sum_{k=n}^{\infty} a_k z^{-k}$, $a_n \neq 0$. We set

$$\mathfrak{I}_k = \{g \in \mathfrak{I} \mid \operatorname{ord}_\infty(g(z) - z) \geqslant 1 - k\}, \quad k \leqslant 1.$$

Then $\mathfrak{I} \supset \mathfrak{I}_1 \supset \mathfrak{I}_0 \supset \mathfrak{I}_{-1} \supset \dots$, and $\mathfrak{I}_k$ is a normal subgroup of $\mathfrak{I}$. If $\Gamma \subset \mathfrak{I}$ is a subgroup, then we will write $\Gamma_k = \Gamma \cap \mathfrak{I}_k$, $k \leqslant 1$.

Theorem 1 [3]. *Let $\Gamma \subset \mathfrak{I}$ be a discrete subgroup. Then:*
1) *at most one of the quotient groups Γ_k/Γ_{k-1} $(k \leqslant 1)$ is not trivial;*
2) *for all $k \leqslant 1$ the group $\Gamma_k/\Gamma_{k-1} \subset \mathfrak{I}_k/\mathfrak{I}_{k-1} \simeq (\mathbb{C}, +)$ is a discrete subgroup of $\mathbb{C}$.*

Definition. A subgroup $\Gamma \subset \mathfrak{I}$ is said to be *formally discrete* if Γ satisfies conditions 1) and 2) of Theorem 1.

Remark 1. The quotient group Γ/Γ_1 is not considered in Theorem 1.

Remark 2. Let $\Gamma \subset \mathfrak{I}$ be a subgroup. If Γ is not soluble, then its orbits are dense in domains with infinity on the boundary (see [14]). Therefore if Γ is discrete, it is soluble. If Γ_1 is soluble, then it is Abelian (see [3], p. 68, Proposition 2.1). This proves part 1) of Theorem 1. In order to prove part 2) we use the following fact from local dynamics. Let $g \in \mathfrak{I}_1$. If g is not embeddable in a flow, then the centralizer of g in $\mathfrak{I}_1$ is isomorphic to $\mathbb{Z}$, and if it is embeddable in a flow, then it coincides with the corresponding flow (see [13], p. 26, Theorem 3, as well as p. 23, Theorem 1).

Let X be a Riemann surface and ∞ a point of X. We denote by $\mathfrak{I}(X, \infty)$ the group of germs of conformal maps $(X, \infty) \to (X, \infty)$. The choice of local parameter at the point $\infty \in X$ enables us to identify $\mathfrak{I}(X, \infty)$ and $\mathfrak{I}$. Let Y be another Riemann surface and assume that f holomorphically maps a deleted neighbourhood of infinity in X into Y. Then we define a group $T_f = \{g \in \mathfrak{I}(X, \infty) \mid f \circ g = f\}$.

We consider a pair of compact Riemann surfaces X, Y with marked points $\infty \in X$, $\infty \in Y$. We recall that a *generalized polynomial* is a holomorphic map $p : (X, \infty) \to (Y, \infty)$ such that $p^{-1}(\{\infty\}) = \{\infty\}$. It is easy to verify that if p is a generalized polynomial, then T_p is a cyclic group of order $\deg p$.

We consider the following diagram of generalized polynomials:

$$\begin{tikzcd} & (X,\infty) \arrow[dl, "p"'] \arrow[dr, "q"] & \\ (Y,\infty) & & (Z,\infty) \end{tikzcd} \tag{2}$$

Let f and g be non-constant meromorphic functions in deleted neighbourhoods of infinity in Y and Z, respectively. We assume that

$$f \circ p = g \circ q. \tag{3}$$

Then T_p and T_q generate a discrete subgroup of $\mathfrak{I}(X, \infty)$. Conversely, if T_p and T_q generate a discrete group, then there are functions f and g as above for which (3) holds.

In fact in this paper we find all the pairs of generalized polynomials p and q for which T_p and T_q generate an infinite *formally* discrete group.

Remark 3. We denote by $\overline{\mathfrak{I}}$ the group of all formal diffeomorphisms $(\mathbb{CP}^1, \infty) \to (\mathbb{CP}^1, \infty)$, that is, $\overline{\mathfrak{I}} = \{z \mapsto a_1 z + a_0 + a_{-1}z^{-1} + \cdots \mid a_i \in \mathbb{C}, a_1 \neq 0\}$ with composition as the operation. The subgroups $\overline{\mathfrak{I}}_k \subset \overline{\mathfrak{I}}$ are defined just like the $\mathfrak{I}_k \subset \mathfrak{I}$. We have $\mathfrak{I} \subset \overline{\mathfrak{I}}$. If $\Gamma \subset \overline{\mathfrak{I}}$ is a subgroup such that $\Gamma \not\subset \mathfrak{I}$, then one can no longer say that the group Γ is discrete, but one can talk about formal discreteness. In this sense formal discreteness is an algebraic property of Γ.

Everywhere we shall denote by $\mathcal{M}(X)$ the field of meromorphic functions on the Riemann surface X.

Let $p : (X, \infty) \to (Y, \infty)$ be a generalized polynomial. Then $\mathcal{M}(Y) \subset \mathcal{M}(X)$. Let F be an intermediate field, that is, $\mathcal{M}(Y) \subset F \subset \mathcal{M}(X)$, and let W be a model for it, that is, a compact Riemann surface such that F is isomorphic to $\mathcal{M}(W)$ over $\mathbb{C}$. We obtain a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "p"] \arrow[dr, "p_1"'] & & Y \\ & W \arrow[ur, "p_2"'] & \end{tikzcd}$$

We set $\infty_W = p_1(\infty_X)$. Then $p_1 : (X, \infty) \to (W, \infty)$ and $p_2 : (W, \infty) \to (Y, \infty)$ will be generalized polynomials.

The following theorem can be considered as a description of the rational solutions of the functional equation (3).

Theorem 2. *Assume that diagram* (2) *is given. Then there are two alternatives.*

1) *There exists a commutative diagram of generalized polynomials, unique up to isomorphism,*

$$\begin{tikzcd}
 & (X,\infty) \arrow[dl, "p"'] \arrow[d, "f"] \arrow[dr, "q"] & \\
(Y,\infty) \arrow[dr] & (V,\infty) \arrow[l, "p_1"'] \arrow[r, "q_1"] \arrow[d, "g"] & (Z,\infty) \arrow[dl] \\
 & (W,\infty) &
\end{tikzcd} \tag{4}$$

such that $\deg f = \gcd(\deg p, \deg q)$, $\deg g = (\deg p_1)(\deg q_1)$. *Here the groups* T_p *and* T_q *generate the group* $T_{g\circ f}$, $\mathcal{M}(Y)\cap\mathcal{M}(Z) = \mathcal{M}(W)$, $\mathcal{M}(V)$ *is the compositum of the fields* $\mathcal{M}(Y)$ *and* $\mathcal{M}(Z)$.

2) T_p *and* T_q *generate an infinite non-Abelian subgroup of* $\mathfrak{I}(X,\infty)$; *in this case* $\mathcal{M}(Y)\cap\mathcal{M}(Z) = \mathbb{C}$.

The following theorems describe the pairs of generalized polynomials p, q for which T_p and T_q generate an infinite formally discrete subgroup of $\mathfrak{I}(X,\infty)$.

Proposition 1. *Assume that diagram* (2) *is given. Then among the fields* F *such that* $\mathcal{M}(Z)\subset F\subset\mathcal{M}(X)$ *and* $F\cap\mathcal{M}(Y)\neq\mathbb{C}$, *there is a smallest by inclusion. We denote it by* $F_{q,p}$.

Remark 4. From the geometric point of view this means that there exists a commutative diagram of generalized polynomials

$$\begin{tikzcd}
X \arrow[r, "q_1"] \arrow[d] & X_{q,p} \arrow[r, "q_2"] \arrow[d] & Z \\
Y \arrow[r] & Y_{q,p} &
\end{tikzcd}$$

such that $q_2\circ q_1 = q$, and the following universal property holds. For every commutative diagram of generalized polynomials

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "p"] & X' \arrow[r, "h"] \arrow[d] & Z \\
Y \arrow[r] & Y' &
\end{tikzcd}$$

such that $h\circ g = q$, there exists a unique holomorphic map $f\colon X'\to X_{q,p}$ with the property $f\circ g = q_1$.

Definition. The pair p, q in diagram (2) is said to be *irreducible* if $F_{p,q} = F_{q,p} = \mathcal{M}(X)$.

Remark 5. Assume that the pair p, q in diagram (2) is irreducible and that $\deg p$, $\deg q > 1$. Then $\mathcal{M}(Y) \cap \mathcal{M}(Z) = \mathbb{C}$.

Example. We set $X = \mathbb{CP}^1$, $p(z) = z^n$, $q(z) = (z+1)^m$, where n, m are natural numbers. Then the pair p, q is irreducible.

For each pair of generalized polynomials (2) one can canonically construct an irreducible pair in the following way. We set $F = F_{p,q} \cap F_{q,p}$, $F_1 = \mathcal{M}(Y) \cap F$, $F_2 = \mathcal{M}(Z) \cap F$. We denote the compositum of $F_{p,q}$ and $F_{q,p}$ by K. We obtain the diagram of fields

$$\begin{array}{ccccccc}
\mathcal{M}(X) & \supset & K & \supset & F_{q,p} & \supset & \mathcal{M}(Z) \\
 & & \cup & & \cup & & \cup \\
 & & F_{p,q} & \supset & F & \supset & F_2 \\
 & & \cup & & \cup & & \\
 & & \mathcal{M}(Y) & \supset & F_1 & &
\end{array}$$

It corresponds to the following commutative diagram of generalized polynomials:

$$\begin{tikzcd}
X \arrow[r] & V \arrow[r, "h_2"] \arrow[d, "r_2"] & X_{q,p} \arrow[r, "q_1"] \arrow[d, "r_1"] & Z \arrow[d, "r"] \\
 & X_{p,q} \arrow[r, "h_1"] \arrow[d, "p_1"] & W \arrow[r, "\tilde{q}"] \arrow[d, "\tilde{p}"] & W_2 \\
 & Y \arrow[r, "h"] & W_1 &
\end{tikzcd} \tag{5}$$

We call diagram (5) the *canonical diagram.*

It follows from the definition of $F_{p,q}$ (see Proposition 1) that the compositum of the fields F and $\mathcal{M}(Y)$ is equal to $F_{p,q}$. Analogously, $F_{q,p}$ is the compositum of the fields F and $\mathcal{M}(Z)$. It follows from Theorem 2 that $\deg h = \deg h_1 = \deg h_2$, $\deg r = \deg r_1 = \deg r_2$, $\deg p_1 = \deg \widetilde{p}$, $\deg q_1 = \deg \widetilde{q}$, $\gcd(\deg h, \deg \widetilde{p}) = \gcd(\deg r, \deg \widetilde{q}) = \gcd(\deg h, \deg r) = 1$.

Proposition 2. *The pair $\widetilde{p}$, $\widetilde{q}$ is irreducible.*

Proposition 3. *Assume that a diagram* (2) *is given such that* $\mathcal{M}(Y) \cap \mathcal{M}(Z) = \mathbb{C}$. *Let $\widetilde{p}$, $\widetilde{q}$ be the corresponding irreducible pair of generalized polynomials. Then the following conditions are equivalent:*

(i) *T_p and T_q generate a discrete subgroup of* $\mathfrak{I}(X, \infty)$;

(ii) *$T_{\widetilde{p}}$ and $T_{\widetilde{q}}$ generate a discrete subgroup of* $\mathfrak{I}(W, \infty)$.

This assertion remains valid if "discrete" is replaced by "formally discrete".

Theorem 3. *We consider a diagram* (2) *such that the pair* p, q *is irreducible and* $\deg p$, $\deg q > 1$. *Assume that* T_p *and* T_q *generate a formally discrete group.*

Then there exists a commutative diagram

$$\begin{tikzcd}
(Y,\infty) \arrow[d] & (X,\infty) \arrow[l, "p"'] \arrow[r, "q"] \arrow[d] & (Z,\infty) \arrow[d] \\
(\mathbb{CP}^1,\infty) & (\mathbb{CP}^1,\infty) \arrow[l, "p_1"'] \arrow[r, "q_1"] & (\mathbb{CP}^1,\infty)
\end{tikzcd}$$

in which the vertical arrows are isomorphisms and the pair p_1, q_1 *is the following standard pair of polynomials*: $p_1(z) = z^n$, $q_1(z) = (z+1)^m$, *where* $n, m, \operatorname{lcm}(n, m) \in \{2, 3, 4, 6\}$.

Conversely, the pair p_1, q_1 *is irreducible,* T_{p_1} *and* T_{q_1} *generate a discrete subgroup of* $\mathfrak{I}$, $\mathbb{C}(p_1) \cap \mathbb{C}(q_1) = \mathbb{C}$.

Our main result, stated in the Introduction, follows from Theorems 1-3 and Propositions 1-3.

§ 2. Algebraic technique

Let X be a compact Riemann surface and ∞ a point of X. The valuation of the field $\mathcal{M}(X)$ that corresponds to the point ∞ will be denoted by the same symbol ∞. We denote by $\mathcal{M}(X)_\infty$ the completion of $\mathcal{M}(X)$ at ∞. Let p: $(X,\infty) \to (Y,\infty)$ be a generalized polynomial. The restriction of ∞ to $\mathcal{M}(Y)$ is denoted by the same symbol. It is known that $\mathcal{M}(X)_\infty$ is a cyclic Galois extension of degree $\deg p$ of the field $\mathcal{M}(Y)_\infty$. To each $g \in \mathfrak{I}(X,\infty)$ we assign the automorphism of $\mathcal{M}(X)_\infty$, defined by the formula $(gf)(x) = f(g^{-1}x)$, $f \in \mathcal{M}(X)_\infty$. Thus we have obtained an embedding of $\mathfrak{I}(X,\infty)$ into the automorphism group of the topological field $\mathcal{M}(X)_\infty$ over $\mathbb{C}$. This embedding also induces an isomorphism between T_p and $\operatorname{Gal}(\mathcal{M}(X)_\infty/\mathcal{M}(Y)_\infty)$. In what follows we shall identify these two groups.

Lemma 1. *Let* X *and* Y *be compact Riemann surfaces,* f: $X \to Y$ *a holomorphic* n*-sheeted cover. Let* g: $W \to Y$ *be the smallest Galois cover that passes through* X:

$$\begin{tikzcd}
W \arrow[r] \arrow[dr, "g"'] & X \arrow[d, "f"] \\
& Y
\end{tikzcd}$$

Let $f^{-1}(y_0) = \{x_1, \ldots, x_k\} \subset X$ *for* $y_0 \in Y$. *We assume that the multiplicity of* f *at* x_i *is equal to* l_i, *and* $g(w_0) = y_0$ *for* $w_0 \in W$.

Then the multiplicity of g *at* w_0 *is equal to* $\operatorname{lcm}(l_1, \ldots, l_k)$.

The following explicit construction of W is useful in the proof. Let $A \subset Y$ be the set of critical values of f. We set $Y' = Y \setminus A$, $X' = X \setminus f^{-1}(A)$. We denote by Z' the set of pairs (y, φ), where $y \in Y'$ and φ: $f^{-1}(y) \to \{1, \ldots, n\}$ is a bijection.

We denote by g the map from Z' to Y' that maps (y, φ) to y. A complex structure is introduced on Z' in a natural way. The group S_n acts on Z' by biholomorphic transformations, and this action is transitive on the fibres of g. Let Z be a smooth compactification of Z' and W the connected component of Z. Then $g\colon W \to Y$ will be the desired Galois cover.

The rest of the proof is omitted.

Corollary. *Let $p\colon (X, \infty) \to (Y, \infty)$ be a generalized polynomial, let K be a finite Galois extension of $\mathcal{M}(X)$ such that K is not ramified over $\infty \in X$. Let L be the smallest Galois extension of $\mathcal{M}(Y)$ that contains K.*

Then L is also unramified over $\infty \in X$.

We consider the diagram (2) of generalized polynomials. We fix an algebraic closure $\overline{\mathcal{M}(X)}$. We construct a tower of fields $\mathbf{k}_p^m$, $\Bbbk_q^m \subset \overline{\mathcal{M}(X)}$, $m \geqslant 0$, in the following way. We set $\mathbf{k}_p^0 = \Bbbk_q^0 = \mathcal{M}(X)$. Let $\mathbf{k}_p^m$ be the smallest Galois extension of $\mathcal{M}(Y)$ that contains $\Bbbk_q^{m-1}$, and $\mathbf{k}_q^m$ the smallest Galois extension of $\mathcal{M}(Z)$ that contains $\Bbbk_p^{m-1}$. By induction we check that $\mathbf{k}_p^m \supset \mathbf{k}_p^{m-1}$ and $\Bbbk_q^m \supset \Bbbk_q^{m-1}$. We set $E = \bigcup_m \mathbf{k}_p^m = \bigcup_m \mathbf{k}_q^m$. Then $E \subset \overline{\mathcal{M}(X)}$ is a subfield that contains $\mathcal{M}(X)$ and is normal over both $\mathcal{M}(Y)$ and $\mathcal{M}(Z)$. It is clear that $E \subset \overline{\mathcal{M}(X)}$ is the smallest subfield with this property.

Lemma 2. *For every $m \geqslant 0$ the fields $\mathbf{k}_p^m$ and $\Bbbk_q^m$ are unramified over $\infty \in X$. Therefore E is unramified over $\infty \in X$.*

Proof. Apply the preceding corollary.

We fix a place ∞' of E over $\infty \in X$. The choice of ∞' gives an embedding $E \hookrightarrow \mathcal{M}(X)_\infty = E_{\infty'}$ over $\mathcal{M}(X)$. Since E is normal over $\mathcal{M}(X)$, the image of E in $\mathcal{M}(X)_\infty$ does not depend on the choice of ∞'.

We set $G_p = \mathrm{Gal}(E/\mathcal{M}(Y))$, $G_q = \mathrm{Gal}(E/\mathcal{M}(Z))$, $U = \mathrm{Gal}(E/\mathcal{M}(X))$.

We denote by G the subgroup of $\operatorname{Aut} E$ generated by G_p and G_q. It is well known that for every valuation ω of $\mathcal{M}(X)$ (trivial on $\mathbb{C}$) the action of U on the set of places of E over ω is transitive.

We denote the set of places of E over ∞ by S. The groups G_p and G_q act on S, and therefore G also acts on S. The action of U on S is free, since E is unramified over ∞, and transitive, as explained above.

It is known that the assignment of $\sigma \in \mathrm{Gal}(\mathcal{M}(X)_\infty/\mathcal{M}(Y)_\infty)$ to its restriction to $E \subset E_{\infty'} = \mathcal{M}(X)_\infty$ defines an isomorphism

$$T_p = \mathrm{Gal}(\mathcal{M}(X)_\infty/\mathcal{M}(Y)_\infty) \xrightarrow{\sim} \{g \in G_p \mid g\infty' = \infty'\}.$$

Analogously we have an isomorphism:

$$T_q = \mathrm{Gal}(\mathcal{M}(X)_\infty/\mathcal{M}(Z)_\infty) \xrightarrow{\sim} \{g \in G_q \mid g\infty' = \infty'\}.$$

Let Γ be the subgroup of $\operatorname{Aut} \mathcal{M}(X)_\infty$ generated by T_p and T_q.

Lemma 3. 1) *For every $g \in G_p$ (respectively, $g \in G_q$) there exist unique $h \in T_p$ (respectively, $h \in T_q$) and $\sigma \in U$ such that $g = h\sigma$.*

2) *Restriction to E defines an isomorphism*

$$\Gamma \tilde{\to} \{g \in G \mid g\infty' = \infty'\}.$$

3) *For every $g \in G$ there exist unique $h \in \Gamma$ and $\sigma \in U$ such that $g = h\sigma$.*

Proof. Assertion 1) follows from the facts that S is identified with G_p/T_p or G_q/T_q and that the action of U on S is free and transitive.

We shall prove assertions 2) and 3). We have a homomorphism $f\colon \Gamma \to \{g \in G \mid g\infty' = \infty'\}$. It is injective, so that E is dense in $\mathcal{M}(X)_\infty$. It follows from 1) that for any $g \in G$ there exist $h \in \Gamma$ and $\sigma \in U$ such that $g = f(h)\sigma$. Such h and σ are unique since the action of U on S is free. If $g\infty' = \infty'$, then $\sigma = 1$, so that f is surjective.

Remark 1. We shall consider T_p, T_q and Γ as subgroups of G. At the same time, Γ can be considered as the subgroup of $\mathfrak{I}(X, \infty)$ generated by T_p and T_q. In view of Lemma 3 we have a bijection $G/U \leftrightarrow \Gamma$. The group U acts on G/U by left translations, and hence U acts on the set Γ, without preserving the group structure on Γ. This action plays a decisive role in this paper and has the following analytic meaning. The elements of Γ can be considered as germs of algebraic functions at $\infty \in X$. Analytic continuation along closed paths defines the monodromy action of H on Γ, where H is the inverse limit of the groups $\pi_1(X \setminus S, \infty)$, $S \subset X \setminus \{\infty\}$, $\#S < \infty$. There is a canonical homomorphism $f\colon H \to U$ with a dense image, and the monodromy action of H on Γ arises from the action of U on Γ. Thus, the action of U on Γ is an algebraic version of the monodromy action used in § 4 of [3].

Remark 2. We notice that E is a field of transcendence degree 1 over $\mathbb{C}$, and, in general, E is not generated over $\mathbb{C}$ by a finite number of elements. At the same time there exists a finite subset $A \subset E$ such that every subfield E' of E that contains A and is invariant under $\operatorname{Aut}_{\mathbb{C}} E$ coincides with E. (Let A be the set of generators of $\mathcal{M}(X)$ over $\mathbb{C}$. Then E is generated by $\bigcup_{g\in\Gamma} gA$ over $\mathbb{C}$.) Fields with such properties were studied in [5].

We now consider the following abstract situation. Let G be an abstract group and let U and Γ be subgroups of G. Let $G = \Gamma \cdot U$, $\Gamma \cap U = 1$ (then $U\Gamma = (\Gamma U)^{-1} = G$). We obtain a bijection $\Gamma \leftrightarrow G/U$. The group G acts on G/U by left translation and therefore G acts on Γ.

Lemma 4. *Let A and B be subsets of Γ.*

1) *If A is invariant under the action of U, then A^{-1} is also invariant.*

2) *A is invariant $\Leftrightarrow UA \subset AU \Leftrightarrow AU \subset UA \Leftrightarrow UA = AU$.*

3) *If A and B are invariant, then AB is also invariant.*

Proof. It is clear that A is invariant $\Leftrightarrow UA \subset AU$ and A^{-1} is invariant $\Leftrightarrow UA^{-1} \subset A^{-1}U \Leftrightarrow AU \subset UA$. Therefore, in order to prove 1) and 2) it suffices to show that if $UA \subset AU$, then $AU \subset UA$. Let $a \in A$, $\sigma \in U$. Then $a\sigma = \sigma' a'$ for some

$\sigma' \in U, a' \in \Gamma$. We claim that $a' \in A$: $a' = (\sigma')^{-1} a \sigma \in UAU \subset AUU = AU$. Thus, $a' \in AU \cap \Gamma = A$.

In order to prove 3) we notice that if $UA \subset AU$ and $UB \subset BU$, then $UAB \subset AUB \subset ABU$.

Lemma 5. *Let $\Delta \subset \Gamma$ be a subgroup. Then the following conditions are equivalent:*
1) *$U\Delta$ is a subgroup;*
2) *ΔU is a subgroup;*
3) *Δ is invariant under the action of U.*

Proof. Conditions 1) and 2) are equivalent, because $\Delta U = (U\Delta)^{-1}$. 1) means that $U\Delta U\Delta \subset U\Delta$ and $(U\Delta)^{-1} \subset U\Delta$. Each of these inclusions is equivalent to $\Delta U \subset U\Delta$, that is, it is equivalent to condition 3).

In § 5 we shall need the following description. Let $A_1, \dots, A_k, B_1, \dots\dots, B_n \subset \Gamma$ be U-invariant subsets.

Definition. A relation of the form

$$A_1 \dots A_k = B_1 \dots B_n \tag{6}$$

is a $k+n$-tuple

$$(a_1, \dots, a_k, b_1, \dots, b_n) \in A_1 \times \dots \times A_k \times B_1 \times \dots \times B_n$$

such that $a_1 \dots a_k = b_1 \dots b_n$.

We define an action of U on the set of relations of the form (6) in the following way. Let $\sigma \in U$ and let $(a_1, \dots, a_k, b_1, \dots, b_n)$ be a relation of the form (6). There are unique $a_1' \in A_1$ and $\sigma_1 \in U$ such that $\sigma a_1 = a_1' \sigma_1$. There are unique $a_2' \in A_2$ and $\sigma_2 \in U$ such that $\sigma_1 a_2 = a_2' \sigma_2$, and so on. We obtain $a_1' \in A_1, \dots, a_k' \in A_k$ and $\sigma_1, \dots, \sigma_k \in U$ such that $\sigma a_1 \dots a_k = a_1' \dots a_k' \sigma_k$. In an analogous way, we find $b_1' \in B_1, \dots, b_n' \in B_n$ and $\overline{\sigma}_1, \dots, \overline{\sigma}_n \in U$ such that $\sigma b_1 \dots b_n = b_1' \dots b_n' \overline{\sigma}_n$. Since $a_1 \dots a_k = b_1 \dots b_n$, we have $a_1' \dots a_k' \sigma_k = b_1' \dots b_n' \overline{\sigma}_n$, from which we find that $a_1' \dots a_k' = b_1' \dots b_n'$. Thus $(a_1', \dots, a_k', b_1', \dots, b_n')$ is a relation of the form (6). By definition, σ maps the set $(a_1, \dots, a_k, b_1, \dots, b_n)$ into the set $(a_1', \dots, a_k', b_1', \dots, b_n')$. This is in fact an action, since for any $i \leqslant k$, $j \leqslant n$ the product $a_1' \dots a_i'$ is the result of the action of σ on $a_1 \dots a_i \in \Gamma = G/U$ and $b_1' \dots b_j'$ is the result of the action of σ on $b_1 \dots b_j$.

§ 3. The canonical diagram

We need the following simple generalization of Artin's theorem from Galois theory.

Lemma 6. *Let K be a Galois extension of a field F, $U = \mathrm{Gal}(K/F)$. Let G be a subgroup of $\mathrm{Aut}\, K$ such that $G \supset U$ and $[G:U] = n$. We set $\mathbf{k} = K^G$. Then K is a Galois extension of $\mathbf{k}$, $[F:\mathbf{k}] = n_1$, $\mathrm{Gal}(K/\mathbf{k}) = G$.*

The proof of this is a word-for-word repetition of the argument given in [6], Chapter 7, Theorem 2.

Proof of Theorem 2. Diagram (4), in which $\deg f = \gcd(\deg p, \deg q)$, $\deg g = (\deg p_1)(\deg q_1)$, is unique if it exists, because $\mathcal{M}(W) = \mathcal{M}(Y) \cap \mathcal{M}(Z)$ and $\mathcal{M}(V)$ is the compositum of the fields $\mathcal{M}(Y)$ and $\mathcal{M}(Z)$. In fact, $\deg p_1 = \deg p / \gcd(\deg p, \deg q)$ and $\deg q_1 = \deg q / \gcd(\deg p, \deg q)$ are relatively prime. Therefore $\mathcal{M}(V)$ is the compositum of the fields $\mathcal{M}(Y)$ and $\mathcal{M}(Z)$. Since $[\mathcal{M}(Y) : \mathcal{M}(W)] = \deg g / \deg p_1 = \deg q_1$ and $[\mathcal{M}(Z) : \mathcal{M}(W)] = \deg p_1$ are relatively prime, we have $\mathcal{M}(W) = \mathcal{M}(Y) \cap \mathcal{M}(Z)$.

In the rest of the proof we use the notation of § 2. If Γ is infinite then it is not Abelian (if T_p and T_q commute, then Γ is finite). Obviously, $\mathcal{M}(Y) \cap \mathcal{M}(Z) = \mathbb{C}$ in this case.

If Γ is finite, then Γ_1 is trivial, since $\mathfrak{I}_1$ is torsion-free. Therefore $\Gamma = \Gamma/\Gamma_1$ is cyclic of order $d = \operatorname{lcm}(\deg p, \deg q)$. By Lemma 3 $[G : U] = d$. We set $F = E^G$. Using Lemma 6, we will obtain $[\mathcal{M}(X) : F] = d$. It is clear that $\mathcal{M}(Y) \cap \mathcal{M}(Z) = F$. Let W denote a model of the field F. We have a diagram

$$\begin{tikzcd} & X \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[dd, "r"] & \\ Y \arrow[dr, "p_0"'] & & Z \arrow[dl, "q_0"] \\ & W & \end{tikzcd} \tag{7}$$

We set $\infty_W = r(\infty_X)$. Passing to completions, we will find

$$\begin{array}{ccc} \mathcal{M}(X)_\infty & \supset & \mathcal{M}(Z)_\infty \\ \cup & & \cup \\ \mathcal{M}(Y)_\infty & \supset & F_\infty \end{array}$$

Since $\deg r = d$, we have $[\mathcal{M}(X)_\infty : F_\infty] \leqslant d$. But $[\mathcal{M}(X)_\infty : \mathcal{M}(Y)_\infty]$ and $[\mathcal{M}(X)_\infty : \mathcal{M}(Z)_\infty]$ divide $[\mathcal{M}(X)_\infty : F_\infty]$. Hence, d divides $[\mathcal{M}(X)_\infty : F_\infty]$. Therefore $d = [\mathcal{M}(X)_\infty : F_\infty]$, that is, $r\colon (X, \infty) \to (W, \infty)$ is a generalized polynomial, and p_0 and q_0 are also generalized polynomials. Moreover, T_p and T_q generate a subgroup in T_r of order d, that is, they generate T_r.

We notice that $\deg p_0 = \operatorname{lcm}(\deg p, \deg q)/\deg p$ and $\deg q_0 = \operatorname{lcm}(\deg p, \deg q)/\deg q$ are relatively prime. Applying Remark 1 from the Introduction to the diagram $(Y,\infty) \to (W,\infty) \leftarrow (Z,\infty)$, we obtain a commutative diagram

$$\begin{tikzcd}
& (V,\infty) \arrow[dl, "p_1"'] \arrow[dr, "q_1"] \arrow[dd, "g"] & \\
(Y,\infty) \arrow[dr, "p_0"'] & & (Z,\infty) \arrow[dl, "q_0"] \\
& (W,\infty) &
\end{tikzcd} \tag{8}$$

in which V is the normalization of $Y \times_W Z$, $\deg p_1 = \deg q_0$, $\deg q_1 = \deg p_0$. It follows from the definition of $Y \times_W Z$ that the diagrams (7) and (8) can be embedded in a diagram of the form (4).

Remark. On X there are two equivalence relations:

$$R_p = \{(x_1, x_2) \in X \times X \mid p(x_1) = p(x_2)\},$$
$$R_q = \{(x_1, x_2) \in X \times X \mid q(x_1) = q(x_2)\}.$$

We let R denote the equivalence relation that they generate. If Γ is finite, then it is easy to show that R is an algebraic curve on $X \times X$. The essential part of the proof of Theorem 2 is the construction of the Riemann surface X/R. This was done using Lemma 6 (in actuality we constructed the field $\mathcal{M}(X/R)$). The Riemann surface X/R can also be constructed directly, using the results of Grauert [7]. In addition, it is possible to construct the algebraic curve X/R in the framework of algebraic geometry using Theorem 4.1 from p. 262 of [8].

Corollary. *Let $p_i\colon (X,\infty) \to (Y_i,\infty)$ be generalized polynomials, $i \in \{1,2,3\}$. We have a diagram of fields*

$$\begin{array}{ccccc}
\mathcal{M}(Y_1) & \subset & \mathcal{M}(X) & \supset & \mathcal{M}(Y_3) \\
& & \cup & & \\
& & \mathcal{M}(Y_2) & &
\end{array}$$

Let $\mathcal{M}(Y_i) \cap \mathcal{M}(Y_j) \neq \mathbb{C}$ for any i, j. Then $\mathcal{M}(Y_1) \cap \mathcal{M}(Y_2) \cap \mathcal{M}(Y_3) \neq \mathbb{C}$.

Proof. By Theorem 2 the elements of T_{p_i} and T_{p_j} commute, and we have a diagram of generalized polynomials

$$\begin{tikzcd} & X \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[dd, "r"] & \\ Y_1 \arrow[dr] & & Y_2 \arrow[dl] \\ & W & \end{tikzcd}$$

such that $\mathcal{M}(W) = \mathcal{M}(Y_1) \cap \mathcal{M}(Y_2)$, T_{p_1} and T_{p_2} generate T_r. The elements of T_r and T_{p_3} commute. Applying Theorem 2 to the pair r, p_3 we will obtain $\mathcal{M}(Y_1) \cap \mathcal{M}(Y_2) \cap \mathcal{M}(Y_3) \neq \mathbf{C}$.

Proposition 1 now follows immediately from this corollary.

We consider a commutative diagram of generalized polynomials

$$\begin{tikzcd} X \arrow[rr, "h_1"] \arrow[dr, "h_2"'] & & Y \arrow[dl] \\ & Z & \end{tikzcd}$$

T_{h_1} is a subgroup of T_{h_2}. The right factor h_1 of the map h_2 is recovered from the group T_{h_1} as follows: $\mathcal{M}(Y) = \mathcal{M}(X) \cap \mathcal{M}(X)_\infty^{T_{h_1}}$. Thus, some subgroups of T_{h_2} (not necessarily all subgroups) correspond to right factors of h_2.

We return to diagram (2). In § 1 the intermediate field $\mathcal{M}(Z) \subset F_{q,p} \subset \mathcal{M}(X)$ was introduced. The following theorem indicates the subgroup of T_q corresponding to this field. There exists a commutative diagram of generalized polynomials

$$\begin{tikzcd} X \arrow[r, "q_1"] \arrow[d, "p"] & X_{q,p} \arrow[r, "q_2"] \arrow[d] & Z \\ Y \arrow[r] & Y_{q,p} & \end{tikzcd}$$

such that $q_2 \circ q_1 = q$, $\mathcal{M}(X_{q,p}) = F_{q,p}$, $\mathcal{M}(Y_{q,p}) = F_{q,p} \cap \mathcal{M}(Y)$. We set $H_{q,p} = \{\sigma \in T_q \mid \tau\sigma = \sigma\tau \text{ for all } \tau \in T_p\}$.

Theorem 4. $T_{q_1} = H_{q,p}$.

We need two lemmas to prove this theorem.

Lemma 7. *Let $H \subset \mathfrak{I}$ be a finite subgroup. Then H is cyclic.*

Proof. Since $\mathfrak{I}_1$ is torsion-free, we have $H \cap \mathfrak{I}_1 = 1$. Therefore, $H \simeq H/H_1 \hookrightarrow \mathfrak{I}/\mathfrak{I}_1 \simeq \mathbb{C}^*$. A finite subgroup of $\mathbb{C}^*$ is cyclic.

We use the constructions and notation of § 2.

Lemma 8. *Let $L \subset \Gamma$ be a subset that is invariant under the action of U, that is, $uLU = LU$ for all $u \in U$. We set*

$$N_L = \{\sigma \in T_q \mid L\sigma = L\}.$$

Then N_L is a subgroup of T_q that is invariant under the action of U.

Remark. The subgroups T_p and T_q of Γ are invariant under U.

Proof. It is clear that N_L is a subgroup of T_q. In correspondence with Lemma 4 it remains to prove that $UN_L \subset N_L U$.

Let $\sigma \in N_L$, $\gamma \in U$, $\gamma\sigma = \sigma'\gamma'$, where $\sigma' \in T_q$, $\gamma' \in U$. We will show that $L\sigma' = L$. Using Lemma 4, we have $LU \subset UL$, and thus $LUN_L \subset ULN_L \subset UL \subset LU$. Therefore $L\sigma' = L\gamma\sigma(\gamma')^{-1} \subset LU$. On the other hand, $L\sigma' \subset \Gamma$ and $\Gamma \cap LU = L$, and hence $L\sigma' \subset L$. Since σ is of finite order, we have $L\sigma' = L$.

Proof of Theorem 4. We set $L = T_qT_p$. By part 3) of Lemma 4, L is U-invariant. By Lemma 8 we obtain a subgroup N_L $(= N)$ of T_q that is invariant under the action of U. By Lemma 5, NU will be a subgroup of G, $[NU : U] = \#N$. In addition, $\mathcal{M}(Z) \subset E^{NU} \subset \mathcal{M}(X)$, $[\mathcal{M}(X) : E^{NU}] = \#N$. This means that N corresponds to a right factor of q. Obviously, $H_{q,p} \subset N$. We will show that $H_{q,p} = N$. We have $T_pN \subset T_qT_p$. Let $\tau\sigma = \sigma'\tau'$, where $\tau, \tau' \in T_p$, $\sigma \in N$, $\sigma' \in T_q$. Then

$$T_qT_p\sigma' = T_qT_p\sigma(\tau')^{-1} = T_qT_p(\tau')^{-1} = T_qT_p.$$

Thus, $\sigma' \in N$. We have shown that $T_pN \subset NT_p$, and hence, NT_p is a subgroup of Γ. By Lemma 7 NT_p is Abelian. Therefore $N \subset H_{q,p}$.

We have proved that $H_{q,p}$ corresponds to a right factor of q. Using Theorem 2, it is easy to see that $H_{q,p}$ corresponds to $F_{q,p}$.

By definition, the pair p, q in (2) is irreducible if $F_{q,p} = F_{p,q} = \mathcal{M}(X)$. By Theorem 4 this is equivalent to the property $H_{p,q} = H_{q,p} = 1$.

Example. We consider the pair of polynomials $p(z) = z^n$, $q(z) = (z+1)^m$. We have $T_p = \{z \mapsto \varepsilon z \mid \varepsilon^n = 1\}$, $T_q = \{z \mapsto \delta z + (\delta - 1) \mid \delta^m = 1\}$. Now it is easy to verify that $H_{p,q} = H_{q,p} = 1$, that is, the pair p, q is irreducible.

Lemma 9. *We consider a diagram* (2) *in which $\mathcal{M}(Y) \cap \mathcal{M}(Z) \neq \mathbb{C}$. Let K be an intermediate field: $(\mathcal{M}(Y) \cap \mathcal{M}(Z)) \subset K \subset \mathcal{M}(Y)$. Let $\widetilde{K}$ denote the compositum of K and $\mathcal{M}(Z)$. Then $\widetilde{K} \cap \mathcal{M}(Y) = K$.*

Proof. We set $F = \mathcal{M}(Y) \cap \mathcal{M}(Z)$. The corresponding diagram of fields has the form

$$\begin{array}{ccccc} \mathcal{M}(X) & \supset & \widetilde{K} & \supset & \mathcal{M}(Z) \\ \cup & & \cup & & \cup \\ \mathcal{M}(Y) & \supset & K & \supset & F \end{array}$$

By Theorem 2 $[\mathcal{M}(Y) : F]$ and $[\mathcal{M}(Z) : F]$ are relatively prime. Therefore $[K : F]$ and $[\mathcal{M}(Z) : F]$ are also relatively prime, and hence, $[\widetilde{K} : K] = [\mathcal{M}(Z) : F]$. From this it follows that $[\mathcal{M}(Y) : K]$ and $[\widetilde{K} : K]$ are relatively prime, from which we obtain $K = \widetilde{K} \cap \mathcal{M}(Y)$.

Proof of Proposition 2. This follows immediately from Lemma 9.

Proof of Proposition 3. The assertion is non-trivial in the case of formal discreteness. Using Theorem 2, it suffices to prove the following result.

Lemma 10. *We consider a diagram of generalized polynomials*

$$\begin{tikzcd} & X \arrow[d, "r"'] & \\ & Y \arrow[dl, "p"'] \arrow[dr, "q"] & \\ Z & & W \end{tikzcd}$$

Let Γ be the group generated by T_p and T_q, and let Γ' be the group generated by T_{por} and T_{qor}. Γ is formally discrete if and only if Γ' is formally discrete.

Proof. We set $\mathfrak{J}(X,Y,\infty) = \{(g_X, g_Y) \mid g_X \in \mathfrak{J}(X,\infty),\ g_Y \in \mathfrak{J}(Y,\infty),\ g_Y \circ r = r \circ g_X\}$. Then $\mathfrak{J}(X,Y,\infty)$ is a subgroup of $\mathfrak{J}(X,\infty) \times \mathfrak{J}(Y,\infty)$. We consider the projections $\pi\colon \mathfrak{J}(X,Y,\infty) \to \mathfrak{J}(Y,\infty)$ and $j\colon \mathfrak{J}(X,Y,\infty) \to \mathfrak{J}(X,\infty)$. It is easy to verify that π is surjective, $\operatorname{Ker}\pi = T_r$ and j is injective, $T_{por} = j(\pi^{-1}(T_p))$, $T_{qor} = j(\pi^{-1}(T_q))$. Therefore $\Gamma' = j(\pi^{-1}(\Gamma))$.

We choose a meromorphic function of z in a neighbourhood of $\infty \in Y$ with a first-order pole at ∞ (so that z^{-1} is the local coordinate at ∞). We choose an analogous function of ζ in a neighbourhood of $\infty \in X$ so that $r^*(z) = \zeta^n$, $n = \deg r$ (that is, in terms of z and ζ the map r is written as $z = \zeta^n$). Then the elements $g_X \in \mathfrak{J}(X,\infty)$, $g_Y \in \mathfrak{J}(Y,\infty)$ can be written in the form

$$g_X(\zeta) = \sum_{j=-1}^{\infty} a_j \zeta^{-j}, \qquad g_Y(z) = \sum_{k=-1}^{\infty} b_k z^{-k}, \qquad a_{-1} \neq 0, \quad b_{-1} \neq 0.$$

Further, the relation $g_Y \circ r = r \circ g_X$ acquires the form

$$\Bigl(\sum_{j=-1}^{\infty} a_j\zeta^{-j}\Bigr)^n = \sum_{k=-1}^{\infty} b_k\zeta^{-kn},$$

that is,

$$\sum_{j=-1}^{\infty} a_j\zeta^{-j} = \zeta\bigl(b_{-1} + b_0\zeta^{-n} + b_1\zeta^{-2n} + \cdots\bigr)^{1/n}.$$

Now it is clear that the subgroup $\Gamma \subset \mathfrak{I}(Y,\infty)$ is formally discrete if and only if $j(\pi^{-1}(\Gamma))$ is formally discrete.

This proves Lemma 10, and also Proposition 3.

§ 4. Irreducible pairs of generalized polynomials

Theorem 5. *We consider a diagram* (2) *in which the pair* p, q *is irreducible and* $\deg p > 1$, $\deg q > 1$. *We denote by* Γ *the group generated by* T_p *and* T_q. *Assume that* Γ_1 *is Abelian. We set* $n = \deg p$, $m = \deg q$, $\widetilde{p}(z) = z^n$, $\widetilde{q}(z) = (z+1)^m$; $\widetilde{p}, \widetilde{q} \in \mathbb{C}[z]$. *Let* $\widetilde{\Gamma}$ *be the group generated by* $T_{\widetilde{p}}$ *and* $T_{\widetilde{q}}$.

Then there exists an isomorphism $\varphi\colon \Gamma \tilde{\to} \widetilde{\Gamma}$ *such that* $\varphi|_{T_p}\colon T_p \tilde{\to} T_{\widetilde{p}}$, $\varphi|_{T_q}\colon T_q \tilde{\to} T_{\widetilde{q}}$ *and* $\varphi|_{\Gamma_1}\colon \Gamma_1 \tilde{\to} \widetilde{\Gamma}_1$ *are also isomorphisms. Moreover,* Γ *is formally discrete if and only if* $\widetilde{\Gamma}$ *is formally discrete.*

For the proof we shall need the following fact. Let k be a natural number. We denote by $g^t_{z^{k+1}}$ the time-t transformation for the flow of the holomorphic vector field $z^{k+1}\frac{d}{dz}$. This is a germ of a conformal map $(\mathbb{C},0) \to (\mathbb{C},0)$. The set of germs $G(k) = \bigl\{\lambda g^t_{z^{k+1}} \mid \lambda \in \mathbb{C}^*, t \in \mathbb{C}\bigr\}$ forms a group under composition. We abbreviate $\lambda g^t_{z^{k+1}}$ to (λ, t). Then the multiplication table for $G(k)$ has the following form:

$$(\lambda, t) \times (\mu, s) = (\lambda\mu, t\mu^k + s). \tag{9}$$

The subgroup $C(k) = \bigl\{\lambda \in \mathbb{C} \mid \lambda^k = 1\bigr\}$ is the centre of $G(k)$. We set $G_d(k) = \bigl\{\lambda g^t_{z^{k+1}} \in G(k) \mid \lambda^d = 1\bigr\}$. Then $G_d(k)$ is a subgroup of $G(k)$. It is easy to verify that if $h \in G_k(k)$ is an element of finite order, then $h \in C(k)$.

Theorem A ([9], Theorem 2.2, p. 66). *A finitely generated non-Abelian soluble group of germs of conformal maps* $(\mathbb{C},0) \to (\mathbb{C},0)$ *is formally equivalent to a finitely generated subgroup of* $G(k)$ *for some* k.

Proof of Theorem 5. We set $d = \operatorname{lcm}(n,m)$. We choose a local parameter z at $\infty \in X$ and identify $\mathfrak{I}(X,\infty)$ with the group of germs of conformal maps $(\mathbb{C},0) \to (\mathbb{C},0)$. By Remark 5 of § 1 $\mathcal{M}(Y) \cap \mathcal{M}(Z) = \mathbb{C}$. By Theorem 2 Γ is non-Abelian. On the other hand Γ_1 is Abelian and therefore Γ is soluble. By Theorem A, Γ is formally equivalent to a subgroup of $G_d(k)$ for some k. We have $T_p \cap G_k(k) \subset C(k)$, $T_q \cap G_k(k) \subset C(k)$. By Theorem 4, $H_{p,q} = H_{q,p} = 1$, and therefore $T_p \cap G_k(k) = T_q \cap G_k(k) = 1$. Hence $\gcd(n,k) = \gcd(m,k) = 1$, and hence $\gcd(d,k) = 1$. We define a map $f\colon G_d(k) \to G_d(1)$ by the formula $\lambda g^t_{z^{k+1}} \mapsto \lambda^k g^t_{z^2}$. The condition $\gcd(d,k) = 1$ implies that f is bijective. The multiplication table (9) shows that f is an isomorphism.

Let h_p (respectively h_q) be a generator of T_p (respectively T_q). Let $\varepsilon g_{z^2}^{t_1}$, $\delta g_{z^2}^{t_2}$ be the elements of $G_d(1)$ corresponding to h_p and h_q; then ε (respectively δ) is a primitive nth root of unity (respectively a primitive mth root of unity). Using conjugation in $G_d(1)$ we may assume that $t_1 = 0$. Then $t_2 \neq 0$. We notice that the map $\lambda g_{z^2}^{t} \mapsto \lambda g_{z^2}^{ct}$ $(c \in \mathbb{C}^*)$ is an automorphism of $G_d(1)$. Finally the pair of generators takes the form ε, $\delta g_{z^2}^{1}$. It is clear that $\widetilde{\Gamma}$ is formally discrete if and only if Γ is formally discrete.

Lemma 11. *We set* $p(z) = z^n$, $q(z) = (z+1)^m$, $n, m \geqslant 2$. *Let* Γ *be the group generated by* T_p *and* T_q. *The group* Γ *is formally discrete if and only if* $\operatorname{lcm}(n,m) \in \{2,3,4,6\}$.

Proof. Let ε (respectively, δ) be a primitive nth (respectively, mth) root of unity. We have $\varepsilon\Gamma_1 = \Gamma_1$, $\delta\Gamma_1 = \Gamma_1$ (here Γ_1 is considered as a subgroup of $\mathbb{C}$). Therefore $e^{2\pi i/d}\Gamma_1 \subset \Gamma_1$, where $d = \operatorname{lcm}(n,m)$. Hence, the discreteness of Γ implies that $d \in \{2,3,4,6\}$. On the other hand, $\Gamma_1 \subset \mathbb{Z}[e^{2\pi i/d}]$, and if $d \in \{2,3,4,6\}$, then Γ_1 is discrete and Γ is formally discrete.

Corollary. *Under the hypotheses of Theorem* 5 *the group* Γ *is formally discrete if and only if* $\operatorname{lcm}(n,m) \in \{2,3,4,6\}$.

Lemma 12 (the fundamental group-theoretic lemma). *We set* $p(z) = z^n$, $q(z) = (z+1)^m$, n, $m \geqslant 2$. *Let* Γ *be the group generated by* T_p *and* T_q. *Let* G *be an abstract group, and let* U *be a subgroup of* G. *Assume that* Γ *is embedded in* G *and* $\Gamma U = G$, $\Gamma \cap U = 1$. *We assume that* T_pU *and* T_qU *are subgroups of* G. *Let* $(n,m) \in P_1 \cup P_2 \cup P_3$, *where* $P_1 = \{(n,m) \mid n = m\}$, $P_2 = \{(n,m) \mid n = 2$ *or* $m = 2\}$, *and* P_3 *consists of the pairs* $(3,6)$ *and* $(6,3)$.

Then there exists a subgroup U' *of* U *such that* $[U : U'] < \infty$ *and* U' *is a normal subgroup of* G.

This lemma will be proved in the following section.

Remark. If $\operatorname{lcm}(n,m) \in \{2,3,4,6\}$ and n, $m \geqslant 2$, then $(n,m) \in P_1 \cup P_2 \cup P_3$.

Theorem 6. *We consider a diagram* (2) *in which the pair* p, q *is irreducible and* $\deg p > 1$, $\deg q > 1$. *Let* Γ *be the group generated by* T_p *and* T_q. *We assume that* Γ_1 *is Abelian. We set* $n = \deg p$, $m = \deg q$, $\widetilde{p}(z) = z^n$, $\widetilde{q}(z) = (z+1)^m$. *If* $(n,m) \in P_1 \cup P_2 \cup P_3$, *then there exists a commutative diagram*

$$\begin{array}{ccccc}
(Y,\infty) & \xleftarrow{\;p\;} & (X,\infty) & \xrightarrow{\;q\;} & (Z,\infty) \\
\downarrow & & \downarrow & & \downarrow \\
(\mathbb{CP}^1,\infty) & \xleftarrow{\;\widetilde{p}\;} & (\mathbb{CP}^1,\infty) & \xrightarrow{\;\widetilde{q}\;} & (\mathbb{CP}^1,\infty)
\end{array}$$

in which the vertical arrows are isomorphisms.

Proof. We use the notation of § 2. By Theorem 5 we can apply the group-theoretic lemma to obtain a subgroup U' of U such that $[U : U'] < \infty$ and U' is normal in G.

The field $E^{U'}$ is normal over both $\mathcal{M}(Y)$ and $\mathcal{M}(Z)$. Therefore $E^{U'} = E$, that is, $U' = 1$. Thus, $\#U < \infty$, and we obtain a diagram

$$\begin{tikzcd}
 & W \arrow[d, "r"] & \\
 & X \arrow[dl, "p"'] \arrow[dr, "q"] & \\
Y & & Z
\end{tikzcd}$$

in which W is a compact Riemann surface, $\mathcal{M}(W) = E$, r is a non-constant holomorphic map, $p \circ r$ and $q \circ r$ are Galois covers. Since $G \subset \operatorname{Aut} W$, we have $\# \operatorname{Aut} W = \infty$, and hence, the genus of W is equal to zero or one. G acts on the finite set $S = r^{-1}(\infty) \subset W$. If the genus of W is one, then the group $\{g \in \operatorname{Aut} W \mid gw = w\}$ is finite for every point $w \in W$. Therefore the group $\{g \in \operatorname{Aut} W \mid gS = S\}$ is also finite. Thus, the genus of W is equal to zero.

We set $G_0 = \{g \in G \mid g(s) = s \text{ for every } s \in S\}$. Then $[G : G_0] < \infty$, and $\#G_0 = \infty$. This means that $\#S \leq 2$. By Lemma 2, r is unramified over $\infty \in X$. Assume that $\#S = 2$. By part 2) of Lemma 3 we have $\Gamma = G_0$. We may assume that $W = \mathbb{CP}^1$, $S = \{0, \infty\}$. Then $\Gamma \subset \{g \in \operatorname{Aut} \mathbb{CP}^1 \mid g(0) = 0, g(\infty) = \infty\} \simeq \mathbb{C}^*$, and hence Γ is Abelian, a contradiction. Thus, $\#S = 1$, that is, r is an isomorphism, and p and q are Galois covers. This completes the proof.

Theorem 3 is a special case of Theorem 6.

§ 5. Proof of the fundamental group-theoretic lemma

The group-theoretic lemma was formulated in § 4.

Remark. The idea of the proof is to study the action of U on the set of relations of a certain form in Γ. (This action was defined at the end of § 2.) We set $A = T_p \setminus \{\mathrm{id}\}$, $B = T_q \setminus \{\mathrm{id}\}$. We shall use relations of the form

$$B \cdot A = A \cdot B, \tag{10}$$

and also of the form

$$A \cdot B \cdot A = B \cdot A \cdot B. \tag{11}$$

Proposition 4. *The fundamental group-theoretic lemma holds in the case $n = m$.*

Proof. We set $G_p = T_p U$, $G_q = T_q U$, $U_p = \{\sigma \in U \mid \sigma\tau U = \tau U \text{ for all } \tau \in T_p\}$, $U_q = \{\sigma \in U \mid \sigma\tau U = \tau U \text{ for all } \tau \in T_q\}$. The group U_p is the kernel of the action of G_p on G_p/U. Therefore U_p is normal in G_p and $[U : U_p] < \infty$. Since U_q is normal in G_q, it suffices to prove that $U_p = U_q$ (then U_p will be a normal subgroup of G).

We set $h_p(z) = \varepsilon z$, $h_q(z) = \varepsilon z + (\varepsilon - 1)$. Here $h_p \in T_p$, $h_q \in T_q$, and ε is a primitive nth root of unity. We consider two cases.

Case 1: n is even.

(a) If $n = 2$, then $U = U_p = U_q$, and there is nothing to prove.

(b) Assume that $n \geqslant 4$. We consider the set of relations of the form (10). If $h_q^{l_1} h_p^{l_2} = h_p^{k_1} h_q^{k_2}$ then $\varepsilon^{l_1+l_2} = \varepsilon^{k_1+k_2}$ and $\varepsilon^{l_1} - 1 = \varepsilon^{k_1}(\varepsilon^{k_2} - 1)$.

Now it is easy to verify that the relations of the form (10) are exactly the following:

$$h_q^l h_p^{-l+\frac{n}{2}} = h_p^{l+\frac{n}{2}} h_q^{-l}, \tag{12}$$

where $2l \not\equiv 0 \mod n$. We notice that all the h_p^s $(s \not\equiv 0 \mod \frac{n}{2})$ appear in the right-hand sides of the relations (12). We shall show that $U_p = U_q$. Assume that $\sigma \in U_q$. Then σ preserves each relation (12), and therefore σ preserves h_p^i for every $i \not\equiv 0 \mod \frac{n}{2}$. Since σ preserves T_p and id, σ preserves $h_p^{n/2}$. Thus, $\sigma \in U_p$. Analogously, $U_p \subset U_q$.

Case 2: n is odd.

Since Γ_1 is Abelian, $h_q^{-l} h_p^l$ and $h_q^{-s} h_p^s$ commute. This gives relations

$$h_p^{l_1} h_q^{l_2} h_p^{l_3} = h_q^{-l_3} h_p^{-l_2} h_q^{-l_1}, \tag{13}$$

where $l_i \not\equiv 0 \mod n$, $l_1 + l_2 + l_3 \equiv 0 \mod n$. The relations of the form (11) contain the relations (13). We shall not find all the relations of the form (11), but we shall prove the following lemma.

Lemma 13. *We assume that*

$$h_p^{l_1} h_q^{l_2} h_p^{l_3} = h_q^{k_1} h_p^{k_2} h_q^{k_3}; \qquad l_i, k_i \not\equiv 0 \mod n. \tag{14}$$

Then $l_1 \not\equiv k_1 \mod n$ (recall that n is odd).

Derivation of Case 2 *from Lemma* 13. We see from the formulae (13) that for any $l_1 \not\equiv 0 \mod n$, $k_1 \not\equiv 0 \mod n$, $k_1 \not\equiv l_1 \mod n$ there exists a relation of the form (14) with given l_1, k_1. Let $\sigma \in U$, $\sigma h_p^s U = h_p^t U$ for some s, t. Then σ maps the set of relations of the form (14) such that $h_p^{l_1} = h_p^s$ into the set of those relations for which $h_p^{l_1} = h_p^t$. Therefore σ maps the set $\{h_q^j \mid j \not\equiv s \mod n\}$ into the set $\{h_q^j \mid j \not\equiv t \mod n\}$, and hence $\sigma h_q^s U = h_q^t U$. Thus, for $\sigma \in U$ we have $\sigma h_q^s U = h_q^t U \Leftrightarrow \sigma h_p^s U = h_p^t U$. Therefore $U_p = U_q$.

It remains to prove Lemma 13.

We set $\alpha_l = h_q^l h_p^{-l}$. Then $\alpha_l(z) = z + (\varepsilon^l - 1)$. Let $h_p^{l_1} h_q^{l_2} h_p^{l_3} = h_q^{k_1} h_p^{k_2} h_q^{k_3}$; $l_i, k_i \not\equiv 0 \mod n$. Then $k_2 + k_3 = l_2 + l_3$. Hence,

$$\left(h_q^{-l_1} h_p^{l_1}\right)\left(h_q^{l_2} h_p^{-l_2}\right) = \left(h_p^{k_2} h_q^{-k_2}\right)\left(h_q^{k_2+k_3} h_p^{-k_2-k_3}\right),$$

that is, $\left(\varepsilon^{-l_1} - 1\right) + \left(\varepsilon^{l_2} - 1\right) = \left(\varepsilon^{k_2+k_3} - 1\right) - \left(\varepsilon^{k_2} - 1\right)$. Therefore it suffices to prove the following lemma.

Lemma 14. *Assume that n is odd, and that ε is a primitive nth root of unity. Then the equation*

$$\varepsilon^{\gamma_1} + \varepsilon^{\gamma_2} + \varepsilon^{\gamma_3} = \varepsilon^{\mu} + 2 \tag{15}$$

has no solutions such that $\gamma_1, \gamma_2, \gamma_3 \not\equiv 0 \mod n$, where $\gamma_i, \mu_i \in \mathbb{Z}$ are unknown.

Proof. If $\mu \equiv 0 \mod n$, then $\varepsilon^{\gamma_1} = \varepsilon^{\gamma_2} = \varepsilon^{\gamma_3} = 1$, so that $\mu \not\equiv 0 \mod n$. The idea is to average over the action of $\mathrm{Gal}(\mathbb{Q}(\varepsilon)/\mathbb{Q})$.

We set $K = \bigcup_m \mathbb{Q}(\sqrt[m]{1})$. For every $m \in \mathbb{N}$ we define a $\mathbb{Q}$-linear functional $T_m \colon \mathbb{Q}(\sqrt[m]{1}) \to \mathbb{Q}$ by the formula

$$T_m(z) = \frac{1}{\#H} \sum_{h \in H} h(z),$$

where $H = \mathrm{Gal}(\mathbb{Q}(\sqrt[m]{1})/\mathbb{Q})$.

If $m' \mid m$, then $T_m\big|_{\mathbb{Q}(\sqrt[m']{1})} = T_{m'}$. Therefore the formula $T\big|_{\mathbb{Q}(\sqrt[m]{1})} = T_m$ gives a well-defined $\mathbb{Q}$-linear functional $T \colon K \to \mathbb{Q}$.

We recall that if δ is a primitive mth root of unity, then $\mathrm{Tr}_{\mathbb{Q}}^{\mathbb{Q}(\delta)}(\delta) = \mu(m)$, where μ is the Möbius function. Therefore $T\delta = \frac{\mu(m)}{\varphi(m)}$. We have

$$\varphi(1) = 1 = \varphi(2), \qquad \varphi(3) = \varphi(4) = \varphi(6) = 2,$$

$\varphi(m) > 2$ for $m \neq 1, 2, 3, 4, 6$. Therefore for odd $m > 1$ we have $-1/2 \leqslant T\delta < 1/2$. Applying T to (15), we find

$$\frac{3}{2} \leqslant 2 + T(\varepsilon^{\mu}) = T(\varepsilon^{\gamma_1}) + T(\varepsilon^{\gamma_2}) + T(\varepsilon^{\gamma_3}) < \frac{3}{2},$$

a contradiction.

This proves Proposition 4.

Proposition 5. *The fundamental group-theoretic lemma holds for $n = 3$, $m = 6$.*

Proof. Let ω be a primitive 6th root of unity. We set $h_p(z) = \omega^2 z$, $h_q(z) = \omega z + (\omega - 1)$. It is easy to show that in this case there are two relations of the form (10):

$$h_q h_p = h_p^2 h_q^5, \qquad h_q^5 h_p^2 = h_p h_q. \tag{16}$$

The group U acts on the set of these two relations. Therefore the set $A = \{h_q, h_q^5\}$ is U-invariant. By part 3) of Lemma 4, $A \cdot A = \{h_q^2, h_q^4, h_q^6 = \mathrm{id}\}$ is also U-invariant. We let Γ' denote the subgroup of Γ generated by T_p and $\{\mathrm{id}, h_q^2, h_q^4\}$. Then Γ' is invariant under U, and therefore $G' = \Gamma' U$ will be a subgroup of G. Moreover, $[G : G'] = [\Gamma : \Gamma'] < \infty$. By Proposition 4 the group-theoretic lemma holds for G' (in this case $n = m = 3$). We will obtain a subgroup U' of U such that $[U : U'] < \infty$ and U' is normal in G'. Now $\bigcap_{g \in G/G'} gU'g^{-1}$ will be the desired subgroup of U.

Proposition 6. *The fundamental group-theoretic lemma holds for $n = 2$ and arbitrary m.*

Proof. Let $\sigma \in T_p$, $\sigma \neq 1$. Then the group U preserves σ, so that $\sigma T_q \sigma$ is U-invariant. We let Γ' denote the subgroup of Γ generated by T_q and $\sigma T_q \sigma$. Then $G' = \Gamma' U$ is a subgroup of G, and $[G : G'] = [\Gamma : \Gamma'] < \infty$. By Proposition 4 the group-theoretic lemma holds for G', and the desired subgroup of U is constructed as in Proposition 5.

Bibliography

[1] L. Flatto, "A theorem on level curves of harmonic functions", *J. London Math. Soc.* **1** (1969), 470–472.

[2] L. J. Hansen and H. S. Shapiro, "Graphs and functional equations", *Ann. Acad. Sci. Fennicae Ser. A. I. Math.* **18** (1993), 125–146.

[3] S. A. Lysenko, "On the functional equation $f(p(z)) = g(q(z))$, where f and g are meromorphic functions and p and q are polynomials", *Mat. Fiz., Anal., Geom.* **2**:1 (1995), 68–86. (Russian)

[4] F. Gross, "Factorization of meromorphic functions and some open problems", in: *Lect. Notes in Math.*, vol. 599, Springer, Berlin 1977, pp. 51–67.

[5] I. I. Pyatetskii-Shapiro and I. R. Shafarevich, "Galois theory of transcendental extensions and uniformization", *Izv. Akad. Nauk SSSR Ser. Mat.* **30** (1966), 671–704. (Russian)

[6] S. Lang, *Algebra*, Addison-Wesley, Reading, Mass. 1966; Russian transl., Mir, Moscow 1968.

[7] H. Grauert, "On meromorphic equivalence relations", *Contributions to Several Complex Variables, Aspects of Math.* **E9** (1986), 115–145.

[8] M. Demazure and A. Grothendieck, Séminaire de Géométrie *Algèbrique du Bois Marie* 1962/64 (SGA3): *Schémas en Groupes* I, Lecture Notes in Math. **151**, Springer, Berlin 1970.

[9] P. J. Elizarov, Yu. S. Il'yashenko, A. A. Shcherbakov, and S. M. Voronin, "Finitely generated groups of germs of one-dimensional conformal mappings and invariants for complex singular points of analytic foliations of the complex plane", Advances in Soviet Math. **14**, Amer. Math. Soc., Providence, RI 1993, pp. 57–105. (English)

[10] J. R. Ritt, "Prime and composite polynomials", *Trans. Amer. Math. Soc.* (1922), 51–66.

[11] W. H. J. Fuchs and F. Gross, "Generalization of a theorem of A. and C. Rényi on periodic functions", *Acta. Sci. Math.* **32** (1971), 83–86.

[12] H. S. Shapiro, "The functional equation $f(P(z)) = g(Q(z))$ and a problem of A. and C. Rényi", *Studia Sci. Math. Hungar.* **1** (1966), 255–259.

[13] Yu. S. Il'yashenko, "Nonlinear Stokes phenomena", Advances in Soviet Math., vol. 14, Amer. Math. Soc., Providence, RI 1993, pp. 1–55. (English)

[14] A. A. Shcherbakov, "On the denseness of orbits of the pseudogroup of conformal mappings and a generalization of the Huday-Verenov theorem", *Vestnik Mosk. Gos. Univ. Ser.* 1, *Mat.-Mekh.* **1982** no. 2, 10–15; English transl. in *Moscow Univ. Math. Bull.* **37** (1982).

Khar'kov

Received 8/FEB/96
Translated by J. S. JOEL

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TeX