WINTER LOCAL LANGLANDS Wh-5

Abstract. The purpose of this talk (Wh-5) is two-fold: first we will set up the context needed to attack the FLE; second, we will address material needed for the local-to-global compatibilities to be discussed in Ja-4.

1. The formal setting

We begin by addressing the formal setting: Recall that we have introduced two forms of the FLE (at positive and negative levels respectively).

We begin at the negative level[1]; when we set out to prove the FLE we want to relate the Kazhdan-Lusztig category for G and Whittaker category for $\mathbf{Gr}_{\check{G}}$ to the corresponding categories for the Torus; this relation takes the form of a diagram:

$$\begin{array}{ccc} \mathbf{KL}(G)_{-\kappa} & {}^{FLE_G} & \mathbf{Whit}_{-\check{\kappa}}(\mathbf{Gr}_{\check{G}}) \\ \downarrow j_!^{KM,Lus} & & \downarrow j_!^{Whit,Lus} \\ \mathbf{KL}(T)_{-\kappa} & \xrightarrow{FLE_T} & \mathbf{Whit}_{-\check{\kappa}}(\mathbf{Gr}_{\check{T}}) \end{array}$$

The desired vertical functors should make the diagram commute, be factorizable and by applying these functors to the units on both sides we expect to obtain the factorization algebras constituting the subject of the last talk. Morevover, the factorizable algebras should match up under the bottom isomorphism, but it will **not** be the case that the induced map on factorization module categories will be an equivalence.

The essential thrust of the last talk will be a conjectural bootstrapping of an equivalence given what structure we do have apparent here.

Remark 1.1. *We saw in* Ja-*3 the definition of the functor* $j_!^{KM,Lus}$. *Note the designation of Lus in contrast to the existence of a more naive functor corresponding to the de-Concini-Kac form of the quantum group, which will not produce the correct diagram.*

[1] Recall that by convention κ denotes positive level, hence $-\kappa$ is used for negative level.

We still need a definition of the functor $j_!^{Whit,Lus}$. First, we'll drop the notation Lus[2] and define another functor $j_{!*}^{Whit}$; this new functor also has an incarnation on the Kac-Moody side as $j_{!*}^{KM}$, giving a diagram:

$$\begin{array}{ccc}
\mathbf{KL}(G)_{-\kappa} & \scriptstyle{FLE_G} & \mathbf{Whit}_{-\check{\kappa}}(\mathbf{Gr}_{\check{G}}) \\
\big\downarrow j_!^{KM} \big\downarrow j_{!*}^{KM} & & \big\downarrow j_!^{Whit} \big\downarrow j_{!*}^{Whit} \\
\mathbf{KL}(T)_{-\kappa} & \xrightarrow{FLE_T} & \mathbf{Whit}_{-\check{\kappa}}(\mathbf{Gr}_{\check{T}})
\end{array}$$

We can similarly express the corresponding diagram at positive level:

$$\begin{array}{ccc}
\mathbf{KL}(G)_{\kappa} & \scriptstyle{F\tilde{L}E_G} & \mathbf{Whit}_{\check{\kappa}}(\mathbf{Gr}_{\check{G}}) \\
\big\downarrow j_!^{KM} \big\downarrow j_{!*}^{KM} & & \big\downarrow j_!^{Whit} \big\downarrow j_{!*}^{Whit} \\
\mathbf{KL}(T)_{\kappa} & \xrightarrow{F\tilde{L}E_T} & \mathbf{Whit}_{\check{\kappa}}(\mathbf{Gr}_{\check{T}})
\end{array}$$

As we have seen, for the torus there is no disctinction between FLE for positive and negative levels (recall positivity refers to the killing form on simple factors.) However, on the Kac-Moody side, these functors will differ by a Cartan involution. In contrast, since the Whittaker category is geometric, it does not see the distinction between positive and negative level; the main features of these functors on the Whittaker side will be the same as at the negative level. As we saw in JA-3, these functors on the Kac-Moody side strongly depend on the positivity or negativity of the level, while on the Whittaker side they do not.

Remark 1.2. *What was referred to as j_* in* JA*-3 (in relation to C_* semi-infinite cohomology) at positive level is here $j_!$.*

In the next section we will recall definitions of a subset of these functors (for the rest, and in greater detail, see JA-3).

Remark 1.3. *We can express the plan for* JA*-4 in terms of these FLE diagrams for negative and positive level: to see this, note that all of these local categories are related to the corresponding global categories ($\mathbf{Bun}(G), \mathbf{Bun}(\check{G})\mathbf{Bun}(T), \mathbf{Bun}(\check{T})$, and similarly for parabolics and their levi factors etc.), such that there analogously exist functors relating these global categories.* JA*-4 will describe the interaction of these global categories equipped with global functors and their compatabilities and interactions with variants of the Eisenstein series and constant*

[2]reserving this notation for the factorization algebra as in Ω_q^{Lus} arising as the image of the unit under the functor $j_!^{KM}$

term functors. We will see that the structure amounts in the global setting to a series of commutative cubes.

2. Definitions of the functors $j_!$ and $j_{!*}$ on the Kac-Moody Side

2.1. $j^{KM}_{\{!,!*\}}$ at the positive level. Let us begin by recalling from JA-3 the semi-infinite cohomology functor

$$C_* = C^{\frac{\infty}{2}}(\mathfrak{L}(\mathfrak{n}^-)^+, -),$$

in terms of which we can define $j_!^{KM}$ and $j_{!*}^{KM}$, via the following diagram:

$$\begin{array}{ccccc} j^{KM}_{\{!,!*\}} : \hat{\mathfrak{g}} - \mathbf{Mod}_\kappa^{\mathfrak{L}\mathfrak{G}^+} & \overset{\phi_!}{\underset{\phi_{!*}}{\rightrightarrows}} & (\hat{\mathfrak{g}} - \mathbf{Mod}_\kappa)_{\mathfrak{L}(\mathfrak{N}^-)\mathfrak{L}\mathfrak{T}^+} & \xrightarrow{C_*} & (\hat{\mathfrak{t}} - \mathbf{Mod}_\kappa)_{\mathfrak{L}\mathfrak{T}^+} \\ \| & & & & \| \\ \mathbf{KL}(G)_\kappa & & & & \mathbf{KL}(T)_\kappa \end{array}$$

Here we have used the fact that invariants and co-invariants (conjecturally) coincide via an equivalance, as well as the fact that C_* is well-behaved at the positive level. Thus, at the positive level we have reduced the definition of $j_!^{KM}$ and $j_{!*}^{KM}$ to the specification of $\phi_!$ and $\phi_{!*}$.

2.2. The functors $\phi_!$ and $\phi_{!*}$ by convolution. The functors $\phi_!$ and $\phi_{!*}$ map $\mathfrak{L}\mathfrak{G}+$ invariants to $\mathfrak{L}(\mathfrak{N}^-)\mathfrak{L}\mathfrak{T}^+$-coinvariants, and hence exhibit a universal nature: both are given by convolution with an object of

$$\mathbf{Dmod}(\mathbf{Gr}_G)_{\mathfrak{L}(\mathfrak{N}^-)\mathfrak{L}\mathfrak{T}^+} \xrightarrow{\sim} \mathbf{Dmod}(\mathbf{Gr}_G)^{\mathfrak{L}\mathfrak{N}\mathfrak{L}\mathfrak{T}^+}.$$

The appearance of the affine grassmannian as underlying geometry for our **Dmod** category corresponds to the fact that in the domain of the functors $\phi_{\{!,!*\}}$ we have passed to $\mathfrak{L}\mathfrak{G}^+$-invariants; by the action of convolution by anything in this category, we can pass to $\mathfrak{L}(\mathfrak{N}^-)\mathfrak{L}\mathfrak{T}^+$-coinvariants so as to land in the desired codomain.

Remark 2.1. *Given that we do not yet have the factorizable equivalence between invariants and co-invariants used above, it is worth noting that we will only need the functor in one direction, namely, the actual diagram we need is* $\mathbf{Dmod}(\mathbf{Gr}_G)_{\mathfrak{L}(\mathfrak{N}^-)\mathfrak{L}\mathfrak{T}^+} \longleftarrow \mathbf{Dmod}(\mathbf{Gr}_G)^{\mathfrak{L}\mathfrak{N}\mathfrak{L}\mathfrak{T}^+}$.

In order to specify these two functors, we can thus specify two factorizable objects we will convolve against. To this end, consider the inclusion

$$S^0_{Ran} \overset{j}{\hookrightarrow} \overline{S^0_{Ran}}$$

of the zero semi-infinite orbit into its closure (where Ran here again denotes the factorizable/global version). We have an inclusion of the corresponding **Dmod** categories:

$$\begin{array}{ccc} \mathbf{Dmod}(\mathbf{Gr}_G)_{\mathfrak{L}(\mathfrak{N}^-)\mathfrak{L}\mathfrak{T}^+} & \longleftarrow & \mathbf{Dmod}(\mathbf{Gr}_G)^{\mathfrak{L}\mathfrak{N}\mathfrak{L}\mathfrak{T}^+}. \\ & & \big\uparrow \\ & & \mathbf{Dmod}(\overline{S^0})^{\mathfrak{L}\mathfrak{N}\mathfrak{L}\mathfrak{T}^+} \end{array}$$

One of our objects will nominally be built from the dualizing object.

Remark 2.2. *Recall that in order to even define a restriction functor over the Ran space—which is a colimit—we have to consider maps*

$$\begin{array}{c} S^0_{X^I} \\ \big\downarrow \\ X^I \end{array}$$

for finite sets I over a smooth curve X. In general, there is no base change between shriek-pushforward and shriek-pullback, but one can prove a theorem here that in fact base-change is well-defined; $j_!$ over the entire Ran space will in effect realize $j_!$ over any given X^I—i.e. $j_!$ behaves well with respect to the given pullback diagrams, hence our desired object is factorizable.

The second object will be the semi-infinite $\mathbf{IC}^{\frac{\infty}{2}}$ sheaf introduced in a previous talk, hence our two functors will be built from convolution against:

$$j_!(\omega_{S^0_{Ran}}) \qquad\qquad \mathbf{IC}^{\frac{\infty}{2}}$$

Remark 2.3. *The $j_!$ functor (at the positive level) realizes semi-infinite cohomology: to see this, note we took the object yielding our $j_!$ functor, considered it as part of the $\mathfrak{L}\mathfrak{N}\mathfrak{L}\mathfrak{T}^+$ invariant category, moved it to the $\mathfrak{L}(\mathfrak{N}^-)\mathfrak{L}\mathfrak{T}^+$ co-invariant category and then convolved against it. This produces a functor equivalent to forgetting $\mathfrak{L}\mathfrak{G}^+$ equivariance followed by projection.*

Next we will define the analagous functors on the Whittaker category.

Remark 2.4. *We have not said anything on the a-priori conceptual reason for expecting these functors as-defined on either side (KM and Whittaker) to match up? One possible answer: follow local-to-global compatabilities. Another way might be to see it via the original 2-categorical formalism.*

2.3. $j^{whit}_{\{!,!*\}}$ **on the Whittaker side.** We want a functor from the Whittaker category on the affine grassmannian[3] to the Whittaker category for the affine grassmannian of the torus[4].

$$\begin{array}{c} \mathbf{Whit}^!_\kappa(\mathbf{Gr}_{\check{G}}) \\ \Big\downarrow j^{whit}_{!*} \Big\downarrow j^{whit}_{!} \\ \mathbf{Dmod}(\mathbf{Gr}_{\check{T}}) \end{array}$$

Remark 2.5. *Once again, we must reproduce the analogous structure from the Kac-Moody side factorazably. Here we will only indicate the structure at a point, but since the objects introduced posess a factorization structure, it should be clear how to extend the structure over the Ran space.*

The affine grassmannian for the torus $\check{T}$ is a discrete set indexed by the coweights of $\check{T}$; hence we will define our functors $j_{\{!,!*\}}$ separately for each coweight.

2.4. $j^{whit,\lambda}_!$ **and** $j^{whit,\lambda}_{!*}$**.** Recall our underlying geometry of sheaves over the affine grassmannian with an equivariance condition. Once again we start with the zero semi-infinite orbit S^0, but in addition we will also consider the various translates by coweights λ, denoted $S^{0,\lambda}$. In lieu of $\mathfrak{L}\mathfrak{N}$, we need to consider equivariance with respect to $\mathfrak{L}(\mathfrak{N}^-)$, which we'll denote by $S^{-,\lambda}$. The analogous inclusion of orbits into their closures, along with the two sheaves we will consider for convolution, becomes:

$$S^{-,\lambda} \xhookrightarrow{\;j\;} \overline{S^{-,\lambda}}$$

$$j_!(\omega_{S^{-,\lambda}}) \qquad \mathbf{IC}^{\frac{\infty}{2}+\lambda}$$

We stress that we have translated the analogous objects from the zero-semi-infinite orbit; in particular, here $\mathbf{IC}^{\frac{\infty}{2}+\lambda}$ is the fiber of the

[3]As remarked earlier, since the Whittaker category is geometric the twisting parameter will play no role

[4]Since the torus has no unipotent part, the Whit functor is vacuous

sheaf introduced in JA–5, restricted to a copy of the affine grassmannian over one point of the curve.

Remark 2.6. *We warn again that* $\mathbf{IC}^{\frac{\infty}{2}+\lambda}$ *is* ***not*** *the* **IC***-extension in the t-structure of a single copy of the grassmannian. One needs to perform the extension globally over the Ran space (i.e. allowing the points to move), followed by restriction to the fiber over a given point of the curve and a subsequent shift by* λ*.*

The functors $j^{whit}_{\{!,!*\}}$ are then given by mapping any given Whittaker sheaf $\mathcal{F}$ to the de Rahm cohomology along the affine grassmannian of the shriek-tensor-product of $\mathcal{F}$ with either of the two objects introduced above.

$$\mathcal{F} \mapsto \qquad \Gamma_{dR}(\mathbf{Gr}_G, \mathcal{F} \otimes^! -)$$

In particular, starting with an $\mathfrak{LN}$ equivariant sheaf (hence a geometric object with truly infinite dimensional support), we intersect it with something equivariant with respect to $\mathfrak{L}(\mathfrak{N}^-)$. On compact objects this has the effect of producing a geometric object supported on the closure of the intersection of particular $\mathfrak{LN}$ and $\mathfrak{L}(\mathfrak{N}^-)$ orbits. This intersection is finite dimensional, hence the above de Rahm cohomology is taken over a finite dimensional variety.

These are the Jacquet functors.