Towards a geometrization of the minimal automorphic representations for $\mathbb{SO}_{2n}$

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1. **Motivations** The theory of minimal representations of reductive groups exists (at least since 1989) in several settings, over finite, local and global fields (cf. [1] for a recent survey). They are of interest especially because they allow to prove in the classical setting some particular cases of Langlands functoriality via "generalized theta correspondences". With Vincent Lafforgue we are trying to geometry these representations, our initial motivation was to further obtain out of it some new cases of the geometric Langlands functoriality.

1.1 Case of finite field $k=\mathbb{F}_q$. Let G/k be split reductive connected, $B\subset G$ a Borel. Then $\operatorname{Ind}_{B(k)}^{G(k)}=\operatorname{Fun}(G(k)/B(k))$ is a module over $G(k)\times A_q$, where A_q is the finite Hecke algebra at q. For $q=1$ one gets $A_1\tilde{\to}\bar{\mathbb{Q}}_\ell[W]$, where W is the Weyl group of G, and there is a canonical bijection (at least, for q lying in some open subset of $\mathbb{C}$) $\check{A}_q\tilde{\to}\check{A}_1$. Here $\check{A}$ denotes the set of irreducible representations of a finite-dimensional algebra A, similarly let $\check{W}$ be the set of irreducible finite-dimensional representations of W. Then

$$\operatorname{Fun}(G(k)/B(k))\tilde{\to}\oplus_{\chi\in\check{W}}R_\chi\otimes\chi_q$$

as a $G(k)\times A_q$-module. Here R_χ is an irreducible $G(k)$-module (over $\bar{\mathbb{Q}}_\ell$), and $\chi_q\in\check{A}_q$ corresponds to χ via the above bijection. See [6].

Examples: a) If χ is trivial then R_χ is trivial.

b) Let χ be the reflection representation on the root space. Assume G simple adjoint such that all roots have the same length (that is, G is of type D_m with $m\geq 4$, A_m, E_m). Then χ is irreducible and R_χ is called reflection representation of $G(k)$. If $\mathcal{O}_{min}$ is the minimal nontrivial nilpotent orbit in $\mathfrak{g}=\operatorname{Lie}G$ then $\dim R_\chi=q^{\frac{1}{2}\dim\mathcal{O}_{min}}+\dots$.

For example, if $G=\mathrm{SL}_n$ then R_χ is the space of functions f on $\mathbb{P}^{n-1}(k)$ such that $\int_{\mathbb{P}^{n-1}(k)}fdx=0$.

One may address the problem: construct a category with an action of G, which is a geometric analog of the reflection representations.

Example: consider the case $H=\mathbb{SO}(V)$, where V is an orthogonal split vector space over k with $\dim V=2n$. Then

$$R_\chi=\{f:V(k)\to\bar{\mathbb{Q}}_\ell\mid f=\operatorname{Four}_\psi(f),\ f(0)=0,\ f\text{ is extension by zero from }Y(k)\}$$

Here $Y=\{v\in V\mid\langle v,v\rangle=0\}$. Actually, the condition $f=\operatorname{Four}_\psi(f)$ follows from the fact that both f and $\operatorname{Four}_\psi(f)$ are extensions by zero from $Y(k)$.

Geometric version. Let $\tilde{\mathcal{R}}_\chi$ be the category

$$\tilde{\mathcal{R}}_\chi=\{K\in P(Y)\mid\text{equipped with }\operatorname{Four}_\psi(K)\tilde{\to}K\}$$

here $P(Y)$ is the category of perverse sheaves on Y. Then H acts on this category. Note that $\mathrm{Four}_\psi(\mathrm{IC}(Y)) \widetilde{\to} \mathrm{IC}(Y)$, and the subcategory $\tilde{\mathcal{R}}_0$ generated by $\mathrm{IC}(Y)$ is "the trivial representation" of H. Then $\tilde{\mathcal{R}}_\chi/\tilde{\mathcal{R}}_0$ is probably a reasonable geometric analog of R_χ in this case. I mention this as in the global case there is also a problem of "separating" the minimal representation of the trivial one.

1.2 Case of a local field. I'll just mention some of ideas, there is plenty of litterature.

1.2.1 Let F be a local non archimedian field of characteristic zero. Let G/F be simple, split, simply-connected. Let (π, V) be an irreducible smooth representation of $G(F)$. It has a character distribution (viewed as a distribution Ch_π is a neighbourhood of $0 \in \mathfrak{g}(F)$). Let f be a locally constant function supported on a sufficiently small open neighbourhood of $0 \in \mathfrak{g}(F)$. Then $\int_{\mathfrak{g}(F)} \pi(expx)f(x)dx$ is of finite rank and

$$Ch_\pi(f) = \mathrm{tr} \int_{\mathfrak{g}(F)} \pi(expx)f(x)dx$$

Harish-Chandra and Howe: there is $c_{\mathcal{O}} \in \bar{\mathbb{Q}}_\ell$ for each nilpotent orbit $\mathcal{O} \subset \mathfrak{g}^*(F)$ such that for any f as above

$$Ch_\pi(f) = \sum_{\mathcal{O}} c_{\mathcal{O}} \int_{\mathfrak{g}^*(F)} \hat{f}\mu_{\mathcal{O}},$$

here $\hat{f}$ is the Fourier transform of f, $\mu_{\mathcal{O}}$ is a $G(\mathcal{O})$-invariant measure on $\mathcal{O}$.

On the set of nilpotent orbits one has the order $\mathcal{O}_1 \le \mathcal{O}_2$ iff $\mathcal{O}_1 \subset \overline{\mathcal{O}_2}$. Then the minimal nontrivial orbit is the one through the highest root (if one identifies $\mathfrak{g}^*$ with $\mathfrak{g}$ via a Killing form).

Definition 1. $d = \max_{c_{\mathcal{O}} \neq 0} \frac{1}{2} \dim \mathcal{O}$ is called Gelfand-Kirillov dimension of π. The wave front set $WF(\pi)$ is the set of maximal elements in $\{\mathcal{O} \mid c_{\mathcal{O}} \neq 0\}$. Then π is minimal iff $WF(\pi) = \mathcal{O}_{min}$.

Let $\phi : \mathrm{SL}_2 \to \check{G}$ correspond to the subregular nilpotent orbit in the Langlands dual group. Let π_ϕ be the spherical representation of $G(F)$ with the Satake parameter

$$\phi \begin{pmatrix} |\, t\,|^{1/2} & 0 \\ 0 & |\, t\,|^{-1/2} \end{pmatrix}.$$

Here $t \in F$ is a uniformizer.

Fact 1. ([14]) If G is of type A_{2n-1}, D_n, E_n then π_ϕ is a minimal representation. For GL_{2n}, $\pi_\phi = ind^G_{P_{2n-1,1}}(triv)$, the unitary induction.

Fact 2. 1) In types D_n, E_n the minimal representation exists and is unique.
2) In types $(B_n, n > 3)$ and G_2 a split group has no minimal representation.

3) In types B_3 (that is, for $\mathbb{SO}_7$), F_4 and C_n the minimal representation exists as a representation of a 2-fold metaplectic cover of $G(F)$. In type C_n this is the Weil representation of $\widetilde{\mathbb{S}\mathrm{p}}_{2n}(F)$.
4) In types $B_n, n > 3$ no metaplectic cover admits a minimal representation (Vogan?)
5) The 3-fold metaplectic cover of G_2 admits a min representation.

This claim becomes clearer in view of Arthur conjectures (see 1.2.5 and 1.2.3.4).

1.2.2 Minimality is reflected in vanishing of many (twisted) Jacquet modules in local setting (in global setting: of many Fourier coefficients).

Let $\mathcal{O} \subset \mathfrak{g}(F)$ be a nilpotent orbit, let $Y \in \mathcal{O}$. There is $H \in \mathfrak{g}$ semisimple such that $[H, Y] = -2Y$, and the eigenvalues of H on $\mathfrak{g}$ are integers (for example, coming from a SL_2-triple). Let $\mathfrak{g} = \sum_i \mathfrak{g}_i$, where

$$\mathfrak{g}_i = \{x \in \mathfrak{g} \mid [H, x] = ix\}$$

Let $\mathfrak{n}' = (Z_{\mathfrak{g}}(Y) \cap \mathfrak{g}_1) + \sum_{i\geq 2} \mathfrak{g}_i$ and $N' = exp(\mathfrak{n}')$ be the corresponding subgroup. Let $\psi : F \to \bar{\mathbb{Q}}_\ell$ be a nontrivial additive character. Moeglin-Waldspurger ([12]) explain that (Y, H) yields a character $\psi_{Y,H} : N' \to \bar{\mathbb{Q}}_\ell^*$, $exp(x) \mapsto \psi(\langle X, Y\rangle)$. Here $\langle ., .\rangle$ is the Killing form. Let $J_{\psi_{Y,H}}(V)$ denote the twisted Jacquet module of V for $(N', \psi_{Y,H})$. Set

$$Wh_\psi(V) = \{\mathcal{O} \mid J_{\psi_{Y,H}}(V) \neq 0 \text{ for some } H\}$$

Fact 3. ([12]) $WF(\pi)$ equals the set of maximal elements of $Wh_\psi(V)$.

The condition $J_{\psi_{Y,H}}(V) \neq 0$ means that V admits a generalized Whittaker model for these data.

In the global setting one can similarly attach to an automorphic representation a collection of nilpotent orbits (see [2] and Arthur conjectures on the unipotent representations).

1.2.3 One more definition of minimality. Let G/F be simple, simply-connected, split. Let $T \subset B \subset G$ be a maximal torus and Borel. Let α be the highest root. Pick an SL_2-triple $(h_\alpha, e_\alpha, e_{-\alpha})$, where e_α is in the root space of the max root, $h_\alpha = [e_\alpha, e_{-\alpha}]$. Let

$$\mathfrak{g}(i) = \{x \in \mathfrak{g} \mid [h_\alpha, x] = ix\}$$

Then $\mathfrak{g}(i) = 0$ for $\mid i \mid > 2$, and $\oplus_{i\geq 0}\mathfrak{g}(i)$ is a parabolic subalgebra, its nilpotent radical is the Heisenberg Lie algebra $\mathfrak{n} = \mathfrak{g}(1) \oplus \mathfrak{g}(2)$ with the center $\mathfrak{z} = \mathfrak{g}(2)$, this is the root space of α. Let $Z = exp(\mathfrak{z}) \subset G$ be the corresponding unipotent subgroup.

Let $P = MN$ be the corresponding parabolic (the Heisenberg parabolic). Assume

$$(C1) \qquad M \text{ acts transitively on } \mathfrak{z}(F) - \{0\} \text{ with stabilizor } M_{ss} = [M, M]$$

(this is equivalent to requiring that G is not of type A_n or C_n). For C_n this is not a very reasonable assumption, as we know ([8]) that in the geometric setting (over an algebraically closed base field) the Weil representation does not depend on the additive character ψ any more.

On $V = \mathfrak{g}(1)$ one gets a non degenerate symplectic form $x, y \mapsto \omega(x, y)$ given by $[x, y] = \omega(x, y)e_\alpha$. So, N is the Heisenberg group. Let $P_{ss} = M_{ss}N$, the group M_{ss} preserves ω, hence a morphism $M_{ss} \to \mathbb{S}\mathrm{p}(V)$. The group $N(F)$ admits a unique smooth irreducible representation ρ_ψ with given central character ψ. Let $Mp(V) \to \mathbb{S}\mathrm{p}(V)$ be the corresponding 2-fold metaplectic covering, then ρ_ψ is the Weil representation of $Mp(V) \rtimes N(F)$.

Assume

$(C2)$ There is a splitting $M_{ss}(F) \to Mp(V)$ (not to work with the covering groups)

Split groups of types G_2, D_n, E_n satisfy both C1,C2. Since M_{ss} is simply-connected, $M_{ss}(F)$ is perfect, so this splitting is unique. Let (ρ_ψ, W_ψ) denote the obtained representation of $P_{ss}(F)$. Set $W = ind_{P_{ss}(F)}^{P(F)}(W_\psi)$, smooth compact induction.

Definition 2. ([1]) If π is a smooth representation of $G(F)$, set

$$\pi_{Z,\psi} = \pi / \langle \pi(z)v - \psi(z)v, z \in Z, v \in \pi \rangle,$$

the twisted Jacquet module. This is a $P_{ss}(F)$-module, and Z acts via ψ on it. One checks that

$$\pi_{Z,\psi} \widetilde{\to} \mathrm{Hom}_{N(F)}(W_\psi, \pi_{Z,\psi}) \otimes W_\psi$$

as $N(F)$-modules. Then $\mathrm{Hom}_{N(F)}(W_\psi, \pi_{Z,\psi})$ is naturally a $M_{ss}(F)$-module, where $m \in M_{ss}(F)$ sends $T : W_\psi \to \pi_{Z,\psi}$ to the composition

$$W_\psi \stackrel{m^{-1}}{\to} W_\psi \stackrel{T}{\to} \pi_{Z,\psi} \stackrel{m}{\to} \pi_{Z,\psi}$$

If π is irreducible, say that π is *minimal* iff

$$\mathrm{Hom}_{N(F)}(W_\psi, \pi_{Z,\psi}) \widetilde{\to} \bar{\mathbb{Q}}_\ell$$

as $M_{ss}(F)$-modules

Fact 4. ([1]) $P(F)$ admits a unique smooth representation, namely W, such that

$$W_{Z,\psi'} \widetilde{\to} \begin{cases} W_{\psi'}, & \text{if } \psi' \neq 1 \\ 0, & \text{if } \psi' = 1 \end{cases}$$

In particular, W is independent of ψ.

Remark 1. If G is of type D_n, E_n then W is irreducible as a $P(F)$-module ([5], Proposition 3). The dual of W is $ind_{P_{ss}(F)}^{P(F)} W_{\psi^{-1}}$.

Assume for a moment that $\bar{\mathbb{Q}}_\ell$ is replaced by $\mathbb{C}$, assume ψ of absolut value one. Pick a polarization $\mathfrak{g}(1) = \mathfrak{g}(1)^+ \oplus \mathfrak{g}(1)^-$. Then we can identify W_ψ with the Schwarz space $\mathcal{S}(\mathfrak{g}(1)^+)$ of locally constant functions with compact support. This yields a conjugate linear map $W_\psi \to W_{\psi^{-1}}$, $f \mapsto \bar{f}$ in this model, hence an hermitian product on W_ψ, because $W_\psi^* \widetilde{\to} W_{\psi^{-1}}$. In turn, this yields a $P(F)$-invariant hermitian product $f_1, f_2 \mapsto \int_{P_{ss}(F)\backslash P(F)} \{f_1(p), f_2(p)\} dp$, where $\{.,.\}$ is the hermitian product on W_ψ. One may show this is a unique (up to a scalar) nontrivial $P(F)$-invariant hermitian form on W.

Fact 5. 1) Under C1,C2, the unitary completion $\hat{W}$ of W admits at most one extension as a representation of $P(F)$ to that of $G(F)$.
2) If G is of type D_n, E_n then it does extend to a unitary irreducible representation of $G(F)$.

One can further take smooth vectors in $\hat{W}$ and get a smooth irreducible representation of $G(F)$, this is the minimal one for D_n, E_n. The suspace $W \subset \hat{W}$ is not stable under $G(F)$.

The minimal representations π_{min} of $G(F)$ all appear as degenerate principal series: there is a maximal parabolic $Q \subset G$ and a nonramified character $\chi : Q(F) \to \bar{\mathbb{Q}}_\ell^*$ such that π_{min} appears as a nonramified irreducible subquotient of $ind_{Q(F)}^{G(F)}\chi$ (and this induced representation is not irreducible). For D_n, E_n π_{min} is actually a representation of $G_{ad}(F)$.

Example: $G = \mathrm{S}\mathbb{O}_{2n}(F)$, $F = k((t))$. Let $P \subset G$ be the Siegel parabolic, its Levi quotient is $M = \mathrm{GL}_n$. Let $\alpha : P(F) \to M(F) \stackrel{det}{\to} F^*$ then π_{min} is the unique irreducible nonramified submodule in $ind_{P(F)}^{G(F)} \mid \alpha \mid$.

Remark 2. 1) An important **open problem** is to construct a geometric analog of degenerate principal series (as a category with an action of $G(F)$), geometry should make visible the action of SL_2 of Arthur.
2) If G is of type D_n, E_n then there is a unique simple coroot $\check{\beta}$ such that $\langle \check{\beta}, \alpha \rangle \neq 0$, here α is the highest root. Then W is irreducible already as a $\check{\beta}(F^*)N(F)$-module ([5]), here $\check{\beta} : \mathbb{G}_m \to T$ is this coroot. Also one has $M = \check{\beta}(F^*)M_{ss}$, and P is the Heisenberg parabolic is the standard maximal parabolic corresponding to β. Here β is the root corresponding to $\check{\beta}$.

1.2.3.2 Let H_G denote the Iwahori-Hecke algebra of $G(F)$. The irreducible representations of H_G corresponding to the minimal representations of $G(F)$ are constructed by Lusztig ([7]). In his paper there is a claim that looks like a local analog of the global Arthur's conjecture (Fact 6 below), namely that if $(s, u) \in G^L \times \mathrm{Lie}\, G^L$ with s semi-simple (and $sus^{-1} = q^{-1}u$) corresponds to a square-integrable representation of H_G then (s, u) is not centralized by any torus in G^L. (That is, u is not contained in a proper Levi of G^L).

1.2.4 Models of minimal representations Let G/F be simple, split, here $F = k((t))$. Let $Q \subset G$ be a maximal parabolic corresponding to a simple root τ. Write $Q = MN$, where M is the Levi, N is the unipotent radical. Assume N abelian (it suffices that τ appears in the highest root with multiplicity one). Examples:

$\mathfrak{g}$	D_n	D_n	E_6	E_7
M_{ss}	A_{n-1}	D_{n-1}	D_5	E_6

Let $\bar{N}$ be the opposite unipotent radical, $\bar{\mathfrak{n}} = \operatorname{Lie}\bar{N}$. There is a unique nonzero minimal $M_{ss}(F)$-orbit Ω_1 on $\bar{\mathfrak{n}}(F)$. The group $Q(F)$ acts on $C^\infty(\Omega_1)$ by

$$(nf)(y) = f(y)\psi(-\langle n, y\rangle),\ n \in N(F), y \in \Omega_1$$

$$(mf)(y) = \chi(m)f(m^{-1}ym),\ y \in \Omega_1, m \in M(F)$$

here $\langle .,. \rangle$ is the pairing induced by the Killing form, $\psi : F \to \bar{\mathbb{Q}}_\ell^*$ is an additive character, $\chi : M(F) \to (M/M_{ss})(F) \to \bar{\mathbb{Q}}_\ell^*$ is some unramified character.

Then π_{min} of $G(F)$ appears as a $Q(F)$-submodule in $C^\infty(\Omega_1)$, it strictly contains $C_c^\infty(\Omega_1)$. At the level of L^2, $\hat{\pi}_{min} = L^2(\Omega_1)$ for a suitable measure.

Examples. Let $G = \mathbb{SO}_{2n}$, Q denote Siegel parabolic, write U for the standard representation of the Levi $M = \mathrm{GL}_n$ then $\mathfrak{n} = \wedge^2 U$, $\bar{\mathfrak{n}} = \wedge^2 U^*$ and

$$\Omega_1 = \{y \in \wedge^2 U^*(F) \mid y : U(F) \to U^*(F) \text{is of rank}\, 2\}$$

If Q is the parabolic preserving a 1-dimensional isotropic subspace L in the standard representation, write L_{-1} for its orthogonal complement and $V' = L_{-1}/L$. Then $\mathfrak{n} = V' \otimes L$ and $\Omega_1 \subset V' \otimes L^*(F)$ is the cone of isotropic vectors.

1.2.5 Global case. Let X/k be a smooth projective geometrically irreducible curve, $k = \mathbb{F}_q$. Let $\mathbb{A}$ be the adeles of F, $\mathcal{O} \subset \mathbb{A}$ the entire adeles, G simple, simply-connected split of type D_n, $n \geq 4$ or E_n. Let $\pi_v = \pi_{v,min}$ for each place v. Then $\pi = \otimes'_v \pi_v$ is automorphic, square-integrable ([3]). This is proved via realization of π as a residual representation for some Eisenstein series. If G is of type D_n then π appears in $L^2(G(F)\backslash G(\mathbb{A}))$ discretely with multiplicity one. In addition to the proof of [3], there is a generalization by Moeglin, she proved the following particular case of Arthur conjectures.

Classification of min representation should be seen from the perspective of Arthur conjectures.

Fact 6. ([10]) Let G/k be split, one of the groups $\mathbb{SO}_{2n+1}, \mathbb{Sp}_{2n}, \mathbb{SO}_{2n}$. Let $\check{G}$ be the Langlands dual. There is a bijection

$$\left\{ \begin{array}{c} \text{irred. repr. } \pi = \otimes'_v \pi_v \\ \text{appearing discretely in } L^2(G(F)\backslash G(\mathbb{A})) \\ \text{with a } G(\mathcal{O}) - \text{fixed vector} \\ \text{and satisfying (C) below} \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{unipotent } \check{G} - \text{orbits in } \check{G}(\bar{\mathbb{Q}}_\ell) \\ \text{that do not intersect} \\ \text{any proper Levi} \end{array} \right\}$$

(C): representations appearing in $ind_{B(\mathbb{A})}^{G(\mathbb{A})}(\chi)$, where $\chi : T(\mathbb{A})/T(\mathcal{O}) \to \bar{\mathbb{Q}}_\ell^*$ is a character that decomposes as $T(\mathbb{A})/T(\mathcal{O}) \widetilde{\to} \mathrm{Div}(X) \otimes \Lambda \stackrel{\deg \otimes \mathrm{id}}{\to} \Lambda \to \bar{\mathbb{Q}}_\ell^*$. Let $\phi : \mathrm{SL}_2 \to \check{G}$ correspond to a given unipotent orbit, let π be the corresponding subrepresentation in $L^2(G(F)\backslash G(\mathbb{A}))$ then for any place v, π_v is spherical with Langlands parameter

$$\phi\begin{pmatrix} \mid t_x \mid^{1/2} & 0 \\ 0 & \mid t_x \mid^{-1/2} \end{pmatrix},$$

here $t_x \in F_x$ is a uniformizer.

If G is split of type D_n, E_n, B_n then the subregular nilpotent orbit in $\check{G}$ does not intersect any proper Levi. The corresponding representation of D_n in the sense of Arthur conjectures is minimal (for E_n it should also be so, I don't know if this Arthur conjecture is proved for E_n). However, for B_n the representation corresponding to the subregular nilpotent orbit in $\check{G}$ is not minimal! This is related with the fact that the minimal nilpotent orbit in $\mathrm{S}\mathbb{O}_{2n+1}$ is not special (in the sense of Luzstig-Spaltenstein [15]). The subregular nilpotent orbit in $\mathbb{S}\mathrm{p}_{2n}$ is given by the partition $(2n-2, 2)$.

Fact 7. ([11]) If G is of type B, C, D then any element of $WF(\pi)$ is a special nilpotent orbit.

(This result of Moeglin does not exclude the possibility that a metaplectic covering group of $G(F)$ admits a minimal representations.) One should also compare this with the notion of an *admissible nilpotent coadjoint orbit* for p-adic group introduced in [13].

Purpose Find a geometric analog of the minimal representation of $\mathrm{S}\mathbb{O}_{2n}(\mathbb{A})$.

1.2.7 Explicit formulas Let $H = \mathrm{S}\mathbb{O}_{2n}$ split, let $P \subset H$ be the Siegel parabolic. Write $P = MN$, where $M = \mathrm{GL}_n$ is the Levi, and N is the unipotent radical. Rewriting the formulas of ([3], [4]) geometrically, one gets the following. For $s \in \mathbb{C}$ let $\mathbb{E}_{P,s} : \mathrm{Bun}_H \to \bar{\mathbb{Q}}_\ell$ be the function

$$V \mapsto \sum_{U \subset V} q^{s \deg U},$$

the sum being taken over all rank n isotropic subbundles subbundles U of V. This function is absolutely convergent for Res large enough and extends to a meromorphic function to $\mathbb{C}$. Its center of symmetry of the functional equation is $\frac{n-1}{2}$. It has simple poles at $s = 0, 1, n-1, n-2$ (and maybe at other points?). The residual representation at $s = 0, n-1$ is trivial 1-dimensional, and the residual representation at $1, n-2$ is the minimal one (it is self-dual). So, the unique up to a multiple minimal Hecke eigenform on Bun_H is

$$V \mapsto \operatorname*{Res}_{s=n-2} \mathbb{E}_{P,s}(V) ds$$

Let Bun_P^d be the connected component given by $\deg U = d$. Write $\nu^d : \mathrm{Bun}_P^d \to \mathrm{Bun}_H$ for the projection, set $R^d = (\nu^d)_!\bar{\mathbb{Q}}_\ell$. Then

$$\mathbb{E}_{P,s} = \sum_{d\in\mathbb{Z}} q^{sd}\,\mathrm{tr}(\mathrm{Fr}_V, (R^d)_V) \tag{1}$$

Note that $\dim.\mathrm{rel}(\nu^d) = const - (n-1)d$. Roughly, by Riemann-Roch, the top cohomology of $(R^d)_V$ is $\bar{\mathbb{Q}}_\ell[2const + 2(n-1)d](const + (n-1)d]$. This explains the pole at $n-1$, and makes essentially shows that the corresponding residual representation is trivial.

One more remark, one may formally consider the complex $\mathcal{R} = \sum_{d\in\mathbb{Z}} R^d[-2d]$, and similarly $\sum_{d\in\mathbb{Z}} R^d[-2(n-2)d]$. They essentially satisfy the Hecke property corresponding to a subregular unipotent orbit in $\check{H} = \mathbb{SO}_{2n}$. This means that for the standard representation W of $\check{H}$, one has $\mathrm{H}_H^{\leftarrow}(W, \mathcal{R}) \widetilde{\to} \mathcal{E} \otimes \mathcal{R}[1]$, where

$$\mathcal{E} = (\oplus_{i=-1}^{1}\bar{\mathbb{Q}}_\ell[2i]) \oplus (\oplus_{i=2-n}^{n-2}\bar{\mathbb{Q}}_\ell[2i])$$

(actually, one has to stratify the Hecke stack a little for this to be true, so more precisely, this is true in a suitable Grothendieck group (maybe completed?)).

One has a similar picture for the parabolic $Q \subset H$ preserving a 1-dimensional isotropic subspace in the standard representation. The stack Bun_Q classifies $L \in \mathrm{Bun}_1, V' \in \mathrm{Bun}_{SO_{2n-2}}$ and an exact sequence $0 \to L \to ? \to V' \to 0$ on X. Write Bun_Q^d for the substack given by $\deg L = d$. Let $\nu_Q^d : \mathrm{Bun}_Q^d \to \mathrm{Bun}_H$ be the projection, set $R_Q^d = (\nu_Q^d)_!\bar{\mathbb{Q}}_\ell$. Let $\mathbb{E}_{Q,s} : \mathrm{Bun}_H \to \bar{\mathbb{Q}}_\ell$ be given by

$$V \mapsto \sum_{\nu_Q(\mathcal{F}_Q)=V} q^{s\deg L} \tag{2}$$

where the Q-torsor $\mathcal{F}_Q$ on X is given by the above exact sequence. Here the symmetry point of the functional equation is $n-1$. It has simple poles at $s = 0, n-2, n, 2n-2$. The poles at $0, 2n-2$ give trivial residual representation, and those at $n-2, n$ give the minimal one. One may also define the complexes $\mathcal{R}$ as above and check their Hecke property. Again, the relative dimension $\dim.\mathrm{rel}(\nu_Q^d) = const - (2n-2)d$, this explains the pole at $2n-2$.

2. **Reminders**

2.1 Local Howe correspondence Let $F = k((t))$, $k = \mathbb{F}_q$. Consider the Weil representation $\mathcal{S}_\psi$ of $Mp(V)(F)$. Let $H_1, H_2 \subset \mathbb{S}\mathrm{p}(V)$ be a dual pair (split reductive connected groups, centralizers of each other). The only example we need is $(\mathbb{SO}_{2m}, \mathbb{S}\mathrm{p}_{2n})$ in $\mathbb{S}\mathrm{p}_{4nm}$. In this case the metaplectic cover splits over $(H_1 \times H_2)(F)$.

Let $R_\psi(H_i) = \{\pi \text{ irred. repr. of } H_i(F) \mid \mathrm{Hom}_{H_i}(\mathcal{S}_\psi, \pi) \neq 0\}$. Given $\pi_1 \in R_\psi(H_1)$ there is a biggest quotient $\mathcal{S}_\psi \to \pi_1 \otimes \Theta(\pi_1)$, where $\Theta(\pi_1)$ is a representation of $H_2(F)$. Moreover, $\Theta(\pi_1)$ is of finite length and admits a unique irreducible quotient

$\Theta(\pi_1) \to \theta(\pi_1)$. Local Howe correspondence sends π_1 to $\theta(\pi_1)$. It gives (at least, if the residual characteristic is not 2) a bijection $R_\psi(H_1)\widetilde{\to}R_\psi(H_2)$.

If π_1 is the trivial representation of $\mathrm{SL}_2(F)$ then for the dual pair $(\mathrm{SL}_2, \mathrm{S}\mathbb{O}_{2n})$ the representation $\Theta(\pi_1) = \theta(\pi_1)$ is the minimal representation. That is, $\theta(\pi_1)$ are the $\mathrm{SL}_2(F)$-coinvariants in $\mathcal{S}_\psi$.

2.2 Theta-sheaf The global analog of the Howe correspondence is the theta-lifting. I recall it in the geometric setting.

Now k is algebraically closed of characteristic $p > 2$, we use ℓ-adic sheaves (for $char k = 0$ there is a version with D-modules).

Let X/k be a smooth projective connected curve. Let $n \geq 1$, let G_n be the sheaf of automorphisms of $\mathcal{O}_X^n \oplus \Omega^n$ preserving a symplectic form $\wedge^2(\mathcal{O}_X^n \oplus \Omega^n) \to \Omega$. Here Ω is the canonical line bundle on X.

Bun_{G_n} = stack of G_n-torsors on X = stack classifying $M \in \mathrm{Bun}_{2n}$ with a symplectic form $\wedge^2 M \to \Omega$. Let $\mathcal{A}_{G_n}$ be the line bundle on Bun_{G_n} with fibre $\det \mathrm{R}\Gamma(X, M)$ at M. Write $\widetilde{\mathrm{Bun}}_{G_n}$ for the gerb of square roots of $\mathcal{A}_{G_n}$ over Bun_{G_n}.

Let ${}_i\mathrm{Bun}_{G_n} \subset \mathrm{Bun}_{G_n}$ be the locally closed substack given by $\dim \mathrm{H}^0(X, M) = i$. One has a line bundle ${}_i\mathcal{B}$ on it with fibre $\det \mathrm{H}^0(X, M)$. Then ${}_i\mathcal{B}^2 \widetilde{\to} \mathcal{A}_{G_n} \mid_{{}_i\mathrm{Bun}_{G_n}}$, hence a section $\rho : {}_i\mathrm{Bun}_{G_n} \to {}_i\widetilde{\mathrm{Bun}}_{G_n}$.

Let ${}_i\mathrm{Aut} = \mathrm{Hom}_{S_2}(sign, \rho_!\bar{\mathbb{Q}}_\ell)$, a local system of rank one, order two.

Definition 3. Aut_g is the intermediate extension of ${}_0\mathrm{Aut}[\dim \mathrm{Bun}_{G_n}]$, Aut_s is the intermediate extension of ${}_1\mathrm{Aut}[\dim \mathrm{Bun}_{G_n} - 1]$ to $\widetilde{\mathrm{Bun}}_{G_n}$. The theta-sheaf is $\mathrm{Aut} = \mathrm{Aut}_g \oplus \mathrm{Aut}_s$.

This is a Hecke eigen-sheaf on $\widetilde{\mathrm{Bun}}_{G_n}$. Let $P_n \subset G_n$ be the Siegel parabolic preserving $\mathcal{O}_X^n$. Let $\nu : \mathrm{Bun}_{P_n} \to \mathrm{Bun}_{G_n}$ send $(L \subset M)$ to $(M, \mathcal{B})$, where $\mathcal{B} = \det \mathrm{R}\Gamma(X, L^* \otimes \Omega)$ is equipped with $\mathcal{B}^2 \widetilde{\to} \det \mathrm{R}\Gamma(X, M)$ coming from the exact sequence $0 \to L \to M \to L^* \otimes \Omega \to 0$.

Explicit formula for $\nu^* \mathrm{Aut}$. Let $\mathcal{V} = \{L \in \mathrm{Bun}_n, s : L \to \Omega\}$. Let $\mathcal{V}_2 = \{L \in \mathrm{Bun}_n, t : \mathrm{Sym}^2 L \to \Omega^2\}$. Let $\pi : \mathcal{V} \to \mathcal{V}_2$ over Bun_n send s to $s \otimes s$. This is a finite morphism. The stack Bun_{P_n} classifies: $L \in \mathrm{Bun}_n$ and an exact sequence $0 \to \mathrm{Sym}^2 L \to ? \to \Omega \to 0$. So, we have a diagram of dual generalized vector bundles

$$\mathcal{V}_2 \to \mathrm{Bun}_n \leftarrow \mathrm{Bun}_{P_n}$$

Let $\psi : \mathbb{F}_p \to \bar{\mathbb{Q}}_\ell^*$ be a nontrivial additive character. Let $S_{P,\psi} = \mathrm{Four}_\psi(\pi_!\bar{\mathbb{Q}}_\ell[\dim \mathcal{V}])$.

Fact 8. ([8, 9]) There is an isomorphism $\nu^* \mathrm{Aut}[\dim.\mathrm{rel}(\nu)] \widetilde{\to} S_{P,\psi}$

2.3 Geometric theta-lifting Let $n, m \geq 1$, let $G = G_n$, $H = \mathrm{S}\mathbb{O}_{2m}$ split. The stack Bun_H classifies $V \in \mathrm{Bun}_{2m}$ with a nondegenerate symmetric form $\mathrm{Sym}^2 V \to \mathcal{O}$ and a compatible trivialization $\det V \widetilde{\to} \mathcal{O}$. Let $\mathcal{A}_H$ be the line bundle with fibre $\det \mathrm{R}\Gamma(X, V)$ at V. Write $\mathrm{Bun}_{G,H} = \mathrm{Bun}_G \times \mathrm{Bun}_H$.

Let $\tau : \mathrm{Bun}_{G,H} \to \mathrm{Bun}_{G_{2nm}}$ send (M, V) to $M \otimes V$ with the natural symplectic form. One has canonically $\tau^* \mathcal{A}_{G_{2nm}} \widetilde{\to} \mathcal{A}_H^{2n} \otimes \mathcal{A}_{G_n}^{2m} \otimes \det \mathrm{R}\Gamma(X, \mathcal{O})^{-4nm}$. Extend τ to a map

$$\tilde{\tau} : \mathrm{Bun}_{G,H} \to \widetilde{\mathrm{Bun}}_{G_{2nm}}$$

sending (M, V) to $(M, V, \mathcal{B})$, where

$$\mathcal{B} = \det \mathrm{R}\Gamma(X, V)^n \otimes \det \mathrm{R}\Gamma(X, M)^m \otimes \det \mathrm{R}\Gamma(X, \mathcal{O})^{-2nm}$$

equipped with $\mathcal{B}^2 \widetilde{\to} \det \mathrm{R}\Gamma(X, M \otimes V)$. Set

$$\mathrm{Aut}_{G,H} = \tilde{\tau}^* \mathrm{Aut}[\dim.\mathrm{rel}(\tau)] \in \mathrm{D}^{\prec}(\mathrm{Bun}_{G,H})$$

Notation: If S is a stack locally of finite type over k, $\mathrm{D}(S)$ is the unbounded derived category of etale $\bar{\mathbb{Q}}_\ell$-sheaves on S, $\mathrm{D}^{\prec}(S) \subset \mathrm{D}(S)$ is the full subcategory of objects K such that for any open $U \subset S$ of finite type, $K\mid_U \in D^-(U)$. Further, $\mathrm{D}^-(S)_! \subset \mathrm{D}^-(S)$ is the full subcategory of objects that are extension by zero from an open substack of finite type.

Remark 3. Hecke functors act on $\mathrm{D}^{\prec}(\mathrm{Bun}_G)$, one may also ask about its spectral decomposition.

Consider the diagram of projections $\mathrm{Bun}_H \stackrel{\mathfrak{q}}{\leftarrow} \mathrm{Bun}_{G,H} \stackrel{\mathfrak{p}}{\to} \mathrm{Bun}_G$.

Definition 4. Define functors $F_G : \mathrm{D}^-(\mathrm{Bun}_H)_! \to \mathrm{D}^{\prec}(\mathrm{Bun}_G)$ and $F_H : D^-(\mathrm{Bun}_G)_! \to \mathrm{D}^{\prec}(\mathrm{Bun}_H)$ by

$$F_G(K) = \mathfrak{p}_!(\mathrm{Aut}_{G,H} \otimes \mathfrak{q}^* K)[-\dim \mathrm{Bun}_H] \quad F_H(K) = \mathfrak{q}_!(\mathrm{Aut}_{G,H} \otimes \mathfrak{p}^* K)[-\dim \mathrm{Bun}_G]$$

Remind that $\check{G} \widetilde{\to} \mathbb{SO}_{2n+1}$, $\check{H} \widetilde{\to} \mathbb{SO}_{2m}$ over $\bar{\mathbb{Q}}_\ell$.

1) For $m \le n$ define κ as the composition $\check{H} \times \mathbb{G}_m \stackrel{\mathrm{id} \times 2\rho}{\to} \check{H} \times \mathbb{SO}_{2n-2m+1} \to \check{G}$, where the second map is given by the direct sum of representations. We used the fact that the restriction of the irreducible representation $\mathrm{SL}_2 \to \mathbb{SO}_{2n-2m+1}$ to $\mathbb{G}_m \subset \mathrm{SL}_2$ is given by 2ρ. Then F_G commutes with Hecke functors with respect to κ.

2) For $m > n$ define κ as the composition $\check{G} \times \mathbb{G}_m \stackrel{\mathrm{id} \times \rho}{\to} \check{G} \times \mathbb{SO}_{2m-2n-1} \to \check{H}$, where the second map is given by the direct sum. Then F_H commutes with Hecke functors with respect to κ.

We extended here the usual definition of the Hecke functors to

$$\mathrm{H}^{\leftarrow}_G : \mathrm{Rep}(\check{G} \times \mathbb{G}_m) \times \mathrm{D}(\mathrm{Bun}_G) \to \mathrm{D}(\mathrm{Bun}_G)$$

via adding shifts. If St is the standard representation of $\mathbb{G}_m$, $\mathcal{S} \in \mathrm{Rep}(\check{G})$ then $\mathrm{H}^{\leftarrow}_G(\mathcal{S} \otimes St^r, \cdot) = \mathrm{H}^{\leftarrow}_G(\mathcal{S}, \cdot)[r]$.

Actually theta-lifting operators (classically and geometrically) are defined on a bigger category then $\mathrm{D}^-(\cdot)_!$. In 1965 paper about Siegel-Weil formula A. Weil has proved a convergence criterium for theta-series, that actually geometrizes well.

Proposition 1. *1) If $m > n+1$ then $F_G(\bar{\mathbb{Q}}_\ell) \in \mathrm{D}^{\prec}(\mathrm{Bun}_G)$ is absolutely convergent. 2) If $n = 1$ then $F_H(\bar{\mathbb{Q}}_\ell)$ is absolutely convergent iff $m = 1$.*

Here 'absolutely convergent' for 1) means the following: for any open substack of finite type $U \subset \mathrm{Bun}_G$ and any $N \in \mathbb{Z}$ the contribution of all but finite number of Shatz strata of Bun_H to the complex $\mathfrak{p}_! \mathrm{Aut}_{G,H} \mid_U$ lives in $\mathrm{D}^{\leq N}(U)$, the usual t-structure. So, if we are interested only in $\mathrm{H}^N(\mathfrak{p}_! \mathrm{Aut}_{G,H} \mid_U)$ then only a finite number of Shatz strata contribute. Proof: calculation of dimensions.

3. **Trying to geometrize the minimal representation for** $\mathbb{SO}_{2n}$

3.1 We are interested in the case $n = 1$, $m \geq 4$, in which there is no absolut convergence of $F_H(\bar{\mathbb{Q}}_\ell)$. If $F_H(\bar{\mathbb{Q}}_\ell)$ was absolutely convergent, it would be the Hecke eigensheaf with the Hecke eigenvalue corresponding to the subregular nilpotent orbit. Indeed, the subregular SL_2-triple is $\mathrm{SL}_2 \stackrel{diag}{\to} \mathrm{SL}_2 \times \mathrm{SL}_2 \to \mathbb{SO}_3 \times \mathbb{SO}_{2m-3} \to \mathbb{SO}_{2m}$, the last map is the direct sum, and the previous one is the product of regular nilpotent SL_2-triples. Let $q : \mathrm{Bun}_{G_1,H} \to \mathrm{Bun}_H$ be the projection. [1]

Still we will analyse the complex $F_H(\bar{\mathbb{Q}}_\ell)$ and try to "regularize" it with a hope to get the corresponding automorphic sheaf.

Classically there are 2 approaches to such "regularization":

- for number fields, use an archimedian place and apply a Kazimir operator at this place to make the series absolutely convergent;
- replace $\bar{\mathbb{Q}}_\ell$ on Bun_{G_1} by the Eisenstein series $\mathrm{Eis}_B^{G_1}(\bar{\mathbb{Q}}_\ell)$, its residue is $\bar{\mathbb{Q}}_\ell$, so calculate $F_H(\mathrm{Eis}_B^{G_1}(\bar{\mathbb{Q}}_\ell))$ first and then pass to the residue. This essentially amount to getting the minimal representation of $H(\mathbb{A})$ as the residual one out of $\mathrm{Eis}_Q^H(\bar{\mathbb{Q}}_\ell)$. Here $Q \subset H$ is the parabolic preserving a 1-dimensional isotropic subspace in the standard representation.

Neither of them is evident to geometrize.

From now on $n \geq 4$ and $H = \mathbb{SO}_{2n}$ split. We will present a conjectural construction of a perverse sheaf $\mathcal{K}$ irreducible on each connected component of Bun_H that is a kind of regularization of $F_H(\bar{\mathbb{Q}}_\ell)$. We expect that the automorphic sheaf corresponding to the subregular nilpotent orbit in $\check{H}$, should be a combination of $\mathcal{K}$ eventually with the constant sheaf. We conjecture that the full triangulated subcategory of $\mathrm{D}^{\prec}(\mathrm{Bun}_H)$ generated by $\mathcal{K}$ and $\bar{\mathbb{Q}}_\ell$ is stable under the Hecke functors, and $\mathcal{K}$ becomes a Hecke eigensheaf in the quotient category $\mathrm{D}^{\prec}(\mathrm{Bun}_H)/\mathrm{D}_0$. Here $\mathrm{D}_0 \subset \mathrm{D}^{\prec}(\mathrm{Bun}_H)$ is the full triangulated subcategory generated by $\bar{\mathbb{Q}}_\ell$.

There will be 3 different constructions of $\mathcal{K}$. But let us first explain how we have guessed the first construction.

[1] It may happen that $\mathrm{Aut}_{G_1,H}$ is a compact object of the category $\mathrm{D}^{\prec}(\mathrm{Bun}_{G_1,H})$, we don't know if this is true, this would imply that $q_! \mathrm{Aut}_{G_1,H}$ can be defined, this is another version of convergence.

Let $P \subset H$ be the Siegel parabolic. Recall that Bun_P classifies $U \in \mathrm{Bun}_n$ and an exact sequence

$$0 \to \wedge^2 U \to ? \to \mathcal{O}_X \to 0$$

Write $\mathcal{Y}_P = \{U \in \mathrm{Bun}_n,\ v : \wedge^2 U \to \Omega\}$. So, $\mathcal{Y}_P$ and Bun_P are dual generalized vector bundles over Bun_n.

The functor $\mathrm{D}^-(\mathrm{Bun}_{G_1})_! \to \mathrm{D}^{\prec}(\mathcal{Y}_P)$ given by $K \mapsto \mathrm{Four}_\psi\, \nu_P^* F_H(K)[\dim.\mathrm{rel}(\nu_P)]$ is easier to understand then F_H.

Namely, let $\mathcal{S}_P$ be the stack classifying $U \in \mathrm{Bun}_n, M \in \mathrm{Bun}_{G_1}$ and a section $s : U \to M$. Let $\pi : \mathcal{S}_P \to \mathcal{Y}_P$ be the map sending (U, M, s) to $(U, \wedge^2 s)$. Let $\mathrm{pr}_{G_1} : \mathcal{S}_P \to \mathrm{Bun}_{G_1}$ send (U, M, s) to M. Then

$$\mathrm{Four}_\psi\, \nu_P^* F_H(K) \widetilde{\to} \pi_!\, \mathrm{pr}_{G_1}^* K$$

up to a shift. The complex $\pi_! \bar{\mathbb{Q}}_\ell$ is 'bad' over the whole of $\mathcal{Y}_P$ but it becomes much better over the open substack $\overset{\circ}{\mathcal{Y}}_P \subset \mathcal{Y}_P$ given by $v \neq 0$.

Let $\mathcal{Z}_P \subset \mathcal{Y}_P$ be the closed substack classifying (U, v) such that the generic rank of $v : U \to U^* \otimes \Omega$ is at most 2. Then π factors as $\mathcal{S}_P \to \mathcal{Z}_P \hookrightarrow \mathcal{Y}_P$.

Let $\overset{\circ}{\mathcal{S}}_P$ and $\overset{\circ}{\mathcal{Z}}_P$ be the preimages of $\overset{\circ}{\mathcal{Y}}_P$ under π.

For $d \geq 0$ let $X^{(d)}$ denote the d-th symmetric power of X. Stratify $\overset{\circ}{\mathcal{Z}}_P$ by locally closed substacks $\mathcal{Z}_{P,m}$ indexed by $m \geq 0$. Here $\mathcal{Z}_{P,m}$ is the stack classifying $D \in X^{(m)}$, $U \in \mathrm{Bun}_n$, $M' \in \mathrm{Bun}_2$ with an isomorphism $\det M' \widetilde{\to} \Omega(-D)$, and a surjection $s : U \to M'$ of $\mathcal{O}_X$-modules.

The restriction $\pi : \overset{\circ}{\mathcal{S}}_P \to \overset{\circ}{\mathcal{Z}}_P$ of π is representable and proper, this is an isomorphism over $\mathcal{Z}_{P,0}$. The stack $\overset{\circ}{\mathcal{S}}_P$ is smooth.

Proposition 2. *1) If $n \geq 4$ then $\pi : \overset{\circ}{\mathcal{S}}_P \to \overset{\circ}{\mathcal{Z}}_P$ is small, so*

$$\pi_!\, \mathrm{IC}(\overset{\circ}{\mathcal{S}}_P) \widetilde{\to} \mathrm{IC}(\overset{\circ}{\mathcal{Z}}_P)$$

2) If $n = 3$ then this map is semi-small, and $\oplus_{m \geq 0} \mathrm{IC}(\mathcal{Z}_{P,m})$ is a direct summand of $\pi\, \mathrm{IC}(\overset{\circ}{\mathcal{S}}_P)$.

Proof The stack $\mathcal{Z}_{P,m}$ is smooth for any m. Its connected component $\mathcal{U}$ containing a given point (U, M, s, D) is of dimension $m(1-n) - 2\deg U + (n^2+3)(g-1)$. So, the codimension of $\mathcal{U}$ in the corresponding connected component of $\overset{\circ}{\mathcal{Z}}_P$ equals $m(n-1)$. The fibre of $\pi : \overset{\circ}{\mathcal{S}}_P \to \overset{\circ}{\mathcal{Z}}_P$ over (U, M, s, D) is the scheme of upper modifications $M' \subset M$ such that $\mathrm{div}(M/M') = D$. This fibre is connected of dimension m. $\square$

Our idea was to replace $\pi_! \bar{\mathbb{Q}}_\ell$ over $\mathcal{Y}_P$ by $\mathrm{IC}(\mathcal{Z}_P)$.

3.2 P-MODEL. Let ${}^e\mathrm{Bun}_n \subset \mathrm{Bun}_n$ be the open substack given by two conditions $\mathrm{H}^0(X, \wedge^2 U) = 0$ and $\mathrm{H}^0(X, \Omega \otimes \wedge^2 U) = 0$ for $U \in \mathrm{Bun}_n$. Let ${}^e\mathcal{S}_P$, ${}^e\mathrm{Bun}_P$, ${}^e\mathcal{Y}_P$ and so on be the preimages of ${}^e\operatorname{Bun}_n$ in the corresponding stacks. Write $\nu_P : \operatorname{Bun}_P \to \operatorname{Bun}_H$ for the natural map. Its restriction ${}^e\operatorname{Bun}_P \to \operatorname{Bun}_H$ is smooth.

Let

$$K_{P,\psi} = \operatorname{Four}_\psi(\operatorname{IC}(\mathcal{Z}_P)) \in \mathrm{D}^{\prec}(\operatorname{Bun}_P),$$

the Fourier transfrom here is the functor $\mathrm{D}^{\prec}(\mathcal{Y}_P) \to \mathrm{D}^{\prec}(\operatorname{Bun}_P)$.

We conjecture that for $n \geq 4$ there is a (defined up to a unique isomorphism) perverse sheaf $\mathcal{K}$ on Bun_H irreducible on each connected component such that the Properties 1,2,3 below hold.

Property 1. There is an isomorphism over ${}^e\operatorname{Bun}_P$

$$\nu_P^*\mathcal{K}[\dim.\operatorname{rel}(\nu_P)] \widetilde{\to} K_{P,\psi}$$

Remark 4. The definition of $K_{P,\psi}$ extends (by 'the same formula') to the stack $\operatorname{Bun}_{\bar{P}}$, where $\bar{P}$ is the Siegel parabolic in $\mathrm{P}\mathbb{SO}_{2n}$, and $\mathcal{K}$ should be a perverse sheaf on $\operatorname{Bun}_{\mathrm{P}\mathbb{SO}_{2n}}$, $n \geq 4$.

3.3 Q-model. Let $Q \subset H$ be a parabolic preserving a 1-dimensional isotropic subspace in the standard representation of H. The stack Bun_Q classifies $W \in \operatorname{Bun}_1, V' \in \operatorname{Bun}_{\mathbb{SO}_{2n-2}}$ and an exact sequence $0 \to W \to ? \to V' \to 0$ on X.

Let $\mathcal{Y}_Q$ be the stack classifying $W \in \operatorname{Bun}_1, V' \in \operatorname{Bun}_{\mathbb{SO}_{2n-2}}$ and $t : W \to V' \otimes \Omega$. So, $\mathcal{Y}_Q$ and Bun_Q are generalized vector bundles over $\operatorname{Bun}_1 \times \operatorname{Bun}_{\mathbb{SO}_{2n-2}}$.

Let $\mathcal{Z}_Q \subset \mathcal{Y}_Q$ be the closed substack given by the condition that the composition

$$W^{\otimes 2} \stackrel{t\otimes t}{\to} \operatorname{Sym}^2(V' \otimes \Omega) \to \Omega^2$$

vanishes, that is, the image of t is isotropic.

Set

$$K_{Q,\psi} = \operatorname{Four}_\psi(\operatorname{IC}(\mathcal{Z}_Q)) \in \mathrm{D}^{\prec}(\operatorname{Bun}_Q)$$

Let ${}^u(\operatorname{Bun}_1 \times \operatorname{Bun}_{\mathbb{SO}_{2n-2}}) \subset \operatorname{Bun}_1 \times \operatorname{Bun}_{\mathbb{SO}_{2n-2}}$ be the open substack given by $\mathrm{H}^0(X, V' \otimes W) = 0$ and $\mathrm{H}^0(X, V' \otimes W \otimes \Omega) = 0$ for $W \in \operatorname{Bun}_1, V' \in \operatorname{Bun}_{\mathbb{SO}_{2n-2}}$.

Write ${}^u\mathcal{Y}_Q$, ${}^u\operatorname{Bun}_Q$, ${}^u\mathcal{Z}_Q$ and so on for the preimages of ${}^u(\operatorname{Bun}_1 \times \operatorname{Bun}_{\mathbb{SO}_{2n-2}})$ in the corresponding stack. Let $\nu_Q : \operatorname{Bun}_Q \to \operatorname{Bun}_H$ be the natural map.

Property 2. There is an isomorphism over ${}^u\operatorname{Bun}_Q$

$$\nu_Q^*\mathcal{K}[\dim.\operatorname{rel}(\nu_Q)] \widetilde{\to} K_{Q,\psi}$$

Of coarse, these two models reflect the local models described in 1.2.4.

3.4 In order to formulate the R-model for the Heisenberg parabolic we need first to introduce the extended theta-sheaf.

This is something we have found ourself, we are not aware of its classical analog.

3.5 Extended theta-sheaf Let now $n > 0$, remind that G_n is the group scheme of automorphisms of $M_0 = \mathcal{O}_X^n \oplus \Omega^n$ preserving the symplectic form $\wedge^2(M_0) \to \Omega$.

Let $H_n = M_0 \oplus \Omega$ be the corresponding Heisenberg group scheme on X with operation

$$(m_1, \omega_1)(m_2, \omega_2) = (m_1 + m_2, \omega_1 + \omega_1 + \frac{1}{2}\langle m_1, m_2\rangle)$$

The group scheme G_n acts on H_n by group automorphisms, write $\mathbb{G}_n = G_n \ltimes H_n$ for the corresponding semi-direct product.

The stack $\operatorname{Bun}_{\mathbb{G}_n}$ classifies $M_1 \in \operatorname{Bun}_{2n+2}$ with symplectic form $\wedge^2 M_1 \to \Omega$ and a section $v : \Omega \hookrightarrow M_1$ whose image is a subbundle. For a point of $\operatorname{Bun}_{\mathbb{G}_n}$ write L_{-1} for the orthogonal complement to Ω, so $M = L_{-1}/\Omega \in \operatorname{Bun}_{G_n}$. Set $\widetilde{\operatorname{Bun}}_{\mathbb{G}_n} = \widetilde{\operatorname{Bun}}_{G_n} \times_{\operatorname{Bun}_{G_n}} \operatorname{Bun}_{\mathbb{G}_n}$.

Let ${}_0\operatorname{Bun}_{\mathbb{G}_n} \subset \operatorname{Bun}_{\mathbb{G}_n}$ be the open substack given by $\mathrm{H}^0(X, M) = 0$, define ${}_0\widetilde{\operatorname{Bun}}_{G_n}$, ${}_0\widetilde{\operatorname{Bun}}_{\mathbb{G}_n}$ similarly. Write $\operatorname{Bun}_\Omega$ for the stack classifying exact sequences

$$0 \to \Omega \to ? \to \mathcal{O} \to 0 \tag{3}$$

on X. Let $ev_\Omega : \operatorname{Bun}_\Omega \to \mathbb{A}^1$ be the map sending (3) to the corresponding element of $\mathrm{H}^1(X, \Omega)$. We have canonically

$${}_0\widetilde{\operatorname{Bun}}_{\mathbb{G}_n} \widetilde{\to} {}_0\widetilde{\operatorname{Bun}}_{G_n} \times \operatorname{Bun}_\Omega \tag{4}$$

Definition 5. Write $\operatorname{Aut}^e_\psi$ for the intermediate extension of

$$(\operatorname{Aut} \boxtimes ev_\Omega^* \mathcal{L}_\psi)[1-g]$$

under the open immersion ${}_0\widetilde{\operatorname{Bun}}_{\mathbb{G}_n} \hookrightarrow \widetilde{\operatorname{Bun}}_{\mathbb{G}_n}$. Here e stands for 'extended', we call $\operatorname{Aut}^e_\psi$ *the extended theta-sheaf.*

Now we will present an explicit formula for $\operatorname{Aut}^e_\psi$ in the Schrödinger model, which generalizes the corresponding formula for Aut.

Remind that $P_n \subset G_n$ is the parabolic group subscheme preserving $\mathcal{O}^n$.

Set $\mathbb{P}_n = P_n \ltimes H_n$. The stack $\operatorname{Bun}_{\mathbb{P}_n}$ classifies $\mathcal{L} \in \operatorname{Bun}_n$ included into an exact sequence on X

$$0 \to \Omega \to \bar{\mathcal{L}} \to \mathcal{L} \to 0 \tag{5}$$

and an exact sequence on X

$$0 \to \operatorname{Sym}^2 \bar{\mathcal{L}} \to ? \to \Omega \to 0 \tag{6}$$

Let $\mathcal{T}_n$ be the stack classifying $\mathcal{L} \in \operatorname{Bun}_n$ and an exact sequence (5) on X. Let $\mathcal{Z}_{\mathcal{T}_n}$ be the stack classifying a point of $\mathcal{T}_n$ and a splitting $s : \bar{\mathcal{L}} \to \Omega$ of (5). Of coarse, $\mathcal{Z}_{\mathcal{T}_n}$ identifies with Bun_n.

Write $\mathcal{Z}_{2,\mathcal{T}_n}$ for the stack classifying a point of $\mathcal{T}_n$ as above and a section $\bar{s}: \operatorname{Sym}^2 \bar{\mathcal{L}} \to \Omega^2$. The map $h_{\mathcal{T}} : \mathcal{Z}_{\mathcal{T}_n} \to \mathcal{Z}_{2,\mathcal{T}_n}$ over $\mathcal{T}_n$ given by $\bar{s} = s \otimes s$ is a closed immersion. One has a diagram of dual generalized vector bundles over $\mathcal{T}_n$

$$\mathcal{Z}_{2,\mathcal{T}_n} \to \mathcal{T}_n \leftarrow \operatorname{Bun}_{\mathbb{P}_n}$$

Set

$$K_{\mathbb{P}_n,\psi} = \operatorname{Four}_\psi h_{\mathcal{T}!} \bar{\mathbb{Q}}_\ell[\dim \operatorname{Bun}_n]$$

We lift the natural map $\operatorname{Bun}_{\mathbb{P}_n} \to \operatorname{Bun}_{\mathbb{G}_n}$ to a morphism $\tilde{\nu}_{\mathbb{P}} : \operatorname{Bun}_{\mathbb{P}_n} \to \widetilde{\operatorname{Bun}}_{\mathbb{G}_n}$ sending the above point of $\operatorname{Bun}_{\mathbb{P}_n}$ to $(\Omega \hookrightarrow M_1, \mathcal{B})$. Here $\mathcal{B} = \det \mathrm{R}\Gamma(X, \mathcal{L}^* \otimes \Omega)$ is equipped with

$$\mathcal{B}^2 \widetilde{\to} \det \mathrm{R}\Gamma(X, M)$$

given by the exact sequence $0 \to \mathcal{L} \to M \to \mathcal{L}^* \otimes \Omega \to 0$, and $M = L_{-1}/\Omega$.

Let ${}^0\operatorname{Bun}_n \subset \operatorname{Bun}_n$ be the open substack classifying $\mathcal{L} \in \operatorname{Bun}_n$ such that $\mathrm{H}^0(X, \operatorname{Sym}^2 \mathcal{L}) = 0$. Write ${}^0\operatorname{Bun}_{\mathbb{P}_n}$ for the preimage of ${}^0\operatorname{Bun}_n$ under the map $\operatorname{Bun}_{\mathbb{P}_n} \to \operatorname{Bun}_n$ sending the above point to $\mathcal{L}$. The restriction $\tilde{\nu}_{\mathbb{P}} : \operatorname{Bun}_{\mathbb{P}_n} \to \widetilde{\operatorname{Bun}}_{\mathbb{G}_n}$ is smooth.

Proposition 3. *There is an isomorphism over* ${}^0\operatorname{Bun}_{\mathbb{P}_n}$

$$\tilde{\nu}_{\mathbb{P}}^* \operatorname{Aut}^e_\psi[\dim.\operatorname{rel}(\tilde{\nu}_{\mathbb{P}})] \widetilde{\to} K_{\mathbb{P}_n,\psi} \tag{7}$$

Remark 5. We conjecture that the isomorphism (7) holds over the whole of $\operatorname{Bun}_{\mathbb{P}_n}$.

There is also a finite-dimensional analog of the extended theta-sheaf. In the case $k = \mathbb{F}_q$ we show that $\operatorname{Aut}^e_\psi$ is a geometric analog of the following matrix coefficient of the Weil representation. Let $\chi : \Omega(\mathbb{A})/\Omega(F) \to \bar{\mathbb{Q}}_\ell^*$ be the character

$$\chi(\omega) = \psi(\sum_{x \in X} \operatorname{tr}_{k(x)/k} \operatorname{Res} \omega_x) \tag{8}$$

Denote by $(\rho, \mathcal{S}_\psi)$ a (unique up to isomorphism) irreducible representation of $H_n(\mathbb{A})$ over $\bar{\mathbb{Q}}_\ell$ with central character χ. It gives rise to the metaplectic extension $\hat{G}_n(\mathbb{A})$ of $G_n(\mathbb{A})$, and $\mathcal{S}_\psi$ is a representation of $\hat{\mathbb{G}}_n(\mathbb{A}) = \hat{G}_n(\mathbb{A}) \ltimes H_n(\mathbb{A})$.

Now $\mathcal{S}_\psi$ admits a unique up to a multiple non zero $H_n(F)$-invariant functional $\Theta : \mathcal{S}_\psi \to \bar{\mathbb{Q}}_\ell$, and a unique up to a multilple $H_n(\mathcal{O})$-invariant vector v_0. This allows to see $\mathbb{G}_n(F)$, $\mathbb{G}_n(\mathcal{O})$ as subgroups in $\hat{\mathbb{G}}_n(\mathbb{A})$. Let

$$\phi : \mathbb{G}_n(F) \backslash \hat{\mathbb{G}}_n(\mathbb{A}) / \mathbb{G}_n(\mathcal{O}) \to \bar{\mathbb{Q}}_\ell \tag{9}$$

be given by $\phi(g) = \Theta(g v_0)$, $g \in \hat{\mathbb{G}}_n(\mathbb{A})$. Then $\widetilde{\operatorname{Bun}}_{\mathbb{G}_n}$ can be thought of as a geometric analog of

$$\mathbb{G}_n(F) \backslash \hat{\mathbb{G}}_n(\mathbb{A}) / \mathbb{G}_n(\mathcal{O}),$$

and Aut^e_ψ is a geometric counterpart of ϕ.

3.6 R-MODEL. Let $R \subset H$ be the parabolic subgroup preserving a 2-dimensional isotropic subspace in the standard representation of H. The stack Bun_R classifies $V \in \mathrm{Bun}_H$ with an isotropic subbundle $U_2 \subset V$, where $U_2 \in \mathrm{Bun}_2$. For $(U_2 \subset V) \in \mathrm{Bun}_R$ write $V' = V_{-2}/U_2 \in \mathrm{Bun}_{\mathbb{SO}_{2n-4}}$, where V_{-2} is the orthogonal complement of U_2 in V.

Let $\mathcal{Y}_R$ be the stack classifying $(U_2 \subset V) \in \mathrm{Bun}_R$ and a section $v_2 : \wedge^2 U_2 \to \Omega$. Let $f_R : \mathcal{Y}_R \to \mathrm{Bun}_R$ be the projection forgetting v_2. Write $\overset{\circ}{\mathcal{Y}}_R \hookrightarrow \mathcal{Y}_R$ for the open substack given by $v_2 \neq 0$.

Let $\mathcal{X}_R$ be the stack classifying $(U_2 \subset V) \in \mathrm{Bun}_R$ and an upper modification $s_2 : U_2 \subset M$ equipped with $\det M \widetilde{\to} \Omega$. Here $M \in \mathrm{Bun}_2$ and s_2 is an inclusion of coherent $\mathcal{O}_X$-modules. Let

$$\pi_R : \mathcal{X}_R \to \overset{\circ}{\mathcal{Y}}_R$$

be the map over Bun_R given by $v_2 = \wedge^2 s_2$. The map π_R is representable and proper.

3.6.1 Our immediate purpose is to define a map

$$\tilde{\rho}_R : \mathcal{X}_R \to \widetilde{\mathrm{Bun}}_{\mathbb{G}_{2n-4}} \tag{10}$$

Given a vector bunbdle $\mathcal{M}$ on X, a line bundle $\mathcal{A}$ on X and a symplectic form $\wedge^2\mathcal{M} \to \mathcal{A}$, we write $H(\mathcal{M}) = \mathcal{M} \oplus \mathcal{A}$ for the Heisenberg group scheme on X with operation

$$(m_1, a_1)(m_2, a_2) = (m_1 + m_2, a_1 + a_2 + \frac{1}{2}\langle m_1, m_2\rangle)$$

Given $(U_2, V') \in \mathrm{Bun}_2 \times \mathrm{Bun}_{\mathbb{SO}_{2n-4}}$, the vector bundle $U_2 \otimes V'$ is equipped with a natural symplectic form $\wedge^2(U_2 \otimes V') \to \wedge^2 U_2$. Now Bun_R identifies canonically with the stack classifying $(V_2, V') \in \mathrm{Bun}_2 \times \mathrm{Bun}_{\mathbb{SO}_{2n-4}}$ and a torsor on X under the group scheme $H(U_2 \otimes V')$.

Write Mod_2 for the stack classifying $U_2 \in \mathrm{Bun}_2$ with an upper modification $s_2 : U_2 \hookrightarrow M$. Consider $\mathrm{Mod}_2 \times_{\mathrm{Bun}_2} \mathrm{Bun}_R$ classifying $(U_2 \subset V) \in \mathrm{Bun}_R$ and $(s_2 : U_2 \hookrightarrow M) \in \mathrm{Mod}_2$. Define

$$e_R : \mathrm{Mod}_2 \times_{\mathrm{Bun}_2} \mathrm{Bun}_R \to \mathrm{Bun}_R$$

as follows. For a point of the source write $V' = V_{-2}/U_2$, where V_{-2} is the orthogonal complement of U_2 in V. The map s_2 yields an inclusion of coherent $\mathcal{O}_X$-modules $H(U_2 \otimes V') \subset H(M \otimes V')$, which is a homomorphism of group schemes over X. View $(U_2 \subset V) \in \mathrm{Bun}_R$ as a triple $(U_2, V', \mathcal{F})$, where $\mathcal{F}$ is a torsor on X under $H(U_2 \otimes V)$. Let $\tilde{\mathcal{F}}$ be the torsor under $H(M \otimes V')$ on X obtained from $\mathcal{F}$ by the extension of the structure group $H(U_2 \otimes V') \to H(M \otimes V')$. Then $(M, V', \tilde{\mathcal{F}}) \in \mathrm{Bun}_R$ is given by some pair $(M \subset \tilde{V}) \in \mathrm{Bun}_R$. By definition, e_R sends $(U_2 \subset V, U_2 \subset M)$ to $(M \subset \tilde{V})$.

Definition 6. Given $(U_2 \subset V) \in \mathrm{Bun}_R$, the vector bundle $U_2 \otimes V$ is equipped with a symplectic form $\wedge^2(U_2 \otimes V) \to \wedge^2 U_2$. Consider then $M_1 = (\mathrm{Sym}^2 U_2)^{\perp} / \mathrm{Sym}^2 U_2$, where $(\mathrm{Sym}^2 U_2)^{\perp}$ is the orthogonal complement of $\mathrm{Sym}^2 U_2$ in $U_2 \otimes V$. So, M_1 is equipped with a symplectic form $\wedge^2 M_1 \to \wedge^2 U_2$ and a line subbundle $\wedge^2 U_2 \subset M_1$. We will refer to M_1 with these structures as *the symplectic-Heisenberg bundle* associated to $(U_2 \subset V) \in \mathrm{Bun}_R$.

Consider $(U_2 \subset V, U_2 \stackrel{s_2}{\hookrightarrow} M) \in \mathcal{X}_R$. Let $(M \subset \tilde{V}) \in \mathrm{Bun}_R$ be the image of this point under e_R. By definition, ρ_R sends the above point of $\mathcal{X}_R$ to the symplectic-Heisenberg bundle $(\det M \subset M_1)$ associated to $(M \subset \tilde{V})$. Since we are given an isomorphism $\det M \widetilde{\to} \Omega$, this symplectic-Heisenberg bundle is a point of $\mathrm{Bun}_{\mathbb{G}_{2n-4}}$.

One has a canonical ($\mathbb{Z}/2\mathbb{Z}$-graded) isomorphism

$$\det \mathrm{R}\Gamma(X, M \otimes V') \widetilde{\to} \det \mathrm{R}\Gamma(X, V')^2 \otimes \det \mathrm{R}\Gamma(X, M)^{2n-4} \otimes \det \mathrm{R}\Gamma(X, \mathcal{O})^{8-4n} \quad (11)$$

Then ρ_R is lifted to $\tilde{\rho}_R$ sending the above point to $(\Omega \subset M_1, \mathcal{B}_1)$, where

$$\mathcal{B}_1 = \det \mathrm{R}\Gamma(X, V') \otimes \det \mathrm{R}\Gamma(X, M)^{n-2} \otimes \det \mathrm{R}\Gamma(X, \mathcal{O})^{4-2n} \quad (12)$$

and $\mathcal{B}_1^2$ is identified with $\det \mathrm{R}\Gamma(X, M_1)$ via (11).

3.6.2 Let ${}^w(\mathrm{Bun}_2 \times \mathrm{Bun}_{\mathbb{SO}_{2n-4}}) \subset \mathrm{Bun}_2 \times \mathrm{Bun}_{\mathbb{SO}_{2n-4}}$ be the open substack given, informally, by the condition that U_2 is sufficiently 'negative' and V' is 'sufficiently stable' compared to U_2. The precise definition is too technical for a talk.

Write ${}^w\overset{\circ}{\mathcal{Y}}_R$, ${}^w\mathcal{Y}_R$, ${}^w\,\mathrm{Bun}_R$ and so on for the preimage of ${}^w(\mathrm{Bun}_2 \times \mathrm{Bun}_{\mathbb{SO}_{2n-4}})$ in the corresponding stack.

> Precise definition that I omitted: write $P_{n-2} \subset \mathbb{SO}_{2n-4}$ for the Siegel parabolic. Let ${}^\flat(\mathrm{Bun}_2 \times \mathrm{Bun}_{P_{n-2}}) \subset \mathrm{Bun}_2 \times \mathrm{Bun}_{P_{n-2}}$ be the open substack given by
>
> $$\mathrm{H}^0(X, \mathrm{Sym}^2(U_2 \otimes U')) = \mathrm{H}^0(X, \Omega \otimes \wedge^2 U') = \mathrm{H}^0(X, U_2 \otimes U'^* \otimes \Omega) = 0 \quad (13)$$
>
> for $(U_2, U' \subset V') \in \mathrm{Bun}_2 \times \mathrm{Bun}_{P_{n-2}}$. The projection is smooth
>
> $$\mathrm{id} \times \nu_P : {}^\flat(\mathrm{Bun}_2 \times \mathrm{Bun}_{P_{n-2}}) \to \mathrm{Bun}_2 \times \mathrm{Bun}_{\mathbb{SO}_{2n-4}} \quad (14)$$
>
> Let ${}^w(\mathrm{Bun}_2 \times \mathrm{Bun}_{\mathbb{SO}_{2n-4}}) \subset \mathrm{Bun}_2 \times \mathrm{Bun}_{\mathbb{SO}_{2n-4}}$ be the intersection of the image of (14) with the open substack given by the conditions
>
> $$\mathrm{H}^0(X, \wedge^2 U_2) = \mathrm{H}^0(X, \Omega \otimes \wedge^2 U_2) = \mathrm{H}^0(X, \Omega \otimes U_2 \otimes V') = 0 \quad (15)$$

Proposition 4. *The complex*

$$(\pi_R)_! \tilde{\rho}_R^* \mathrm{Aut}^e_\psi[\dim.\mathrm{rel}(\tilde{\rho}_R)] \quad (16)$$

is perverse over the open substack ${}^w\overset{\circ}{\mathcal{Y}}_R \subset \overset{\circ}{\mathcal{Y}}_R$.

Let $\mathcal{F}_{R,\psi}$ be the intermediate extension of (16) under the open immersion ${}^w\overset{\circ}{\mathcal{Y}}_R \hookrightarrow {}^w\mathcal{Y}_R$. For the projection $f_R : {}^w\mathcal{Y}_R \to {}^w \mathrm{Bun}_R$ we set

$$K_{R,\psi} = (f_R)_! \mathcal{F}_{R,\psi} \in \mathrm{D}^{\prec}({}^w\mathrm{Bun}_R) \tag{17}$$

Proposition 5. *The complex (17) is a perverse sheaf on* ${}^w \mathrm{Bun}_R$ *irreducible on each connected component.*

Let $\nu_R : \mathrm{Bun}_R \to \mathrm{Bun}_H$ be the natural map, its restriction to ${}^w \mathrm{Bun}_R$ is smooth.

Property 3. There exists an isomorphism over ${}^w \mathrm{Bun}_R$

$$\nu_R^* \mathcal{K}[\dim.\mathrm{rel}(\nu_R)] \widetilde{\to} K_{R,\psi} \tag{18}$$

Remark 6. As we will see, the generic rank of $\mathcal{K}$ must depend on the genus of X, in this sense the problem of constructing of $\mathcal{K}$ is not 'competely local' as was the case for the theta-sheaf Aut.

3.7 Comparing P and Q models. Assume that P, Q, R are standard parabolics with the same Borel. Consider the diagram

$$\mathrm{Bun}_Q \overset{\nu_{P,Q}}{\leftarrow} \mathrm{Bun}_{P\cap Q} \overset{\nu_{Q,P}}{\to} \mathrm{Bun}_P$$

The stack $\mathrm{Bun}_{P\cap Q}$ classifies exact sequences $0 \to W \to U \to U' \to 0$, $0 \to \wedge^2 U \to ? \to \mathcal{O}_X \to 0$ with $W \in \mathrm{Bun}_1, U' \in \mathrm{Bun}_{n-1}$. Write ${}^\diamond(\mathrm{Bun}_1 \times \mathrm{Bun}_{n-1}) \subset \mathrm{Bun}_1 \times \mathrm{Bun}_{n-1}$ for the open substack given by

$$\begin{gathered} \mathrm{H}^0(X, U' \otimes W) = \mathrm{H}^0(X, \wedge^2 U') = \mathrm{Hom}(U', W) = 0 \\ \mathrm{Hom}(U', W \otimes \Omega) = \mathrm{H}^0(X, \Omega \otimes \wedge^2 U') = 0 \end{gathered} \tag{19}$$

for $W \in \mathrm{Bun}_1$, $U' \in \mathrm{Bun}_{n-1}$. Write ${}^\diamond\mathrm{Bun}_{P\cap Q}$ for the preimage of ${}^\diamond(\mathrm{Bun}_1 \times \mathrm{Bun}_{n-1})$ in $\mathrm{Bun}_{P\cap Q}$.

Proposition 6. *There exists an isomorphism on* ${}^\diamond\mathrm{Bun}_{P\cap Q}$

$$\nu_{P,Q}^* K_{Q,\psi}[\dim.\mathrm{rel}(\nu_{P,Q})] \widetilde{\to} \nu_{Q,P}^* K_{P,\psi}[\dim.\mathrm{rel}(\nu_{Q,P})] \tag{20}$$

Remark 7. $K_{P,\psi}$ is perverse over the open substack of Bun_P given by $\mathrm{H}^0(X, \wedge^2 U) = 0$, and $K_{Q,\psi}$ is perverse over the open substack of Bun_Q given by $\mathrm{H}^0(X, W \otimes V') = 0$. The restrictions of $\nu_{P,Q}$ and of $\nu_{Q,P}$ to ${}^\diamond\mathrm{Bun}_{P\cap Q}$ are smooth, so both sides of (20) are perverse. Actually both sides of (20) are irreducible over each connected component of ${}^\diamond\mathrm{Bun}_{P\cap Q}$.

The proof is too technical for a talk. Here is the idea. Let $U_P \subset P, U_Q \subset Q$ be the unipotent radicals, we use that fact that $U_Q \cap U_P$ is 'big enough'. Set $S = (P \cap Q)/(U_P \cap U_Q)$. Then ${}^{\diamond}\mathrm{Bun}_{P\cap Q} \to {}^{\diamond}\mathrm{Bun}_S$ is a torsor under some vector bundle E over ${}^{\diamond}\mathrm{Bun}_S$. For the purpose of the talk imagine that this torsor is trivial (or twist things). Then we actually establish some isomorphism in $\mathrm{D}(E^*)$ and take Fourier transform. The advantage is that over E^* the sheaf we get is of local nature (indepedent of the curve X), namely it is the IC-sheaf of some rank one local system over some locally closed substack of E^*. However, during the Fourier transform along E something of global nature happens, and the result on ${}^{\diamond}\mathrm{Bun}_{P\cap Q}$, at generic points of its support, becomes a local system of 'big' rank depending on the genus of X.

We could eventually define the perverse sheaf $K_{P\cap Q,\psi}$ on ${}^{\diamond}\mathrm{Bun}_{P\cap Q}$ corresponding to (20) by some explicit formula (not knowing about P or Q but directly in terms of $P \cap Q$), but its definition is more complicated than that of $K_{P,\psi}, K_{Q,\psi}$ precisely because the unipotent radical of $P \cap Q$ is not commutative.

3.8 Comparing P and R-models. It relies on the 2-commutativity of the following diagram

$$\begin{array}{ccc} \mathcal{X}_R & \stackrel{\tilde\rho_R}{\to} & \widetilde{\mathrm{Bun}}_{\mathbb{G}_{2n-4}} \\ \uparrow & & \uparrow \tilde\nu_{\mathbb{P}} \\ \mathcal{X}_R \times_{\mathrm{Bun}_R} \mathrm{Bun}_{P\cap R} & \stackrel{\nu_{\mathbb{P},R}}{\to} & \mathrm{Bun}_{\mathbb{P}_{2n-4}}, \end{array}$$

where $\nu_{\mathbb{P},R}$ is as follows. Given

$$(U_2 \subset U \subset V, s_2 : U_2 \to M, \det M \widetilde{\to} \Omega) \in \mathcal{X}_R \times_{\mathrm{Bun}_R} \mathrm{Bun}_{P\cap R} \tag{21}$$

let $0 \to M \to \tilde U \to U' \to 0$ be the push-forward of $0 \to U_2 \to U \to U' \to 0$ by $s_2 : U_2 \to M$. Let $(M \subset \tilde V) \in \mathrm{Bun}_R$ be the image of $(U_2 \subset V, U_2 \stackrel{s_2}{\hookrightarrow} M)$ by e_R. Let $(\Omega \subset M_1)$ be the symplectic-Heisenberg bundle associated to $(M \subset \tilde V)$. Then $\bar{\mathcal{L}} = (M \otimes \tilde U)/\mathrm{Sym}^2 M$ is a lagrangian subbundle in M_1 fitting into an exact sequence $0 \to \Omega \to \bar{\mathcal{L}} \to M \otimes U' \to 0$. By definition, $\nu_{\mathbb{P},R}$ sends (21) to $(\Omega \subset \tilde{\mathcal{L}} \subset M_1)$.

Consider the natural diagram

$$\mathrm{Bun}_P \stackrel{\nu_{R,P}}{\leftarrow} \mathrm{Bun}_{P\cap R} \stackrel{\nu_{P,R}}{\to} \mathrm{Bun}_R$$

We define some open substack ${}^{\flat}\mathrm{Bun}_{P\cap R} \subset \mathrm{Bun}_{P\cap R}$, I don't want to give a precise definition but the restrictions of $\nu_{R,P}$ and $\nu_{P,R}$ to it are smooth.

> Precise def that I omitted: ${}^{\flat}\mathrm{Bun}_{P\cap R}$ is the preimage of ${}^{\flat}(\mathrm{Bun}_2 \times \mathrm{Bun}_{P_{n-2}})$ under the map $\mathrm{Bun}_{P\cap R} \to \mathrm{Bun}_2 \times \mathrm{Bun}_{P_{n-2}}$ sending $(U_2 \subset U \subset V)$ to $(U_2, U' \subset V')$. Here $U' = U/U_2$, and $V' = V_{-2}/U_2$ with V_{-2} being the orthogonal complement to U_2.

To prove Proposition 4, we establish an explicit formula for the restriction of (16) under the smooth (surjective) projection $\mathrm{pr}_1 : {}^{w}\mathring{\mathcal{Y}}_R \times_{\mathrm{Bun}_R} {}^{\flat}\mathrm{Bun}_{P\cap R} \to {}^{w}\mathring{\mathcal{Y}}_R$.

Write ${}^{w\flat}\mathrm{Bun}_{P\cap R}$ for the preimage of ${}^{w}\mathrm{Bun}_R$ under $\nu_{P,R} : {}^{\flat}\mathrm{Bun}_{P\cap R} \to \mathrm{Bun}_R$.

Proposition 7. *Over* ${}^{w\flat}\mathrm{Bun}_{P\cap R}$ *there exists an isomorphism*

$$\nu_{R,P}^* K_{P,\psi}[\dim.\mathrm{rel}(\nu_{R,P})] \widetilde{\to} \nu_{P,R}^* K_{R,\psi}[\dim.\mathrm{rel}(\nu_{P,R})]$$

The proof is based on the explicit formula (7) for Aut_ψ^e over ${}^0\mathrm{Bun}_{\mathbb{P}_n}$ and is relatively easy once you have guessed the claim.

3.9 Pointwise Euler characteristics The maps $\nu_P : {}^e\mathrm{Bun}_P \to \mathrm{Bun}_H$ and $\nu_Q : {}^u\mathrm{Bun}_Q \to \mathrm{Bun}_H$ are surjective. The P and Q-compatibility already implies the following.

Corollary 1. *There is a function* $E_{\mathcal{K}} : \mathrm{Bun}_H(k) \to \mathbb{Z}$ *with the following properties.*
1) For any k-point $\eta \in {}^e\mathrm{Bun}_P$ *over* $V \in \mathrm{Bun}_H(k)$ *one has*

$$\chi(K_{P,\psi}\mid_\eta) = (-1)^{\dim.\mathrm{rel}(\nu_P)} E_{\mathcal{K}}(V)$$

2) For any k-point $\eta \in {}^u\mathrm{Bun}_Q$ *over* $V \in \mathrm{Bun}_H(k)$ *one has*

$$\chi(K_{Q,\psi}\mid_\eta) = (-1)^{\dim.\mathrm{rel}(\nu_Q)} E_{\mathcal{K}}(V)$$

3.10 $\mathcal{K}$ via the theta lifting. For $r > \max\{2g-2, 0\}$ write ${}_{un,r}\mathrm{Bun}_{G_1}$ for the substack of $M \in \mathrm{Bun}_{G_1}$ admitting a line subbundle $L \subset M$ of degree r. Such subbundle is unique if it exists, and the stack ${}_{un,r}\mathrm{Bun}_{G_1}$ classifies $L \in \mathrm{Bun}_1^r$ and an exact sequence $0 \to L \to M \to L^* \otimes \Omega \to 0$.

Let ${}_r\mathrm{Bun}_{G_1} \subset \mathrm{Bun}_{G_1}$ be the open substack of $M \in \mathrm{Bun}_{G_1}$ such that for any $L \in \mathrm{Bun}_1$ if $\deg L > r$ then $\mathrm{Hom}(L, M) = 0$. It is of finite type, and (provided $r > \max\{2g-2, 0\}$) its complement is stratified by ${}_{un,s}\mathrm{Bun}_{G_1}$ with $s > r$.

Write ${}_r\mathrm{Bun}_n \subset \mathrm{Bun}_n$ for the open substack of U such that for $L \in \mathrm{Bun}_1$ if $\deg L > r$ then $\mathrm{Hom}(U, L^* \otimes \Omega) = 0$. Write ${}^e_r\mathrm{Bun}_P$, ${}_r\mathcal{Y}_P$ and so on for the preimage of ${}_r\mathrm{Bun}_n$ in the corresponding stack.

Let ${}_rq : {}_r\mathrm{Bun}_{G_1} \times \mathrm{Bun}_H \to \mathrm{Bun}_H$ be the projection. Though $F_H(\bar{\mathbb{Q}}_\ell)$ is divergent,

$${}_rK = {}_rq_! \mathrm{Aut}_{G_1,H} \in \mathrm{D}^{\prec}(\mathrm{Bun}_H)$$

is well-defined.

IDEA: obtain the desired Hecke eigensheaf out of the sequence ${}_rK$ by passing to the limit by r in some sense.

Conjecture 1. *For any* $r \geq \max\{2g-2, 0\}$ *the* 0*-th perverse cohomology sheaf* ${}^p\mathcal{H}^0({}_rK)$ *admits a canonical irreducible subquotient denoted* ${}_r\mathcal{K}$ *such that there exists an isomorphism over* ${}^e_r\mathrm{Bun}_P$

$$\nu_P^*({}_r\mathcal{K})[\dim.\mathrm{rel}(\nu_P)] \widetilde{\to} K_{P,\psi} \qquad (22)$$

The irredicible perverse sheaves ${}_r\mathcal{K}$ *for all* $r \geq \max\{2g-2, 0\}$ *are defined up to a unique isomorphism and canonically isomorphic one to another.*

Then the desired $\mathcal{K}$ would be defined as ${}_r\mathcal{K}$ for any $r \geq \max\{2g-2, 0\}$.

The explicit formula for the restriction of $\mathrm{Aut}_{G_1,H}$ to $\mathrm{Bun}_{G_1,P}$ immediately yields the following.

Proposition 8. *For all r there is an isomorphism over ${}_r\overset{\circ}{\mathcal{Y}}_P$*

$$\mathrm{Four}_{\psi}^{-1}\,\nu_P^*({}_rK)[\dim.\mathrm{rel}(\nu_P)] \widetilde{\to} \mathrm{IC}(\mathcal{Z}_P) \tag{23}$$

2) The isomorphism (23) still holds over ${}^e_r\overset{\circ}{\mathcal{Y}}_P$ with ${}_rK$ replaced by ${}^p\mathcal{H}^0({}_rK)$.

Definition 7. Let $d, r \in \mathbb{Z}$ be such that ${}^e_r\mathrm{Bun}_n^d$ is nonempty, so ${}^e_r\overset{\circ}{\mathcal{Y}}{}_P^d$ is also nonempty. The right superscript d means everywhere that we take the preimage of Bun_n^d in the corresponding stack. By lemma below, there is a unique irreducible subquotient ${}_r\mathcal{K}^d$ of the perverse sheaf ${}^p\mathcal{H}^0({}_rK)$ equipped with an isomorphism

$$\mathrm{Four}_{\psi}^{-1}\,\nu_P^*({}_r\mathcal{K}^d)[\dim.\mathrm{rel}(\nu_P)] \widetilde{\to} \mathrm{IC}(\mathcal{Z}_P) \tag{24}$$

over ${}^e_r\overset{\circ}{\mathcal{Y}}{}_P^d$. The perverse sheaf ${}_r\mathcal{K}^d$ is defined up to a unique isomorphism. We can not conclude for the moment that (24) holds over ${}^e_r\mathcal{Y}_P^d$, as the LHS could apriori be a non irreducible perverse sheaf.

Lemma 1. *Let $f : \mathcal{A} \to \mathcal{B}$ be an exact functor between abelian categories. Let F, F' be two objects of $\mathcal{A}$ which are of finite length and $\alpha : F \to F'$ a morphism in $\mathcal{A}$. Assume that $R = f(F)$ is an irreducible object of $\mathcal{B}$, and $f(\alpha) : f(F) \to f(F')$ is an isomorphism. Then F admits a biggest subobject F_0 such that $f(F_0) = 0$, let $F'_0 \subset F'$ be the corresponding biggest subobject of F'. Then F/F_0 admits a unique irreducible subobject F_1, and $f(F_1) \widetilde{\to} R$. Define $F'_1 \subset F'/F'_0$ similarly. Then $\alpha : F_0 \to F'_0$ and the induced map $\alpha : F_1 \to F'_1$ is an isomorphism. We refer to F_1 as* the subquotient canonically associated to (f, F).

Note that ${}_r\mathrm{Bun}_{G_1} \subset {}_{r+1}\mathrm{Bun}_{G_1}$, hence a canonical morphism ${}_rK \to {}_{r+1}K$ and ${}^p\mathcal{H}^0({}_rK) \to {}^p\mathcal{H}^0({}_{r+1}K)$. By Lemma 1, it induces an isomorphism ${}_r\mathcal{K}^d \widetilde{\to} {}_{r+1}\mathcal{K}^d$ for r large enough. Define a perverse sheaf $\mathcal{K}^d$ on Bun_H as ${}_r\mathcal{K}^d$ for any r large enough. It is defined up to a unique isomorphism.

To prove Conjecture 1 one should show that $\mathcal{K}^d$ are all isomorphic to each other.

Theorem 1. *If the ground field is of characteristic zero and we are working with D-modules then $\mathcal{K}^d$ are all isomorphic to each other.*

I will explain the idea of the proof later, now we continue in positive characteristic.

3.11 Towards Hecke property Let $\mathcal{U}_r \subset \mathrm{Bun}_H$ be the open substack of V such that for all $L \in \mathrm{Bun}_1$ with $\deg L \geq r$ one has $\mathrm{H}^0(L^* \otimes V \otimes \Omega) = 0$.

The same explicit formula for $\mathrm{Aut}_{G_1,H}$ as above yields the following.

Lemma 2. *For $r \geq \max\{2g-2, 0\}$ the $*$-restriction of* $\mathrm{Aut}_{G_1,H}$ *to* ${}_{un,r}\mathrm{Bun}_{G_1} \times \mathcal{U}_r$ *identifies with*

$$\bar{\mathbb{Q}}_\ell[\dim \mathrm{Bun}_{G_1,H} - 2n(r+1-g)]$$

Let ${}_rL$ be defined by the distinguished triangle ${}_{r-1}K \to {}_rK \to {}_rL$ on Bun_H*. If $r \geq \max\{2g-2, 0\}$ then $({}_rL)\mid_{\mathcal{U}_r}$ is the constant complex placed in usual degrees $\leq (2n-4)r + (5g-5) + 2n(1-g) - \dim \mathrm{Bun}_H$. Actually,*

$$({}_rL)\mid_{\mathcal{U}_r} \widetilde{\to} \mathrm{R}\Gamma(\underline{\mathrm{Pic}}^r X, \bar{\mathbb{Q}}_\ell) \otimes \mathrm{R}\Gamma_c(B(\mathbb{G}_m), \bar{\mathbb{Q}}_\ell)[(4-2n)r+3-3g+2n(g-1)+\dim \mathrm{Bun}_H],$$

here $\underline{\mathrm{Pic}}^r X$ is the Picard scheme of degree r line bundles on X.

We have $\mathcal{U}_r \subset \mathcal{U}_{r+1}$ and $\cup_r \mathcal{U}_r = \mathrm{Bun}_H$. Write $\mathrm{D}(\mathcal{U}_r)_0 \subset \mathrm{D}^{\prec}(\mathcal{U}_r)$ for the full triangulated subcategory generated by the constant complex $\mathrm{R}\Gamma_c(B(\mathbb{G}_m), \bar{\mathbb{Q}}_\ell)$. Let ${}^{st}_rK$ be the image of ${}_rK$ in $\mathrm{D}^{\prec}(\mathcal{U}_r)/\mathrm{D}(\mathcal{U}_r)_0$. Then ${}^{st}K = ({}^{st}_rK)$ form an object of the 2-limit

$$2-\varprojlim_r (\mathrm{D}^{\prec}(\mathcal{U}_r)/\mathrm{D}(\mathcal{U}_r)_0)$$

The Hecke functors act naturally on this 2-limit.

From the properties of the theta-lifting (Section 2.3) it 'almost follows' that ${}^{st}K$ is a Hecke eigen-sheaf in $2-\varprojlim_r(\mathrm{D}^{\prec}(\mathcal{U}_r)/\mathrm{D}(\mathcal{U}_n)_0)$ with the eigenvalue (1). More precisely, it is easy to prove the following. Write W for the standard representation of $\check{H} \widetilde{\to} \mathbb{SO}_{2n}$.

Proposition 9. *For each $r > \max\{2g-1, 1\}$ there is a complex $K_W \in \mathrm{D}^{\prec}(\mathrm{Bun}_H)$, an isomorphism*

$${}_x\mathrm{H}^{\leftarrow}_H(W, K^r) \widetilde{\to} K_W \oplus (\bigoplus_{i=2-n}^{n-2} K^r[2i]),$$

and distinguished triangles $K^{r+1}[-2] \to K_W \to \tilde{K}_W$ and $K^r \to \tilde{K}_W \to K^{r-1}[2]$ in $\mathrm{D}^{\prec}(\mathrm{Bun}_H)$. Here for simplicity we considered the Hecke functor at some fixed point $x \in X$.

3.12.1 Residues of geometric Eisenstein series We will see that $\mathcal{K}$ in general is not a Hecke eigensheaf (already for genus zero). It is tempting to realize both $\mathcal{K}$ and the automorphic sheaf corresponding to the minimal representation as a residue of some geometric Eisenstein series.

Write $\overline{\mathrm{Bun}}^m_Q$ for the stack classifying $V \in \mathrm{Bun}_H$ together with an isotropic subsheaf $L \subset V$, where $L \in \mathrm{Bun}^m_1$, that is, $\deg L = m$. Write $\bar{\nu}^m_Q : \overline{\mathrm{Bun}}^m_Q \to \mathrm{Bun}_H$ for the projection. Set $\mathcal{S}^m = (\bar{\nu}^m_Q)_! \bar{\mathbb{Q}}_\ell[\dim \overline{\mathrm{Bun}}^m_Q]$. Remind the map $\nu : \mathrm{Bun}_P \to \mathrm{Bun}_H$.

Lemma 3. *The complex*

$$\bar{\mathcal{S}}^m = \mathrm{Four}_\psi(\nu_P^* \mathcal{S}^m)[\dim.\mathrm{rel}(\nu_P)] \in \mathrm{D}(\mathcal{Y}_P)$$

is the extension by zero under $\mathcal{Z}_P \to \mathcal{Y}_P$*. If* $g = 0$ *then assume* $m \le 1$*, for* $g \ge 1$ *assume* $m \le 2 - 2g$*. Then for any* d *small enough* ${}^p\mathcal{H}^{3-3g-2m}(\bar{\mathcal{S}}^m) \mid_{\mathcal{Y}_P^d}$ *contains a unique irreducible subquotient isomorphic to* $\mathrm{IC}(\mathcal{Z}_P)$ *over* $\mathcal{Y}_P^d$*. Remind that here the superscript* d *denotes the preimage of* Bun_n^d *in* $\mathcal{Y}_P$*.*

We have taken here $\bar{\mathbb{Q}}_\ell$ on the non smooth stack $\overline{\mathrm{Bun}}_Q^m$ in order to make the proof easy. We can not prove the same for $\mathrm{IC}(\overline{\mathrm{Bun}}_Q^m)$ or $\bar{\mathbb{Q}}_\ell \mid_{\mathrm{Bun}_Q^m}$, but we expect that these sequences would give the same residues.

We see that the perverse sheaf $\mathcal{K}$ we are looking for should appear with multiplicity one in the perverse sheaf ${}^p\mathcal{H}^{3-3g-2m}(\mathcal{S}^m)$ for all m satisfying the assumptions of Lemma 3. The function $m \mapsto 3 - 3g - 2m$ corresponds to the point $s = n$ in (2).

There is a relation between ${}_rK$ and $\mathcal{S}^m$ that I don't explain, but it assures that the residue we get is not 'new', but that of Conjecture 1.

The important difficulty with the residues of Eisenstein series (as opposed to the theta-lifting) is that we don't see any natural transition maps between these perverse sheaves for different m (one would like to have eventually a direct or projective system of compexes indexed by m in order to pass to the limit).

However, the following framework for the residues of geometric Eisenstein series is possibly reasonable.

3.12.2 Let G be a simple, simply-connected group, $P \subset G$ a maximal parabolic with Levi M. Note that $\pi_1(M) \widetilde{\to} \mathbb{Z}$. Let $\mathcal{S}^m = (\nu_P^m)_! \bar{\mathbb{Q}}_\ell$, where $\nu_P^m : \mathrm{Bun}_P^m \to \mathrm{Bun}_G$ is the natural map.

Consider the induced representation $ind_{P(\mathbb{A})}^{G(\mathbb{A})}(\chi^s)$, $s \in \mathbb{C}$, where $\chi : \mathbb{P}(\mathbb{A}) \to (M/[M,M])(\mathbb{A}) \widetilde{\to} \mathbb{A}^* \to \mathbb{Q}^*$. Here the last map sends $a \in \mathbb{A}^*$ to $\mid a \mid$. The simplest residual representations appear inside these induced representations as non ramified subquotients at points $s \in \mathbb{C}$ of reducibility, where the corresponding Eisenstein series $E_P^G(s)$, $s \in \mathbb{C}$ has a simple pole.

We propose that for such s there is an affine function $\alpha : \mathbb{Z} \to \mathbb{Z}$, $\alpha(m) = am + b$ such that for m small enough, the perverse sheaf ${}^p\mathcal{H}^{\alpha(m)}(\mathcal{S}^m)$ stabilizes (or at least, contains the same irreducible perverse sheaf $\mathcal{K}$ as a subquotient). Then say that the sequence $\mathcal{S}^r$ *has a residue in the direction* α.

In the cases $g = 0$ and $g = 1$ we have calculated the sheaf $\mathcal{K}$ (from Conjecture 1) as the above residues.

3.13 Case $g = 0$. Apply the previous lemma with $m = 1$ then ${}^p\mathcal{H}^1(\mathcal{S}^1)$ is easy to calculate.

Remind the Shatz stratification of Bun_H for genus zero. Fix $0 \le m \le n$ and a collection $\bar{a} = (a_1, \dots, a_m)$ with $a_1 \ge \dots \ge a_m > 0$. The stratum $Shatz_{\bar{a}}$ classifies orthogonal vector bundles of the form $W \oplus V' \oplus W^*$, the sum of V' with $W \oplus W^*$ being orthogonal. Here $V' \widetilde{\to} \mathcal{O}^{2n-2m}$ with the standard symmetric form $\mathrm{Sym}^2 V' \to \mathcal{O}$, $W \widetilde{\to} \mathcal{O}(a_1) \oplus \dots \oplus \mathcal{O}(a_m)$, and the form on $W \oplus W^*$ is given by the pairing between W and W^*.

Write $Shatz^b \subset \mathrm{Bun}_H^b$ for the open Shatz stratum in the component Bun_H^b, $b \in \mathbb{Z}/2\mathbb{Z}$.

For each $b \in \mathbb{Z}/2\mathbb{Z}$ the stack $\mathrm{Bun}_H^b - Shatz^b$ is irreducible, its open Shatz stratum corresponds to $\bar{a} = (1,1)$ for $b = 0$, and to $\bar{a} = (1,1,1)$ for $b = 1$.

Proposition 10. *Assume $g = 0$. The perverse sheaf $\mathcal{K}$ identifies with* $\mathrm{IC}(Shatz_{\bar{a}})$*, $\bar{a} = (1,1)$ over Bun_H^0, and with* $\mathrm{IC}(Shatz_{\bar{a}})$*, $\bar{a} = (1,1,1)$ over Bun_H^1.*

Lemma 4. *Assume $g = 0$.*
1) Assume $d \in \mathbb{Z}$ even such that ${}^e\mathrm{Bun}_P^d$ is not empty. Let $\bar{a} = (1,1)$. Then the fibre of ${}^e\mathrm{Bun}_P^d \to \mathrm{Bun}_H$ over $V \in Shatz_{H,\bar{a}}$ is irreducible.
2) Assume d odd such that ${}^e\mathrm{Bun}_P^d$ is not empty. Let $\bar{a} = (1,1,1)$. Then the fibre of ${}^e\mathrm{Bun}_P^d \to \mathrm{Bun}_H$ over $V \in Shatz_{H,\bar{a}}$ is irreducible.

This lemma shows that for $g = 0$ Property 1 already determines $\mathcal{K}$ up to a unique isomorphism!

3.14 Case $g = 1$. Remind the following phenomenon that holds only for $g = 1$.

Let G be any connected split reductive group, $T \subset G$ a maximal torus, W the Weyl group of (G,T). Write Bun_G^0, Bun_T^0 for the 0-th connected component of the corresponding stack. Then, over a suitable open substack of Bun_G^0, the map $f : \mathrm{Bun}_T^0 \to \mathrm{Bun}_G^0$ is a Galois covering with Galois group W. So, to each irreducible representation R of the Weyl group one attaches an irreducible perverse sheaf K_R on Bun_G^0, which generically is the direct summand in $f_! \,\mathrm{IC}(\mathrm{Bun}_T)$ on which W acts by R.

Write S_n for the symmetric group on n elements. In our case of $\mathbb{SO}_{2n}$ one has a natural surjection $W \to S_n$, let R be the standard $n-1$-dimensional representation of S_n viewed as a representation of W. Then the perverse sheaf $\mathcal{K}$ over Bun_H^0 identifies with K_R.

To describe $\mathcal{K}$ over Bun_H^1 we need more notations.

First, there is an open substack of $\mathrm{Bun}_{\mathbb{SO}_4}^1$ classifying orthogonal bundles of the form $V = \sum_{\alpha \in \mathrm{H}^1(X, \mathbb{Z}/2\mathbb{Z})} \mathcal{A}_\alpha$, where $\mathcal{A}_\alpha$ is the corresponding line bundle on X equipped with the quadratic form $\mathcal{A}_\alpha^2 \widetilde{\to} \mathcal{O}$. The sum is orthogonal.

For $n \geq 3$ consider the map $f : \mathrm{Bun}_{\mathbb{SO}_4}^1 \times \mathrm{Bun}_{\mathbb{SO}_{2n-4}}^0 \to \mathrm{Bun}_H^1$ sending (W,V) to $W \oplus V$, where the direct sum is orthogonal. This map is generically étale.

Now for $n \geq 4$ the restriction $f^*(\mathcal{K} \mid_{\mathrm{Bun}_H^1})$ identifies, generically, with the perverse sheaf $\mathrm{IC}(\mathrm{Bun}_{\mathbb{SO}_4}^1) \boxtimes K_R$, where now R is the following representation of the Weyl group $W(\mathbb{SO}_{2n-4})$ of $\mathbb{SO}_{2n-4}$. We have a surjection $W(\mathbb{SO}_{2n-4}) \to S_{n-2}$, and R is the standard $n-3$-dimensional representation of S_{n-2} restricted to $W(\mathbb{SO}_{2n-4})$.

This is obtained, with some additional work, from Lemma 3 taking $g = 1$ and $m = 0$.

Remark 8. For any g the generic fibres of both $\nu_P : \mathrm{Bun}_P^d \to \mathrm{Bun}_H$ and of $\nu_Q : \mathrm{Bun}_Q^d \to \mathrm{Bun}_H$ are connected for d small enough. So, for $g = 1$ any of Properties 1 or 2 already determines $\mathcal{K}$ up to a unique isomorphism.

3.15 Case of characteristic zero. In this case work with D-modules. The proof of Theorem 1 is based on the properties of the characteristic varieties of $\mathcal{K}^d$.

3.15.1 The cotangent bundle $T^*\operatorname{Bun}_H$ classifies $V \in \operatorname{Bun}_H$ and $\sigma \in \mathrm{H}^0(X, \mathfrak{h}_V^* \otimes \Omega) = \mathrm{H}^0(X, \wedge^2 V \otimes \Omega)$.

Let $\bar{\mathcal{O}}_{min} \subset \mathfrak{h}^*$ be the closure of the minimal nilpotent orbit. Let $\mathcal{C} \subset T^*\operatorname{Bun}_H$ be the substack classifying (V, σ) such that σ is a section of $(\bar{\mathcal{O}}_{min})_V^\Omega$, the twist by Ω and V over X. Then $\mathcal{C}$ contains the zero section $\operatorname{Bun}_H \hookrightarrow \mathcal{C}$. Let $\mathcal{C}'$ be the complement of the zero section in $\mathcal{C}$.

For $m \geq 0$ write $\mathcal{C}^m$ for the stack classifying: flags $(V_2 \subset V_{-2} \subset V)$, $V \in \operatorname{Bun}_H$, V_2 is an isotropic subbundle, V_{-2} its orthogonal complement, $D \in X^{(m)}$ and an isomorphism $\det(V/V_{-2}) \widetilde{\rightarrow} \Omega(-D)$. Then $\mathcal{C}^m$ is a locally closed substack in $T^*\operatorname{Bun}_H$ and $\mathcal{C}^m$, $m \geq 0$ form a stratification of $\mathcal{C}'$. The stack $\mathcal{C}^m$ is smooth of dimension $\dim \operatorname{Bun}_H - (2n-4)m$.

We prove that $\mathcal{C}^0$ is lagrangian. This is a consequence of the lemma below (applied to $Z = \mathcal{O}_{min}$).

Lemma 5. *Let G be a reductive group, (Z, ω) a smooth symplectic variety with an action of $\mathbb{G}_m$ such that ω is 1-homogeneous. Assume Z is equipped with a symplectic action of G commuting with the action of $\mathbb{G}_m$ (symplectic means that ω is preserved by the action of G). Then the image of $\Pi_{X/pt}([Z/G]^\Omega)$ in $T^*\operatorname{Bun}_G$ is isotropic.*

Here $[Z/G]$ denotes the quotient stack (equipped with an action of $\mathbb{G}_m$), and $[Z/G]^\Omega$ is its twist by Ω over X, and $\Pi_{X/pt}([Z/G]^\Omega)$ is the classifying stack of global sections of $[Z/G]^\Omega$ over X. The map to $T^*\operatorname{Bun}_G$ comes from the moment map $Z \to \mathfrak{g}^*$.

We expect that the characteristic variety of the minimal automorphic sheaf on Bun_H is the closure of $\mathcal{C}^0$ (eventually with the zero section of $T^*\operatorname{Bun}_H$ added).

3.15.2 For $n \geq 4$ then the stack $\mathcal{C}^0$ is smooth and irreducible (in each connected component of $T^*\operatorname{Bun}_H$). If $n = 2$ then $\mathcal{C}^0$ is contained in $T^*(\operatorname{Bun}_H^0)$ and is irreducible. For $n = 3$ the stack $\mathcal{C}^0$ is not irreducible. Write $\bar{\mathcal{C}}^0$ for the image of $\mathcal{C}^0$ under the projection $T^*\operatorname{Bun}_H \to \operatorname{Bun}_H$.

Assume $g \geq 2$. The proof of the following Proposition is beautiful!

Proposition 11. *1) If $n = 2$ then $\bar{\mathcal{C}}^0 \subset \operatorname{Bun}_H^0$ is of codimension one and irreducible. 2) Let $n \geq 4$ then for each $a \in \mathbb{Z}/2\mathbb{Z}$ the substack $\bar{\mathcal{C}}^0 \cap \operatorname{Bun}_H^a \subset \operatorname{Bun}_H^a$ is of codimension one and irreducible.*

Thus, we get some canonically defined divisor on Bun_H, it is different from the usual Pfaffian divisor! (This holds for any *char* $k \neq 2$).

Proposition 12. *Let $n \geq 4$, $d, r \in \mathbb{Z}$ such that ${}^e_r\overset{\circ}{\mathcal{Y}}{}^d_P$ is not empty. Assume that $\mathcal{F}$ is an irreducible perverse sheaf on $\operatorname{Bun}_H^{d \bmod 2}$ such that*

$$\operatorname{Four}_\psi \nu_P^* \mathcal{F}[\dim.\operatorname{rel}(\nu_P)] \widetilde{\rightarrow} \operatorname{IC}(\mathcal{Z}_P)$$

over ${}^e_r\overset{\circ}{\mathcal{Y}}{}^d_P$. *Then the preimage of* $\mathrm{Bun}_H^{d \bmod 2}$ *in* $\mathcal{C}^0$ *is contained in the characteristic variety of* $\mathcal{F}$.

Proof (sketch). Remind that ${}^e\mathcal{Z}_{P,0}$ classifies (U, M, s) with $U \in {}^e\mathrm{Bun}_n$, $M \in \mathrm{Bun}_{G_1}$ and a surjection $s : U \to M$. Denote by ${}^e\mathcal{Z}^0_{P,0}$ the open substack of ${}^e\mathcal{Z}_{P,0}$ defined by the condition $H^0(X, U \otimes M) = 0$.

Remind that ${}^e\mathrm{Bun}_P$ and ${}^e\mathcal{Y}_P$ are dual vector bundles over ${}^e\mathrm{Bun}_n$. There is a natural isomorphism $\iota : T^*({}^e\mathcal{Y}_P) \to T^*({}^e\mathrm{Bun}_P)$.

Let $N^*_{{}^e\mathcal{Z}^0_{P,0}}({}^e\mathcal{Y}_P)$ be the conormal bundle of ${}^e\mathcal{Z}^0_{P,0}$ in ${}^e\mathcal{Y}_P$. Then

$$\iota(N^*_{{}^e\mathcal{Z}^0_{P,0}}({}^e\mathcal{Y}_P)) \subset T^*({}^e\mathrm{Bun}_P) \subset T^*\mathrm{Bun}_P$$

is included in $T^*\mathrm{Bun}_H \times_{\mathrm{Bun}_H} \mathrm{Bun}_P$ and, in fact, it is included in $\mathcal{C}^0 \times_{\mathrm{Bun}_H} \mathrm{Bun}_P$. □

Lemma 6. *Suppose that* ${}^e_r\mathrm{Bun}^d_n$ *is non empty for some* $r, d \in \mathbb{Z}$. *Let* $\mathcal{F}$ *be an irreducible perverse sheaf on* $\mathrm{Bun}_H^{d \bmod 2}$ *whose restriction to* ${}^e_r\mathrm{Bun}^d_P$ *is non zero and descends under the map* ${}^e_r\mathrm{Bun}^d_P \to {}^e_r\mathrm{Bun}^d_n$. *Then* $\mathcal{C}^0$ *is not included into the characteristic variety of* $\mathcal{F}$.

Combining the two, we get the following.

Proof of Theorem 1

Let r be as in Conjecture 1. The perverse sheaf ${}^p\mathcal{H}^0({}_rK)$ admits exactly one irreducible subquotient such that its characteristic variety contains $\mathcal{C}^0$. This property is independent of d, so the perverse sheaves $\mathcal{K}^d$ (see Def.7) are all isomorphic to each other. □

3.15.3 Assume $n \geq 4$, $g \geq 2$ and g odd. Then we can prove (using a criterium of Braden for characteristic varieties) that the D-module $\mathcal{K}$ from Theorem 1 is non zero at the generic point of Bun^a_H for each $a \in \mathbb{Z}/2\mathbb{Z}$, and so K is determined by Property 1 up to a unique isomorphism in this case.

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