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Modelling of fatigue crack propagation using Piecewise-Deterministic Markov Processes

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Abstract

Fatigue crack propagation is a stochastic phenomenon in nature due to the inherent uncertainties coming from material properties, environmental conditions and loads. Stochastic modelling offers an appropriate framework for modelling crack propagation since it is intended to include sources variabilities. In this work, we propose to model crack propagation dynamics as Piecewise Deterministic Markov Processes (PDMP) using random crack growth models. Paris law commonly used in literature seems inadequate when the crack extends in a rapid manner. To overcome this drawback, a new modelling that consists to combine Paris and Forman laws is proposed. Empirical curves from literature is used to adjust the parameters associated to fatigue crack growth models. Regime-switching models seem very attractive since assessed crack growth rates are very close to experimental values. Moreover, asymptotic behaviour before failure is better captured.

Keywords: Fatigue crack propagation; Uncertainties; Markov processes; Crack growth models.

1 Introduction

Fatigue crack growth (FCG) in materials exhibits a wide range of scatter even under identical experimental conditions [1, 2]. The sources of uncertainties come from material inhomogeneity, severe environmental conditions and loads sequences. To take into account the effect of uncertainties, stochastic modelling strategies seems more pertinent than classical deterministic approaches. Several papers have been published with the purpose of quantifying and incorporating variabilities with different modelling for fatigue crack growth and uncertainties quantifications [3]. As commonly known in fatigue life, we distinguish : crack initiation and crack growth periods. The variabilities sources can be essentially different for each period. In the present work, our attention is focused on crack growth period. This period can be divided into two stages : stable region (stage I) and unstable region (stage II) when FCG extends in a rapid manner without an increase in load or applied energy. Several models of FCG have been proposed in literature as Paris and Forman laws widely used in fatigue crack modelling due to their simplicity and few parameters to identify. The previous discussed models are deterministic, several strategies are proposed in literature to treat FCG from a probabilistic viewpoint. Yang and Manning [4] proposed a random Paris-Erdogan model by adding a noise function while Mattrand and Bourinet [5] used Markov chain to propagate the variability affecting non-proportional load sequences. Chiquet et al. [6] used jump Markov process to model FCG as degradation mechanism evolves continuously in time and growth rate changes at random time. In the present work, stochastic form of Paris and Forman laws are used, uncertainties are integrated through material parameters. To incorporate variabilities, we suppose that material parameters can change at random time. It is important to mention that evolution in each stage is considered as deterministic. With the previous considerations, FCG dynamics can be modelled as Piecewise Deterministic Markov Processes (PDMP). Two different modelling will be proposed : the first one consists to suppose that FCG dynamics evolves with the same fatigue law before and after jump time. In the second modelling, regime-switching model is proposed in stage II. Paris law is used to model FCG in both regimes for the first modelling, while combination between Paris and Forman laws is proposed for the second one. Along the paper, “Paris-Paris modelling” notation refers to the first modelling and “Paris-Forman modelling” to the second one.

The organization of the paper is as follows. In Section 2, we give an overview about the main crack growth laws and how to model FCG by PDMP. In section 3, statistical analysis will be done to adjust experimental data and to compare both modelling. Results are presented and discussed in section 4. In the final section, conclusions are drawn and future research directions are proposed.

2 Crack growth modelling

2.1 Deterministic laws

In the literature several laws have been proposed to model FCG. Paris and Erdogan [7] law given by Eq. (1) is widely used for steel and aluminium materials.

$$\frac{da}{dN} = C(\Delta K)^m \tag{1}$$

where:

a	:	length of fatigue crack;
N	:	number of cycles;
$\frac{da}{dN}$	:	crack growth rate per cycle;
(C, m)	:	material parameters;
ΔK	:	stress intensity factor.

The stress intensity factor can be interpreted as a quantity which characterize the stress distribution around the crack tip. In general, it can be expressed in the following form:

$$\Delta K = \beta \Delta\sigma \sqrt{\pi a} \tag{2}$$

where β is the geometric correction factor and $\Delta\sigma = \sigma_{max} - \sigma_{min}$ is the stress range.
Eq. (1) has some limitations, it does not account the stress ratio effect on crack growth, neither the asymptotic behaviour that characterize the second stage. To tackle these limitations, Forman et al. [8] proposed a model given by Eq. (3) that consists to include physical parameters as stress ratio and fracture toughness.

$$\frac{da}{dN} = \frac{C(\Delta K)^m}{(1-R)k_c - \Delta K} \tag{3}$$

where:

R	:	stress ratio $R = \frac{\sigma_{min}}{\sigma_{max}}$;
k_c	:	fracture toughness.

In Eq. (3) k_c represents the critical value of the stress intensity factor required to cause failure. As the stress intensity factor approaches the fracture toughness, crack growth increases leading to failure. In practice, it is very difficult to access to the fracture toughness value as it depends upon several parameters and is very sensitive to changes in environmental conditions and rolling direction. Throughout the manuscript, fracture thoughness is assumed deterministic, the value given to k_c is equal to 44 MPa$\sqrt{m}$ according to Hertz [9].
In this work, both Paris and Forman laws are used to model FCG. One of the aim of this work is to show if combination between the above models allow us to better describe fatigue crack behaviour during the whole fatigue life. To make random FCG laws, we suppose that material parameters defined by $(m, logC)$ can take values in a finite state space $\mathbb{E}$ composed of N couples of $(m, logC)$. Transition time of crack propagation between stages is supposed random and denoted by T^*.

2.2 PDMP for crack propagation

Markov processes offer an attractive area of research since it is a flexible tool for modelling and can cover a wide range of multidisciplinary domains. In this subsection, a brief description is given to illustrate the use of PDMP in FCG. For more details concerning theoretical aspects of PDMP, the reader can refer to Davis works [10]. Begining from the fact that fatigue crack growth can be divided into two regimes with different crack propagation rates, this means that parameters governing crack propagation laws can change when regime-switching occurs. Transition between regimes may happen at an arbitrary random time. In our contribution, let suppose that FCG evolves deterministically with $(m_1, logC_1)$ parameters until random time denoted by T^*. After T^* the process continues (with or without switching laws) with new values of $(m_2, logC_2)$ as illustrated in Fig. 1.

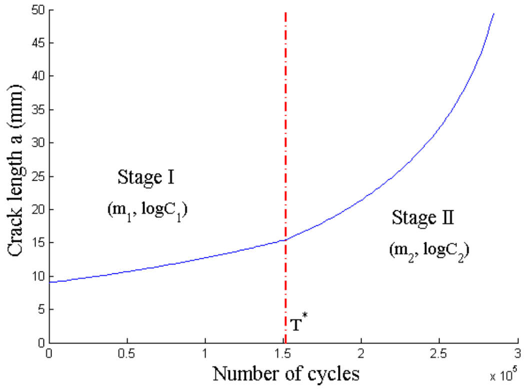

Figure 1: Possible path of crack propagation

The basic ingredients of this PDMP model consists to define :

- a state space $\mathbb{E}$ for the parameters m and $logC$ in each regime;
- the initial law of these parameters given by a probability distribution on the state space;
- the parameter of the exponential distribution of the jump time for each regime of propagation;
- the law of transition between the first regime and the second one given by a transition matrix on $\mathbb{E}$.

3 Statistical analysis

3.1 Experimental data

In this study, experimental data provided by Virkler et al. [1] are used to adjust material parameters. It consists on a set of 68 identical centre-cracked aluminium alloy specimens (152 mm wide and 2.54 mm thick) subjected to constant amplitude loading $\Delta\sigma = 48.28$ MPa at a stress ratio R = 0.2. The number of loading cycles for the tip of the crack to advance a predetermined increment Δa was recorded from an initial crack length of 9 mm to a final length of 49.8 mm. Fig. 2 gives the crack length as a function of the number of load cycles which shows the wide variation in the number of cycles required to grow the cracks.

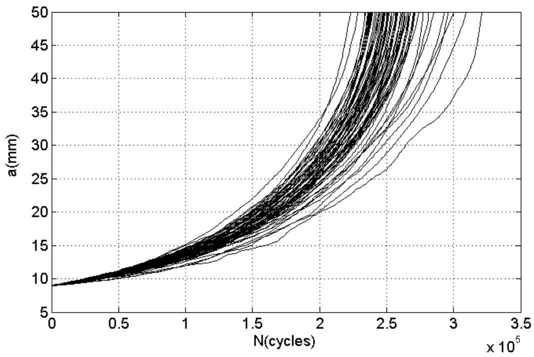

Figure 2: Virkler empirical data set

3.2 Fitting

The aim of this subsection is to determine optimal parameters needed to properly fit the experimental data. Modelling FCG by PDMP with unit jump consists to determine $(m, logC)$ parameters in each regime as the optimum jump time T^*. Consequently, the vector of parameters is denoted by $X = (m_1, logC_1, T^*, m_2, logC_2)$. The proposed objective function consists to minimize the distances between crack length given by empirical curve and numerical prediction for the 68 specimens. The optimization problem can be stated as follows:

$$\begin{cases} \underset{X}{\text{Minimize}} & f(X) = \sum_{i=1}^{164} (a^i_{num}(X) - a^i_{exp})^2 \\ \text{subject to :} & X \in [X_L, X_U] \end{cases} \tag{4}$$

where $f(X)$ is the objective function to be minimized, a^i_{num} and a^i_{exp} represent the numerical prediction and experimental crack length in each increment (the total number of increments is 164). X_L and X_U are vectors of lower and upper bounds of parameters, respectively. Initiallly, the values assigned to the material parameters are randomly selected from the range of values. Simulated annealing (SAN) algorithm, a powerful tool for global minimization is chosen to minimize the objective function since this one presents several local minima. The term *annealing* comes from analogies to the controlled cooling of physical substances to achieve a type of optimal state for the substance. The efficiency of the algorithm depends on the implementation of the annealing shedule and choice of sampling required for generating a new candidate point. Optimal values obtained show a strong correlation between material parameters in each regime. The linear relationship property can be exploited in the modelling.

3.3 Objective function analysis

In order to compare objective function values for both modelling, we plot in Fig. 3 the histograms which give the occurrence number as a function of the normalized objective function values. Normalization consists to divide all the objective functions by their maximum value. Obviously, Paris-Paris modelling yields to optimal objective functions compared to Paris-Forman modelling. The preliminary observation is true if we interest to the whole crack propagation path. But as commonly known, crack propagation phenomenon becomes critical from a certain crack length denoted by a_c. To make a judgement about the fitting quality before failure, we fix a_c at 45 mm and we assess the Partial Objective Function (POF) for both modelling. Fig. 4 shows the partial objective function for some cracks. Black areas correspond to the POF values for Paris-Paris modelling while white areas are the POF values for Paris-Forman modelling. We can observe that the POF values for Paris-Forman modelling is lesser compared to those obtained with Paris-Paris modelling for the majority of cracks. Coupling Paris and Forman laws seems very attractive since it provide a goodness of fit in stage II very important since critical region must be accurately modelled to predict the critical crack length.

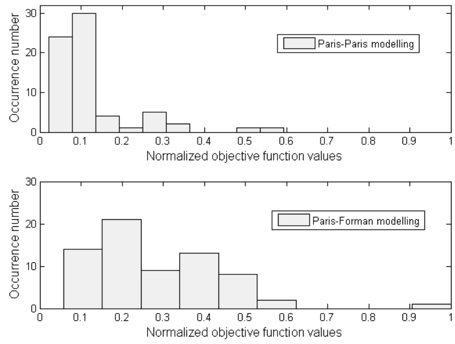

Figure 3: Comparison of global normalized objective function

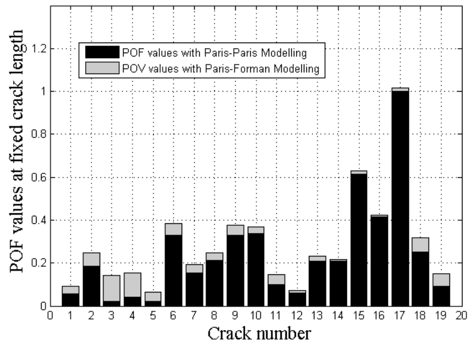

Figure 4: Comparison of partial objective function at fixed crack length (a_c=45 mm)

To better understand the benefits of Forman law at the last part of fatigue crack propagation path, we select two cracks numbered 28 and 48. Crack 28 reaches the final crack length after 263254 cycles while crack 48 reaches the same length after 295745 cycles. Based on Figs. 5 and 6, we can see that Paris-Forman modelling capture well the asymptotic behaviour than Paris-Paris modelling. Obtained results show the benefits of regime-switching models at the last period of fatigue crack propagation by incorporating physical parameters as the stress ratio and fracture toughness. By such modelling, conservative design can be avoided since critical crack length can be estimated with reduced confidence interval.

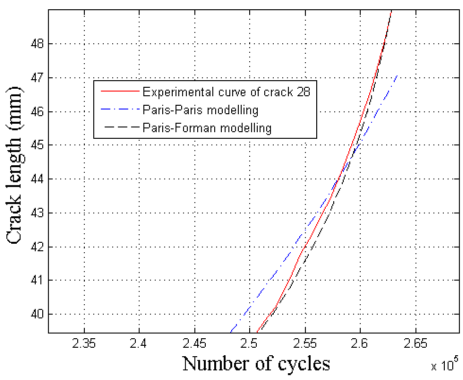

Figure 5: Comparison of fitting : rapid crack case

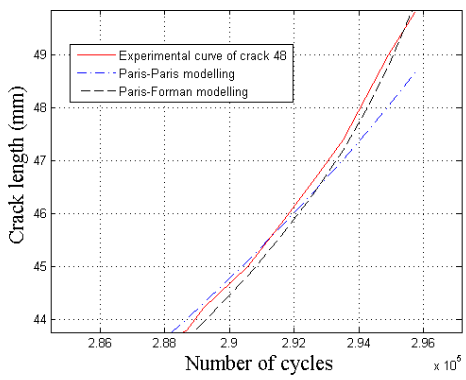

Figure 6: Comparison of fitting : slow crack case

4 Discussion

4.1 Jump times

The aim of this part is to analyse statistically the differences between optimum jump times obtained by both modelling. Table 1 presents statistical parameters of the optimum jump times T^* for each modelling. Based on the optimal collected data of T^*, we can observe that Paris-Paris modelling predicts early the jump times compared to Paris-Forman modelling. Results obtained by Paris-Forman modelling seem to have physical meaning and more realistic since in fatigue, crack propagation behaviour becomes usually unstable after a high number of cycles. Other important result, obtained jump times with Paris-Forman modelling have a lesser scatter (19184 cycles v.s 51792 cycles).

Jump time $T^*(cycles)$	Mean	st.dev	min	max
Paris-Paris modelling	121013	51792	33737	264077
Paris-Forman modelling	241401	19184	192389	296091

Table 1: Statistical analysis of the optimum jump time

4.2 Crack growth rates

Fatigue crack growth rates is evaluated based on the theoretical models and is compared to the FCG rates assessed based on experimental data. The measure of crack growth rate per cycle is approximated as follows:

$$\frac{da}{dN} \approx \frac{\Delta a}{\Delta N} = \frac{a_{i+1} - a_i}{N_{i+1} - N_i} \tag{5}$$

where N_i and N_{i+1} are the number of cycles at which ith and (i+1) th measurements, respectively, and a_i and a_{i+1} are the values of crack lengths at $N = N_i$ and $N = N_{i+1}$, respectively.

For the first modelling, comparison of the approximated results with the experimental values shows a significant difference. Indeed, Paris-Paris modelling underestimates the values of crack growth rates and is unable to predict precisely the changes of crack growth rates observed approximately at 40 mm crack length based on empirical data (see Fig. 7). However, as shown in Figs. 7 and 8 the modelling that combines Paris and Forman laws seems more interesting and predicts precisely the FCG rates, moreover transition region is distinguishable to sign the critical crack length when FCG rates increase rapidly and present an asymptotic behaviour. The obtained results show the efficiency of regime-switching laws to assess precisely FCG rates and to distinguish transient behaviour very meaningful in fatigue life inspection and in probabilistic approaches to assess critical values.

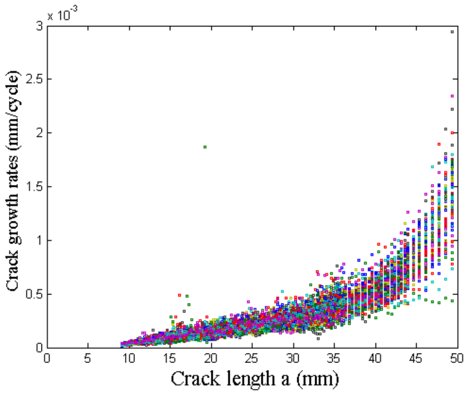

Figure 7: Crack growth rates : experimental data

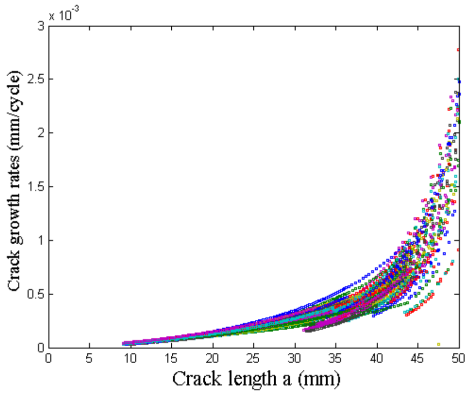

Figure 8: Crack growth rates : Paris-Forman modelling

5 Conclusion

In this paper, Piecewise Deterministic Markov Process was successfully applied to model Fatigue Crack Growth mechanism. Two different modelling was proposed based on Paris and Forman laws. It is shown that modelling FCG with Paris law underestimates crack growth rates and yields to optimum jump time at early stage. Paris-Forman modelling overcomes these limitations, crack growth rates assessed is very close to experimental data. Moreover, the jump time is predicted later with reduced standard deviation that seems more realistic. Proposed regime-switching model seems an interesting strategy since it allow us to capture the transient behaviour and to predict failure with high accuracy with few parameters in the model. Results obtained with constant amplitude loading seem significant, neverthless conditions are certaintly different when the structure is in service. Therefore, it is important to integrate variabilities that can affect loading conditions. Other point that should be developed about prediction of fatigue crack propagation based on few and spaced inspection measures of crack lengths.

Acknowledgements

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