Random growth models with possible extinction

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Random growth models

Random growth models: cells, crystals, epidemics...

Question

Description the asymptotic behaviour of the growth model ?

- Eden's model [Eden 61]

  In $\mathbb{Z}^2$, start from a single occupied site. At each step, choose a site uniformly among empty neighbours of occupied sites, and fill it.
- Richarson's model [Richardson 73]

  Continuous time analogue for Eden's model.
- First-passage percolation [Hammersley–Welsh 65]

  Random perturbation of the graph distance on $\mathbb{Z}^d$.

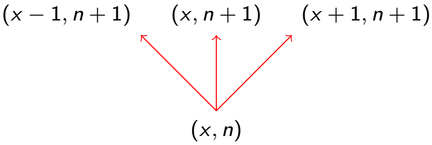
Random growth models with possible extinction:
to allow sites to swap back and forth between two states:

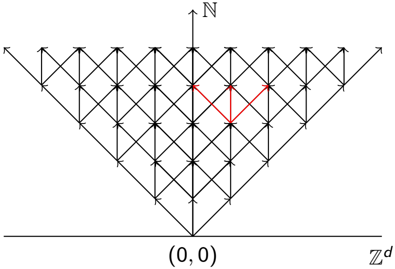
- Oriented percolation [Durrett 84]
- Contact process [Harris 1974]

  Continuous time analogue for oriented percolation.

Oriented percolation in dimension $d + 1$

The oriented graph $\mathbb{Z}^d \times \mathbb{N}$.
Each vertex has $2d + 1$ children:





Randomness.
Each edge is independently kept with probability $p \in (0, 1)$.
$\mathbb{P}_p$: corresponding probability measure.

Oriented percolation: pictures

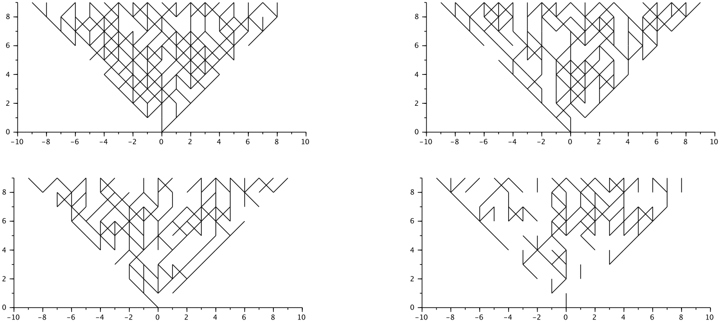
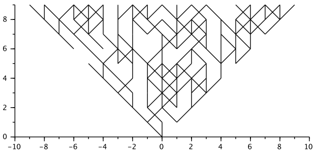


Figure: Examples with $p = 0.7,\ 0.6,\ 0.5,\ 0.4$.

Oriented percolation in dimension $d+1$

Phase transition:



Does there exist infinite open paths?
$\Omega_\infty = \{(0,0) \to \infty\}$

$$\mathbb{P}_p(\Omega_\infty) > 0 \quad \Leftrightarrow \quad p > \overrightarrow{p_c}(d+1).$$

Typical questions:

1. Where are typically the extremities of open paths with length n ?

$$\xi_n = \{x \in \mathbb{Z}^d : \; (0,0) \to (x,n)\}.$$

$\rightsquigarrow$ Shape Theorem for the set ξ_n.

2. At time n, to what extent ξ_n depend on the initial configuration ?
$\rightsquigarrow$ Shape Theorem for the coupled zone.

3. How many open paths with length n can we expect ?

Problem: counting open paths in oriented percolation

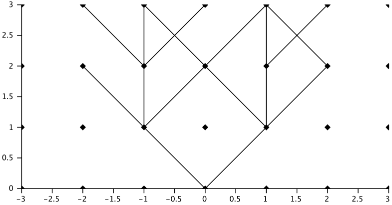


Figure: $n = 3$, $p = 0.6$.

$N_{x,n}$: number of open paths from $(0,0)$ to (x,n)

$N_n = \sum_{x \in \mathbb{Z}^d} N_{x,n}$: number of open paths from $(0,0)$ to level n.

$$(N_{x,n})_{x,n} = \begin{pmatrix} 0 & 1 & 3 & 1 & 4 & 1 & 0 \\ 0 & 1 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad (N_n)_n = \begin{pmatrix} 10 \\ 6 \\ 2 \\ 1 \end{pmatrix}$$

Question

Asymptotic behaviour of N_n ?

Counting open paths: mean behaviour and martingale

- Mean behaviour: $\mathbb{E}_p(N_n) = (2d+1)^n p^n$;

$$\frac{1}{n} \log \mathbb{E}_p(N_n) = \log((2d+1)p).$$

- $\left(\frac{N_n}{((2d+1)p)^n} \right)$ is a non-negative martingale: [Darling 91]

$$\exists W \geq 0 \quad \lim_{n \to +\infty} \frac{N_n}{((2d+1)p)^n} = W \quad \mathbb{P}_p - a.s.$$

- on the event $\{W > 0\}$: $\lim_{n \to +\infty} \frac{1}{n} \log N_n = \log((2d+1)p)$.

On $\{W > 0\}$, $(N_n)_n$ has the same exponential growth rate as $(\mathbb{E}_p(N_n))_n$.

Question

When does $\{W > 0\}$ occur ? And what if $W = 0$?

[Think about the Kesten–Stigum theorem for the Galton-Watson process 66]

Counting open paths: Mean behaviour and martingale

On the event $\{W > 0\}$: $\lim_{n\to+\infty} \frac{1}{n} \log N_n = \log((2d+1)p)$.

- it is possible that $\mathbb{P}_p(\Omega_\infty) > 0$ and $\mathbb{P}_p(W = 0) = 1$:

  [dimension 1 and 2: Yoshida 08]

- it is possible that, on the percolation event,
  - $\overline{\lim}_{n\to+\infty} \frac{1}{n} \log N_n < \log((2d+1)p)$ for some p's,
  - $\overline{\lim}_{n\to+\infty} \frac{1}{n} \log N_n = \log((2d+1)p)$ for some p's.

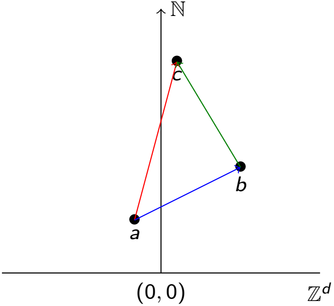
  [Spread out percolation and dimension large enough: Lacoin 12]

Question

a.s. asymptotic behaviour of $\frac{1}{n} \log N_n$ *on the percolation event ?*

Conditional probability: $\overline{\mathbb{P}}_p(.) = \mathbb{P}_p(.|\Omega_\infty)$.

Counting open paths: supermultiplicativity property



$a, b, c \in \mathbb{Z}^d \times \mathbb{N}$ such that $a \to b \to c$:

$$N_{a,c} \geq N_{a,b} N_{b,c}$$
$$(-\log N_{a,c}) \leq (-\log N_{a,b}) + (-\log N_{b,c}).$$

- subadditivity
- stationarity :
  $N_{b,c}$ has the same law as $N_{0,c-b}$
- independence:
  $N_{b,c}$ is independent from $N_{a,b}$

$\left(\frac{1}{n}\log N_n\right)_n$ should converge.

Subadditive ergodic theorems ? [Kingman 68,73; Hammersley 74...]

No: $\log N_{a,b}$ can be infinite, and thus is not integrable...

Convergence is proved for ρ-percolation [Comets–Popov–Vachkovskaia 08]
[Kesten–Sidoravicius 10]

Summary

Counting open paths with length n in oriented percolation:

- Mean behaviour: $\mathbb{E}_p(N_n) = (2d+1)^n p^n$.
- $(-\log N_{a,c}) \leq (-\log N_{a,b}) + (-\log N_{b,c})$:

$$\left(\frac{1}{n}\log N_n\right)_n \text{ should converge.}$$

- Because of possible extinction, infinite quantities appear.

Question: How do we prove convergence results in this context ?

Global convergence result

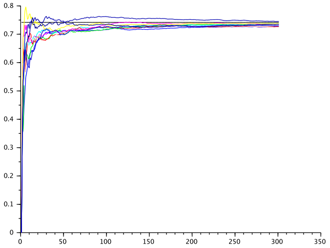
Behaviour in mean:

$$\frac{1}{n}\log \mathbb{E}_p(N_n) = \log((2d+1)p).$$

Almost-sure convergence on Ω_∞:

Theorem (Garet–Gouéré–Marchand)

$$\lim_{n\to+\infty} \frac{1}{n}\log N_n = \tilde{\alpha}_p(0) \quad \overline{\mathbb{P}}_p - a.s.$$



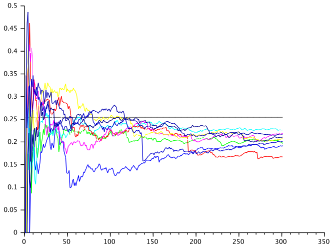
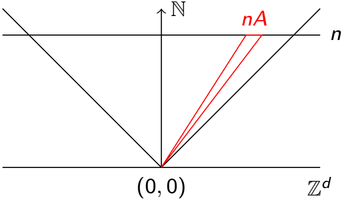


Figure: Representation of $\frac{1}{n}\log N_n$, as a function of n. Values: $n_{\max} = 300$ and $p = 0.7,\ 0.43$. Black line: $\log((2d+1)p)$.

Directional convergence result



$$N_{nA,n} = \sum_{x \in nA} N_{x,n}.$$

Theorem (Garet–Gouéré–Marchand 15)

There exists a concave function $\tilde{\alpha}_p$ such that, for "every" set A

$$\lim_{n\to+\infty} \frac{1}{n} \log N_{nA,n} = \sup_{x\in A} \tilde{\alpha}_p(x) \quad \overline{\mathbb{P}}_p - a.s.$$

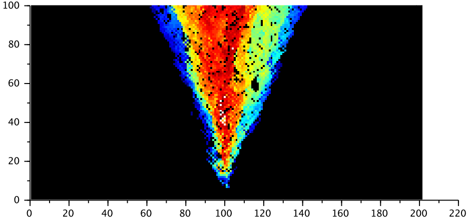


Figure: $n = 100$, $p = 0.6$. Color of pixel (x, k) proportional to $\frac{1}{k} \log N_{x,k}$.

Directional convergence result: p slightly supercritical

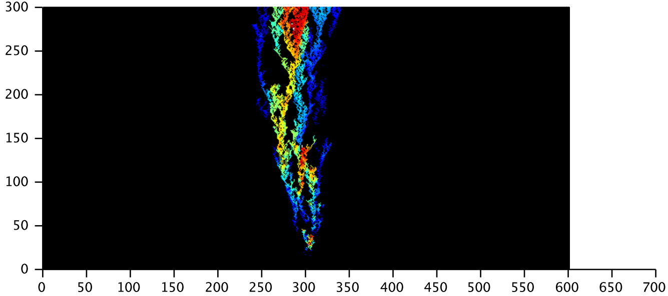
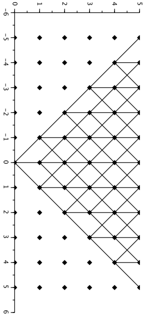


Figure: $n = 300$, $p = 0.45$.

Interpretation as a special case of polymers

Random walk with length n:
a path at random among paths

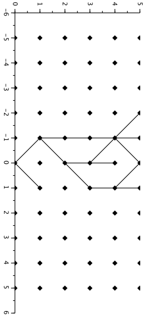
$$\mathbb{P}_n(\gamma) = \frac{1}{(2d+1)^n}$$



Polymer in random potential ω:
a path at random among open paths

$$\mathbb{P}_{n,\omega}(\gamma) = \frac{\mathbf{1}_{\gamma \text{ open in } \omega}}{N_n(\omega)}.$$

$N_n(\omega)$: quenched partition function

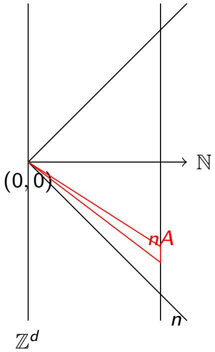


Quenched polymer measure

$$\mathbb{P}_{n,\omega}(\gamma) = \frac{\mathbf{1}_{\gamma \text{ open in } \omega}}{N_n(\omega)}.$$

- Global convergence → ω-a.s. existence of the quenched free energy:
  $$\lim_{n\to+\infty} \frac{1}{n} \log N_n(\omega) = \tilde{\alpha}_p(0).$$
- Directional convergence → LDP for the quenched polymer measure:

$$\lim_{n\to+\infty} \frac{1}{n} \log \mathbb{P}_{n,\omega}(\gamma_n \in nA) = \lim_{n\to+\infty} \frac{1}{n} \log \frac{N_{nA,n}(\omega)}{N_n(\omega)}$$
$$= -\inf_{x\in A} \left(\tilde{\alpha}_p(0) - \tilde{\alpha}_p(x)\right).$$



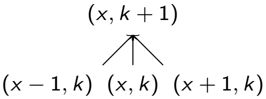
Open questions

- *Is it true that $\forall x \backslash \{0_{\mathbb{R}^d}\} \quad \tilde{\alpha}_p(x) < \tilde{\alpha}_p(0)$?*
- *Is $\tilde{\alpha}_p$ strictly concave ?*
- *Is $\tilde{\alpha}_p$ continuous in p ?*
- *quenched free energy=annealed free energy ?*

Extension to Linear Stochastic Equation (LSE)

- **Counting all paths :** Deterministic linear recurrence equations.

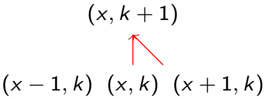
$$N_{x,k+1} = \sum_{y \sim x} N_{y,k}$$



"Pascal's triangle"

- **Counting open paths :** Linear stochastic recurrence equations.

$$N_{x,k+1} = \sum_{y \sim x} a^k_{y,x} N_{y,k}$$



"Pascal's triangle" with iid Bernoulli defects.

- **General Linear Stochastic Equations :** [Yoshida 08]

$$N_{x,k+1} = \sum_{y \sim x} a^k_{y,x} N_{y,k}$$

iid non-negative coefficients

Application : Existence of the quenched free energy for polymer in random potential with values in $\mathbb{R}_+ \cup \{+\infty\}$.

[Garet-Gouéré-Marchand 15]

Convergence results for the number of open paths

Our global convergence result

Theorem

$$\lim_{n\to+\infty} \frac{1}{n}\log N_n = \tilde{\alpha}_p(0) \quad \overline{\mathbb{P}}_p - a.s.$$

relies on the tools we built for proving shape theorems in oriented percolation...

Oriented percolation on $\mathbb{Z}^d \times \mathbb{N}$ with $p > \overrightarrow{p_c}(d+1)$

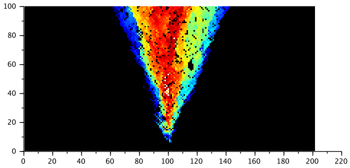


Figure: Percolation cone, dimension $1+1$.

- $\xi_n = \{x \in \mathbb{Z}^d : \ (0,0) \to (x,n)\}$.
- Hitting time : $t(x) = \inf\{n \geq 0 : \ x \in \xi_n\}$.
- Already visited sites : $H_n = \{x \in \mathbb{Z}^d : \ t(x) \leq n\}$. $(H_n)_n$: non-decreasing sequence of random sets.

Theorem (Shape theorem)

There exists a norm μ_p on $\mathbb{R}^d$ (unit ball: A_{μ_p}), such that

$$\overline{\mathbb{P}}_p\left(\exists N > 0 \quad \forall n \geq N \quad (1-\varepsilon)A_{\mu_p} \subset \frac{H_n + [0,1]^d}{n} \subset (1+\varepsilon)A_{\mu_p}\right) = 1.$$

[Durrett–Griffeath 82, Bezuidenhout–Grimmett 90, Durrett 91, Garet–Marchand 12]

General strategy for proving a shape theorem:

- Find a quantity $s(x)$ characterizing the growth in a direction x with

  Subadditivity + Stationarity + Integrability.

- Subadditive ergodic theorem [Kingman 68,73; Hammersley 74; Liggett 85] to obtain directional limits :

$$\mu(x) = \lim_{n\to+\infty} \frac{s(nx)}{n} = \inf_{n\geq 1} \frac{\mathbb{E}s(nx)}{n}.$$

- Prove the convergence is uniform in $\frac{x}{\|x\|}$.

Examples: [Eden 61]

- First-passage percolation: [Richardson 73; Cox–Durrett 81, Boivin 90]
- Brownian motion in random potential: [Sznitmann 94, Mourrat 12]
- "Moving particles": [Alves-Machado-Popov 02, Kesten–Sidoravicius 05,08]

Specific difficulty here: extinction is possible.
Conditioning on non-extinction can for instance destroy independence.

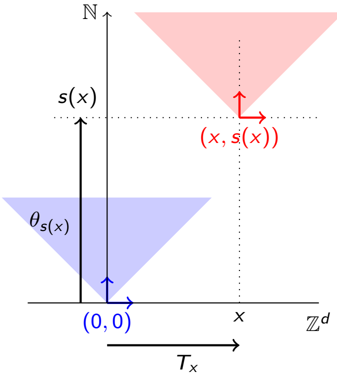
Looking for the good quantity

We work with $\overline{\mathbb{P}}_p(.) = \mathbb{P}_p(.|\Omega_\infty)$.
We're looking for $s(x)$ with : Subadditivity + Stationarity + Integrability.

1. $t(x) = \inf\{n, (0,0) \to (x,n)\}$: **no.**
2. $\tilde{t}(x) = \inf\{n, (0,0) \to (x,n) \to +\infty\}$: **no.**
3. We build $\sigma(x)$, a **regenerating time**:
   - $(0,0) \to (x, \sigma(x)) \to +\infty$;
   - $\overline{\mathbb{P}}_p$ is invariant under $\tilde{\theta}_x = T_x \circ \theta_{\sigma(x)}$;
   - Under $\overline{\mathbb{P}}_p$, $\sigma(x) \circ \tilde{\theta}_x$ et $\sigma(x)$ are i.id. and integrable;
   - σ is (almost) subadditive:

     $$\sigma((n+p)x) \leq \sigma(nx) + \sigma(px) \circ \tilde{\theta}_{nx} + r_x(n,p).$$

   - σ and t are close.



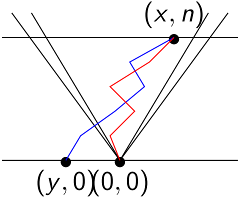
$\rightsquigarrow$ **Shape theorem for** σ;
$\rightsquigarrow$ **Shape theorem for** t.

Shape theorem for the coupled zone

Question.

Markov chain: $\begin{cases} \xi_0 \subset \mathbb{Z}^d, \\ \xi_n = \{x \in \mathbb{Z}^d : \exists x_0 \in \xi_0 : (x_0, 0) \to (x, n)\}. \end{cases}$

How does ξ_n depend on the initial configuration ξ_0 ?



Coupled zone. K_n^0 is the set of points whose state at time n is the same whether $\xi_0 = \{0\}$ or $\xi_0 = \mathbb{Z}^d$.
It is the region where the initial condition is forgotten.

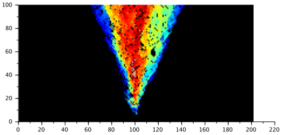
If $x \in K_n^0$, and $\exists y$ such that $(y, 0) \to (x, n)$,
then $(0, 0) \to (x, n)$.

Theorem (Shape theorem for the coupled zone)

$$\overline{\mathbb{P}}_p \left(\exists N \, \forall n \geq N \quad (1-\varepsilon) A_{\mu_p} \subset \frac{(H_n \cap K_n^0) + [0,1]^d}{n} \subset (1+\varepsilon) A_{\mu_p} \right) = 1.$$

Summary: Shape theorem for oriented percolation

Oriented percolation is a typical growth model with possible extinction.



We replaced the hitting time $t(x)$ with a regenerating time $\sigma(x)$: [similar idea in Kucek 89]

$\oplus$ good invariance and ergodicity properties;

$\ominus$ an extra error term.

We can then apply (almost) subadditive ergodic theorems, and follow the classical road.

Applications: [Garet, Gouéré, Marchand, Théret]

- Shape theorem for contact process in random environment,
- Large deviations inequalities for contact process in random environment,
- Continuity of the shape with respect to the infection parameter,
- Number of open paths.

Open questions

Prove that $p \mapsto \mu_p$ is strictly decreasing.

Global convergence result

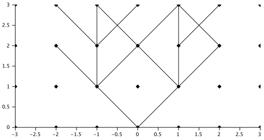


Figure: $n = 3,\ p = 0.6$.

$N_{x,n}$: number of open paths from $(0,0)$ to (x,n)

$N_n = \sum_{x \in \mathbb{Z}^d} N_{x,n}$: number of open paths from $(0,0)$ to level n.

Theorem (Garet–Gouéré–Marchand)

$$\lim_{n \to +\infty} \frac{1}{n} \log N_n = \tilde{\alpha}_p(0) \quad \overline{\mathbb{P}}_p - a.s.$$

Strategy :

1. Use some regenerating times, apply subadditive ergodic theorems and obtain directional limits along random subsequences of times.
2. Use the coupled zone of oriented percolation to come back to full convergence.

1. Directional limits along sequences of regenerating times

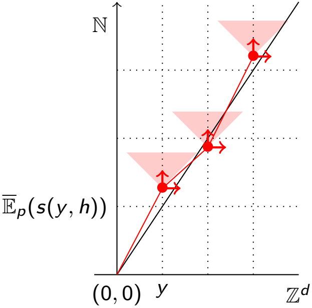
Fix $(y,h) \in \mathbb{Z}^d \times \mathbb{N}^*$. Regenerating time $s(y,h)$, translation $\hat{\theta}$:

- $(0,0) \to (y, s(y,h)) \to \infty$;
- $\overline{\mathbb{P}}_p$ is invariant under $\hat{\theta}$;
- $(s(y,h) \circ (\hat{\theta}^j)_{j\geq 0}$ are iid integrable.

Iteration : sequence of regenerating times

$$S_n = \sum_{k=0}^{n-1} s(y,h) \circ \hat{\theta}^k \sim n\overline{\mathbb{E}}_p(s(y,h)).$$

- $(0,0) \to (y, S_1) \to (2y, S_2) \to \dots$
- $N_{(ny,S_n)}.N_{(py,S_p)} \circ \hat{\theta}^n \leq N_{((n+p)y,S_{n+p})}$.
- $0 \leq \log N_{(ny,S_n)} \leq S_n \log(2d+1)$.



$\bullet$: (ny, S_n).

Subadditive ergodic theorem applied to $f_n = -\log N_{(ny,S_n)}$:

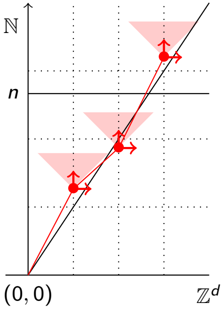
$$\exists \alpha_p(y,h) > 0 \quad \lim_{n\to+\infty} \frac{1}{S_n(y,h)} \log N_{(ny,S_n)} = \alpha_p(y,h) \quad \overline{\mathbb{P}}_p - a.s.$$

2. From directional limits to global convergence

Directional limits: $\lim_{n\to+\infty} \frac{1}{S_n(y,h)} \log N_{(ny,S_n)} = \alpha_p(y,h)$.

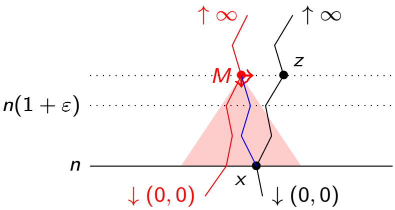
Maximal contribution: $\alpha_p = \sup\left\{\alpha_p(y,h) :\ (y,h) \in \mathbb{Z}^d \times \mathbb{N}^*\right\}$.

1. It is sufficient to work with $\overline{N_n}$:
   open paths that are the beginning of infinite paths.
   Advantage: $\overline{N_n}$ is non-decreasing.
2. Easy part: $\underline{\lim}_{n\to+\infty} \frac{1}{n} \log \overline{N_n} \geq \alpha_p$.
   $\rightsquigarrow$ $\overline{N_n}$ is non-decreasing + renewal theory.
3. Difficult part: $\overline{\lim}_{n\to+\infty} \frac{1}{n} \log \overline{N_n} \leq \alpha_p$.
   $\rightsquigarrow$ Use the coupled zone.



2bis. Use of the coupled zone

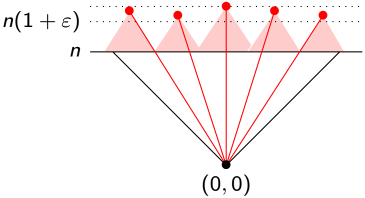
Idea: With the coupled zone, compare numbers of paths coming to close points.



The black path contributes to $\overline{N}_{(x,n)}$:

- M is a point of the sequence associated to (y,h).
- In pink: coupled zone K issued from M, backwards in time.
- Looking backwards from M: $x \in K$ and $z \to x$: so $M \to x$!

$$\text{So } \overline{N}_{(x,n)} \leq \overline{N}_M.$$



Approximation with D directions:
$\rightsquigarrow$ level n covered with D coupled zones:

$$\overline{N}_n \leq \sum_{\bullet} \overline{N}_{\bullet}.$$

$$\rightsquigarrow \quad \overline{\lim_{n \to +\infty}} \frac{1}{n} \log \overline{N_n} \leq \alpha_p.$$

Conclusion

Random growth model with extinction:
By constructing a good regenerating time,
we can rely on the classical
(almost) subadditive ergodic machinery.

1. For oriented percolation/contact process,
   - Shape theorems;
   - Large deviations inequalities;
   - Continuity of the asymptotic shape with respect to the percolation parameter;
   - Asymptotics for the number of open paths in any direction...
2. Shape theorem for variations of the contact process [Deshayes 15]
   - Two stage contact process; [Krone 99]
   - Boundary modified contact process; [Durrett–Schinazi 00]
   - Contact process in randomly evolving contact process; [Broman 07...]
   - Contact process with aging [Deshayes 14]

Thank you for your attention !