Memory-Based Persistence in A Counting Random Walk Process

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Keywords: persistence, random walk, insurance, Markov chains
1. Introduction

The search for non-random patterns in apparently random data is of concern in a broad variety of fields and an extremely large number of techniques and approaches are used for that purpose. In actuarial insurance and financial modeling for example, both jump processes (CTRW-Continuous Time Random Walks) and stochastic differential equations are profusely used to capture the random character of risk events and prices (see for example Scalas 2006 as well as an extended list of references in the paper, Ben Avraham and Havlin 2000; Weiss 1994, 2002; Patlak 1953 and Taylor, 1921). In actuarial science in particular much use is made of the Poisson process (or processes based on similar assumptions) while in finance the Brownian motion is used to represent the behavior of an underlying process uncertainty. These models assume that “uncertainty” is independent of its past, presuming that memory has no effect on future events uncertainty. However, in some cases, this may be too strong an assumption, and thereby the importance to structure uncertainty in terms of the types of dependence that beset randomness. Seminal papers and research on persistence (Weiss 1994, 2002) have contributed to the concern that certain random phenomena do not behave as a simple random walk and thereby have pointed out to both the theoretical and practical implications of assuming that randomness can be “biased” in some manner (see also Balinth, 1986; Masoliver et al., 1993, 2003, 2007; Ben Avraham and Havlin, 2000; Pottier, 2006, Claes and van den Broeck 1987; Cresson et al. 2007; Dechadilok and Deen 2006; Weiss and Rubin 1983; Wu, Springer and Neil 2000; Viswanathan et al. 1999). The purpose of this paper is to consider a memory-based persistent counting random walk, based on a Markov memory of the last event in a random process which is a different model than the Weiss persistent random walk model. At the same time, we shall point out to some preliminary results focused on the effects of short time memory. In particular, we shall provide an explicit expression for the mean and the variance of the underlying memory-based persistent process and discuss the usefulness to some problems in insurance and finance.

This paper’s memory differs in several manners from the memory presumed in several physics papers and other applications areas where memory is deemed important. For example, Telesca and Lovallo (2006) while studying the advent of terrorist attacks suggest that “memory” is associated to correlated events—in the sense that interevents times are not statistically independent. Such an observation implies explicitly that terror events do not have a Poisson distribution. Ferguson and Bazant (2005) while providing a number of avenues to model polymers in solution characterized by long flexible chains as Random walk models of various complexities, suggest also the potential use of a memory imbedded in a correlation between
subsequent steps in a random walk. In this case, “persistence” (in their sense) still exhibits a “normal diffusion”. In the longer run, fractal analysis of random walks, defined by a Hurst exponent, seeks to capture a long run memory, based as well on some “correlation” within the walk that lead to “sub or superdiffusion” processes. Such approaches assume usually that short term memory is well accounted for by Markov models (Cresson et al., 2007) while long-range memory typically gives rise to Non-Markovian walks (Hurst 1951; Mandelbrot 1982; Bunde and Havlin, 1991; Huillet 2002; Tapiero and Vallois, 1996; Peter 1995). This is in contradiction to our analysis of a short term memory model, based on a concept of memory of the previous random event and not only the memory of the previous random state. These “memory processes” remember only the past state and not the random movement the particles had previously as we assume in this paper. For example, “average persistence of random walks” studied by Rieger and Igloi (1999) is essentially based on a birth-death (B-D) random walk (their equation 1) which assumes adsorbing boundaries applied to the B-D random walk and which leads to a “persistence” defined by the probability that expected walker’s position has not returned to its initial position. Persistence of past direction in a random walk or internal bias, was indicated in an early paper by Clifford Patlak (1953) who meant that the process will travel in a given direction need not be the same for all directions but depends solely on the particle’s previous direction of motion. An external bias then arises from an anisotropy of the external force on the particle. Application of such a model is then used to the study of diffusion and long chain polymers.

The motivation for this paper’s attention to short term “memory” arose from the counting of events (whether rare or not) in insurance that presume that events are time independent and therefore based on the Poisson distribution for counting these events and our concern (as in Weiss’s concept of memory) for a memory defined by the “quality” of the previous event (that is whether the previous event was “good” or “bad”, increase or decreased etc. and not only its position). For example, in counting the number of claims that an individual has over time, there is an increased interest in following previous claims in reaching insurance granting decisions. In fact, increasingly, individuation of insurance contracts and the time record of an individual’s history are both useful and essential to measure the propensity of an insured to claim and whether to meet or not his demand for insurance and at what price. For example, individual insured that are healthy initially may eventually become ill, altering the probabilities of their subsequent claims. Health history is therefore needed to predict future claims. By the same token, a driver who has an accident may be more careful in the year that follows the accident but be less
concerned in a subsequent year when there were no accidents to report etc. These are of course hypotheses to be tested against actual records of insurance firms data and insured propensity to claim. Similarly, in weather related insurance, there is an increased interest in counting the number of Hurricanes over a period of time as well as a the propensity of such Hurricanes to recur after “good” and “bad” periods, etc. For example, to what extent do years of intense Hurricane activity follow each other, or vice versa, what would be the propensity of an active Hurricane period to follow a period of non-activity? In many theoretical models, we presume that these events are statistically independent, meaning that there is no statistical memory, justifying thereby counting processes with linear time growth. In our analysis, we shall show that short term memory, unlike Markov models of a random walk, produces nonlinear mean and volatility (variance) evolutions.

Similarly in financial time series, (such as the S&P, the Euro-Dollar exchange rate), we found evidence of short term memory. For example, over a period of 759 consecutive days, we found that the probability of a price increase on the S&P will follow a price increase was 0.4774 while the probability of such an increase following an actual price decline in the previous day was 0.50964. By the same token, the probability of a price decline following a price increase was 0.5254 while a decline following a decline was found to have a probability of 0.4820. Analysis of other time series (such as on intraday data) is likely to lead to more pronounced results. Over shorter periods of time (51 days), these probabilities were found to be even more pronounced (0.5925 and 0.4583) and (0.4074 and 0.54166). These observations lead to some financial traders to devise trading strategies based on the Weiss persistent random walk (see for example Damien, at www.cetrivial.com) and profit from financial markets incompleteness.

In this vein, our paper defines first and explicitly a memory-based persistence for discrete time random walk processes. Subsequently, we point out to the process time nonlinearity. In this sense, the physical random walk model considered is a Non Markov random walk, although in the longer run it may behave as such. While we focus our attention on a counting process (each time assuming a value of 1 or 0), similar results are obtained for a random walk assuming each time +1 or -1 values. To simplify our presentation, we summarize our results in propositions with their extensive proofs given in the appendix. Ergodic results for the mean and the variance processes are outlined.
2. The Memory-based persistent counting process

Assume that an event in any given period of time can be in one of two states: 0,1. The first state « 0 » states that the event has not occurred within the period while « 1 » states that the event has occurred in the period. For example, an insured has claimed this year or he has not. A Hurricane has struck the US South East coast this week or not, etc. Data can be gathered to determine the probability of an event occurring conditional on its past realization and which we denote by a probability \( \alpha > 0 \). By the same token, if the event has not occurred in the period, then the probability of the event occurring in the following year would be \( \beta > 0 \). These observations, recurring in many professions and contexts have motivated the modeling of persistent random processes. Foremost among these processes is the persistence. In particular, Weiss (1995, p. 33) states:

“The outcome of the toss determines the direction of the immediately following step. In the persistent random walk, the random mechanism consists of a biased coin which come heads with probability \( \alpha > 0 \) and tails with probability \( \beta > 0, \beta = 1 - \alpha \). The outcome of any toss determines whether the random walker will continue to move in the same direction as it did in the immediately preceding step (with probability \( \alpha \)) or whether it reverses direction with probability \( \beta \).

A representation of this persistence can be expressed as follows: Let \( x_t \) be the number of events occurring in the time interval \([0,t]\) and let \( y_t \) be the random event, which occurs at time \( t \). Then:

\[
(1) \quad x_{t+1} = x_t + y_{t+1}
\]

Next using the biased coin of Weiss, we have the following outcomes \( P(\epsilon_{t+1} = 1) = \alpha \) and \( P(\epsilon_{t+1} = 0) = \beta \) where \( \{\epsilon_t\} \) are independently distributed random variables. Then, persistence is defined by random events:

\[
(2) \quad y_{t+1} = \epsilon_{t+1} y_t + (1 - \epsilon_{t+1})(1 - y_t)
\]

Note that in the Weiss persistence model we have \( \beta = 1 - \alpha \) and the random walk is obtained as a special case when \( \beta = \alpha = 0.5 \). For the characteristics of such model, references such as Weiss (2002) can be consulted.
In our context, we assume a memory defined by a two-states Markov chain, which is given by:

\[
\mathbf{P} = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}, \quad 0<\alpha,\beta<1
\]

Thus, if we denote by \( y_t \) the value of the random event at time \( t \) then:

\[
\begin{align*}
P(y_t=1|y_{t-1}=0) &= \alpha, \quad P(y_t=0|y_{t-1}=0)=1-\alpha \\
P(y_t=1|y_{t-1}=1) &= 1-\beta, \quad P(y_t=0|y_{t-1}=1)=\beta
\end{align*}
\]

Or

\[
\left[ P(y_t=0) \quad P(y_t=1) \right] = \left[ P(y_{t-1}=0) \quad P(y_{t-1}=1) \right] \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}
\]

Over a period of time \( t \), the total number of events (Hurricanes, claims, upward or downward stock price movements etc.), is therefore given by:

\[
x_t = \sum_{j=0}^{t} y_j
\]

These movements are conditional on the initial event (the current memory of the event), denoted by: \( y_0 = 0 \) or \( y_0 = 1 \). In this memory-based persistent random walk, we calculated the mean and the variance of (6) which are summarized in Proposition 1 below with a proof given in Appendix 1.

**Proposition 1**

Let \( \{x_t, t\geq 0\} \) be a counting random variable of the number of events in a time interval \((0,t)\). And let \((1-\beta,\alpha)\) be the probabilities that an event occurs at time \( t \), conditional on its current (or not) occurrence in the previous period \( t-1 \). Define \( \rho=1-\alpha-\beta \), then the expected number of events in the time period \((0,t)\), are given by:

\[
\begin{align*}
E(x_t|x_0=0) &= \frac{\alpha}{1-\rho} \left[ t + 1 - \frac{1}{1-\rho} (1-\rho^{t+1}) \right] \\
E(x_t|x_0=1) &= \frac{1}{1-\rho} \left[ \alpha(t+1) + \frac{(1-\alpha-\rho)(1-\rho^{t+1})}{1-\rho} \right]
\end{align*}
\]
Further, the second moments of the process counting the number of events that occur in a time interval \((0,t)\) are:

\[
(9) \quad E\left\{x^2(t)|x_0=0\right\} = \alpha \left[ \frac{t(t+1)}{6} \left[ -2\beta t^2 + (3\beta - \alpha + 2)t + \alpha - \beta + 1 \right] + \sigma_3 \left[ -2\beta(\alpha + \beta)t + \beta^2 - \alpha^2 - 4\beta + 2\alpha \right] \right]
\]

\[
(10) \quad E\left\{x^2(t)|x_0=1\right\} = \left( \frac{t+1}{6} \right) \left[ 6+t \left[ -\beta^2 - 3\beta \alpha - 5\beta + 6 \right] + t^2 \left[ -\beta^2 + 3\beta \alpha - 4\beta \right] + 2\beta^2 t^3 \right] + \sigma_3 \beta \left[ 2\beta(\alpha + \beta)t + \beta^2 + 3\alpha^2 + 4\alpha\beta + 2\beta - 4\alpha \right]
\]

Where \( \sigma_3 = \sigma_3(\alpha + \beta, t) \) with \( \sigma_3 = \sigma_3(s, t) \) a real valued term defined by the following relation:

\[
(11) \quad 1 - (1-s)^{t+1} = s(t+1) - t(t+1) \frac{s^2}{2} + \frac{t(t+1)}{6} s^3 + s^4 \sigma_3(s, t)
\]

**Proof:** See Appendix 1.

The results outlined in this proposition indicate both a non-linear mean rate of growth and a nonlinear volatility as equations (7)-(10) indicate. For a random walk we have \( \rho=0 \) and therefore (7) and (8) are reduced to:

\[
E(x|x_0=0) = \alpha t \quad \text{and} \quad E(x|x_0=1) = 1 + \alpha t.
\]

When there is a small and a positive deviation \( \rho \) from the random walk, then there is a "nonlinearity in the mean evolution as equations (7) and (8) attest. In this case, the mean effects of such a memory compared to the random walk (in case of (7)) leads to:

\[
(12) \quad E(x|\rho) - E(x|\rho=0) = \frac{\alpha \rho}{1 - \rho} \left[ (1 - \rho)t - (1 - \rho') \right]
\]

Elementary manipulations indicate in this case that:

\[
(13) \quad \Delta E(x|x_0=0, \rho) = \begin{cases} > 0 & \forall \rho \in [0,1] \\ < 0 & \forall \rho \in [-1,0] \end{cases}
\]

As a result, persistence increases or decreases the mean rate of growth of the underlying random walk process as equation (13) points out. A similar observation is made when consider the means differences.
\( \Delta E(x_i|x_0=1, \rho) \). The effects of persistence on the random walk are more complex however as we shall see below. Further, when there is no persistence, \( \rho=0 \), we have as expected the Binomial random walk as indicated in the following Proposition 2.

**Proposition 2:**
Let \( \alpha+\beta=1 \) and let \( B(t,\alpha) \) be a binomial process with parameter \( \alpha \), then: \( x_i \sim B(t,\alpha) \) if \( x_0=0 \), \( x_i \sim 1+\hat{x}_i \), \( \hat{x}_i \sim B(t,\alpha) \) if \( x_0=1 \) and:

\[
\begin{align*}
(14) \quad & E[x_i|x_0=0] = t\alpha, \quad E[x_i|x_0=1] = 1 + t\alpha, \\
(15) \quad & Var[x_i|x_0=0] = Var[x_i|x_0=1] = \alpha(1-\alpha)t.
\end{align*}
\]

Proof: See Appendix 2

These results are important for two reasons. On the one hand they confirm our general result stated in Proposition 1. On the other, they indicate a number of potential approximations to the underlying probability distributions for these memory-based (persistent) processes. Explicitly, if the parameter \( \alpha \) is not too small but \( \alpha+\beta=1 \), then the Binomial random walk (and its normal approximation) provide a “good” approximation to the persistent counting process. When \( \alpha \) is sufficiently small such that \( \alpha \gg \alpha^2 \) holds, then a Poisson approximations to the persistent counting process is also appropriate. Of course, for other parameters, both the binomial and the Poisson distributions may turn out to be poor approximations (albeit over the short rather than in the long run). The implications of such an observation are numerous. For example, an option price (due to memory effects) may be relatively larger (or smaller) the smaller the option remaining time to its exercise (since the memory effect would “kick in”. Of course, over “the longer run”, and expectedly, these effects are dissipated as indicated in Proposition 3 (with a proof given in Appendix 3).
Proposition 3:

The asymptotic mean and variance of the persistent-one period memory random walk is given by:

\( E\left( \frac{x_t}{t} | x_0 = 0 \right) = \frac{\alpha}{1-\rho} \left( 1 - \frac{\rho}{1-\rho} t + \frac{1}{1-\rho} \rho^{+1} \right) \)  

\( E\left( \frac{x_t}{t} | x_0 = 1 \right) = \frac{1}{1-\rho} \left( \alpha + \frac{1-\alpha-\rho}{1-\rho} t - \frac{1-\alpha-\rho}{1-\rho} \rho^{+1} \right) \)  

And by

\( \frac{1}{t} Var\left( x_t | x_0 = 0 \right) = \frac{\alpha}{(1-\rho)^3} \left( 1 - \alpha - 5\alpha\rho - \rho^2 \right) + \)

\( + \frac{\alpha\rho(4\alpha - 3 + (\alpha + 2)\rho + \rho^2)}{(1-\rho)^4} \frac{1}{t} + o(1). \)  

\( \frac{1}{t} Var\left( x_t | x_0 = 1 \right) = \frac{\alpha(1-\rho-\alpha)(1+\rho)}{(1-\rho)^3} \)

\( - \frac{(1-\rho-\alpha)\rho(4\alpha - 1 + \rho(1+\alpha))}{(1-\rho)^4} \frac{1}{t} + o(1). \)  

where \( t \to \infty. \)

Proof: See Appendix 2

These results are of course consistent with well known results in stochastic processes, while at the same time they indicate that memory effects remain and alter the structure of the random walks evolutions. Of course, over the long run, the underlying process variance has a linear variance growth. Further, when \( \rho = 0, \) we have also and as expected:

\( Var\left( x_t | x_0 = 0 \right) = \alpha(1-\alpha)t. \) The effects of a memory-based persistent random walk in the short run are therefore transient, just as opportunities for arbitrage in financial markets are short lived. Explicitly, when there is persistence, the principal term in the variance difference as stated in equation (18) can be written as follows:
\[
(20) \quad \Delta \left( \frac{1}{t} \text{Var}(x_t|x_0=0) \right) = \frac{1}{t} \text{Var}(x_t|x_0=0, \rho) - \frac{1}{t} \text{Var}(x_t|x_0=0)
\]

Where:

\[
(21) \quad \Delta(.) = \frac{\alpha}{(1-\rho)^2} (1-\alpha -5\alpha \rho - \rho^2) - \alpha(1-\alpha)
\]

A solution for \( \rho \) in terms of \( \alpha \) and subsequent elementary analysis provide a complete and analytical expression to the effects of persistence on the growth of the process variance \( \Delta(.) \) as a function of \( \alpha \). Explicitly, for \( \Delta(.)=0 \), we have \( \delta(\alpha) = (3\alpha -4)^2 - 4(1-\alpha)(3-8\alpha) = -23\alpha^2 + 20\alpha + 4 \), a discriminant term in the quadratic equation (21) in \( \rho \). There are two roots to this discriminant, one negative and the other positive. For a real solution we maintain the positive solution given by: \( \alpha^* = \frac{10 + 8\sqrt{3}}{23} \approx 1.04 \).

Since \( \alpha \in [0,1] \), we deduce that \( \delta(\alpha) \geq 0 \forall \alpha \in [0,1] \). As a result, there are two roots in the quadratic equation (21) when \( \Delta(.)=0 \). These roots are given:

\[
\rho_1 = \frac{4 - 3\alpha - \sqrt{\delta(\alpha)}}{2(1-\alpha)}, \quad \rho_2 = \frac{4 - 3\alpha + \sqrt{\delta(\alpha)}}{2(1-\alpha)}; \quad (\rho_1 < \rho_2)
\]

As a result, and consequently, we obtain the following and complete solution of the persistence summarized by the following:

For \( 0 < \rho < 1 \) we have \( \Delta(.) < 0 \iff \rho_1 < \rho < \rho_2 \)

\(-1 < \rho < 0 \) we have \( \Delta(.) < 0 \iff \rho < \rho_1 \) or \( \rho > \rho_2 \)

A MAPLE aided analysis in case \( \rho_1 < 1 \) indicates as well that:

\[
0 < \rho_1 \quad \text{for} \quad 0 < \alpha < 3/8
\]

\[
-1 < \rho_1 < 0 \quad \text{for} \quad 3/8 < \alpha < 2/3
\]

\[
\rho_1 < -1 \quad \text{for} \quad \alpha > 2/3
\]

In the case \( \rho_2 \geq 1, \alpha \in [0,1] \), our analysis indicates that:

\[
\text{for} \quad 0 < \alpha \leq 3/8, \quad \Delta(.) < 0 \iff \rho \in [-1.0[ \cup [\rho_1, 1[]
\]

\[
\text{for} \quad 3/8 < \alpha \leq 2/3, \quad \Delta(.) < 0 \iff \rho \in [-1, \rho_1[ \cup [0,1[]
\]

\[
\text{for} \quad 2/3 < \alpha \leq 1, \quad \Delta(.) < 0 \iff \rho \in [0,1]
\]
Throughout this analysis, we clearly see that persistence does matter in assessing a process behavior. These observations lead us to conclude that persistence has transient and steady effects on both the mean and the volatility (variance) of a random walk process. Persistence has built in biases, due to short lived behavioral and other events that linger between two equilibrium states, provide an opportunity based on “the market persistence”, affecting both the underlying process trend and its volatility (and therefore the pricing of assets based on such volatility). By the same token, behavioral assumptions regarding individual insured exhibiting a persistent behavior (based on past events) and characterized by our parameters \((\alpha, \beta)\) may provide a better indication to insurers in assessing individuals risks and calculate the risk exposure these insured have. For example, if \(\xi_t = \sum_{j=0}^{\infty} Z_j\) denotes the total claim that a memory-based persistence individual insured has over a period of time \(t\), then the insured mean claim and its variance can be calculated based on our results and an insured risk exposure calculated. Such results are indicative of some of the practical implications our persistence model has which will be considered in far greater detail in subsequent research.

3. Conclusion

The existence and the modeling of long run memory effects have been the subject of considerable research (for example, Hurst, 1951; Mandelbrot 1982, Huillet, 2002, Peter, 1995; Tapiero and Vallois, 1996). Short term persistence has also been outlined by Weiss as stated earlier (Ptalak, 1953, Weiss and Rubin, 1983), with numerous articles pointing out to their applications and specific results of convergence. The purpose of this paper was to focus and complement the study of short term memory (persistence) processes by considering an elementary counting process. Of course, the model we have studied has additional properties that we have not studied yet, including calculating its generating function, its distribution and its limiting distributions. The results we have obtained although cumbersome to prove and summarized in appendices have been tested numerically as well and have indicated a number of intuitive observations. For example, integrating a presumed knowledge of memory (parameters \((\alpha, \beta)\)), can reduce or augment the variance of the underlying uncertainty relative to a random walk, depending
on the relative values of the growth probabilities and as equation (23) indicated.

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Appendix 1:

Proof of the First Moments (Equations (7) and (8))

The mean of the memory based random walk will use the following terms:

\[(1.1)\quad m_0(t) = E\left[ x_t \mid x_0 = 0 \right], \quad m_1(t) = E\left[ x_t \mid x_0 = 1 \right],\]

and,

\[(1.2)\quad m(t) = m_0(t) + m_1(t), \quad \overline{m}(t) = m_0(t) - m_1(t).\]

Lemma 1.1:

\[
\begin{pmatrix}
(m(t)) \\
(\overline{m}(t))
\end{pmatrix} = \begin{pmatrix} +1 \\ -1 \end{pmatrix} + A \begin{pmatrix}
(m(t-1)) \\
(\overline{m}(t-1))
\end{pmatrix}; \quad A = \begin{pmatrix} 1 & \beta - \alpha \\ 0 & \rho \end{pmatrix}
\]

where \(t \geq 1, \quad \rho = 1 - \alpha - \beta\)

Proof:

Using the Markov property of the underlying random walk, we have:

\[
m_0(t) = E\left[ y_0 + y_1 + \ldots + y_t \mid y_0 = 0 \right] = E_0 \left[ y_1 + \ldots + y_t \right]
\]

\[
= E_0 \left[ 1_{\{y_1 = 0\}} (y_1 + \ldots + y_t) \right] + E_0 \left[ 1_{\{y_1 = 1\}} (y_1 + \ldots + y_t) \right]
\]

\[
= (1 - \alpha) E_0 \left[ y_0 + y_1 + \ldots + y_{t-1} \right] + \alpha E_0 \left[ y_0 + y_1 + \ldots + y_{t-1} \right]
\]

Where \(E_0\) is an expectation taken with respect to the Markov chain starting at state 0, while \(E_1\) is an expectation taken with respect to the Markov chain starting at state 1. Therefore,

\[(1.3)\quad m_0(t) = (1 - \alpha) m_0(t-1) + \alpha m_1(t-1).\]

Similarly,

\[(1.4)\quad m_1(t) = 1 + \beta m_0(t-1) + (1 - \beta) m_1(t-1).\]

As a result, we deduce that:

\[(1.5)\quad m(t) = 1 + (1 - \alpha + \beta) m_0(t-1) + (\alpha + 1 - \beta) m_1(t-1) = 1 + m(t-1) + (\beta - \alpha) \overline{m}(t-1)
\]

\[
\overline{m}(t) = -1 + (1 - \alpha - \beta) m_0(t-1) + (\alpha - 1 + \beta) m_1(t-1) = -1 + (1 - \alpha - \beta) \overline{m}(t).
\]

Lemma 1.2:

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Proof:

The proof of this result is determined by recurrence. Explicitly, we have:

\[(1.7) \quad A^0 = \text{Id}; \quad A^1 = \begin{pmatrix} 1 & \beta - \alpha \\ 0 & \rho \end{pmatrix}\]

As a result, we can verify (1.6) by matrix multiplication that \(A^{n+1} = A^n A\).

\[\square\]

**Lemma 1.3:**

Let \(b(t)_{t \in \mathbb{N}}\) be a vector family in \(\mathbb{R}^2\). Define the vector \(a(t)\) by letting \(a(0) = b(0)\) and the recurrence relation:

\[(1.8) \quad a(t) = b(t) + Aa(t-1), \quad t \geq 1.\]

Thus,

\[(1.9) \quad a(t) = \sum_{i=0}^{t} A^i b(t-i).\]

The proof is straightforward and left to the reader.

These lemmas are used next in proving the essential result for the mean process as stated in Proposition 1. Explicitly, we state to prove next equations (7) and (8), or:

\[(1.10) \quad E(x_t | x_0 = 0) = \frac{\alpha}{\alpha + \beta} (t+1) - \frac{\alpha}{(\alpha + \beta)^2} (1 - \rho^{t+1})\]

\[(1.11) \quad E(x_t | x_0 = 1) = \frac{\alpha}{\alpha + \beta} (t+1) + \frac{\beta}{(\alpha + \beta)^2} (1 - \rho^{t+1}).\]

To do so, let \(b(t) = (+1, -1)^t\) and \(a(t) = (m(t), \bar{m}(t))^t; \quad t \geq 0.\) As a result:

\[(1.12) \quad m_0(0) = E(x_t | x_0 = 0) = 0, \quad m_1(0) = E(x_t | x_0 = 1) = 1.\]
Therefore, \( m_0(0) = 1 \), \( m_0(0) = -1 \), and \( a(0) = b(0) \). These conditions can be used to reinterpret Lemma 1.1 by:

\[
0 \ 0 \\
(0) \ 1, \ (0) \ 1 \\
m \ m \\
= = - \\
, \text{ and} \\
(0) \ (0) \\
= a \ b
\]

While from Lemmas 1.3 and Lemma 1.2, we have:

\[
0 \ 1 1 \\
1 \ 0 \\
\]

\[
(1.13) \quad a(t) = b(t) + Aa(t-1), \ t \geq 1; \ a(0) = b(0)
\]

And explicitly written by:

\[
0 \ 1 1 \\
1 \ 0 \\
\]

\[
(1.14) \quad a(t) = \sum_{i=0}^{t} \left( \begin{array}{c}
1 \\
0 \\
\beta - \alpha \\
\alpha + \beta \\
(1-\rho^i) \\
\rho^i \\
\end{array} \right) + \left( \begin{array}{c}
1 \\
0 \\
1 - \rho^i \\
\alpha + \beta \\
\end{array} \right) - 1
\]

Thus:

\[
(1.15) \quad a(t) = \left\{ \begin{array}{ll}
0 & t+1 \\
1 & t+1 - \frac{1-\rho^i}{\alpha + \beta}
\end{array} \right.
\]

Thus:

\[
m(t) = t+1 - \frac{\beta - \alpha}{\alpha + \beta} \left( t+1 - \frac{1-\rho^i}{\alpha + \beta} \right)
\]

\[
\tilde{m}(t) = \frac{1-\rho^i}{\alpha + \beta}
\]

As a result, be the definition of \( m_0(t) \) and \( m_1(t) \), we have:

\[
m_0(t) = \frac{\alpha}{\alpha + \beta} (t+1) - \frac{\alpha}{(\alpha + \beta)^2} (1-\rho^i)
\]

\[
m_1(t) = \frac{\alpha}{\alpha + \beta} (t+1) + \frac{\beta}{(\alpha + \beta)^2} (1-\rho^i)
\]

Which proves equations (7) and (8) once we replace \( \alpha + \beta \) by \( 1-\rho \).

\[
\square
\]

**Proof of the Second Moment: Equations (9) and (10)**

In a similar manner, we can prove the second moments specified in Proposition 1 by equations (9) and (10). To do so, we define again the following variables:

\[2.1 \ M_0(t) = E[x_i^2 | x_0 = 0], \ M_1(t) = E[x_i^2 | x_0 = 1], \]

and,
Lemma 2.1:

\[
\begin{align*}
M(t) & = M_0(t) + M_1(t), \quad \overline{M}(t) = M_0(t) - M_1(t).
\end{align*}
\]

Proof:

Again, using the Markov property of the underlying Markov chain random, we have:

\[
M_0(t) = E\left[(y_0 + y_1 + \ldots + y_t)^2 \mid y_0 = 0\right] = (1 - \alpha)M_0(t-1) + \alpha M_1(t-1).
\]

Further,

\[
M_1(t) = E\left[(1 + y_1 + \ldots + y_t)^2 \mid y_0 = 1\right] = 1 + 2E\left[y_1 + \ldots + y_t \mid y_0 = 1\right] + E\left[(y_1 + \ldots + y_t)^2 \mid y_0 = 1\right]
\]

\[
= 1 + 2E\left[y_0 + y_1 + \ldots + y_t - 1 \mid y_0 = 1\right] + \beta M_0(t-1) + (1 - \beta)M_1(t-1).
\]

And as a result,

\[
M_1(t) = -1 + 2m_1(t) + \beta M_0(t-1) + (1 - \beta)M_1(t-1).
\]

From these results, we deduce:

\[
\begin{align*}
M(t) & = -1 + 2m_1(t) + M(t-1) + (\beta - \alpha)\overline{M}(t-1), \\
\overline{M}(t) & = 1 - 2m_1(t) + (1 - \alpha - \beta)\overline{M}(t-1).
\end{align*}
\]

Lemma 2.2:

\[
M(t) = \sum_{i=0}^{t} q(i,t), \quad \overline{M}(t) = \sum_{i=0}^{t} \overline{q}(i,t),
\]

With

\[
q(i,t) = \left(-1 + 2m_1(t - i)\right) \left(1 - \frac{\beta - \alpha}{\alpha + \beta} (1 - \rho)\right), \quad \overline{q}(i,t) = \rho^i \left(1 - 2m_1(t - i)\right)
\]

As well as:
\[(2.10) \quad M_0(t) = \frac{1}{2}(M(t) + \bar{M}(t)), \quad M_1(t) = \frac{1}{2}(M(t) - \bar{M}(t)) \]

Proof:
Let
\[(2.11) \quad b(t) = (-1 + 2m_i(t))(+1,-1)', t \geq 0.\]

From Lemmas 1.3 and 1.2 and (2.7b), we have:
\[
\begin{align*}
\begin{pmatrix} M(t) \\ \bar{M}(t) \end{pmatrix} &= \sum_{i=0}^{\ell} A^i b(t-i) = \sum_{i=0}^{\ell} (-1 + 2m_i(t-i)) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} \beta - \alpha (1 - \rho^i) \\ \alpha + \beta \end{pmatrix} \begin{pmatrix} +1 \\ -1 \end{pmatrix} \\
&= \sum_{i=0}^{\ell} (-1 + 2m_i(t-i)) \begin{pmatrix} 1 - \beta - \alpha (1 - \rho^i) \\ \alpha + \beta \end{pmatrix} \begin{pmatrix} -\rho^i \\ 1 - \rho^i \end{pmatrix}
\end{align*}
\]

From which we conclude that:
\[
\begin{align*}
(2.13) \quad M(t) &= \sum_{i=0}^{\ell} (-1 + 2m_i(t-i)) \begin{pmatrix} 1 - \beta - \alpha (1 - \rho^i) \\ \alpha + \beta \end{pmatrix} \begin{pmatrix} -\rho^i \\ 1 - \rho^i \end{pmatrix}, \\
\bar{M}(t) &= \sum_{i=0}^{\ell} \rho^i (1 - 2m_i(t-i))
\end{align*}
\]

Since, \( M(t) = M_0(t) + M_1(t), \quad \bar{M}(t) = M_0(t) - M_1(t) \), we obtain the proposition result.

Lemma 2.3

\[
(2.14) \sum_{i=0}^{\ell} (t-i) = \frac{t(t+1)}{2}, \quad \sum_{i=0}^{\ell} i\rho^i = -\frac{1}{(\alpha + \beta)^2} \left( (\alpha + \beta) t \rho^{i+1} + \rho^{i+1} - \rho \right), \quad \sum_{i=0}^{\ell} \rho^i = \frac{1 - \rho^{i+1}}{\alpha + \beta}
\]

The proof is straightforward.

Lemma 2.4:

\[
(2.15) \quad \bar{M}(t) = \frac{1}{(\alpha + \beta)^2} \left[ (-\alpha^2 + \beta^2 - 2\beta + 2\alpha - 2\beta(\alpha + \beta)(t+1)) \frac{1 - \rho^{i+1}}{\alpha + \beta} + 2(\beta - \alpha)(t+1) \right]
\]

Proof:
Note that:
\[ \rho' (1-2m_1(t-i)) = \rho \left( 1 - \frac{2\alpha}{\alpha + \beta}(t-i+1) - \frac{2\beta}{(\alpha + \beta)^2}(1 - \rho^{-i+1}) \right) \]

(2.16) \[ = \frac{\rho' \alpha}{(\alpha + \beta)^2} \left( \beta^2 - \alpha^2 - 2\beta - 2\alpha(\alpha + \beta)t + 2\alpha(\alpha + \beta)t + 2\beta \rho'^{i+1} \right) \]

\[ = \frac{1}{(\alpha + \beta)^2} \left( \beta^2 - \alpha^2 - 2\beta - 2\alpha(\alpha + \beta)t \right) \rho' + 2\alpha(\alpha + \beta)i \rho' + 2\beta \rho'^{i+1} \]

Using Lemmas 2.2 and Lemma 2.3, we obtain:

\[ \bar{M}(t) = \frac{1}{(\alpha + \beta)^2} \left( \frac{\beta^2 - \alpha^2}{\alpha + \beta} - \frac{2\alpha(\alpha + \beta)}{\alpha + \beta} \right) \rho' + 2\alpha(\alpha + \beta)t \rho' + 2\beta \rho'^{i+1} \]

(2.17) \[ = \frac{1}{(\alpha + \beta)^2} \left( \beta^2 - \alpha^2 - 2\beta - 2\alpha(\alpha + \beta)t \right) \rho' + 2\alpha(\alpha + \beta)i \rho' + 2\beta \rho'^{i+1} \]

Replace \( \rho'^{i+1} \) by \( \rho'^{i+1} - 1 + 1 \) and \( \rho' - \rho \) by \( \rho'^{i+1} - 1 + \alpha + \beta \), then:

\[ \bar{M}(t) = \frac{1}{(\alpha + \beta)^2} \left( \beta^2 - \alpha^2 - 2\beta - 2\alpha(\alpha + \beta)(t+1) \right) \rho' + 2(\beta - \alpha)(t+1) \rho' \]

(2.18) \[ \bar{M}(t) = \frac{1}{(\alpha + \beta)^2} \left( \beta^2 - \alpha^2 - 2\beta - 2\alpha(\alpha + \beta)(t+1) \right) \rho' + 2(\beta - \alpha)(t+1) \rho' \]

Lemma 2.5

\[ M(t) = \frac{2\alpha}{(\alpha + \beta)^3} \left( \beta^2 - \alpha^2 + 2\beta + 2\alpha(\alpha + \beta)t + 2\beta \rho'^{i+1} \right) \frac{1}{\alpha + \beta} + \alpha - \beta \bar{M}(t) \]

(2.19) \[ M(t) = \frac{2\alpha}{(\alpha + \beta)^3} \left( \beta^2 - \alpha^2 + 2\beta + 2\alpha(\alpha + \beta)t + 2\beta \rho'^{i+1} \right) \frac{1}{\alpha + \beta} + \alpha - \beta \bar{M}(t) \]

Proof:

Note that from (2.16), we have:

\[ q(i,t) = (-1 + 2m_1(t-i)) \left( 1 - \frac{\beta - \alpha}{\alpha + \beta} (1 - \rho') \right) \]

\[ = \frac{2\alpha}{\alpha + \beta} (-1 + 2m_1(t-i)) + \frac{\beta - \alpha}{\alpha + \beta} \rho' (-1 + 2m_1(t-i)) \]

(2.20) \[ = \frac{2\alpha}{(\alpha + \beta)^3} \left( \beta^2 - \alpha^2 + 2\beta + 2\alpha(\alpha + \beta)t - 2\alpha(\alpha + \beta)t - 2\beta \rho'^{i+1} \right) \rho' (-1 + 2m_1(t-i)) \]

Using Lemmas 2.2 and 2.3, we obtain:
\[(2.21) M(t) = \frac{2\alpha}{(\alpha + \beta)^3} \left\{ (t+1)(-\beta^2 + \alpha^2 + 2\beta) + \alpha(\alpha + \beta)t(t+1) - 2\beta \rho \left( \frac{1 - \rho^{t+1}}{1 - \rho} \right) \right\} + \frac{\alpha - \beta}{\alpha + \beta} \bar{M}(t) \]

This implies (2.19).

To obtain the second moment estimate, we define for convenience the function, \( \sigma_3 = \sigma_3(\alpha + \beta, t) \) where \( \sigma_3 = \sigma_3(s, t) \) is a real valued term defined by the following relation:

\[(2.22) 1 - (1 - s)^{t+1} = s(t+1) - t(t+1) \frac{s^2}{2} + t(t^2 - 1) \frac{s^3}{6} = s^4 \sigma_3(s, t). \]

Then we have the essential proposition of the paper regarding the second order moments which we summarize and prove below.

To prove equations (9) and (10) of Proposition 1 we proceed as follows. We utilize first Lemma 2.5, then:

\[(2.23) M_0(t) = \frac{1}{2} \left( M(t) + \bar{M}(t) \right) \]

By Lemma (2.4), we have however:

\[(2.24) M_0(t) = \frac{\alpha}{(\alpha + \beta)^3} \left\{ \frac{1 - \rho^{t+1}}{\alpha + \beta} \right\} \]

\[
\begin{align*}
\left[ (\beta^2 - \alpha^2 - 2\beta + 2\alpha - 2\beta(\alpha + \beta)(t+1)) \left( \frac{1 - \rho^{t+1}}{\alpha + \beta} \right) \right] + 2(\beta - \alpha)(t+1) + \left( -\beta^2 + \alpha^2 + 2\beta \right)(t+1) + \alpha(\alpha + \beta)t(t+1) - 2\beta \rho \left( \frac{1 - \rho^{t+1}}{\alpha + \beta} \right) \\
\left( \beta^2 - \alpha^2 - 2\beta + 2\alpha - 2\beta(\alpha + \beta)(t+1) \right) \left( \frac{1 - \rho^{t+1}}{\alpha + \beta} \right) + 2(\beta - \alpha)(t+1) + \left( -\beta^2 + \alpha^2 + 2\beta \right)(t+1) + \alpha(\alpha + \beta)t(t+1)
\end{align*}
\]
\[
\frac{\alpha}{(\alpha + \beta)^3} \left\{ \frac{1 - \rho^{t+1}}{\alpha + \beta} \left[ (-2\beta st + s(\beta - \alpha) - 4\beta + 2\alpha) \right] + (t+1)(\alpha st + (\alpha - \beta)s - 2\alpha + 4\beta) \right\}
\]

Where \( s = \alpha + \beta \) and where we write:

\[
(2.25) \quad \frac{1 - \rho^{t+1}}{\alpha + \beta} = \frac{t+1 - \frac{s}{2}t(t+1) + \frac{t(t^2-1)}{6}s^2 + s^3\sigma_3}{s}
\]

Thus:

\[
(2.26) \quad M_0(t) = \frac{\alpha}{(\alpha + \beta)^3} \left\{ \left( t+1 - \frac{s}{2}t(t+1) + \frac{t(t^2-1)}{6}s^2 + s^3\sigma_3 \right) (-4\beta + 2\alpha + s(\beta - \alpha - 2\beta t)) + (t+1)(-2\alpha + 4\beta + s(\alpha - \beta + \alpha t)) \right\}
\]

\[
= \frac{\alpha}{(\alpha + \beta)^3} \{ q_0 + q_1s + q_2s^2 + q_3s^3 \}
\]

Where

\[
q_0 = (t+1)(2\alpha - 4\beta) + (t+1)(-2\alpha + 4\beta) = 0
\]

\[
q_1 = (t+1)(-\alpha + \beta - 2\beta t) - \frac{t(t+1)}{2}(2\alpha - 4\beta) + (t+1)(\alpha - \beta + \alpha t)
\]

\[
= (t+1)\left[ -\alpha + \beta - 2\beta t + (2\beta - \alpha)t - \beta + \alpha + \alpha t \right] = 0
\]

\[
q_2 = \frac{t(t^2-1)}{6}(2\alpha - 4\beta) - \frac{t(t+1)}{2}(-\alpha + \beta - 2\beta t)
\]

\[
= \frac{t(t+1)}{6}\left[ (2\alpha - 4\beta)(t-1) - 3(-\alpha + \beta - 2\beta t) \right]
\]

\[
= \frac{t(t+1)}{6}\left[ (2\alpha + 2\beta)t + \beta + \alpha \right] = \frac{t(t+1)}{6}s(2t+1).
\]

\[
q_3 = \sigma_3 \left[ -4\beta + 2\alpha + s(-\alpha + \beta - 2\beta t) \right] + \frac{t(t^2-1)}{6}(-\alpha + \beta - 2\beta t).
\]

Then
\[
M_0(t) = \alpha \left\{ \frac{t(t+1)}{6} (2t+1) + \frac{t(t^2-1)}{6} (\beta - \alpha - 2\beta t) \right\} + \sigma_3 \left( -4\beta + 2\alpha + s(\beta - \alpha - 2\beta t) \right)
\]
(2.27)

\[
M_0(t) = \alpha \left\{ \frac{t(t+1)}{6} \left[ -2\beta t^2 + (3\beta - \alpha - 2)t + \alpha - \beta + 1 \right] \right\} + \sigma_3 \left( -2\beta(\alpha + \beta)t + \beta^2 - \alpha^2 - 4\beta + 2\alpha \right)
\]

We turn next to the calculation of

\[
M_1(t) = \frac{1}{2} \left( M(t) - \overline{M}(t) \right)
\]
(2.28)

\[
= \frac{\alpha}{(\alpha + \beta)} \left\{ \left[ (-\beta^2 + \alpha^2 + 2\beta)(t+1) \right] - 2\beta \rho \left[ \frac{1-\rho \alpha}{\alpha + \beta} \right] - \frac{\beta}{\alpha + \beta} \overline{M}(t) \right\}
\]

By Lemma 2.4:

\[
M_1(t) = \frac{1}{(\alpha + \beta)^3} \left\{ \left[ \frac{1-\rho \alpha}{\alpha + \beta} \right] \left[ -\beta^2 + \alpha^2 + 2\beta - 2\alpha + 2\beta(\alpha + \beta)(t+1) - 2\alpha(1-\alpha - \beta) \right] \right\}
\]

\[
+ 2(\beta - \alpha)\beta(t+1) + \alpha \left( -\beta^2 + \alpha^2 + 2\beta \right)(t+1) + \alpha^2 (\alpha + \beta)t(t+1)
\]

\[
= \frac{1}{(\alpha + \beta)^3} \left\{ \left[ \frac{1-\rho \alpha}{\alpha + \beta} \right] \left( 2\beta st + s(\alpha - \beta) + 2s(\alpha + \beta) + 2\beta - 4\alpha \right) \right\}
\]

\[
+ (t+1) \left( \alpha^2 st + s(\alpha - \beta) + 4s \alpha \beta - 2\beta \right)
\]

\[
= \frac{1}{(\alpha + \beta)^3} \left\{ \beta \left[ t+1 - \frac{st(t+1)}{2} + \frac{t(t^2-1)}{6} s^2 + s^3 \sigma_3 \right] \left( 2\beta - 4\alpha + s(2\beta t + 3\alpha + \beta) \right)
\]

\[
+ (t+1) \left( 4s \alpha \beta - 2\beta t + s(\alpha^2 t + \alpha(\alpha - \beta)) \right)
\]

\[
= \frac{1}{(\alpha + \beta)^3} \left\{ \hat{q}_0 + \hat{q}_1s + \hat{q}_2s^2 + \hat{q}_3s^3 \right\}
\]

where
\[ q_0 = \beta(t+1)(2\beta - 4\alpha) + (t+1)\beta(4\alpha - 2\beta) = 0 \]
\[ q_1 = -\frac{\beta t(t+1)}{2}(2\beta - 4\alpha) + \beta(t+1)(2\beta t + 3\alpha + \beta) + \beta t(t+1)(\alpha^2 t + \alpha(\alpha - \beta)) \]
\[ = (t+1)[-\beta t(2\alpha) + \beta(2\beta t + 3\alpha + \beta) + (\alpha^2 t + \alpha(\alpha - \beta))] \]
\[ = (t+1)s^2(t+1) = (t+1)^2 s^2 \]
\[ q_2 = \left[ \frac{t(t^2 - 1)}{6}(2\beta - 4\alpha) - \frac{t(t+1)}{2}(3\alpha + \beta + 2\beta t) \right] \beta \]
\[ = \frac{t(t+1)}{6}[(2\beta - 4\alpha)(t-1) - 3(3\alpha + \beta + 2\beta t)] \beta \]
\[ = -\frac{t(t+1)}{6}[4t + 5] \beta s, \]
\[ q_3 = \sigma_3 \left[ 2\beta - 4\alpha + s(3\alpha + \beta + 2\beta t) \right] \beta + \frac{\beta t(t^2 - 1)}{6}(3\alpha + \beta + 2\beta t). \]

As a result,
\[ M_1(t) = (t+1)^2 - \frac{t(t+1)}{6}(4t + 5)\beta + \frac{\beta t(t^2 - 1)}{6}(2\beta t + 3\alpha + \beta) \]
\[ + \sigma_3 \left[ 2\beta - 4\alpha + s(3\alpha + \beta + 2\beta t) \right] \beta \]
\[ = \frac{(t+1)}{6}[6t + 6 - t(4t + 5)\beta + \beta t(t-1)(2\beta t + 3\alpha + \beta)] \]
\[ + \sigma_3 \beta \left[ 2\beta(\alpha + \beta)t + 3\alpha^2 + \beta^2 + 4\alpha + 2\beta - 4\alpha \right] \]
\[ \left(2.30\right) \]
\[ = \frac{(t+1)}{6} \left[ 6 + t(-\beta^2 - 3\alpha\beta - 5\beta + 6) + t^2(-\beta^2 + 3\alpha\beta - 4\beta) + 2\beta^2 t^3 \right] \]
\[ + \sigma_3 \beta \left[ 2\beta(\alpha + \beta)t + 3\alpha^2 + \beta^2 + 4\alpha + 2\beta - 4\alpha \right] \]

These last result, confirms our first proposition where \( \rho = 1 - \alpha - \beta \) expresses a deviation from the standard random walk in which case, \( \alpha + \beta = 1 \).

Appendix 2

To prove the results of Proposition 2 we use again:
\[ (3.1) \quad E\left[ x_t | x_0 = 0 \right] = m_0(t), \quad E\left[ x_t | x_0 = 1 \right] = m_1(t), \]

Where
\[ (3.2) \quad m_0(t) = \frac{\alpha}{\alpha + \beta} (t+1) - \frac{\alpha}{(\alpha + \beta)^2} (1 - \rho^{t+1}) \]
(3.3) \[ m_t(t) = \frac{\alpha}{\alpha + \beta} (t+1) + \frac{\beta}{(\alpha + \beta)^2} (1 - \rho^{t+1}) \]

Since \( \alpha + \beta = 1 \), therefore, \( \rho = 0 \) and

(3.4) \[ m_0(t) = \alpha(t+1) - \alpha = \alpha t, \]

(3.5) \[ m_t(t) = \alpha(t+1) + \alpha = \alpha t + 1. \]

Which proves our equations. The proof for the second order terms is more complicated however.

**Appendix 3: Asymptotic results**

Consider first the expected value \( E(x_t) \). Asymptotically, we expect that:

(4.1) \[ \lim_{t \to +\infty} \frac{E(x_t)}{t} = c_1 \]

Where \( c_1 = \int x \nu(dx) \), and \( \nu \) is the invariant probability measure. Since,

\( \nu(dx) = \frac{\beta}{\alpha + \beta} \delta_0(dx) + \frac{\alpha}{\alpha + \beta} \delta_1(dx) \)

then

(4.2) \[ c_1 = \frac{\alpha}{\alpha + \beta} \]

From (7), we deduce that:

(4.3) \[ E\left( \frac{x_t}{t} | x_0 = 0 \right) = \frac{\alpha}{\alpha + \beta} + \frac{\alpha(\alpha + \beta - 1)}{(\alpha + \beta)^2} \frac{1}{t} + \frac{\alpha}{(\alpha + \beta)^2} \frac{\rho^{t+1}}{t} \]

And therefore (4.1) is obtained. However, in this asymptotic development, the second term in \( \frac{1}{t} \) is accounting for the “memory” as well. In fact, when we have a random walk then \( 1 = \alpha + \beta, \rho = 0 \), and we get exactly: \( E\left( \frac{x_t}{t} | x_0 = 0 \right) = \frac{\alpha}{\alpha + \beta} \).

From (8), we also have:
which yields of course (4.1). Note that the second term depends on whether \( x_0 = 0 \) or \( x_0 = 1 \) and finally, \( \frac{\rho^{t+1}}{t} \) is negligible compared to \( \frac{c}{t} \) since \(-1 < \rho < 1\).

Similarly, we can study the asymptotic behavior of the second moment \( E(x_t^2), \ t \to \infty \). Note that by the ergodic theorem \( \left( \frac{x_t^{t'}}{t} \to c_2 \right): \)

\[
\lim_{t \to \infty} E\left( \frac{x_t^2}{t^2} \right) = c_2 = \frac{\alpha^2}{(\alpha + \beta)^2}
\]

Extensive analysis will show first that:

\[
E\left\{ x_t^2 \Big| x_0 = 0 \right\} = \frac{\alpha^2}{(\alpha + \beta)^2} t^2 + \frac{\alpha(2\alpha^2 + \alpha \beta - \beta^2 + 2\beta - 2\alpha)}{(\alpha + \beta)^3} t + \frac{\alpha(1-\alpha-\beta)(\beta^2-\alpha^2-4\beta+2\alpha)}{(\alpha + \beta)^4}
\]

\[
+ \frac{2\alpha}{(\alpha + \beta)^4}(2\beta(\alpha + \beta)t - \beta^2 + \alpha^2 + 4\beta - 2\alpha)\rho^{t+1}
\]

Similarly,

\[
E\left\{ x_t^2 \Big| x_0 = 1 \right\} = \frac{(t+1)}{6} \left[ \frac{6+t(-\beta^2-3\alpha\beta-5\beta+6)}{t^2(-\beta^2+3\alpha\beta-4\beta)+2t^3\beta^2} + \sigma^2 \beta \right]
\]

Which, when combined with the expectation provides the variances stated in equation (18) in Proposition 3. Similar manipulations applied to (4.7) will yield equation (19). Finally, we note that the formulas for the variance are reduced as well to (4.3) and (4.4). In these transformations we have used as well the relationship \( \rho = 1 - \alpha - \beta \).
ABSTRACT

This paper considers a memory-based persistent counting random walk, based on a Markov memory of the last event. This persistent model is a different than the Weiss persistent random walk model however, leading thereby to different results. We point out to some preliminary result, in particular, we provide an explicit expression for the mean and the variance, both nonlinear in time, of the underlying memory-based persistent process and discuss the usefulness to some problems in insurance, finance and risk analysis. The motivation for the paper arose from the counting of events (whether rare or not) in insurance that presume that events are time independent and therefore based on the Poisson distribution for counting these events.