Mazurkiewicz's Theorem

Mazurkiewicz's Theorem says that there is a subset of $\mathbb{R}^2$ which intersects each line exactly twice. The proof below is taken from the first of the two links

http://www.mathnerds.com/best/Mazurkiewicz/solution.aspx

http://mathoverflow.net/questions/21470/subset-of-the-plane-that-intersects-every-line-exactly-twice

We define the ordinals in such a way that each ordinal a is the set of those ordinals $< a$. For any set S we write $|S|$ for the least ordinal equipotent to S, and call it the *cardinality* of S. Let c (for *continuum*) be the cardinality of $\mathbb{R}$. For any subset S of $\mathbb{R}^2$ we denote by $\langle S\rangle$ the set of lines generated by S (a line being generated by S if it has at least two points in common with S). Let $d \mapsto L_d$ be a bijection from c onto the set of lines in $\mathbb{R}^2$.

For each $d \in c$ we define $Z_d \subset \mathbb{R}^2$, $f(d) \in c$, and $z_d \in \mathbb{R}^2$, as follows. Let z_0 be any point of L_0, put $f(0) = 0$, and let Z_0 be the empty set. Now assume $0 < d < c$, and $f(e), z_e$ already defined for $e < d$. Put $Z_d := \{z_e \mid e < d\}$. As $|\langle Z_d\rangle| < c$ (because $|Z_d| < c$), there is a least $f(d)$ in c such that $L_{f(d)} \notin \langle Z_d\rangle$. Let z_d be any point of the set

$$L_{f(d)} - (Z_d \cup (\cup\langle Z_d\rangle)),$$

which is easily seen to be nonempty.

Let Z be the union of the Z_d and L any line in $\mathbb{R}^2$. We claim $|L \cap Z| = 2$, that is, Z is the sought-for subset of $\mathbb{R}^2$. To prove $|L \cap Z| \leq 2$ $(*)$ we assume by contradiction that there are $g < h < i$ in c such that $z_g, z_h, z_i \in L \cap Z$. We have $z_g, z_h \in Z_i$ by definition of Z_i, and thus $L \in \langle Z_i\rangle$, contradicting the definition of z_i. To prove $|L\cap Z| \geq 2$ we assume by contradiction $|L\cap Z| < 2$. Put $L = L_d$ and let g be in c. The inequality $|L_d \cap Z_g| < 2$ implies $L_d \notin \langle Z_g\rangle$, and thus $f(g) \leq d$ by minimality of $f(g)$. This shows that Z is contained into the union of the L_e such that $e \leq d$. As $|L_e \cap Z| \leq 2$ by $(*)$, we get $|Z| \leq 2|d| + 2 < c$, contradicting the obvious equality $|Z| = c$.

mazurkiewicz.100614

Pierre-Yves Gaillard