Spin$^c$ structures on manifolds

Nicolas Ginoux

Seminar on Seiberg-Witten theory - University of Regensburg

November 17, 2012

Abstract: After introducing the spin$^c$ group and the spinor representation, we discuss spin$^c$ structures and show that every orientable closed smooth 4-dimensional manifold has a spin$^c$ structure. We closely follow [6, App. D] and [1] (see also [2] for a few details).

1 The spin$^c$ group and its representations

1.1 The spin group

Definition 1.1 Let $n$ be a positive integer. The spin group in dimension $n$, denoted by $\text{Spin}_n$, is the non-trivial 2-fold covering of the special orthogonal group $\text{SO}_n$.

The group $\text{Spin}_n$ is a compact $\frac{n(n-1)}{2}$-dimensional Lie group, connected if $n \geq 2$ and simply-connected if $n \geq 3$. In fact, if $\text{Spin}_n \xrightarrow{\xi} \text{SO}_n$ denotes this non-trivial covering map, then $\xi(z) = z^2$ for any $z \in \text{Spin}_2 \cong U^1 = \{z \in \mathbb{C}, |z| = 1\}$ and $\xi$ is the universal covering map if $n \geq 3$. In particular, we have the following short exact sequence of Lie groups:

$$1 \longrightarrow \{\pm 1\} \longrightarrow \text{Spin}_n \xrightarrow{\xi} \text{SO}_n \longrightarrow 1.$$

Examples 1.2

1. For $n = 3$, the spin group $\text{Spin}_3 \cong \text{SU}_2$, where $\xi$ becomes the well-known 2-fold covering map.

2. For $n = 4$, the spin group $\text{Spin}_4 \cong \text{Spin}_3 \times \text{Spin}_3 \cong \text{SU}_2 \times \text{SU}_2.$
This defines the spin group as an abstract Lie group. Actually, the spin group is a Lie subgroup of a natural Lie group, namely the group of units of a Clifford algebra.

**Definition 1.3** Let \( q_C(z, z') := \sum_{j=1}^{n} z_j z'_j \) denote the canonical complex bilinear form on \( \mathbb{C}^n \). The complex Clifford algebra in dimension \( n \) is defined as
\[
\mathbb{C}l_n := \mathbb{C}l(\mathbb{C}^n, q_C) := \bigotimes \mathbb{C}^n / \mathcal{I},
\]
where \( \bigotimes \mathbb{C}^n \) denotes the tensor algebra of \( \mathbb{C}^n \) and \( \mathcal{I} \) the two-sided ideal generated by the elements of the form \( z \otimes w + w \otimes z + 2q_C(z, z') \cdot 1 \), where \( z, w \) run in \( \mathbb{C}^n \).

**Proposition 1.4** Endowed with the so-called Clifford multiplication \( [a] \cdot [b] := [a \otimes b] \), the complex Clifford algebra in dimension \( n \) is an associative algebra with unit which is linearly isomorphic to the exterior algebra \( \bigwedge \mathbb{C}^n \) (hence of complex dimension \( 2^n \)). It can be characterised as the smallest associative complex algebra with unit containing \( \mathbb{C}^n \) and where the relations
\[
z \cdot w + w \cdot z = -2q_C(z, w) \cdot 1
\]
are satisfied for all \( z, w \in \mathbb{C}^n \).

**Proposition 1.5** The spin group in dimension \( n \) can be identified with the following subgroup of the group \( \mathbb{C}l_n^\times \) of units of \( \mathbb{C}l_n \):
\[
\text{Spin}_n \cong \{v_1 \cdot \ldots \cdot v_{2k} \mid v_j \in \mathbb{R}^n, |v_j| = 1, \ k \geq 1\} \subset \mathbb{C}l_n^\times.
\]
Moreover, the 2-fold covering homorphism \( \xi \) can be identified with the restriction of the adjoint map acting on \( \mathbb{R}^n \):
\[
\xi = \text{Ad}_{\text{Spin}_n} \circ \text{Spin}_n \rightarrow \text{Aut}(\mathbb{R}^n), \ u \mapsto (v \mapsto u \cdot v \cdot u^{-1}).
\]

### 1.2 The spin\(^c\) group

**Definition 1.6** Let \( n \) be a positive integer. The spin\(^c\) group in dimension \( n \), denoted by \( \text{Spin}_n^c \), is the subgroup
\[
\text{Spin}_n^c := \{\lambda u \mid \lambda \in U_1, u \in \text{Spin}_n\} \subset \mathbb{C}l_n^\times.
\]

The group homomorphism
\[
\text{Spin}_n \times U_1 \longrightarrow \text{Spin}_n^c \quad (u, \lambda) \mapsto \lambda u
\]
is by definition surjective and its kernel is \{±(1, 1)\} since \(\text{Spin}_n \cap U_1 = \{±1\}\). Therefore,
\[
\text{Spin}^c_n \cong \text{Spin}_n \times U_1/\mathbb{Z}_2,
\]
which is sometimes taken as a definition for the \(\text{spin}^c\) group.

As for the spin group, there is a short exact sequence of Lie groups
\[
1 \rightarrow \{±1\} \rightarrow \text{Spin}^c_n \xrightarrow{\xi^c} \text{SO}_n \times U_1 \rightarrow 1,
\]
where \(\xi^c([u, \lambda]) := (\xi(u), \lambda^2)\). Beware that \(\text{Spin}^c_n\), though connected for \(n \geq 2\), is never simply-connected:
\[
\pi_1(\text{Spin}^c_n) = \begin{cases} 
\mathbb{Z}^2 & \text{if } n = 2 \\
\mathbb{Z} & \text{if } n \geq 3.
\end{cases}
\]

### 1.3 The spinor representation

**Proposition 1.7** Let \(\Sigma_n := \mathbb{C}^{2^{[\frac{n}{2}]}},\) then there exist complex algebra homomorphisms
\[
\mathbb{C}l_n \cong \begin{cases} 
\text{End}_\mathbb{C}(\Sigma_n) & \text{if } n \text{ is even} \\
\text{End}_\mathbb{C}(\Sigma_n) \oplus \text{End}_\mathbb{C}(\Sigma_n) & \text{if } n \text{ is odd}.
\end{cases}
\]

The representation space \(\Sigma_n\) can actually be constructed explicitly as a subspace of \(\mathbb{C}l_n\) itself (on which \(\mathbb{C}l_n\) acts from the left by Clifford multiplication), see [2].

Since any complex matrix algebra is simple, Proposition 1.7 implies that there is up to equivalence only one (non-zero) irreducible complex representation of \(\mathbb{C}l_n\) if \(n\) is even and there are exactly two if \(n\) is odd. To distinguish the two, we introduce the so-called complex volume element
\[
\omega_n^c := i^{\frac{n+1}{2}}e_1 \cdots e_n \in \mathbb{C}l_n,
\]
where \((e_j)_{1 \leq j \leq n}\) is any p.o.n.b of \(\mathbb{R}^n\) with the canonical metric and orientation.

**Lemma 1.8** The complex volume element acts as an isometric involution on \(\Sigma_n\). More precisely,
\[
\omega_n^c \cdot = \begin{cases} 
\text{Id}_{\Sigma^+} \oplus -\text{Id}_{\Sigma^-} & \text{if } n \text{ is even} \\
\text{Id}_{\Sigma^+} \oplus -\text{Id}_{\Sigma^-} & \text{if } n \text{ is odd},
\end{cases}
\]
where \(\Sigma_n^\pm := \text{Ker}(\omega_n^c \cdot \mp \text{Id}_{\Sigma_n}) \subset \Sigma_n\) in the case \(n\) even.
From now on, we denote by $\delta_n : \mathbb{C}\iota_n \rightarrow \text{End}_C(\Sigma_n)$ the representation provided by Proposition 1.7 if $n$ is even and by the factor of $\text{End}_C(\Sigma_n) \oplus \text{End}_C(\Sigma_n)$ on which $\omega^C_n$ acts as the identity if $n$ is odd.

**Definition 1.9** The representation $\delta_n$ is called the **complex spinor representation**.

**Proposition 1.10** The spinor representation satisfies the following:

i) There exists up to scaling only one Hermitian product on $\Sigma_n$ such that each vector in $\mathbb{R}^n$ acts in a skew-Hermitian way on $\Sigma_n$.

ii) In $n$ is even, then $\delta_n|_{\text{Spin}}^\iota_n$ splits into the sum of two inequivalent irreducible complex representations: $\delta_n|_{\text{Spin}}^\iota_n = \delta_n^+ \oplus \delta_n^-$, where $\delta_n^\pm : \text{Spin}_n^\iota \rightarrow \text{Aut}_C(\Sigma_n^\pm)$ are irreducible with $\delta_n^+ \not\approx \delta_n^-$.  

iii) If $n$ is odd, then $\delta_n|_{\text{Spin}}^\iota_n$ is irreducible. Moreover, the restriction of the factor of $\text{End}_C(\Sigma_n) \oplus \text{End}_C(\Sigma_n)$ on which $\omega^C_n$ acts as minus the identity to $\text{Spin}_n^\iota_n$ gives rise to an equivalent representation.

In case $n$ even, the representations $\delta_n^\pm$ are called **half-spinor representations**; $\delta_n^+$ is the positive one and $\delta_n^-$ the negative one. Note in particular that, as a consequence of Proposition 1.10(i) and of Proposition 1.5, the representation $\delta_n$ is unitary.

## 2 Spin$^c$ structures

We denote by $P_{\text{SO}_n}TM \rightarrow M$ the $\text{SO}_n$-principal bundle of positively oriented orthonormal frames on the tangent bundle of an oriented Riemannian manifold $(M^n, g)$.

**Definition 2.1** Let $(M^n, g)$ be an $n$-dimensional oriented Riemannian manifold.

1. A spin structure on $(M^n, g)$ is a reduction of $P_{\text{SO}_n}TM \rightarrow M$ to the spin group. More precisely, a spin structure is given by a $\text{Spin}_n$-principal bundle $P_{\text{Spin}_n}TM \rightarrow M$ together with a 2-fold covering map $P_{\text{Spin}_n}TM \overset{\eta}{\longrightarrow} P_{\text{SO}_n}TM$ such that the following diagramme commutes:

$$
\begin{array}{ccc}
P_{\text{Spin}_n}TM \times \text{Spin}_n & \longrightarrow & P_{\text{Spin}_n}TM \\
\eta \times \xi & \searrow & \\
\downarrow & & M \\
\downarrow & & \\
P_{\text{SO}_n}TM \times \text{SO}_n & \longrightarrow & P_{\text{SO}_n}TM
\end{array}
$$
2. A spin$^c$-structure on $(M^n, g)$ consists of a pair $(P_{\text{Spin}^c_n} TM, P_{U_1})$, where $P_{\text{Spin}^c_n} TM \rightarrow M$ is a Spin$^c_n$-principal bundle, $P_{U_1} \rightarrow M$ is a $U_1$-principal bundle, together with a 2-fold covering map $P_{\text{Spin}^c_n} TM \xrightarrow{\eta^c} P_{SO_n} TM \times P_{U_1}$ such that the following diagram commutes:

\[ P_{\text{Spin}^c_n} TM \times \text{Spin}^c_n \xrightarrow{\eta^c \times \xi^c} P_{\text{Spin}^c_n} TM \xleftarrow{\eta^c} M \]

\[ P_{SO_n} TM \times M P_{U_1} \times (SO_n \times U_1) \xrightarrow{\eta^c \times \xi^c} P_{SO_n} TM \times M P_{U_1} \]

3. The manifold $(M^n, g)$ is called spin (resp. spin$^c$) if and only if it admits a spin- (resp. spin$^c$-)structure.

Any spin structure defines a spin$^c$ structure in an obvious way: take $P_{U_1} := M \times U_1$ to be the trivial $U_1$-bundle and extend the Spin$^c_n$-bundle via the inclusion Spin$^c_n \subset \text{Spin}^c_n$. In particular, any spin manifold is spin$^c$.

The condition to be spin or spin$^c$ a priori depends on the metric (through $P_{SO_n} TM$). It actually only has to do with the topology of the manifold since it may be understood as an orientability condition of second order, as we shall prove next. Denote by $r : \mathbb{Z} \rightarrow \mathbb{Z}_2$ the mod-2-reduction and also by $r : H^q(M; \mathbb{Z}) \rightarrow H^q(M; \mathbb{Z}_2)$ the induced homomorphism in cohomology.

**Proposition 2.2**

i) A smooth manifold $M$ is spin if and only if its first and second Stiefel-Whitney classes vanish, that is, iff $w_1(TM) = 0$ and $w_2(TM) = 0$.

ii) A smooth manifold $M$ is spin$^c$ if and only if its first Stiefel-Whitney class vanishes and its second Stiefel-Whitney class is the mod-2-reduction of an integral class, that is, iff $w_1(TM) = 0$ and $w_2(TM) \in r(H^2(M; \mathbb{Z}))$.

**Proof:** We only prove ii), see e.g. [6] or [3] for i). We follow [6] App. D]. First $M$ has to be orientable in order to be spin$^c$, the orientability of $TM$ being equivalent to $w_1(TM) = 0$, which we assume from now on. The short exact sequence of groups [1] induces the following long exact sequence inČech cohomology:

\[ \ldots \rightarrow H^1(M; \mathbb{Z}_2) \rightarrow H^1(M; \text{Spin}^c_n) \xrightarrow{\xi^c} H^1(M; SO_n) \oplus H^1(M; U_1) \xrightarrow{w_2 + r_{\circ \xi^c}} H^2(M; \mathbb{Z}_2) \rightarrow \ldots, \]
where \( w_2 : H^1(M; \text{SO}_n) \to H^2(M; \mathbb{Z}_2) \) denotes the homomorphism associating the second Stiefel-Whitney class to an equivalence class of \( \text{SO}_n \)-bundles (or, equivalently, of Riemannian vector bundles) and \( c_1 : H^1(M; \mathbb{U}_1) \to H^2(M; \mathbb{Z}) \) denotes the homomorphism associating the first Chern class to an equivalence class of \( \text{U}_1 \)-bundles (or, equivalently, of Hermitian line bundles).

The condition \( M \) to be spin\(^c\) means that there exists a \( \mathbb{U}_1 \)-bundle \( P_{\mathbb{U}_1} \to M \) such that the element \( [(P_{\text{SO}_n} TM, P_{\mathbb{U}_1})] \in H^1(M; \text{SO}_n) \oplus H^1(M; \mathbb{U}_1) \) lies in the image of the map \( \xi^c \). This, in turn, is equivalent to \( [(P_{\text{SO}_n} TM, P_{\mathbb{U}_1})] \) lying in the kernel of \( w_2 + r \circ c_1 \), meaning that \( w_2(P_{\text{SO}_n} TM) = r(c_1(P_{\mathbb{U}_1})) \). Since \( c_1 : H^1(M; \mathbb{U}_1) \to H^2(M; \mathbb{Z}) \) is a group isomorphism, the condition to be spin\(^c\) for \( M \) is therefore equivalent to \( w_2(TM) = w_2(P_{\text{SO}_n} TM) \in r(H^2(M; \mathbb{Z})) \), which was to be shown.

\[ \square \]

Examples 2.3

1. Any 1-dimensional manifold is spin, a circle having two inequivalent spin structures. Any orientable surface is also spin since in that case \( w_2(TM) \in H^2(M; \mathbb{Z}_2) \cong \mathbb{Z}_2 \) is the mod-2-reduction of the Euler characteristic (which is even). Any 3-dimensional orientable 3-dimensional manifold has trivial tangent bundle and hence is spin. The “simplest” example of non-spin manifold is the complex 2-dimensional projective space \( \mathbb{C}P^2 \).

2. The set of (inequivalent) spin structures of a given spin manifold \( M \) can be shown to stand in one-to-one correspondence with its cohomology group \( H^1(M; \mathbb{Z}_2) \). In particular, \( M \) may have more than one spin structure. However, there is only one if e.g. \( M \) is simply-connected.

3. Any \emph{almost-Hermitian} manifold has a natural spin\(^c\) structure, due to the existence of a reduction of \( P_{\text{SO}_{2m}} \) to the unitary group \( \mathbb{U}_m \) and of a lift \( U_m \to \text{Spin}_{2m}^c \) over \( U_m \xrightarrow{\text{incl.} \times \det} \text{SO}_{2m} \times \mathbb{U}_1 \).

4. Spin\(^c\) structures need also not be unique: if \( P_{\mathbb{U}_1}(\alpha) \to M \) is any \( \mathbb{U}_1 \)-bundle (with Chern-class \( \alpha \in H^2(M; \mathbb{Z}) \)) over \( M \), then

\[
P_{\text{Spin}_{2m}^c \times_M P_{\mathbb{U}_1}(\alpha)/\mathbb{U}_1} \to P_{\text{SO}_n \times_M (P_{\mathbb{U}_1} \otimes P_{\mathbb{U}_1}(2\alpha))}
\]

defines a new spin\(^c\) structure, where the new associated \( \mathbb{U}_1 \)-bundle is \( P_{\mathbb{U}_1} \otimes P_{\mathbb{U}_1}(2\alpha) \).

From now on, we shall implicitly assume that, on a given spin\(^c\) manifold, a spin\(^c\) structure is fixed.
**Definition 2.4** Let \((M^n, g)\) be a spin\(^c\) manifold. The spinor bundle of \(M\) is the vector bundle – denoted by \(\Sigma M\) – associated to the Spin\(_{\text{c}}^c\)-bundle via the spinor representation:

\[
\Sigma M := P_{\text{Spin}_{\text{c}}}^c TM \times \delta_n \Sigma_n = P_{\text{Spin}_{\text{c}}}^c TM \times \Sigma_n / \sim,
\]

where \((p, \sigma) \sim (p \cdot u, \delta_n(u^{-1})(\sigma))\) for all \((p, \sigma) \in P_{\text{Spin}_{\text{c}}}^c TM \times \Sigma_n\) and \(u \in \text{Spin}_{\text{c}}^c\).

By definition, the spinor bundle is a complex vector bundle of (complex) rank \(2^{[n/2]}\) over \(M\). Sections of \(\Sigma M\) are called spinor fields or just spinors. Since \(\delta_n\) can be assumed unitary (see above), \(\Sigma M\) can be naturally endowed with a pointwise Hermitian inner product \(\langle \cdot, \cdot \rangle\), turning it into a Hermitian vector bundle. Like the space \(\Sigma_n\), the spinor bundle also admits a Clifford multiplication:

**Proposition 2.5** The spinor representation of \(\mathbb{C}l_n\) induces a linear map

\[
TM \otimes \Sigma M \rightarrow \Sigma M, \quad X \otimes \varphi \mapsto X \cdot \varphi,
\]

satisfying the (pointwise) Clifford relation

\[
X \cdot (Y \cdot \varphi) + Y \cdot (X \cdot \varphi) = -2g(X,Y) \varphi
\]

for all \(X, Y \in TM\) and \(\varphi \in \Sigma M\). Moreover, the Hermitian inner product \(\langle \cdot, \cdot \rangle\) can be defined such that

\[
\langle X \cdot \varphi, \psi \rangle = -\langle \varphi, X \cdot \psi \rangle
\]

for all \(X \in TM\) and \(\varphi, \psi \in \Sigma M\).

As a last important step, any connection 1-form on the auxiliary bundle \(P_{U_1}\) induces, together with the Levi-Civita connection of \((M^n, g)\), a metric connection on \(\Sigma M\):

**Proposition 2.6** Let \(A \in \Omega^1(P_{U_1}, i\mathbb{R})\) be any connection 1-form on \(P_{U_1}\). Then \(A\) and the Levi-Civita connection \(\nabla\) of \((M^n, g)\) together induce a metric covariant derivative \(\nabla^A\) on \(\Sigma M\), which satisfies:

\[
\nabla^A_X(Y \cdot \varphi) = (\nabla_X Y) \cdot \varphi + Y \cdot \nabla^A_X \varphi,
\]

for all \(X, Y \in \Gamma(M, TM)\) and \(\varphi \in \Gamma(M, \Sigma M)\).

**Definition 2.7** The Dirac operator associated to a connection 1-form \(A\) on the auxiliary bundle \(P_{U_1} \rightarrow M\) on a spin\(^c\) manifold \((M^n, g)\) is the operator

\[
D^A : \Gamma(M, \Sigma M) \rightarrow \Gamma(M, \Sigma M), \quad \varphi \mapsto \sum_{j=1}^n e_j \cdot \nabla^A_{e_j} \varphi,
\]

where \(\nabla^A\) is the covariant derivative associated to \(A\) and \((e_j)_{1 \leq j \leq n}\) is any local o.n.b. of \(TM\).
The Dirac-operator is a well-defined, elliptic, formally self-adjoint differential operator of order 1. It is even essentially self-adjoint if \((M^n, g)\) is complete.

**Theorem 2.8 (Schrödinger-Lichnerowicz formula)** For any connection 1-form \(A\) on the auxiliary bundle \(P_{U_1} \to M\) on a spin\(c\) manifold \((M^n, g)\), we have

\[
(D^A)^2 = (\nabla^A)^* \nabla^A + \frac{S}{4} \text{Id} + \frac{F_A}{2} \cdot \text{Id},
\]

where \((\nabla^A)^* \nabla^A := -\text{tr}_g((\nabla^A)^2) = \sum_{j=1}^n \nabla^A_{\nabla e_j e_j} - \nabla^A\nabla^A_{e_j e_j}\) is the connection Laplacian associated to \(\nabla^A\) (here \(\{e_j\}_{1 \leq j \leq n}\) is a local o.n.b. of \(TM\)), \(S\) is the scalar curvature of \((M, g)\) and \(F_A \in \Gamma(A^2 T^* M \otimes i\mathbb{R})\) is the curvature form of \(A\).

**Proof:** Fix a local orthonormal basis \(\{e_j\}_{1 \leq j \leq n}\) of \(TM\). Using the compatibility conditions as well as the Clifford relations, we have, for any \(\varphi \in \Gamma(\Sigma M)\),

\[
(D^A)^2 \varphi = \sum_{j,k=1}^n e_j \cdot \nabla^A_{e_j} (e_k \cdot \nabla^A_{e_k} \varphi)
\]

\[
= \sum_{j,k=1}^n e_j \cdot \nabla e_j e_k \cdot \nabla^A_{e_k} \varphi + e_j \cdot e_k \cdot \nabla^A_{e_j e_k} \varphi
\]

\[
= - \sum_{j,k=1}^n e_j \cdot e_k \cdot \nabla^A_{\nabla e_j e_k} \varphi + \sum_{j,k=1}^n e_j \cdot e_k \cdot \nabla^A_{e_j e_k} \varphi
\]

\[
= \sum_{j=1}^n (\nabla^A_{\nabla e_j e_j} - \nabla^A_{e_j e_j}) \varphi
\]

\[
+ \sum_{1 \leq j < k \leq n} e_j \cdot e_k \cdot (\nabla^A_{e_j e_k} \nabla^A_{e_k} - \nabla^A_{e_k e_k} \nabla^A_{e_j} + \nabla^A_{e_j e_k} + \nabla^A_{e_k e_j}) \varphi
\]

\[
= (\nabla^A)^* \nabla^A \varphi + \sum_{1 \leq j < k \leq n} e_j \cdot e_k \cdot (\nabla^A_{e_j e_k} - \nabla^A_{e_j e_k}) \varphi
\]

\[
= (\nabla^A)^* \nabla^A \varphi + \frac{1}{2} \sum_{j,k=1}^n e_j \cdot e_k \cdot R^A_{e_j e_k} \varphi.
\]

Now locally the connection \(\nabla^A\) and its curvature \(R^A\) can be expressed as follows: choosing local sections \(u\) of \(P_{SO_n} TM \to M\) and \(s\) of \(P_{U_1} \to M\), we obtain a local section \(\tilde{u}\) of \(P_{\text{Spin}_c} TM \to M\) and hence a local trivialization \(\{\psi_\alpha\}_{1 \leq \alpha \leq 2^\frac{n}{2}}\) of \(\Sigma M\). In that case, we have, for all tangent vectors \(X, Y\) (defined locally),

\[
\nabla^A_X \psi_\alpha = \frac{1}{4} \sum_{j,k=1}^n g(\nabla_X e_j, e_k) e_j \cdot e_k \cdot \psi_\alpha + \frac{A(ds(X))}{2} \psi_\alpha.
\]
from which
\[ R_{X,Y}^{\nabla A} = \frac{1}{4} \sum_{j,k=1}^{n} g(R_{X,Y}^{\nabla} e_j, e_k)e_j \cdot e_k \cdot + \frac{1}{2} dA(ds(X), ds(Y)) \]

follows. By definition of the Clifford action of forms
\[ \sum_{j,k=1}^{n} F_A(e_j, e_k)e_j \cdot e_k = 2 \sum_{1 \leq j < k \leq n} F_A(e_j, e_k)e_j \cdot e_k = 2F_A, \]

so that only the action of the curvature of the Levi-Civita connection of \((M, g)\) remains to be determined. The first Bianchi identity and the preceding local expressions of \(\nabla^A\) and \(R_{X,Y}^{\nabla A}\) imply that, for any \(X \in TM\),
\[ \sum_{j,k,l=1}^{n} g(R_{X,e_j}^{\nabla} e_k, e_l)e_j \cdot e_k \cdot e_l \cdot \varphi = - \sum_{j,k,l=1}^{n} g(R_{e_j,e_k}^{\nabla} X, e_l)e_j \cdot e_k \cdot e_l \cdot \varphi \]
\[ - \sum_{j,k,l=1}^{n} g(R_{e_k,X}^{\nabla} e_j, e_l)e_j \cdot e_k \cdot e_l \cdot \varphi \]
\[ = - \sum_{j,k,l=1}^{n} g(R_{X,e_j}^{\nabla} e_k, e_l)(e_k \cdot e_l \cdot e_j - e_k \cdot e_j \cdot e_l) \cdot \varphi, \]

with
\[ e_k \cdot e_l \cdot e_j - e_k \cdot e_j \cdot e_l = - e_k \cdot e_j \cdot e_l - 2\delta_{jl}e_k + e_j \cdot e_k \cdot e_l + 2\delta_{jk}e_l \]
\[ = 2e_j \cdot e_k \cdot e_l + 4\delta_{jk}e_l - 2\delta_{jl}e_k. \]

We deduce that
\[ 3 \sum_{j,k,l=1}^{n} g(R_{X,e_j}^{\nabla} e_k, e_l)e_j \cdot e_k \cdot e_l \cdot \varphi = -4 \sum_{j,l=1}^{n} g(R_{X,e_j}^{\nabla} e_l,e_j) e_l \cdot \varphi + 2 \sum_{j,k=1}^{n} g(R_{X,e_j}^{\nabla} e_k, e_j)e_k \cdot \varphi \]
\[ = -4 \sum_{l=1}^{n} g(Ric(X), e_l) e_l \cdot \varphi - 2 \sum_{k=1}^{n} g(Ric(X), e_k)e_k \cdot \varphi \]
\[ = -6Ric(X) \cdot \varphi, \]
where $\text{Ric}$ denotes the Ricci tensor of $(M,g)$. Therefore,

$$
\frac{1}{2} \sum_{j,k=1}^{n} e_j \cdot e_k \cdot R^A_{ej,ek} \varphi = \frac{1}{8} \sum_{i,j,k,l=1}^{n} g(R^A_{ei,ej} e_i, e_j) e_i \cdot e_j \cdot e_k \cdot e_l \cdot \varphi
+ \frac{1}{4} \sum_{j,k=1}^{n} F_A(e_j, e_k) e_j \cdot e_k \cdot \varphi
$$

$$
= -\frac{1}{4} \sum_{i=1}^{n} e_i \cdot \text{Ric}(e_i) \cdot \varphi + \frac{F_A}{2} \cdot \varphi
$$

$$
= -\frac{1}{4} \sum_{i,j=1}^{n} g(\text{Ric}(e_i), e_j) e_i \cdot e_j \cdot \varphi + \frac{F_A}{2} \cdot \varphi
$$

$$
= \frac{1}{4} \sum_{i=1}^{n} g(\text{Ric}(e_i), e_i) \varphi + \frac{F_A}{2} \cdot \varphi
$$

$$
= S \varphi + \frac{F_A}{2} \cdot \varphi,
$$

which concludes the proof. □

3 The 4-dimensional case

**Theorem 3.1 ([5])** Every closed orientable smooth 4-dimensional manifold is spin$^c$.

**Proof**: We follow [1, pp. 144-145]. We can assume w.l.o.g. that the manifold $M$ is connected. By Proposition 2.2, we have to show that $w_2(TM) \in \text{Im}(r) := r(H^2(M;\mathbb{Z})) \subset H^2(M;\mathbb{Z}_2)$. We define $T := \text{Tor}(H^2(M;\mathbb{Z}))$, the torsion subgroup of $H^2(M;\mathbb{Z})$.

**Claim**: $\text{Im}(r) = \{ \gamma \in H^2(M;\mathbb{Z}_2), \gamma \cup y = 0 \ \forall y \in r(T) \}$.

**Proof**: Let $\Gamma := \{ \gamma \in H^2(M;\mathbb{Z}_2), \gamma \cup y = 0 \ \forall y \in r(T) \}$. If $\gamma \in \text{Im}(r)$, then there exists $\alpha \in H^2(M;\mathbb{Z})$ with $r(\alpha) = \gamma$. Similarly, for any $y \in r(T)$, there exists $\beta \in T$ with $r(\beta) = y$. It follows $\gamma \cup y = r(\alpha \cup \beta)$. But $\alpha \cup \beta \in H^4(M;\mathbb{Z})$ and $H^4(M;\mathbb{Z}) \cong \mathbb{Z}$ because $M$ is orientable; since $\beta$ is a torsion element, there exists an $m \in \mathbb{N} \setminus \{0\}$ with $m\beta = 0$ and hence $m(\alpha \cup \beta) = 0$, which yields $\alpha \cup \beta = 0$ and therefore $\gamma \cup y = 0$. This shows $\text{Im}(r) \subset \Gamma$. The other inclusion will be proven as soon as the $\mathbb{Z}_2$-dimensions of $\text{Im}(r)$ and $\Gamma$ are shown to coincide. Since $\mathbb{Z}_2$ is a field, the cup product $H^2(M;\mathbb{Z}_2) \times H^2(M;\mathbb{Z}_2) \rightarrow H^4(M;\mathbb{Z}_2) \cong \mathbb{Z}_2$ defines a non-degenerate (symmetric) bilinear form (see e.g.
\[ \text{Prop. 3.38}], \text{ therefore} \]
\[
\dim_{\mathbb{Z}_2}(\Gamma) = \dim_{\mathbb{Z}_2}(r(T)^\perp) = \dim_{\mathbb{Z}_2}(H^2(M; \mathbb{Z}_2)) - \dim_{\mathbb{Z}_2}(r(T)).
\]
Thus we have to show that \( \dim_{\mathbb{Z}_2}(\text{Im}(r)) = \dim_{\mathbb{Z}_2}(H^2(M; \mathbb{Z}_2)) - \dim_{\mathbb{Z}_2}(r(T)). \)
The short exact sequence of abelian groups \( 0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{r} \mathbb{Z}_2 \to 0 \)
induces the following long exact sequence in cohomology
\[
\ldots \to H^2(M; \mathbb{Z}) \xrightarrow{2} H^2(M; \mathbb{Z}) \xrightarrow{r} H^2(M; \mathbb{Z}_2) \xrightarrow{\beta} H^3(M; \mathbb{Z}) \to \ldots
\]
where \( \beta \) is the so-called \textit{Bockstein homomorphism}. In particular, \( \text{Im}(r) = \text{Ker}(\beta) \), so that \( \dim_{\mathbb{Z}_2}(\text{Im}(r)) = \dim_{\mathbb{Z}_2}(H^2(M; \mathbb{Z}_2)) - \dim_{\mathbb{Z}_2}(\text{Im}(\beta)) \).
Hence it suffices to show that \( \text{Im}(\beta) \cong r(T) \). Now the universal coefficient theorem (see e.g. \[ \text{4}, \text{Thm 3.2}]) states that there is the following short exact sequence of \( \mathbb{Z} \)-modules:
\[
0 \to \text{Ext}_\mathbb{Z}(H_{q-1}(M; \mathbb{Z}), \mathbb{Z}) \to H^q(M; \mathbb{Z}) \to \text{Hom}_\mathbb{Z}(H_q(M; \mathbb{Z}), \mathbb{Z}) \to 0,
\]
for every \( q \in \mathbb{N} \setminus \{0\} \). Moreover, the free part of \( H_{q-1}(M; \mathbb{Z}) \) does not contribute to \( \text{Ext}_\mathbb{Z} \), more precisely \( \text{Ext}_\mathbb{Z}(H_{q-1}(M; \mathbb{Z}), \mathbb{Z}) \cong \text{Tor}(H_{q-1}(M; \mathbb{Z})) \).
Similarly, the torsion part of \( H_q(M; \mathbb{Z}) \) does not contribute to \( \text{Hom}_\mathbb{Z} \), that is, \( \text{Hom}_\mathbb{Z}(H_q(M; \mathbb{Z}), \mathbb{Z}) \cong \mathbb{Z}^{b_q} \), where \( b_q \in \mathbb{N} \) is the rank of \( H_q(M; \mathbb{Z}) \); in particular, \( \text{Hom}_\mathbb{Z}(H_q(M; \mathbb{Z}), \mathbb{Z}) \cong H_q(M; \mathbb{Z})/\text{Tor}(H_q(M; \mathbb{Z})) \).
Since the latter is free, the short exact sequence above splits and we obtain
\[
H^q(M; \mathbb{Z}) \cong \text{Ext}_\mathbb{Z}(H_{q-1}(M; \mathbb{Z}), \mathbb{Z}) \oplus \text{Hom}_\mathbb{Z}(H_q(M; \mathbb{Z}), \mathbb{Z}) \oplus \text{Tor}(H_{q-1}(M; \mathbb{Z}) \oplus H_q(M; \mathbb{Z})/\text{Tor}(H_q(M; \mathbb{Z})).
\]
As a consequence, \( \text{Tor}(H^q(M; \mathbb{Z})) \cong \text{Tor}(H_{q-1}(M; \mathbb{Z})) \). Since \( M \) is closed and orientable, Poincaré duality implies \( H_{q-1}(M; \mathbb{Z}) \cong H^{n-q+1}(M; \mathbb{Z}) \), where \( n \) is the dimension of the manifold \( M \). Here we obtain, for \( n = 4 \) and \( q = 3 \):
\[
\text{Tor}(H^3(M; \mathbb{Z})) \cong \text{Tor}(H_2(M; \mathbb{Z})) \cong \text{Tor}(H^2(M; \mathbb{Z})).
\]
We deduce that
\[
\text{Im}(\beta) = \text{Ker}(2 \cdot) \setminus \{ \alpha \in H^3(M; \mathbb{Z}) \mid 2\alpha = 0 \}
\setminus \{ \alpha \in \text{Tor}(H^3(M; \mathbb{Z})) \mid 2\alpha = 0 \}
\cong \{ \alpha \in T \mid 2\alpha = 0 \}
\]
Writing \( T = \bigoplus_{j=1}^{n} \mathbb{Z}_{p_j} \), with \( p_j \in \mathbb{N} \) prime and \( k_j \geq 1 \), the subgroup \( \{ \alpha \in T \mid 2\alpha = 0 \} \) of \( T \) is freely generated over \( \mathbb{Z}_2 \) by the elements of the form
$2^{k_j-1} \in \mathbb{Z}_{p_j}$ (only the $p_j = 2$ appear since 2 is invertible in $\mathbb{Z}_{p_j}$ for any prime $p_j > 2$). So is $T/2T$ for the same reasons. Hence $\{\alpha \in T, 2\alpha = 0\} \cong T/2T$ and, using the long exact sequence above, $r(T) \cong T/\text{Im}(2\cdot) \cap T = T/2T$. On the whole, $\text{Im}(\beta) \cong r(T)$, which was to be proven and yields the claim. √

Pick now an arbitrary $y \in r(T)$, then $w_2(TM) \cup y = y^2$ using a formula due to W.-T. Wu [8]. As above, since $y \in r(T)$ is the image of a torsion element, $y^2 = 0$, so that $w_2(TM) \cup y = 0$. This shows $w_2(TM) \in \Gamma$ and, with the claim, $w_2(TM) \in \text{Im}(r)$. This concludes the proof. □

For further aspects of spin$^c$ geometry, we recommend [7].

References


