Spin^c structures on manifolds

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Abstract: After introducing the spin^c group and the spinor representation, we discuss spin^c structures and show that every orientable closed smooth 4-dimensional manifold has a spin^c structure. We closely follow [6, App. D] and [1] (see also [2] for a few details).

1 The spin^c group and its representations

1.1 The spin group

Definition 1.1 *Let n be a positive integer. The* spin group *in dimension n, denoted by Spin_n, is the non-trivial 2-fold covering of the special orthogonal group SO_n.*

The group Spin_n is a compact $\frac{n(n-1)}{2}$-dimensional Lie group, connected if $n \geq 2$ and simply-connected if $n \geq 3$. In fact, if $\mathrm{Spin}_n \xrightarrow{\xi} \mathrm{SO}_n$ denotes this non-trivial covering map, then $\xi(z) = z^2$ for any $z \in \mathrm{Spin}_2 \cong \mathbb{U}^1 = \{z \in \mathbb{C}, |z| = 1\}$ and ξ is the universal covering map if $n \geq 3$. In particular, we have the following short exact sequence of Lie groups:

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Spin}_n \xrightarrow{\xi} \mathrm{SO}_n \longrightarrow 1.$$

Examples 1.2

1. For $n = 3$, the spin group $\mathrm{Spin}_3 \cong \mathrm{SU}_2$, where ξ becomes the well-known 2-fold covering map.

2. For $n = 4$, the spin group $\mathrm{Spin}_4 \cong \mathrm{Spin}_3 \times \mathrm{Spin}_3 \cong \mathrm{SU}_2 \times \mathrm{SU}_2$.

This defines the spin group as an abstract Lie group. Actually, the spin group is a Lie subgroup of a natural Lie group, namely the group of units of a *Clifford algebra*.

Definition 1.3 *Let* $q_{\mathbb{C}}(z, z') := \sum_{j=1}^{n} z_j z'_j$ *denote the canonical complex bilinear form on* $\mathbb{C}^n$*. The* complex Clifford algebra *in dimension* n *is defined as*

$$\mathrm{Cl}_n := \mathrm{Cl}(\mathbb{C}^n, q_{\mathbb{C}}) := \bigotimes \mathbb{C}^n / \mathcal{I},$$

where $\bigotimes \mathbb{C}^n$ *denotes the tensor algebra of* $\mathbb{C}^n$ *and* $\mathcal{I}$ *the two-sided ideal generated by the elements of the form* $z \otimes w + w \otimes z + 2q_{\mathbb{C}}(z, z') \cdot 1$*, where* z, w *run in* $\mathbb{C}^n$*.*

Proposition 1.4 *Endowed with the so-called Clifford mutliplication* $[a] \cdot [b] := [a \otimes b]$*, the complex Clifford algebra in dimension* n *is an associative algebra with unit which is linearly isomorphic to the exterior algebra* $\bigwedge \mathbb{C}^n$ *(hence of complex dimension* 2^n*). It can be characterised as the smallest associative complex algebra with unit containing* $\mathbb{C}^n$ *and where the relations*

$$z \cdot w + w \cdot z = -2q_{\mathbb{C}}(z, w) \cdot 1$$

are satisfied for all $z, w \in \mathbb{C}^n$*.*

Proposition 1.5 *The spin group in dimension* n *can be identified with the following subgroup of the group* $\mathrm{Cl}_n^\times$ *of units of* Cl_n*:*

$$\mathrm{Spin}_n \cong \{v_1 \cdot \ldots \cdot v_{2k} \,|\, v_j \in \mathbb{R}^n \,,\, |v_j| = 1 \,,\, k \geq 1\} \subset \mathrm{Cl}_n^\times.$$

Moreover, the 2*-fold covering homorphism* ξ *can be identified with the restriction of the adjoint map acting on* $\mathbb{R}^n$*:*

$$\xi = \mathrm{Ad}_{|\mathrm{Spin}_n} : \mathrm{Spin}_n \longrightarrow \mathrm{Aut}(\mathbb{R}^n),\ u \longmapsto (v \mapsto u \cdot v \cdot u^{-1}).$$

1.2 The spin^c group

Definition 1.6 *Let* n *be a positive integer. The* spin^c group *in dimension* n*, denoted by* Spin_n^c*, is the subgroup*

$$\mathrm{Spin}_n^c := \{\lambda u \,|\, \lambda \in \mathbb{U}_1, u \in \mathrm{Spin}_n\} \subset \mathrm{Cl}_n^\times.$$

The group homomorphism

$$\begin{array}{rcl} \mathrm{Spin}_n \times \mathbb{U}_1 & \longrightarrow & \mathrm{Spin}_n^c \\ (u, \lambda) & \longmapsto & \lambda u \end{array}$$

is by definition surjective and its kernel is $\{\pm(1,1)\}$ since $\mathrm{Spin}_n \cap \mathbb{U}_1 = \{\pm 1\}$. Therefore,

$$\mathrm{Spin}_n^c \cong \mathrm{Spin}_n \times \mathbb{U}_1 / \mathbb{Z}_2,$$

which is sometimes taken as a definition for the spin^c group.

As for the spin group, there is a short exact sequence of Lie groups

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Spin}_n^c \xrightarrow{\xi^c} \mathrm{SO}_n \times \mathbb{U}_1 \longrightarrow 1, \tag{1}$$

where $\xi^c([u,\lambda]) := (\xi(u), \lambda^2)$. Beware that Spin_n^c, though connected for $n \geq 2$, is never simply-connected:

$$\pi_1(\mathrm{Spin}_n^c) = \begin{cases} \mathbb{Z}^2 & \text{if } n = 2 \\ \mathbb{Z} & \text{if } n \geq 3. \end{cases}$$

1.3 The spinor representation

Proposition 1.7 *Let $\Sigma_n := \mathbb{C}^{2^{[\frac{n}{2}]}}$, then there exist complex algebra homomorphisms*

$$\mathbb{C}\mathrm{l}_n \cong \begin{cases} \mathrm{End}_{\mathbb{C}}(\Sigma_n) & \textit{if } n \textit{ is even} \\ \mathrm{End}_{\mathbb{C}}(\Sigma_n) \oplus \mathrm{End}_{\mathbb{C}}(\Sigma_n) & \textit{if } n \textit{ is odd.} \end{cases}$$

The representation space Σ_n can actually be constructed explicitly as a subspace of $\mathbb{C}\mathrm{l}_n$ itself (on which $\mathbb{C}\mathrm{l}_n$ acts from the left by Clifford multiplication), see [2].

Since any complex matrix algebra is simple, Proposition 1.7 implies that there is up to equivalence only one (non-zero) irreducible complex representation of $\mathbb{C}\mathrm{l}_n$ if n is even and there are exactly two if n is odd. To distinguish the two, we introduce the so-called complex volume element

$$\omega_n^{\mathbb{C}} := i^{[\frac{n+1}{2}]} e_1 \cdot \ldots \cdot e_n \in \mathbb{C}\mathrm{l}_n,$$

where $(e_j)_{1 \leq j \leq n}$ is any p.o.n.b of $\mathbb{R}^n$ with the canonical metric and orientation.

Lemma 1.8 *The complex volume element acts as an isometric involution on Σ_n. More precisely,*

$$\omega_n^{\mathbb{C}} \cdot = \begin{cases} \mathrm{Id}_{\Sigma_n^+} \oplus -\mathrm{Id}_{\Sigma_n^-} & \textit{if } n \textit{ is even} \\ \mathrm{Id}_{\Sigma_n} \oplus -\mathrm{Id}_{\Sigma_n} & \textit{if } n \textit{ is odd,} \end{cases}$$

where $\Sigma_n^{\pm} := \mathrm{Ker}(\omega_n^{\mathbb{C}} \cdot \mp \mathrm{Id}_{\Sigma_n}) \subset \Sigma_n$ in the case n even.

From now on, we denote by $\delta_n : \mathbb{C}\mathrm{l}_n \longrightarrow \mathrm{End}_{\mathbb{C}}(\Sigma_n)$ the representation provided by Proposition 1.7 if n is even and by the factor of $\mathrm{End}_{\mathbb{C}}(\Sigma_n) \oplus \mathrm{End}_{\mathbb{C}}(\Sigma_n)$ on which $\omega_n^{\mathbb{C}}$ acts as the identity if n is odd.

Definition 1.9 *The representation δ_n is called the* complex spinor representation.

Proposition 1.10 *The spinor representation satisfies the following:*

i) There exists up to scaling only one Hermitian product on Σ_n such that each vector in $\mathbb{R}^n$ acts in a skew-Hermitian way on Σ_n.

ii) In n is even, then $\delta_{n|_{\mathrm{Spin}_n^c}}$ splits into the sum of two inequivalent irreducible complex representations: $\delta_{n|_{\mathrm{Spin}_n^c}} = \delta_n^+ \oplus \delta_n^-$, where $\delta_n^\pm : \mathrm{Spin}_n^c \longrightarrow \mathrm{Aut}_{\mathbb{C}}(\Sigma_n^\pm)$ are irreducible with $\delta_n^+ \nsim \delta_n^-$.

iii) If n is odd, then $\delta_{n|_{\mathrm{Spin}_n^c}}$ is irreducible. Moreover, the restriction of the factor of $\mathrm{End}_{\mathbb{C}}(\Sigma_n) \oplus \mathrm{End}_{\mathbb{C}}(\Sigma_n)$ on which $\omega_n^{\mathbb{C}}$ acts as minus the identity to Spin_n^c gives rise to an equivalent representation.

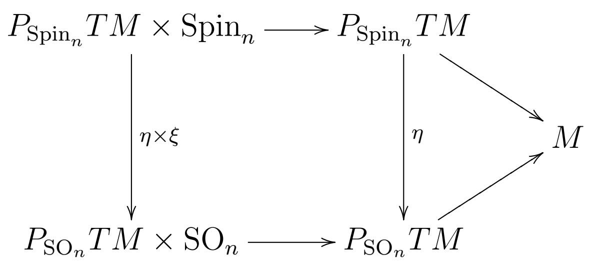
In case n even, the representations $\delta_n^\pm$ are called *half-spinor representations*; δ_n^+ is the positive one and δ_n^- the negative one. Note in particular that, as a consequence of Proposition 1.10.i) and of Proposition 1.5, the representation δ_n is *unitary.*

2 Spin^c structures

We denote by $P_{\mathrm{SO}_n}TM \longrightarrow M$ the SO_n-principal bundle of positively oriented orthonormal frames on the tangent bundle of an oriented Riemannian manifold (M^n, g).

Definition 2.1 *Let (M^n, g) be an n-dimensional oriented Riemannian manifold.*

1. *A* spin structure *on (M^n, g) is a reduction of $P_{\mathrm{SO}_n}TM \longrightarrow M$ to the spin group. More precisely, a spin structure is given by a Spin_n-principal bundle $P_{\mathrm{Spin}_n}TM \longrightarrow M$ together with a 2-fold covering map $P_{\mathrm{Spin}_n}TM \xrightarrow{\eta} P_{\mathrm{SO}_n}TM$) such that the following diagramme commutes:*



2. *A* spin^c-structure *on* (M^n, g) *consists of a pair* $(P_{\mathrm{Spin}_n^c}TM, P_{\mathbb{U}_1})$, *where* $P_{\mathrm{Spin}_n^c}TM \longrightarrow M$ *is a* Spin_n^c*-principal bundle,* $P_{\mathbb{U}_1} \longrightarrow M$ *is a* $\mathbb{U}_1$*-principal bundle, together with a 2-fold covering map* $P_{\mathrm{Spin}_n^c}TM \stackrel{\eta^c}{\longrightarrow} P_{\mathrm{SO}_n}TM \times P_{\mathbb{U}_1}$ *such that the following diagramme commutes:*

$$
\begin{array}{ccccc}
P_{\mathrm{Spin}_n^c}TM \times \mathrm{Spin}_n^c & \longrightarrow & P_{\mathrm{Spin}_n^c}TM & & \\
\downarrow{\scriptstyle \eta^c \times \xi^c} & & \downarrow{\scriptstyle \eta^c} & \searrow & \\
 & & & & M \\
 & & & \nearrow & \\
P_{\mathrm{SO}_n}TM \times_M P_{\mathbb{U}_1} \times (\mathrm{SO}_n \times \mathbb{U}_1) & \longrightarrow & P_{\mathrm{SO}_n}TM \times_M P_{\mathbb{U}_1} & &
\end{array}
$$

3. *The manifold* (M^n, g) *is called spin (resp. spin*^{c}*) if and only if it admits a spin- (resp. spin*^{c}*-)structure.*

Any spin structure defines a spin^c structure in an obvious way: take $P_{\mathbb{U}_1} := M \times \mathbb{U}_1$ to be the trivial $\mathbb{U}_1$-bundle and extend the Spin_n-bundle via the inclusion $\mathrm{Spin}_n \subset \mathrm{Spin}_n^c$. In particular, any spin manifold is spin^c.

The condition to be spin or spin^c *a priori* depends on the metric (through $P_{\mathrm{SO}_n}TM$). It actually only has to do with the topology of the manifold since it may be understood as an orientability condition of second order, as we shall prove next. Denote by $r : \mathbb{Z} \longrightarrow \mathbb{Z}_2$ the mod-2-reduction and also by $r : H^q(M; \mathbb{Z}) \longrightarrow H^q(M; \mathbb{Z}_2)$ the induced homomorphism in cohomology.

Proposition 2.2

i) A smooth manifold M *is spin if and only if its first and second Stiefel-Whitney classes vanish, that is, iff* $w_1(TM) = 0$ *and* $w_2(TM) = 0$.

ii) A smooth manifold M *is spin*^{c} *if and only if its first Stiefel-Whitney class vanishes and its second Stiefel-Whitney class is the mod-2-reduction of an integral class, that is, iff*

$$w_1(TM) = 0 \ \ \text{and} \ \ w_2(TM) \in r(H^2(M; \mathbb{Z})).$$

Proof: We only prove *ii*), see e.g. [6] or [3] for *i*). We follow [6, App. D]. First M has to be orientable in order to be spin^c, the orientability of TM being equivalent to $w_1(TM) = 0$, which we assume from now on. The short exact sequence of groups (1) induces the following long exact sequence in Čech cohomology:

$$\ldots \to H^1(M; \mathbb{Z}_2) \longrightarrow H^1(M; \mathrm{Spin}_n^c) \stackrel{\xi^c}{\to} H^1(M; \mathrm{SO}_n) \oplus H^1(M; \mathbb{U}_1) \stackrel{w_2 + r\circ c_1}{\to} H^2(M; \mathbb{Z}_2) \to \ldots,$$

where $w_2 : H^1(M;\mathrm{SO}_n) \to H^2(M;\mathbb{Z}_2)$ denotes the homomorphism associating the second Stiefel-Whitney class to an equivalence class of SO_n-bundles (or, equivalently, of Riemannian vector bundles) and $c_1 : H^1(M;\mathbb{U}_1) \to H^2(M;\mathbb{Z})$ denotes the homomorphism associating the first Chern class to an equivalence class of $\mathbb{U}_1$-bundles (or, equivalently, of Hermitian line bundles). The condition M to be spin^c means that there exists a $\mathbb{U}_1$-bundle $P_{\mathbb{U}_1} \longrightarrow M$ such that the element $[(P_{\mathrm{SO}_n}TM, P_{\mathbb{U}_1})] \in H^1(M;\mathrm{SO}_n) \oplus H^1(M;\mathbb{U}_1)$ lies in the image of the map ξ^c. This, in turn, is equivalent to $[(P_{\mathrm{SO}_n}TM, P_{\mathbb{U}_1})]$ lying in the kernel of $w_2 + r \circ c_1$, meaning that $w_2(P_{\mathrm{SO}_n}TM) = r(c_1(P_{\mathbb{U}_1}))$. Since $c_1 : H^1(M;\mathbb{U}_1) \to H^2(M;\mathbb{Z})$ is a group isomorphism, the condition to be spin^c for M is therefore equivalent to $w_2(TM) = w_2(P_{\mathrm{SO}_n}TM) \in r(H^2(M;\mathbb{Z}))$, which was to be shown. □

Examples 2.3

1. Any 1-dimensional manifold is spin, a circle having two inequivalent spin structures. Any orientable surface is also spin since in that case $w_2(TM) \in H^2(M;\mathbb{Z}_2) \cong \mathbb{Z}_2$ is the mod-2-reduction of the Euler characteristic (which is even). Any 3-dimensional orientable 3-dimensional manifold has trivial tangent bundle and hence is spin. The "simplest" example of non-spin manifold is the complex 2-dimensional projective space $\mathbb{C}\mathrm{P}^2$.

2. The set of (inequivalent) spin structures of a given spin manifold M can be shown to stand in one-to-one correspondence with its cohomology group $H^1(M;\mathbb{Z}_2)$. In particular, M may have more than one spin structure. However, there is only one if e.g. M is simply-connected.

3. Any *almost-Hermitian* manifold has a natural spin^c structure, due to the existence of a reduction of $P_{\mathrm{SO}_{2m}}$ to the unitary group $\mathbb{U}_m$ and of a lift $\mathbb{U}_m \longrightarrow \mathrm{Spin}^c_{2m}$ over $\mathbb{U}_m \stackrel{\mathrm{incl.}\times\det}{\longrightarrow} \mathrm{SO}_{2m} \times \mathbb{U}_1$.

4. Spin^c structures need also not be unique: if $P_{\mathbb{U}_1}(\alpha) \longrightarrow M$ is any $\mathbb{U}_1$-bundle (with Chern-class $\alpha \in H^2(M;\mathbb{Z})$) over M, then

$$P_{\mathrm{Spin}^c_n} \times_M P_{\mathbb{U}_1}(\alpha)/\mathbb{U}_1 \longrightarrow P_{\mathrm{SO}_n} \times_M (P_{\mathbb{U}_1} \otimes P_{\mathbb{U}_1}(2\alpha))$$

defines a new spin^c structure, where the new associated $\mathbb{U}_1$-bundle is $P_{\mathbb{U}_1} \otimes P_{\mathbb{U}_1}(2\alpha)$.

From now on, we shall implicitely assume that, on a given spin^c manifold, a spin^c structure is fixed.

Definition 2.4 *Let (M^n, g) be a spin^c manifold. The* spinor bundle *of M is the vector bundle – denoted by ΣM – associated to the* Spin_n^c*-bundle via the spinor representation:*

$$\Sigma M := P_{\mathrm{Spin}_n^c} TM \times_{\delta_n} \Sigma_n = P_{\mathrm{Spin}_n^c} TM \times \Sigma_n /_{\sim},$$

where $(p, \sigma) \sim (p \cdot u, \delta_n(u^{-1})(\sigma)$ for all $(p, \sigma) \in P_{\mathrm{Spin}_n^c} TM \times \Sigma_n$ and $u \in \mathrm{Spin}_n^c$.

By definition, the spinor bundle is a complex vector bundle of (complex) rank $2^{[\frac{n}{2}]}$ over M. Sections of ΣM are called *spinor fields* or just *spinors*. Since δ_n can be assumed unitary (see above), ΣM can be naturally endowed with a pointwise Hermitian inner product $\langle \cdot\,, \cdot \rangle$, turning it into a Hermitian vector bundle. Like the space Σ_n, the spinor bundle also admits a Clifford multiplication:

Proposition 2.5 *The spinor representation of $\mathbb{C}\mathrm{l}_n$ induces a linear map $TM \otimes \Sigma M \longrightarrow \Sigma M$, $X \otimes \varphi \longmapsto X \cdot \varphi$, satisfying the (pointwise) Clifford relation*

$$X \cdot (Y \cdot \varphi) + Y \cdot (X \cdot \varphi) = -2g(X, Y)\varphi$$

for all $X, Y \in TM$ and $\varphi \in \Sigma M$. Moreover, the Hermitian inner product $\langle \cdot\,, \cdot \rangle$ can be defined such that

$$\langle X \cdot \varphi, \psi \rangle = -\langle \varphi, X \cdot \psi \rangle$$

for all $X \in TM$ and $\varphi, \psi \in \Sigma M$.

As a last important step, any connection 1-form on the auxiliary bundle $P_{\mathbb{U}_1}$ induces, together with the Levi-Civita connection of (M^n, g), a metric connection on ΣM:

Proposition 2.6 *Let $A \in \Omega^1(P_{\mathbb{U}_1}, i\mathbb{R})$ be any connection 1-form on $P_{\mathbb{U}_1}$. Then A and the Levi-Civita connection ∇ of (M^n, g) together induce a metric covariant derivative ∇^A on ΣM, which satisfies:*

$$\nabla^A_X (Y \cdot \varphi) = (\nabla_X Y) \cdot \varphi + Y \cdot \nabla^A_X \varphi,$$

for all $X, Y \in \Gamma(M, TM)$ and $\varphi \in \Gamma(M, \Sigma M)$.

Definition 2.7 *The* Dirac operator *associated to a connection 1-form A on the auxiliary bundle $P_{\mathbb{U}_1} \longrightarrow M$ on a spin^c manifold (M^n, g) is the operator*

$$D^A : \Gamma(M, \Sigma M) \to \Gamma(M, \Sigma M), \qquad \varphi \mapsto \sum_{j=1}^{n} e_j \cdot \nabla^A_{e_j} \varphi,$$

where ∇^A is the covariant derivative associated to A and $(e_j)_{1 \le j \le n}$ is any local o.n.b. of TM.

The Dirac-operator is a well-defined, elliptic, formally self-adjoint differential operator of order 1. It is even essentially self-adjoint if (M^n, g) is complete.

Theorem 2.8 (Schrödinger-Lichnerowicz formula) *For any connection 1-form A on the auxiliary bundle $P_{\mathbb{U}_1} \longrightarrow M$ on a spin^c manifold (M^n, g), we have*

$$(D^A)^2 = (\nabla^A)^*\nabla^A + \frac{S}{4}\mathrm{Id} + \frac{F_A}{2} \cdot \mathrm{Id},$$

where $(\nabla^A)^*\nabla^A := -\mathrm{tr}_g((\nabla^A)^2) = \sum_{j=1}^n \nabla^A_{\nabla_{e_j}e_j} - \nabla^A_{e_j}\nabla^A_{e_j}$ is the connection Laplacian associated to ∇^A (here $\{e_j\}_{1\leq j\leq n}$ is a local o.n.b. of TM), S is the scalar curvature of (M, g) and $F_A \in \Gamma(\Lambda^2 T^*M \otimes i\mathbb{R})$ is the curvature form of A.

Proof: Fix a local orthonormal basis $\{e_j\}_{1\leq j\leq n}$ of TM. Using the compatibility conditions as well as the Clifford relations, we have, for any $\varphi \in \Gamma(\Sigma M)$,

$$\begin{aligned}
(D^A)^2\varphi &= \sum_{j,k=1}^n e_j \cdot \nabla^A_{e_j}(e_k \cdot \nabla^A_{e_k}\varphi) \\
&= \sum_{j,k=1}^n e_j \cdot \nabla_{e_j}e_k \cdot \nabla^A_{e_k}\varphi + e_j \cdot e_k \cdot \nabla^A_{e_j}\nabla^A_{e_k}\varphi \\
&= -\sum_{j,k=1}^n e_j \cdot e_k \cdot \nabla^A_{\nabla_{e_j}e_k}\varphi + \sum_{j,k=1}^n e_j \cdot e_k \cdot \nabla^A_{e_j}\nabla^A_{e_k}\varphi \\
&= \sum_{j=1}^n (\nabla^A_{\nabla_{e_j}e_j} - \nabla^A_{e_j}\nabla^A_{e_j})\varphi \\
&\quad + \sum_{1\leq j<k\leq n} e_j \cdot e_k \cdot (\nabla^A_{e_j}\nabla^A_{e_k} - \nabla^A_{e_k}\nabla^A_{e_j} - \nabla^A_{\nabla_{e_j}e_k} + \nabla^A_{\nabla_{e_k}e_j})\varphi \\
&= (\nabla^A)^*\nabla^A\varphi + \sum_{1\leq j<k\leq n} e_j \cdot e_k \cdot ([\nabla^A_{e_j}, \nabla^A_{e_k}] - \nabla^A_{[e_j,e_k]})\varphi \\
&= (\nabla^A)^*\nabla^A\varphi + \frac{1}{2}\sum_{j,k=1}^n e_j \cdot e_k \cdot R^{\nabla^A}_{e_j,e_k}\varphi.
\end{aligned}$$

Now locally the connection ∇^A and its curvature R^{∇^A} can be expressed as follows: choosing local sections u of $P_{\mathrm{SO}_n}TM \to M$ and s of $P_{\mathbb{U}_1} \to M$, we obtain a local section $\widetilde{u}$ of $P_{\mathrm{Spin}^c_n}TM \to M$ and hence a local trivialization $\{\psi_\alpha\}_{1\leq\alpha\leq 2^{[\frac{n}{2}]}}$ of ΣM. In that case, we have, for all tangent vectors X, Y (defined locally),

$$\nabla^A_X\psi_\alpha = \frac{1}{4}\sum_{j,k=1}^n g(\nabla_X e_j, e_k)e_j \cdot e_k \cdot \psi_\alpha + \frac{A(ds(X))}{2}\psi_\alpha,$$

from which

$$R^{\nabla^A}_{X,Y} = \frac{1}{4}\sum_{j,k=1}^{n} g(R^{\nabla}_{X,Y}e_j, e_k)e_j\cdot e_k\cdot + \frac{1}{2}\underbrace{dA(ds(X), ds(Y))}_{F_A(X,Y)}$$

follows. By definition of the Clifford action of forms

$$\sum_{j,k=1}^{n} F_A(e_j, e_k)e_j\cdot e_k\cdot = 2\sum_{1\leq j<k\leq n} F_A(e_j, e_k)e_j\cdot e_k\cdot = 2F_A\cdot,$$

so that only the action of the curvature of the Levi-Civita connection of (M, g) remains to be determined. The first Bianchi identity and the preceding local expressions of ∇^A and R^{∇^A} imply that, for any $X \in TM$,

$$\begin{aligned}
\sum_{j,k,l=1}^{n} g(R^{\nabla}_{X,e_j}e_k, e_l)e_j\cdot e_k\cdot e_l\cdot\varphi &= -\sum_{j,k,l=1}^{n} g(R^{\nabla}_{e_j,e_k}X, e_l)e_j\cdot e_k\cdot e_l\cdot\varphi \\
&\quad -\sum_{j,k,l=1}^{n} g(R^{\nabla}_{e_k,X}e_j, e_l)e_j\cdot e_k\cdot e_l\cdot\varphi \\
&= -\sum_{j,k,l=1}^{n} g(R^{\nabla}_{X,e_j}e_k, e_l)(e_k\cdot e_l\cdot e_j - e_k\cdot e_j\cdot e_l)\cdot\varphi,
\end{aligned}$$

with

$$\begin{aligned}
e_k\cdot e_l\cdot e_j - e_k\cdot e_j\cdot e_l &= -e_k\cdot e_j\cdot e_l - 2\delta_{jl}e_k + e_j\cdot e_k\cdot e_l + 2\delta_{jk}e_l \\
&= 2e_j\cdot e_k\cdot e_l + 4\delta_{jk}e_l - 2\delta_{jl}e_k.
\end{aligned}$$

We deduce that

$$\begin{aligned}
3\sum_{j,k,l=1}^{n} g(R^{\nabla}_{X,e_j}e_k, e_l)e_j\cdot e_k\cdot e_l\cdot\varphi &= -4\sum_{j,l=1}^{n} g(R^{\nabla}_{X,e_j}e_j, e_l)e_l\cdot\varphi + 2\sum_{j,k=1}^{n} g(R^{\nabla}_{X,e_j}e_k, e_j)e_k\cdot\varphi \\
&= -4\sum_{l=1}^{n} g(\mathrm{Ric}(X), e_l)e_l\cdot\varphi - 2\sum_{k=1}^{n} g(\mathrm{Ric}(X), e_k)e_k\cdot\varphi \\
&= -6\mathrm{Ric}(X)\cdot\varphi,
\end{aligned}$$

where Ric denotes the Ricci tensor of (M,g). Therefore,

$$\begin{aligned}
\frac{1}{2}\sum_{j,k=1}^{n} e_j\cdot e_k\cdot R^{\nabla^A}_{e_j,e_k}\varphi &= \frac{1}{8}\sum_{i,j,k,l=1}^{n} g(R^{\nabla}_{e_i,e_j}e_k,e_l)e_i\cdot e_j\cdot e_k\cdot e_l\cdot\varphi \\
&\quad +\frac{1}{4}\sum_{j,k=1}^{n} F_A(e_j,e_k)e_j\cdot e_k\cdot\varphi \\
&= -\frac{1}{4}\sum_{i=1}^{n} e_i\cdot \mathrm{Ric}(e_i)\cdot\varphi+\frac{F_A}{2}\cdot\varphi \\
&= -\frac{1}{4}\sum_{i,j=1}^{n} \underbrace{g(\mathrm{Ric}(e_i),e_j)}_{\mathrm{symm.}}\quad \underbrace{e_i\cdot e_j\cdot}_{\mathrm{skew-symm.\ if\ } i\neq j}\quad\varphi+\frac{F_A}{2}\cdot\varphi \\
&= \frac{1}{4}\sum_{i=1}^{n} g(\mathrm{Ric}(e_i),e_i)\varphi+\frac{F_A}{2}\cdot\varphi \\
&= \frac{S}{4}\varphi+\frac{F_A}{2}\cdot\varphi,
\end{aligned}$$

which concludes the proof. □

3 The 4-dimensional case

Theorem 3.1 ([8, 5]) *Every closed orientable smooth 4-dimensional manifold is spin^c.*

Proof: We follow [1, pp. 144-145]. We can assume w.l.o.g. that the manifold M is connected. By Proposition 2.2, we have to show that $w_2(TM)\in \mathrm{Im}(r):=r(H^2(M;\mathbb{Z}))\subset H^2(M;\mathbb{Z}_2)$. We define $T:=\mathrm{Tor}(H^2(M;\mathbb{Z}))$, the torsion subgroup of $H^2(M;\mathbb{Z})$.
Claim: $\mathrm{Im}(r)=\{\gamma\in H^2(M;\mathbb{Z}_2)\,,\,\gamma\cup y=0\ \forall y\in r(T)\}$.
Proof: Let $\Gamma:=\{\gamma\in H^2(M;\mathbb{Z}_2)\,,\,\gamma\cup y=0\ \forall y\in r(T)\}$. If $\gamma\in\mathrm{Im}(r)$, then there exists $\alpha\in H^2(M;\mathbb{Z})$ with $r(\alpha)=\gamma$. Similarly, for any $y\in r(T)$, there exists $\beta\in T$ with $r(\beta)=y$. It follows $\gamma\cup y=r(\alpha\cup\beta)$. But $\alpha\cup\beta\in H^4(M;\mathbb{Z})$ and $H^4(M;\mathbb{Z})\cong\mathbb{Z}$ because M is orientable; since β is a torsion element, there exists an $m\in\mathbb{N}\setminus\{0\}$ with $m\beta=0$ and hence $m(\alpha\cup\beta)=0$, which yields $\alpha\cup\beta=0$ and therefore $\gamma\cup y=0$. This shows $\mathrm{Im}(r)\subset\Gamma$. The other inclusion will be proven as soon as the $\mathbb{Z}_2$-dimensions of $\mathrm{Im}(r)$ and Γ are shown to coincide. Since $\mathbb{Z}_2$ is a field, the cup product $H^2(M;\mathbb{Z}_2)\times H^2(M;\mathbb{Z}_2)\longrightarrow H^4(M;\mathbb{Z}_2)\cong\mathbb{Z}_2$ defines a non-degenerate (symmetric) bilinear form (see e.g.

[4, Prop. 3.38]), therefore

$$\dim_{\mathbb{Z}_2}(\Gamma) = \dim_{\mathbb{Z}_2}(r(T)^{\perp}) = \dim_{\mathbb{Z}_2}(H^2(M;\mathbb{Z}_2)) - \dim_{\mathbb{Z}_2}(r(T)).$$

Thus we have to show that $\dim_{\mathbb{Z}_2}(\mathrm{Im}(r)) = \dim_{\mathbb{Z}_2}(H^2(M;\mathbb{Z}_2)) - \dim_{\mathbb{Z}_2}(r(T))$. The short exact sequence of abelian groups $0 \longrightarrow \mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z} \xrightarrow{r} \mathbb{Z}_2 \longrightarrow 0$ induces the following long exact sequence in cohomology

$$\ldots \to H^2(M;\mathbb{Z}) \xrightarrow{2\cdot} H^2(M;\mathbb{Z}) \xrightarrow{r} H^2(M;\mathbb{Z}_2) \xrightarrow{\beta} H^3(M;\mathbb{Z}) \xrightarrow{2\cdot} H^3(M;\mathbb{Z}) \to \ldots$$

where β is the so-called *Bockstein homomorphism*. In particular, $\mathrm{Im}(r) = \mathrm{Ker}(\beta)$, so that $\dim_{\mathbb{Z}_2}(\mathrm{Im}(r)) = \dim_{\mathbb{Z}_2}(H^2(M;\mathbb{Z}_2)) - \dim_{\mathbb{Z}_2}(\mathrm{Im}(\beta))$. Hence it suffices to show that $\mathrm{Im}(\beta) \cong r(T)$. Now the universal coefficient theorem (see e.g. [4, Thm 3.2]) states that there is the following short exact sequence of $\mathbb{Z}$-modules:

$$0 \longrightarrow \mathrm{Ext}_{\mathbb{Z}}(H_{q-1}(M;\mathbb{Z}),\mathbb{Z}) \longrightarrow H^q(M;\mathbb{Z}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(H_q(M;\mathbb{Z}),\mathbb{Z}) \longrightarrow 0,$$

for every $q \in \mathbb{N} \setminus \{0\}$. Moreover, the free part of $H_{q-1}(M;\mathbb{Z})$ does not contribute to $\mathrm{Ext}_{\mathbb{Z}}$, more precisely $\mathrm{Ext}_{\mathbb{Z}}(H_{q-1}(M;\mathbb{Z}),\mathbb{Z}) \cong \mathrm{Tor}(H_{q-1}(M;\mathbb{Z}))$. Similarly, the torsion part of $H_q(M;\mathbb{Z})$ does not contribute to $\mathrm{Hom}_{\mathbb{Z}}$, that is, $\mathrm{Hom}_{\mathbb{Z}}(H_q(M;\mathbb{Z}),\mathbb{Z}) \cong \mathbb{Z}^{b_q}$, where $b_q \in \mathbb{N}$ is the rank of $H_q(M;\mathbb{Z})$; in particular, $\mathrm{Hom}_{\mathbb{Z}}(H_q(M;\mathbb{Z}),\mathbb{Z}) \cong H_q(M;\mathbb{Z})/\mathrm{Tor}(H_q(M;\mathbb{Z}))$. Since the latter is free, the short exact sequence above splits and we obtain

$$\begin{aligned} H^q(M;\mathbb{Z}) &\cong \mathrm{Ext}_{\mathbb{Z}}(H_{q-1}(M;\mathbb{Z}),\mathbb{Z}) \oplus \mathrm{Hom}_{\mathbb{Z}}(H_q(M;\mathbb{Z}),\mathbb{Z}) \\ &\cong \mathrm{Tor}(H_{q-1}(M;\mathbb{Z})) \oplus H_q(M;\mathbb{Z})/\mathrm{Tor}(H_q(M;\mathbb{Z})). \end{aligned}$$

As a consequence, $\mathrm{Tor}(H^q(M;\mathbb{Z})) \cong \mathrm{Tor}(H_{q-1}(M;\mathbb{Z}))$. Since M is closed and orientable, Poincaré duality implies $H_{q-1}(M;\mathbb{Z}) \cong H^{n-q+1}(M;\mathbb{Z})$, where n is the dimension of the manifold M. Here we obtain, for $n = 4$ and $q = 3$:

$$\mathrm{Tor}(H^3(M;\mathbb{Z})) \cong \mathrm{Tor}(H_2(M;\mathbb{Z})) \cong \mathrm{Tor}(H^2(M;\mathbb{Z})).$$

We deduce that

$$\begin{aligned} \mathrm{Im}(\beta) &= \mathrm{Ker}(2\cdot) \\ &= \left\{\alpha \in H^3(M;\mathbb{Z})\,,\, 2\alpha = 0\right\} \\ &= \left\{\alpha \in \mathrm{Tor}(H^3(M;\mathbb{Z}))\,,\, 2\alpha = 0\right\} \\ &\cong \{\alpha \in T\,,\, 2\alpha = 0\}\,. \end{aligned}$$

Writing $T = \bigoplus_{j=1}^{m} \mathbb{Z}_{p_j^{k_j}}$ with $p_j \in \mathbb{N}$ prime and $k_j \geq 1$, the subgroup $\{\alpha \in T\,,\, 2\alpha = 0\}$ of T is freely generated over $\mathbb{Z}_2$ by the elements of the form

$2^{k_j-1} \in \mathbb{Z}_{2^{k_j}}$ (only the $p_j = 2$ appear since 2 is invertible in $\mathbb{Z}_{p_j^{k_j}}$ for any prime $p_j > 2$). So is $T/_{2T}$ for the same reasons. Hence $\{\alpha \in T\,,\, 2\alpha = 0\} \cong T/_{2T}$ and, using the long exact sequence above, $r(T) \cong T/_{\mathrm{Im}(2\cdot)\cap T} = T/_{2T}$. On the whole, $\mathrm{Im}(\beta) \cong r(T)$, which was to be proven and yields the claim. $\surd$
Pick now an arbitrary $y \in r(T)$, then $w_2(TM) \cup y = y^2$ using a formula due to W.-T. Wu [8]. As above, since $y \in r(T)$ is the image of a torsion element, $y^2 = 0$, so that $w_2(TM) \cup y = 0$. This shows $w_2(TM) \in \Gamma$ and, with the claim, $w_2(TM) \in \mathrm{Im}(r)$. This concludes the proof. $\square$

For further aspects of spin^c geometry, we recommend [7].

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