Moment maps and Noether's theorem

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Abstract: Following mainly [2, Chap. 22, 26 & Sec. 24.1] as well as [5, Sec. 5.2-5.3], we discuss and illustrate the concept of moment maps and their symmetries via Noether's theorem.

Unless otherwise stated, we shall denote by (M,ω) a symplectic manifold and by G a Lie group with Lie algebra $\underline{G}$. For any $f \in C^\infty(M;\mathbb{R})$, we denote by $\mathrm{grad}_\omega(f) \in \Gamma(TM)$ the symplectic gradient vector field associated to f, that is, characterized by $\omega(\mathrm{grad}_\omega(f), X) = df(X)$ for all $X \in TM$. Recall that the *Poisson bracket* of two functions $f,g \in C^\infty(M;\mathbb{R})$ is defined by $\{f,g\} := \omega(\mathrm{grad}_\omega(f), \mathrm{grad}_\omega(g)) \in C^\infty(M;\mathbb{R})$.

1 Moment maps

1.1 Definition and characterisations

Recall that a vector field on M is called *Hamiltonian* iff it is the symplectic gradient of a smooth real-valued function on M.

Definition 1.1 *Let $G \times M \longrightarrow M$ be a smooth group action via symplectomorphisms. A* moment map *for this action is a smooth map $\mu : M \to \underline{G}^*$ such that*

i) (Hamiltonian condition) *for every $X \in \underline{G}$, $\mathrm{grad}_\omega(\mu^X) = X^\sharp$, where $\mu^X : M \to \mathbb{R}$, $x \mapsto \mu(x)(X)$ and $X^\sharp(x) := \frac{d}{dt}(\exp(tX)\cdot x)_{|t=0}$ for all $x \in M$,*

ii) (equivariance) *for all $(g,x) \in G \times X$, $\mu(g\cdot x) = \mathrm{Ad}(g^{-1})^* \circ \mu(x)$, where $u^* := u^t : \theta \mapsto \theta \circ u$ for all $u \in \mathrm{End}(\underline{G})$ and $\theta \in \underline{G}^*$.*

We call *Hamiltonian G-space* any quadruplet (M, ω, G, μ), where G acts smoothly via symplectomorphisms on M and μ is a moment map for that action.

Definition 1.2 *Let $G \times M \longrightarrow M$ be a smooth group action via symplectomorphisms. A* comoment map *for this action is a linear map $\mu^* : \underline{G} \to C^\infty(M; \mathbb{R})$ such that*

i) (Hamiltonian condition) *for every $X \in \underline{G}$, $\text{grad}_\omega(\mu^*(X)) = X^\sharp$, where $X^\sharp$ is defined as above,*

ii) (equivariance) *for all $X, Y \in \underline{G}$, $\mu^*([X,Y]) = \{\mu^*(X), \mu^*(Y)\}$.*

Proposition 1.3 *Let $G \times M \longrightarrow M$ be any smooth group action via symplectomorphisms.*

1. *Any moment map $\mu : M \to \underline{G}^*$ for that action gives rise to a comoment map $\mu^* : \underline{G} \to C^\infty(M; \mathbb{R})$ for the same action via $\mu^*(X)(x) := \mu(x)(X)$ for all $x \in M$ and $X \in \underline{G}$.*

2. *Conversely, if G is connected, any comoment map $\mu^* : \underline{G} \to C^\infty(M; \mathbb{R})$ induces a moment map in the same way.*

Proof: Given any moment map $\mu : M \to \underline{G}^*$ for the symplectic G-action on M, define μ^* as in Proposition 1.3 (and note that μ^* is well-defined and linear). The Hamiltonian condition i) is by definition satisfied by μ^*. Moreover, for all $X, Y \in \underline{G}$ and $x \in M$, we have

$$\begin{aligned}
\mu^*([X,Y])(x) &= \mu^*\left(\frac{d}{dt}\text{Ad}(\exp(tX))(Y)_{|t=0}\right)(x) \\
&= \frac{d}{dt}\left(\mu^*(\text{Ad}(\exp(tX))(Y))(x)\right)_{|t=0} \qquad (\mu^* \text{ is linear}) \\
&= \frac{d}{dt}\left(\mu(x)(\text{Ad}(\exp(tX))(Y))\right)_{|t=0} \\
&\overset{ii)}{=} \frac{d}{dt}\left(\mu(\exp(-tX)\cdot x)(Y)\right)_{|t=0} \\
&= d_x\mu^Y(-X^\sharp(x)) \\
&= -\omega_x(Y^\sharp, X^\sharp) \\
&\overset{i)}{=} \omega_x(\text{grad}_\omega(\mu^X), \text{grad}_\omega(\mu^Y)) \\
&= \{\mu^X, \mu^Y\}(x),
\end{aligned}$$

which shows the equivariance condition ii) for μ^*. This proves 1.
Conversely, assume a comoment map $\mu^* : \underline{G} \to C^\infty(M; \mathbb{R})$ for the symplectic

G-action is given and define μ as above. Then $\mu : M \to \underline{G}^*$ is smooth and obviously satisfies the Hamiltonian condition i). Define $\overline{\mu} : G \times M \to \underline{G}^*$ via $\overline{\mu}(g,x) := \mathrm{Ad}(g)^* \circ \mu(g \cdot x)$ for all $(g,x) \in G \times M$. Again, $\overline{\mu}$ is smooth with $\overline{\mu}(e,x) = \mu(x)$ for all $x \in M$. To prove the equivariance condition ii) for μ, it suffices by connectedness of G to show that $\frac{\partial \overline{\mu}}{\partial g}(g_0, x) = 0$ for all $(g_0, x) \in G \times M$. Pick arbitrary $g_0 \in G$, $x \in M$ and $X, Y \in \underline{G}$. Note that any tangent vector in $T_{g_0}G$ is of the form $d_e L_{g_0}(Z) \in T_{g_0}G$ for some $Z \in \underline{G}$, where $L_{g_0} : G \to G$, $g \mapsto g_0 g$ is the left translation by g_0. We compute

$$\begin{aligned}
\frac{\partial \overline{\mu}}{\partial g}(g_0,x)(d_e L_{g_0}(X))(Y) &= \frac{d}{dt}\left(\overline{\mu}(g_0 \exp(tX), x)(Y)\right)_{|t=0} \\
&= \frac{d}{dt}\left(\mu(g_0 \exp(tX) \cdot x)(\mathrm{Ad}(g_0 \exp(tX))(Y))\right)_{|t=0} \\
&= d_{g_0 \cdot x}\mu(\mathrm{Ad}(g_0)(X)^\sharp(g_0 \cdot x))(\mathrm{Ad}(g_0)(Y)) + \mu(g_0 \cdot x)(\mathrm{Ad}(g_0)([X,Y])) \\
&= d_{g_0 \cdot x}\mu^{\mathrm{Ad}(g_0)(Y)}(\mathrm{Ad}(g_0)(X)^\sharp(g_0 \cdot x)) \\
&\quad + \mu(g_0 \cdot x)([\mathrm{Ad}(g_0)(X), \mathrm{Ad}(g_0)(Y)]) \\
&= \{\mu^*(\mathrm{Ad}(g_0)(Y)), \mu^*(\mathrm{Ad}(g_0)(X))\}(g_0 \cdot x) \\
&\quad + \mu^*([\mathrm{Ad}(g_0)(X), \mathrm{Ad}(g_0)(Y)])(g_0 \cdot x) \\
&= 0,
\end{aligned}$$

where we have used the fact that

$$\begin{aligned}
\frac{d}{dt}\left(g_0 \exp(tX) \cdot x\right)_{|t=0} &= \frac{d}{dt}\big((g_0 \exp(tX) g_0^{-1}) \cdot (g_0 \cdot x)\big)_{|t=0} \\
&= \frac{d}{dt}\left((\exp(t\mathrm{Ad}(g_0)(X))) \cdot (g_0 \cdot x)\right)_{|t=0} \\
&= (\mathrm{Ad}(g_0)(X))^\sharp(g_0 \cdot x).
\end{aligned}$$

This concludes the proof of 2. □

Note 1.4 Given a Hamiltonian G-space (M, ω, G, μ), the subset $\mu^{-1}(\{0\})$ of M is G-invariant by ii). The quotient $\mu^{-1}(\{0\})/G$ – which is not necessarily a smooth manifold – is called *reduced space*, see symplectic reduction in the next talks.

1.2 Examples

Example 1.5 A symplectic action of $G = \mathbb{R}$ on M is nothing but a complete symplectic vector field: given a smooth symplectic action $\phi : \mathbb{R} \times M \to M$, define $X(x) := \frac{\partial \phi}{\partial t}(0, x)$ for all $x \in M$, then X is symplectic because its flow

$(\phi_t := \phi(t,\cdot))_t$ is and is complete since the flow is defined on $\mathbb{R}$. Conversely, if X is symplectic (i.e., $d(X \lrcorner \omega) = 0$) and complete, then its flow defines a symplectic $\mathbb{R}$-action on M. In that case, a moment map for the $\mathbb{R}$-action reduces to a Hamiltonian function for X; in particular, it exists iff X is Hamiltonian. Note that the equivariance condition – which simplifies to an invariance condition since $\underline{\mathbb{R}} = \mathbb{R}$ is abelian – is automatically satisfied by any Hamiltonian function for X. The invariance of a Hamiltonian function along the integral curves of its symplectic gradient can also be obtained from Noether's theorem below. Beware that not every symplectic vector field with periodic flow has a Hamiltonian function. For instance, consider $M := \mathbb{T}^2 = \mathbb{U}_1 \times \mathbb{U}_1$ with the standard symplectic form $\omega = d\theta_1 \wedge d\theta_2$ and $\mathbb{R} \times M \to M$, $(t,(x,y)) \mapsto (e^{it} \cdot x, y)$, the standard action by rotations on the first factor of $\mathbb{T}^2$. For the basis vector $X = 1 \in \underline{\mathbb{R}} = \mathbb{R}$, the associated fundamental vector field on M is given by $X^\sharp = \frac{\partial}{\partial\theta_1}$. Since $X^\sharp \lrcorner \omega = d\theta_2$, the only Hamiltonian functions possible for $X^\sharp$ are those of the form $\theta_2 + c$, $c \in \mathbb{R}$. But since $\theta_2 : \mathbb{R}^2 \to \mathbb{R}$ is not preserved by the $\mathbb{Z}^2$-action, it does not descend to the 2-torus and hence $X^\sharp$ is not Hamiltonian. Note that the Lie-algebra-cohomology group $H^1(\underline{G}; \mathbb{R}) = \mathbb{R}$ does not vanish, see Theorem 1.14 below.

Example 1.6 Similarly, a symplectic action of $G = \mathbb{U}_1$ on M is a periodic symplectic $\mathbb{R}$-action on M. In that case, a moment map for the $\mathbb{U}_1$-action is a Hamiltonian function $M \to \mathbb{R}$ for the induced vector field on M. Again, not every symplectic vector field with periodic flow has a Hamiltonian function. For example, the non-Hamiltonian symplectic $\mathbb{R}$-action above, being 2π-periodic, induces a $\mathbb{U}_1$-action on $\mathbb{T}^2$, but this action has no Hamiltonian function since the above one already has none.

Example 1.7 Let $G = \mathbb{U}_1$ act by multiplication onto $M := \mathbb{C}^n$ with its standard symplectic form $\omega(z,z') := -\mathrm{Im}(\langle z, z'\rangle)$. This action, which is obviously symplectic, is Hamiltonian: one looks for a smooth function $\mu : \mathbb{C}^n \to \mathbb{R} \cong \underline{\mathbb{U}_1}^*$ with $\mathrm{grad}_\omega(\mu) = i^\sharp$, that is, $\mathrm{grad}_\omega(\mu)(z) = iz$ for every $z \in \mathbb{C}^n$, or equivalently

$$d_z\mu(z') = \omega(iz, z') = -\mathrm{Re}(\langle z, z'\rangle)$$

for all $z, z' \in \mathbb{C}^n$. Therefore $\mu(z) := -\frac{1}{2}|z^2| + c$, $c \in \mathbb{R}$, matches.

Example 1.8 Let $G := \mathbb{R}^3$, $M = T^*\mathbb{R}^3 \cong \mathbb{R}^6$ and $G \times M \to M$, $(a,(x,y)) \mapsto (a+x, y)$ be the action induced by that of $\mathbb{R}^3$ on itself by translations. Let $T^*\mathbb{R}^3$ carry its standard symplectic form $\omega = \sum_{i=1}^3 dx_i \wedge dy_i$, which is the canonical form $\omega = -d\Theta$ and $\Theta : \theta \mapsto \pi_{T^*M}(\theta)(d\pi_M(\theta))$ is the canonical 1-form on T^*M (and where $T^*M \xrightarrow{\pi_M} M$ as well as $T(T^*M) \xrightarrow{\pi_{T^*M}} T^*M$). For any $X \in \mathbb{R}^3$, we have $X^\sharp(x,y) = (X,0)$ for all $(x,y) \in \mathbb{R}^3 \times \mathbb{R}^3$ and hence

$$X^\sharp \lrcorner \omega = \sum_{i=1}^{3} X_i dy_i = d(\sum_{i=1}^{3} X_i y_i) = d(\langle X, y\rangle).$$

Therefore one may define $\mu^X(x,y) := \langle X, y\rangle$ for all $X \in \mathbb{R}^3$ and $(x,y) \in M$. Then $\mu : M \to (\mathbb{R}^3)^* \cong \mathbb{R}^3$, $(x,y) \mapsto y$ is obviously a moment map for that G-action. This map is called *linear momentum.*

Example 1.9 Let again $M = T^*\mathbb{R}^3 \cong \mathbb{R}^6$ carry its standard symplectic structure, but this time take $G := \mathrm{SO}_3$, with action induced by its standard operation on $\mathbb{R}^3$, that is, $(A,(x,y)) \mapsto (Ax, Ay)$. Note that the canonical 1-form Θ is preserved by this G-action since $\langle Ax, Ay\rangle = \langle x, y\rangle$ for all $(x,y) \in \mathbb{R}^6$. Identifying $\underline{\mathrm{SO}_3} \cong \mathbb{R}^3$ via $\begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \mapsto (a_1, a_2, a_3)$, the Lie bracket $[A,B] = AB - BA$ becomes the cross product $a \times b$. For any $a \in \mathbb{R}^3 \cong \underline{\mathrm{SO}_3}$, the induced fundamental vector field is given by $a^\sharp(x,y) = (a \times x, a \times y)$ for all $(x,y) \in \mathbb{R}^3 \times \mathbb{R}^3$. Writing the Hamiltonian condition down for this action, one obtains (modulo constants), for all $x, y \in \mathbb{R}^3$:

$$\mu^a(x,y) = \frac{1}{2}(\langle a \times x, y\rangle + \langle a \times y, x\rangle) = \langle a, x \times y\rangle.$$

The map $\mu : M \to \mathbb{R}^3$, $(x,y) \mapsto x \times y$ satisfies the Hamiltonian condition as well as the equivariance one: for any $a, b \in \mathbb{R}^3$ and $x, y \in \mathbb{R}^3$,

$$\mu^{a\times b}(x,y) = \langle a \times b, x \times y\rangle = \langle a, x\rangle\langle b, y\rangle - \langle a, y\rangle\langle b, x\rangle,$$

and on the other hand

$$\begin{aligned} \{\mu^a, \mu^b\}(x,y) &= \omega_{(x,y)}(\mathrm{grad}_\omega(\mu^a), \mathrm{grad}_\omega(\mu^b)) \\ &= \langle \begin{pmatrix} a \times x \\ a \times y \end{pmatrix}, \begin{pmatrix} b \times y \\ -b \times x \end{pmatrix} \rangle \\ &= \langle a, x\rangle\langle b, y\rangle - \langle a, y\rangle\langle b, x\rangle, \end{aligned}$$

that is, $\mu^{a\times b} = \{\mu^a, \mu^b\}$ on M. The map μ is called *angular momentum.*

1.3 Existence and uniqueness of moment maps

Definition 1.10 *For any finite-dimensional (real) Lie algebra $\mathfrak{g}$ and $k \in \mathbb{N}$, the operator $\delta : \Lambda^k\mathfrak{g}^* \to \Lambda^{k+1}\mathfrak{g}^*$ is defined by*

$$(\delta\omega)(X_0, \ldots, X_k) := \sum_{0 \le i < j \le k} (-1)^{i+j}\omega([X_i, X_j], X_0, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_k),$$

for all $\omega \in \Lambda^k\mathfrak{g}^*$ and $X_0, \ldots, X_k \in \mathfrak{g}$ (for $k = 0$, set $\delta := 0$).

It is elementary to show that $\delta^2 = 0$, therefore δ gives rise to a chain complex and hence to the cohomology groups

$$H^k(\mathfrak{g};\mathbb{R}) := {}^{\ker(\delta)\cap\Lambda^k\mathfrak{g}^*}\!\!\big/_{\mathrm{im}(\delta)\cap\Lambda^k\mathfrak{g}^*}$$

for all $k \geq 0$, where by convention $\delta_{|_{\Lambda^{-1}\mathfrak{g}^*}} := 0$. Those groups are by definition the *Lie-algebra-cohomology groups* of $\mathfrak{g}$.

Examples 1.11

1. In case $k = 1$ one has

   $$H^1(\mathfrak{g};\mathbb{R}) = \ker(\delta)\cap\mathfrak{g}^* = \{\theta\in\mathfrak{g}^*\,,\ \theta([X,Y]) = 0\,\forall\, X,Y\in\mathfrak{g}\} = [\mathfrak{g},\mathfrak{g}]^0,$$

   where $[\mathfrak{g},\mathfrak{g}] := \mathrm{Span}([X,Y],\ X,Y\in\mathfrak{g})\subset\mathfrak{g}$ is the derived ideal and, for any $A\subset\mathfrak{g}$, the subset A^0 denotes its polar set in $\mathfrak{g}^*$. In particular, $H^1(\mathfrak{g};\mathbb{R}) = 0$ iff $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$.

2. In case $k = 2$ one has, for any $\omega\in\Lambda^2\mathfrak{g}^*$ and $X,Y,Z\in\mathfrak{g}$,

   $$(\delta\omega)(X,Y,Z) = -\omega([X,Y],Z) - \omega([Y,Z],X) - \omega([Z,X],Y).$$

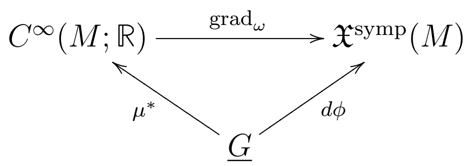
   In particular, $\delta\omega = 0$ iff ω satisfies the analogue of the Jacobi identity. And $H^2(\mathfrak{g};\mathbb{R}) = 0$ iff any such ω is already of the form $(X,Y)\mapsto\theta([X,Y])$ for some $\theta\in\mathfrak{g}^*$.

Theorem 1.12 *Let G be a compact connected Lie group, then there is for every k an isomorphism*

$$H^k(\underline{G};\mathbb{R}) \cong H^k_{dR}(G;\mathbb{R}).$$

The proof of Theorem 1.12 (see e.g. [6, Sec. 8.5]) essentially relies on the fact that, for a compact connected Lie group, the de Rham cohomology groups of G are isomorphic to those built out of *left-invariant* differential forms.

Next we want to give sufficient conditions for the existence and uniqueness of moment maps. The questions may be reformulated in terms of commutative diagrammes as follows. Let $\mathfrak{X}^{\mathrm{symp}}(M)$ and $\mathfrak{X}^{\mathrm{ham}}(M)$ denote the spaces of symplectic vector fields and of Hamiltonian vector fields on M respectively. Given any smooth symplectic group action $\phi : G\times M\to M$, we obtain by differentiating at $e\in G$ the linear map $d\phi : \underline{G}\to\mathfrak{X}^{\mathrm{symp}}(M)$, $d\phi(X) := X^\sharp$, see above. By definition, a comoment map for ϕ is a Lie-algebra-homomorphism $\mu^* : \underline{G} \to C^\infty(M;\mathbb{R})$ making the diagramme



commute, in particular $d\phi(\underline{G}) \subset \mathfrak{X}^{\mathrm{ham}}(M)$. Before we turn to the main result, we state and prove the following lemma, which contains claims from the last talk:

Lemma 1.13 *Let (M,ω) be a symplectic manifold.*

a) For $X, Y \in \mathfrak{X}^{symp}(M)$, one has $d(\omega(Y,X)) = [X,Y]$. In particular, $[X,Y] \in \mathfrak{X}^{ham}(M)$.

b) For any $f, g \in C^\infty(M;\mathbb{R})$, one has $\mathrm{grad}_\omega(\{f,g\}) = -[\mathrm{grad}_\omega(f), \mathrm{grad}_\omega(g)]$. In particular, the map $C^\infty(M;\mathbb{R}) \longrightarrow \mathfrak{X}^{ham}(M)$, $f \mapsto \mathrm{grad}_\omega(f)$, is a Lie-algebra-anti-homomorphism.

c) Let G be a Lie group and $G \times M \to M$ be any smooth symplectic group action on M. Then the map $d\phi : \underline{G} \to \mathfrak{X}^{symp}(M)$, $X \mapsto X^\sharp$, is a Lie-algebra-anti-homomorphism.

Proof: Recall the Cartan identity $L_Z = \iota_Z \circ d + d \circ \iota_Z$ for any $Z \in \mathfrak{X}(M)$, where $\iota_Z := Z \lrcorner$ denotes the inner product by and L_Z the Lie derivative along a vector field Z on M. In particular, a vector field X is symplectic, i.e., $d(\iota_X \omega) = 0$, iff $L_X \omega = 0$.
For any $X, Y \in \mathfrak{X}^{\mathrm{symp}}(M)$, we have

$$\begin{aligned} [X,Y] \lrcorner \omega &= \iota_{L_X Y} \omega \\ &= L_X(Y \lrcorner \omega) - Y \lrcorner L_X \omega \\ &= (\iota_X \circ d + d \circ \iota_X) Y \lrcorner \omega \\ &= d(\omega(Y,X)). \end{aligned}$$

This proves *a*). Statement *b*) is a straightforward consequence of *a*) because Hamiltonian vector fields are symplectic and by the definition of the Poisson bracket. For *c*), we must show that $[X,Y]^\sharp = -[X^\sharp, Y^\sharp]$ for all $X, Y \in \underline{G}$. Let

$f \in C^\infty(M;\mathbb{R})$ be arbitrary, then by definition at any $x \in M$:

$$\begin{aligned}
[X^\sharp, Y^\sharp](x)(f) &= X^\sharp(x)(Y^\sharp(f)) - Y^\sharp(x)(X^\sharp(f)) \\
&= \frac{d}{dt}\big((Y^\sharp(f))(\exp(tX)\cdot x)\big)_{|t=0} - \frac{d}{dt}\big((X^\sharp(f))(\exp(tY)\cdot x)\big)_{|t=0} \\
&= \frac{d}{dt}\left(\frac{d}{ds}\Big(f(\exp(sY)\cdot(\exp(tX)\cdot x))\Big)_{|s=0}\right)_{|t=0} \\
&\quad -\frac{d}{dt}\left(\frac{d}{ds}\Big(f(\exp(sX)\cdot(\exp(tY)\cdot x))\Big)_{|s=0}\right)_{|t=0}.
\end{aligned}$$

By Schwarz' theorem, we can permute both differential operators $\frac{d}{ds}$ and $\frac{d}{dt}$. Using

$$\begin{aligned}
\frac{d}{dt}\left(f(\exp(sY)\cdot\exp(tX)\cdot x)\right)_{t=0} &= \frac{d}{dt}\left(f(\exp(sY)\cdot\exp(tX)\cdot\exp(-sY)\cdot\exp(sY)\cdot x)\right)_{t=0} \\
&= \frac{d}{dt}\left(f(\exp(t\mathrm{Ad}(\exp(sY))(X))\cdot\exp(sY)\cdot x)\right)_{t=0} \\
&= \mathrm{Ad}(\exp(sY))(X)^\sharp(\exp(sY)\cdot x)(f),
\end{aligned}$$

we obtain

$$\begin{aligned}
\frac{d}{ds}\left(\frac{d}{dt}\Big(f(\exp(sY)\cdot(\exp(tX)\cdot x))\Big)_{|t=0}\right)_{|s=0} &= \frac{d}{ds}\big(\mathrm{Ad}(\exp(sY))(X)^\sharp(\exp(sY)\cdot x)(f)\big)_{s=0} \\
&= [Y,X]^\sharp(x)(f) + Y^\sharp(x)(X^\sharp(f)),
\end{aligned}$$

so that, exchanging the roles of X and Y for the second term, we obtain by subtracting

$$[X^\sharp, Y^\sharp](x)(f) = [Y^\sharp, X^\sharp](x)(f) + 2[Y,X]^\sharp(x)(f),$$

which yields the result. □

Theorem 1.14 (Existence and uniqueness of moment maps) *Let G be a connected Lie group with both $H^1(\underline{G};\mathbb{R}) = 0$ and $H^2(\underline{G};\mathbb{R}) = 0$. Then any smooth symplectic G-action on a connected symplectic manifold has a unique moment map.*

Proof: From $H^1(\underline{G};\mathbb{R}) = 0$, we already know that $\underline{G} = [\underline{G}, \underline{G}]$. Let $G \times M \to M$ be any smooth symplectic action of G on a connected symplectic manifold M. Since the commutator of any two symplectic vector fields is Hamiltonian (Lemma 1.13), the map $d\phi : [\underline{G}, \underline{G}] \to \mathfrak{X}^{\mathrm{symp}}(M)$ actually maps into $\mathfrak{X}^{\mathrm{ham}}(M)$. In particular, choosing a basis $\{X_1, \ldots, X_p\}$ of $\underline{G}$, there exist $\tau_1, \ldots, \tau_p \in C^\infty(M; \mathbb{R})$ such that $\mathrm{grad}_\omega(\tau_i) = X_i^\sharp$ for all $1 \leq i \leq p$. Setting $\tau(X_i) := \tau_i$ and extending τ linearly provides a linear map $\tau : \underline{G} \to C^\infty(M; \mathbb{R})$ with $\mathrm{grad}_\omega(\tau(X)) = X^\sharp$ for all $X \in \underline{G}$. The map τ is not necessarily a Lie-algebra-homomorphism, however the fact that $H^2(\underline{G}; \mathbb{R}) = 0$ allows for a slight modification of τ making it into a Lie-algebra-homomorphism. Namely, for any $X, Y \in \underline{G}$, the difference $\tau([X, Y]) - \{\tau(X), \tau(Y)\} \in C^\infty(M; \mathbb{R})$ is actually constant because, by Lemma 1.13,

$$\begin{aligned}
\mathrm{grad}_\omega(\tau([X, Y])) &= [X, Y]^\sharp \\
&= -[X^\sharp, Y^\sharp] \\
&= -[\mathrm{grad}_\omega(\tau(X)), \mathrm{grad}_\omega(\tau(Y))] \\
&= \mathrm{grad}_\omega(\{\tau(X), \tau(Y)\},
\end{aligned}$$

that is, there exist a constant $c(X, Y) \in \mathbb{R}$ with $\tau([X, Y]) - \{\tau(X), \tau(Y)\} = c(X, Y)$. This holds for all $X, Y \in \underline{G}$. Therefore we obtain a map $c : \underline{G} \times \underline{G} \to \mathbb{R}$, which is obviously bilinear and alternate since $(X, Y) \mapsto \tau([X, Y]) - \{\tau(X), \tau(Y)\}$ is. Now since both $[\cdot\,, \cdot]$ and $\{\cdot\,, \cdot\}$ satisfy the Jacobi identity and $c(X, Y)$ is a constant function on M for all $X, Y \in \underline{G}$, we have $\delta c = 0$. By $H^2(\underline{G}; \mathbb{R}) = 0$, there exists a $b \in \underline{G}^*$ such that $\delta b = c$, that is, $c(X, Y) = -b([X, Y])$ for all $X, Y \in \underline{G}$. Setting $\mu^* := \tau + b$, we obtain a linear map $\underline{G} \to C^\infty(M; \mathbb{R})$, still satisfying $\mathrm{grad}_\omega(\mu^*(X)) = X^\sharp$ (because $b(X)$ is constant on M) for any $X \in \underline{G}$ but also

$$\begin{aligned}
\mu^*([X, Y]) &= \tau([X, Y]) + b([X, Y]) \\
&= c(X, Y) + \{\tau(X), \tau(Y)\} + b([X, Y]) \\
&= \{\mu^*(X), \mu^*(Y)\}
\end{aligned}$$

for all $X, Y \in \underline{G}$. Therefore μ^* is a comoment map for the G-action.
Uniqueness trivially follows from $H^1(\underline{G}; \mathbb{R}) = 0$. Namely if μ_1 and μ_2 are two comoment maps for the G-action, then $\mu_1 - \mu_2$ satisfies $\mathrm{grad}_\omega(\mu_1(X) - \mu_2(X)) = X^\sharp - X^\sharp = 0$ for all $X \in \underline{G}$, and since M is connected there is a $b(X) \in \mathbb{R}$ with $\mu_1(X) - \mu_2(X) = b(X)$. Obviously $b \in \underline{G}^*$ and because of $\mu_i([X, Y]) = \{\mu_i(X), \mu_i(Y)\} = \omega(X^\sharp, Y^\sharp)$ for both $i = 1, 2$, we get $(\delta b)(X, Y) = -b([X, Y]) = 0$ for all $X, Y \in \underline{G}$. By $H^1(\underline{G}; \mathbb{R}) = 0$, we obtain $b = 0$ and hence $\mu_1 = \mu_2$. □

Note 1.15 The last argument in the proof of Theorem 1.14 actually shows that, if non-empty, the space of comoment maps for a given smooth symplectic group action of a Lie group G on a connected manifold M is an affine space modelled on $H^1(\underline{G};\mathbb{R})$. Namely, as we have seen above, the difference of any two comoment maps is – provided M is connected – given by an element of $H^1(\underline{G};\mathbb{R})$. Conversely, if $\mu^* : \underline{G} \to C^\infty(M;\mathbb{R})$ is a comoment map, then for any $b \in \ker(\delta) \cap \underline{G}^* = H^1(\underline{G};\mathbb{R})$, the map $\mu^* + b$ is a comoment map for the same group action.

Definition 1.16 *A Lie group is called* semi-simple *iff its Lie algebra contains no non-trivial abelian ideal.*

For instance, any of the classical groups SO_n, SU_n, $\mathrm{SL}(n;\mathbb{R})$ as well as the symplectic group $\mathrm{Sp}(2n;\mathbb{R})$ are semi-simple. A counterexample is given by e.g. $\mathbb{U}_n$, since $i\mathbb{R}\cdot \mathrm{I}_n$ is an abelian ideal in $\underline{\mathbb{U}_n}$.

It is well-known (see e.g. [1, Sec. I.6] or [3, Ch. II]) that a Lie algebra $\mathfrak{g}$ is semi-simple iff its radical (which is the unique maximal solvable ideal in $\mathfrak{g}$) vanishes, which is also equivalent to the Killing form $(X,Y) \mapsto B(X,Y) := \mathrm{tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y))$ being nondegenerate. For compact Lie groups, there is even another very practical characterisation of semi-simplicity:

Proposition 1.17 *A compact Lie group G is semi-simple iff its Lie algebra satisfies $[\underline{G},\underline{G}] = \underline{G}$.*

Proof: The fact that G is compact implies that $\underline{G}$ carries an $\mathrm{Ad}(G)$-invariant inner product $\langle\cdot\,,\cdot\rangle$. In particular, differentiating at $e \in G$ yields

$$\langle \mathrm{ad}(X)(Y), Z\rangle = -\langle X, \mathrm{ad}(Y)(Z)\rangle$$

for all $X,Y,Z \in \underline{G}$. This first implies

$$\underline{G} = [\underline{G},\underline{G}] \underset{\perp}{\oplus} Z(\underline{G}),$$

where $Z(\underline{G}) := \ker(\mathrm{ad}) = \{X \in \underline{G}\,,\ [X,Y] = 0\ \forall\, Y \in \underline{G}\}$ is the centre of the Lie algebra $\underline{G}$. Namely, if $X \in Z(\underline{G})$, then for all $Y, Z \in \underline{G}$,

$$\langle X, [Y,Z]\rangle = -\langle [Y,X], Z\rangle = 0,$$

that is, $X \in [\underline{G},\underline{G}]^\perp$, so that $Z(\underline{G}) \subset [\underline{G},\underline{G}]^\perp$. Conversely, if $X \in [\underline{G},\underline{G}]^\perp$, then for any $Y, Z \in \underline{G}$, one has

$$\langle [X,Y], Z\rangle = \langle X, [Y,Z]\rangle = 0,$$

so that $[X,Y] = 0$ for all $Y \in \underline{G}$, that is, $X \in Z(\underline{G})$. This shows $[\underline{G},\underline{G}]^\perp \subset Z(\underline{G})$ and thus $[\underline{G},\underline{G}]^\perp = Z(\underline{G})$.

If G is semi-simple, then the abelian ideal $Z(\underline{G})$ must vanish, hence $\underline{G} = [\underline{G}, \underline{G}]$. Conversely, if $\underline{G} = [\underline{G}, \underline{G}]$, then $Z(\underline{G}) = 0$. Now if $\mathfrak{h} \subset \underline{G}$ is any abelian ideal, then there exists a connected Lie subgroup H of G_0 (the connected component of the neutral element in G) with $\underline{H} = \mathfrak{h}$. The subgroup H is normal in G_0 because $\mathfrak{h}$ is an ideal and H is a torus since G is compact. In particular, H is contained in a maximal torus of G_0. But since all maximal tori of a given connected compact Lie group are conjugate, H is actually contained in all maximal tori of G_0. Since any element of G_0 is contained in at least one maximal torus, H commutes with each element of G_0 and hence $H \subset Z(G_0)$. By $\underline{Z(G_0)} = Z(\underline{G_0}) = Z(\underline{G}) = 0$, we conclude that $\mathfrak{h} = 0$. This proves that G is semi-simple. □

There is still another characterisation of semi-simplicity in terms of Lie-algebra-cohomology:

Theorem 1.18 (Whitehead's lemmas) *A compact Lie group G is semi-simple iff $H^1(\underline{G}; \mathbb{R}) = 0$ and $H^2(\underline{G}; \mathbb{R}) = 0$.*

See e.g. [4, Thm. III.13] for a proof of Theorem 1.18.

Corollary 1.19 *Let G be a semi-simple compact connected Lie group. Then any smooth symplectic action of G on an arbitrary connected symplectic manifold admits a unique moment map.*

2 Noether's theorem

Theorem 2.1 (Noether's theorem) *Let (M, ω, G, μ) be a Hamiltonian G-space with G connected and $f \in C^\infty(M; \mathbb{R})$ an arbitrary smooth function on M. Then f is G-invariant (i.e., $f(g \cdot x) = f(x)$ for all $(g, x) \in G \times M$) iff μ is constant along the integral curves of $\mathrm{grad}_\omega(f)$.*

Proof: By definition, f is G-invariant iff it is constant on all G-orbits. Since G is connected, this is equivalent to $d_x f(X^\sharp(x)) = 0$ for all $X \in \underline{G}$ and all $x \in M$. But $df(X^\sharp) = \omega(\mathrm{grad}_\omega(f), X^\sharp)$ and, because the G-action is Hamiltonian, $\omega(\mathrm{grad}_\omega(f), X^\sharp) = \omega(\mathrm{grad}_\omega(f), \mathrm{grad}_\omega(\mu^X)) = -d\mu^X(\mathrm{grad}_\omega(f))$. Therefore, $df(X^\sharp) = 0$ on M iff $d\mu^X(\mathrm{grad}_\omega(f)) = 0$ on M. This proves the equivalence. □

Theorem 2.1 generalizes the Noether theorem from last talk ("*Given a Hamiltonian vector field $X = \mathrm{grad}_\omega(\mu)$ and a function $f \in C^\infty(M; \mathbb{R})$, the function f is constant along the integral curves of X iff μ is constant along the integral curves of* $\mathrm{grad}_\omega(f)$") to the case of arbitrary symplectic group actions.

Definition 2.2 *An* integral of motion *for a Hamiltonian G-space* (M, ω, G, μ) *is a G-invariant function $f \in C^\infty(M; \mathbb{R})$. In that case, the* 1*-parameter family of local diffeomorphisms associated to the flow of* $\mathrm{grad}_\omega(f)$ *is called a* symmetry *for* (M, ω, G, μ)*.*

As a consequence, Noether's theorem establishes a bijective correspondence between integrals of motion (modulo constants) and symmetries of Hamiltonian G-spaces.

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