Higher Homotopy groups, cobordisms and Thom spaces

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Abstract: We study the higher homotopy groups of the Thom space of a vector bundle and connect them with the cobordism groups via the Thom isomorphism. We rely heavily on [1, Sec. 4.1] for Section 1, on [2, Sec. 1] and [3, Sec. 17] for Section 2 and on [3, Sec. 18] for Section 3.

1 Higher homotopy groups

In this section let X be any topological space and $I := [0, 1]$ be the unit interval. We call *topological pair* a pair (X, A) where $A \subset X$ is a subset and *topological triple* a triple (X, A, a) where $A \subset X$ is a subset and $a \in A$ is a point in A.

1.1 Homotopy groups

Definition 1.1 *Let $x_0 \in X$ be a point and $n \in \mathbb{N}$.*[1] *Define*

$$\pi_n(X, x_0) := \{f : I^n \longrightarrow X \text{ continuous} \,|\, f(\partial I^n) = \{x_0\}\}/_{\simeq_{\partial I^n}},$$

where $\partial I^n = \{x \in I^n \,|\, x_i \in \{0, 1\}$ for at least one $i\}$ and two continuous maps $f_i : I^n \longrightarrow X$ with $f_i(\partial I^n) = \{x_0\}$ ($i = 0, 1$) satisfy $f_0 \simeq_{\partial I^n} f_1$ if and only if they are homotopic through maps satisfying the same property, i.e., iff there exists $H : [0, 1] \times I^n \longrightarrow X$ continuous with $H(i, \cdot) = f_i$ for both $i = 0, 1$ and $H(t, \partial I^n) = \{x_0\}$ for all $t \in [0, 1]$.

[1] Call this $\mathbb{N}_0$ if you prefer.

A map $f : I^n \longrightarrow X$ with $f(\partial I^n) = x_0$ will be denoted by $f : (I^n, \partial I^n) \longrightarrow (X, x_0)$.[2] In case $n \geq 1$, the set $\pi_n(X, x_0)$ can be identified with that of pointed homotopy classes of continuous maps from the n-dimensional sphere $S^n = I^n/_{\partial I^n}$ to X.

Examples 1.2

1. For $n = 0$ the set $\pi_0(X, x_0)$ can be identified with that of path-connected components of X. This is independent of x_0 because of $\partial I^0 = \varnothing$, in particular it can be denoted by $\pi_0(X)$ instead of $\pi_0(X, x_0)$.

2. For $n = 1$ the set $\pi_1(X, x_0)$ is the usual (pointed) fundamental group.

Proposition 1.3 *Let $x_0 \in X$ and $n \in \mathbb{N} \setminus \{0\}$. Then $\pi_n(X, x_0)$ has a natural group structure which is furthermore abelian if $n \geq 2$.*

From now on $\pi_n(X, x_0)$ will be called (in case $n \geq 1$) the n^{th}-homotopy group of X with basepoint x_0. The group structure will be denoted multiplicatively if $n = 1$ and additively if $n \geq 2$.
We look at the dependence of the homotopy groups upon the basepoint. Given any two points $x_0, x_1 \in X$ joined by a continuous path $c : [0,1] \longrightarrow X$ (where $c(i) = x_i$, $i = 0, 1$), there is a natural map $\Big\{f : (I^n, \partial I^n) \longrightarrow (X, x_0) \text{ continuous}\Big\} \stackrel{\gamma_c}{\longrightarrow} \Big\{f : (I^n, \partial I^n) \longrightarrow (X, x_1) \text{ continuous}\Big\}$, where $\gamma_c(f) : (I^n, \partial I^n) \longrightarrow (X, x_1)$ is obtained as in the (missing) figure. Note that, if $n = 1$, the map $\gamma_c(f)$ is the composition of the paths usually denoted by $cf\bar{c}$ (see below), where $\bar{c}(t) := c(1-t)$. The following lemma shows that γ_c induces a group isomorphism $\pi_n(X, x_0) \longrightarrow \pi_n(X, x_1)$ (where $n \geq 1$) which only depends on the homotopy class of c.

Lemma 1.4 (π_1-action on π_n) *Let $x_0, x_1 \in X$ lie in the same path-connected component of X, let $c : [0,1] \longrightarrow X$ be a continuous path from x_0 to x_1 and $f, f' : (I^n, \partial I^n) \longrightarrow (X, x_0)$ be continuous maps. Assume $n \geq 1$.*

i) If $f \simeq_{\partial I^n} f'$, then $\gamma_c(f) \simeq_{\partial I^n} \gamma_c(f')$. In particular, the map γ_c induces a map $\gamma_c : \pi_n(X, x_0) \longrightarrow \pi_n(X, x_1)$.

ii) $\gamma_c(f \cdot f') \simeq_{\partial I^n} \gamma_c(f) \cdot \gamma_c(f')$. In particular, γ_c is a group homomorphism.

iii) If $c' : [0,1] \to X$ is a continuous path from x_0 to x_1 with $c \simeq_{\partial I} c'$, then $\gamma_c(f) \simeq_{\partial I^n} \gamma_{c'}(f)$. In particular, γ_c only depends on the homotopy class of c (where homotopies fix ∂I).

[2] More generally, the notation $f : (X, A) \longrightarrow (Y, B)$ means f is a map from X into Y sending $A \subset X$ into $B \subset Y$.

iii)' If c is constant, then $\gamma_c = \mathrm{id}_{\pi_n(X,x_0)}$.

iv) If $c' : [0,1] \to X$ is a continuous path from x_1 to $x_2 \in X$, then $\gamma_{cc'}(f) \simeq_{\partial I^n} \gamma_c(\gamma'_c(f))$, where $cc' : I \to X$, $t \mapsto c(2t)$ for $0 \leq t \leq \frac{1}{2}$ and $t \mapsto c'(2t-1)$ for $\frac{1}{2} \leq t \leq 1$, is the natural composition of paths. In particular, γ_c is a group isomorphism.

As a consequence, $[c] \mapsto \gamma_c$ induces a group homomorphism $\pi_1(X, x_0) \longrightarrow \mathrm{Aut}(\pi_n(X, x_0))$, which turns $\pi_n(X, x_0)$ into a $\mathbb{Z}[\pi_1(X, x_0)]$-module.

Note in particular that, of X is path connected (i.e., if $\pi_0(X) = 0$), the groups $\pi_n(X, x_0)$ and $\pi_n(X, x_1)$ are isomorphic for all $x_0, x_1 \in X$ and $n \geq 1$. In that case, they are usually denoted by $\pi_n(X)$ instead of $\pi_n(X, x_0)$.

Like the fundamental group, homotopy groups have the following functorial property.

Lemma 1.5 (π_n is a functor) *Let Y be any topological space with basepoint y_0 and assume $n \geq 1$. Let $\varphi : (X, x_0) \longrightarrow (Y, y_0)$ be any continuous map. Then $[f] \mapsto [\varphi \circ f]$ defines a group homomorphism $\pi_n(\varphi) : \pi_n(X, x_0) \longrightarrow \pi_n(Y, y_0)$ satisfying:*

i) For any continuous map $\psi : (X, x_0) \longrightarrow (Y, y_0)$ homotopic to φ (through maps sending x_0 to y_0), we have $\pi_n(\psi) = \pi_n(\varphi)$.

ii) For any continuous map $\chi : (Y, y_0) \longrightarrow (Z, z_0)$ (where Z is an arbitrary topological space and $z_0 \in Z$), we have $\pi_n(\chi \circ \varphi) = \pi_n(\chi) \circ \pi_n(\varphi)$. Moreover, $\pi_n(\mathrm{id}_X) = \mathrm{id}_{\pi_n(X,x_0)}$.

As a straightforward consequence, the group homomorphism $\pi_n(\varphi)$ is an isomorphism as soon as φ is a homotopy equivalence (with basepoint).

The n^{th} homotopy group of a product is the product of the n^{th} homotopy groups of the factors:

Lemma 1.6 *Let $(X_i)_{i \in I}$ be an arbitrary family of path-connected topological spaces. Then $\pi_n(\prod_{i \in I} X_i) \longrightarrow \prod_{i \in I} \pi_n(X_i)$, $[f] \mapsto ([f_i])_{i \in I}$ is a well-defined group isomorphism.*

Thanks to the lifting property of maps and homotopies through coverings, the higher homotopy groups do not see covering maps:

Lemma 1.7 *If $p : (X, x_0) \longrightarrow (Y, y_0)$ is a covering map, then $\pi_n(p)$ is an isomorphism for all $n \geq 2$.*

Examples 1.8

1. A space X is called *n-simple* if and only if it is path-connected and the π_1-action on $\pi_k(X)$ (see Lemma 1.4) is trivial for all $1 \leq k \leq n$.

2. A space X is called *aspherical* if and only if it is path-connected and $\pi_n(X) = 0$ for all $n \geq 2$. For instance, all contractible spaces are aspherical. More generally, all spaces with contractible universal cover are aspherical.

3. In case $X = S^n$ one has $\pi_m(S^n) = 0$ for $m < n$ and $\pi_n(S^n) = \mathbb{Z}$ as a corollary of Hurewicz theorem below. However, the higher homotopy groups of S^n are only partially known, see table on [1, p.339].

Definition 1.9 *A topological space X is called* n-connected *if and only if $\pi_k(X) = 0$ for all $0 \leq k \leq n$.*

Obviously, a space X is n-connected with $n \geq 1$ if and only if every continuous map $S^n \to X$ is (freely) homotopic to a constant map.

1.2 Hurewicz theorem

Homology and homotopy are related via the famous Hurewicz theorem (the second part can be found in [2, p.207]):

Theorem 1.10 (Hurewicz) *Let X be an $(n-1)$-connected topological space with $n \geq 2$. Then the (well-defined) group homomorphism*

$$\begin{aligned} h : \pi_k(X) &\longrightarrow H_k(X, \mathbb{Z}) \\ [f] &\longmapsto H_k(f)([S^k]) \end{aligned}$$

is an isomorphism for all $1 \leq k \leq n$, where $[S^k] \in H_k(S^k, \mathbb{Z}) \cong \mathbb{Z}$ is the generator fixed by the canonical orientation of S^k. If moreover X is a finite CW-complex, then h has finite kernel and cokernel for all $1 \leq k < 2n - 1$.

For $n = 1$ it is well-known that h is surjective with kernel the commutator subgroup $[\pi_1(X), \pi_1(X)]$ of $\pi_1(X)$, in particular h induces an isomorphism $\pi_1(X)/[\pi_1(X), \pi_1(X)] \longrightarrow H_1(X, \mathbb{Z})$.

There exists a relative version of this theorem: if a topological pair (X, A) is $(n - 1)$-connected with $n \geq 2$ and 1-connected $A \neq \varnothing$ (meaning that $\pi_k(X, A) = 0$ for all $1 \leq k \leq n - 1$ and X is path-connected), then $\pi_k(X, A)$ is canonically isomorphic to $H_k(X, A)$ for all $1 \leq k \leq n$.

1.3 Relative homotopy groups

In this subsection, we denote by $J^{n-1} := \overline{\partial I^n \setminus I^{n-1}} \subset \partial I^n$, where $I^{n-1} = I^{n-1} \times \{0\} \subset \partial I^n$.

Definition 1.11 *Let $A \subset X$ be an arbitrary subset with $x_0 \in A$ and $n \in \mathbb{N} \setminus \{0\}$. Define*

$$\pi_n(X, A, x_0) := \{f : I^n \longrightarrow X \text{ cont.} \,|\, f(\partial I^n) \subset A \text{ and } f(J^{n-1}) = \{x_0\}\}/_{\simeq},$$

where two continuous maps $f_i : I^n \longrightarrow X$ with $f_i(\partial I^n) \subset A$ and $f_i(J^{n-1}) = \{x_0\}$ $(i = 0, 1)$ satisfy $f_0 \simeq f_1$ if and only if they are homotopic through maps satisfying the same property, i.e., iff there exists $H : [0, 1] \times I^n \longrightarrow X$ continuous with $H(i, \cdot) = f_i$ for both $i = 0, 1$ and $H(t, \partial I^n) \subset A$ as well as $H(t, J^{n-1}) = \{x_0\}$ for all $t \in [0, 1]$.

As in the non-relative case, we have the following:

Proposition 1.12 *Let $A \subset X$ with $x_0 \in A$ and $n \in \mathbb{N} \setminus \{0, 1\}$. Then $\pi_n(X, A, x_0)$ has a natural group structure which is furthermore abelian if $n \geq 3$.*

In case $n \geq 2$, the set $\pi_n(X, A, x_0)$ is called n^{th} homotopy group of X *relative to* A. Note that, obviously, $\pi_n(X, \{x_0\}, x_0) = \pi_n(X, x_0)$. Beware there is no natural group structure in case $n = 1$.

The set $\pi_n(X, A, x_0)$ for $n \geq 2$ can be seen as that of all pointed homotopy classes of continuous maps $(D^n, S^{n-1} = \partial D^n, s_0) \to (X, A, x_0)$, where $D^n := \{x \in \mathbb{R}^n \,|\, |x| \leq 1\}$ is the usual closed n-dimensional disk and $s_0 \in S^{n-1}$ is e.g. the North pole.

As before, relative homotopy groups with different basepoints can be related in a natural way provided there is a curve joining them running in the subset A:

Lemma 1.13 (π_1-action on π_n, relative case) *For $x_0, x_1 \in A \subset X$ assume the existence of a continuous map $c : I \longrightarrow A$ with $c(i) = x_i$, $i = 0, 1$. Then for every $n \geq 2$ there is a natural group isomorphism $\gamma_c : \pi_n(X, A, x_0) \to \pi_n(X, A, x_1)$, only depending on the homotopy class of c in A and satisfying $\gamma_{x_0} = \mathrm{id}$ for the constant path x_0 as well as $\gamma_{cc'} = \gamma_c \circ \gamma_{c'}$ for $c' : [0, 1] \longrightarrow X$ continuous with $c(0) = x_1$. In particular, there is a natural group homomorphism $\pi_1(A, x_0) \longrightarrow \mathrm{Aut}(\pi_n(X, A, x_0))$.*

Again, if A is path-connected, the group $\pi_n(X, A, x_0)$ is often denoted simply by $\pi_n(X, A)$. Relative homotopy groups also give rise to a functor:

Lemma 1.14 (π_n is a functor, relative case) *Let (Y, B) be any topological pair, $y_0 \in B$ a point and assume $n \geq 2$. Let $\varphi : (X, A, x_0) \longrightarrow (Y, B, y_0)$ be any continuous map (hence $\varphi(A) \subset B$ and $\varphi(x_0) = y_0$). Then $[f] \mapsto [\varphi \circ f]$ defines a group homomorphism $\pi_n(\varphi) : \pi_n(X, A, x_0) \longrightarrow \pi_n(Y, B, y_0)$ satisfying:*

i) For any continuous map $\psi : (X, A, x_0) \longrightarrow (Y, B, y_0)$ homotopic to φ (through maps $(X, A, x_0) \to (Y, B, y_0)$), we have $\pi_n(\psi) = \pi_n(\varphi)$.

ii) For any continuous map $\chi : (Y, B, y_0) \longrightarrow (Z, C, z_0)$ (where (Z, C, z_0) is an arbitrary topological triple), we have $\pi_n(\chi \circ \varphi) = \pi_n(\chi) \circ \pi_n(\varphi)$. Moreover, $\pi_n(\mathrm{id}_X) = \mathrm{id}_{\pi_n(X,A,x_0)}$.

There is however a particular feature about relative homotopy groups:

Proposition 1.15 (long exact homotopy sequence) *Let (X, A, x_0) be a topological triple. Then the following sequence is exact:*

$$\ldots \longrightarrow \pi_n(A, x_0) \xrightarrow{\pi_n(i)} \pi_n(X, x_0) \xrightarrow{\pi_n(j)} \pi_n(X, A, x_0) \xrightarrow{\partial_n} \pi_{n-1}(A, x_0) \longrightarrow \ldots \longrightarrow \pi_0(X, x_0),$$

where $i : (A, x_0) \to (X, x_0)$ and $j : (X, \{x_0\}, x_0) \to (X, A, x_0)$ are the natural inclusions and $\partial_n([f]) := [f_{|_{I^{n-1}}}]$ (again $I^{n-1} \cong I^{n-1} \times \{0\} \subset \partial I^n$).

Exactness when no group structure is at hand means what it usually means: the set of elements mapped to 0 (which is the homotopy class of the constant map) is the image of the preceding arrow.

The proof of Proposition 1.15 is elementary and relies on the following so-called "compression lemma":

Lemma 1.16 *Let $f(I^n, \partial I^n, J^{n-1}) \longrightarrow (X, A, x_0)$ be a continuous map. Then $[f] = 0 \in \pi_n(X, A, x_0)$ if and only if f is homotopic relatively to $\partial I^n = S^{n-1}$[3] to a continuous map $f' : I^n \longrightarrow A$.*

2 Cobordisms and cobordism groups

Unless otherwise mentioned, all manifolds in this section will be assumed to be smooth - but not necessarily connected!

[3]meaning that the homotopy restricted to ∂I^n does not depend on t

2.1 Cobordisms

Definition 2.1

i) *A (smooth)* manifold triad *is a triple* $(W; V_0, V_1)$*, where* W *is a compact manifold with boundary* $\partial W = V_0 \coprod V_1$ *(hence* V_0 *and* V_1 *are closed hypersurfaces of* W*).*

ii) *A* cobordism *from a manifold* M_0 *to another manifold* M_1 *is a* 5*-tuple* $(W; V_0, V_1; h_0, h_1)$*, where* $(W; V_0, V_1)$ *is a manifold triad and* $h_i : V_i \longrightarrow M_i$ *are (smooth) diffeomorphisms,* $i = 0, 1$*.*

In particular, M_0 and M_1 have to be closed and to have the same dimension in order for a cobordism from M_0 to M_1 to exist.

Examples 2.2

1. Given any closed manifold M, there is always a cobordism from M to itself: just take $(W := [0,1] \times M, V_0 := \{0\} \times M, V_1 := \{1\} \times M; h_0 := p_2, h_1 := p_2)$, where p_2 is the projection onto the second factor.

2. More generally, if $h : M \to M'$ is a diffeomorphism between two closed manifolds M and M', then $(W := [0,1] \times M, V_0 := \{0\} \times M, V_1 := \{1\} \times M; h_0 := p_2, h_1 := h \circ p_2)$ is a cobordism from M to M'.

3. If we accept $\varnothing$ as a closed manifold (of any dimension), then for any compact manifold W, there is a cobordism from the closed manifold $M := \partial W$ to $\varnothing$. In particular, if $(W; V_0, V_1; h_0, h_1)$ is a cobordism from M_0 to M_1, then W can also be seen as a cobordism from $M_0 \coprod M_1$ to $\varnothing$. For instance, using Example 2.2.1 just above, there always exists a cobordism from $M \coprod M$ to $\varnothing$.

Note that, if we did not impose W to be compact, then there would exist a cobordism from *every* boundaryless manifold M to $\varnothing$: just consider $W = M \times [0, \infty[$.

Definition 2.3 *Two cobordisms* $(W; V_0, V_1; h_0, h_1)$ *and* $(W'; V_0', V_1'; h_0', h_1')$ *from a manifold* M_0 *to a manifold* M_1 *are called* equivalent *if and only if there exists a diffeomorphism* $g : W \longrightarrow W'$ *with* $h_i' \circ g = h_i$ *for both* $i = 0, 1$ *(in particular* $g(V_i) = V_i'$*).*

This obviously defines an equivalence relation on all cobordisms from M_0 to M_1. Cobordisms having a boundary piece in common can be glued together thanks to the following

Lemma 2.4 (Gluing cobordisms) *Let $(W; V_0, V_1)$ and $(W'; V_1', V_2)$ be two manifolds triads with $V_1 \neq \varnothing$. Assume the existence of a diffeomorphism $V_1 \xrightarrow{h} V_1'$ and consider $W \cup_h W' := {}^{W \coprod W'}/_{x \sim h(x)}$ with its quotient topology. Then $W \cup_h W'$ admits a smooth manifold structure such that both inclusions $W, W' \hookrightarrow W \cup_h W'$ are smooth embeddings. This smooth structure is unique up to diffeomorphisms leaving $V_0, V_1 \cong V_1'$ and V_2 fixed.*

In a formal way, we can now form a category whose objects are the closed manifolds and whose morphisms from M_0 to M_1 are the equivalence classes of cobordisms from M_0 to M_1. In case the corresponding morphism-sets are non-empty, the composition map $\mathrm{Mor}(M_1, M_2) \times \mathrm{Mor}(M_0, M_1) \to \mathrm{Mor}(M_0, M_2)$ is given by the gluing procedure above (Lemma 2.4), where one should pay attention to the fact that the gluing is still well-defined on the level of cobordism-classes and that the composition we obtain is associative. Each monoid $\mathrm{Mor}(M, M)$ has a neutral element, namely $\iota_M := [([0,1] \times M, \{0\} \times M, \{1\} \times M; p_2, p_2)]$ (from Examples 2.2.1).

Note 2.5 Two cobordism-classes $\mathcal{C}$ and $\mathcal{C}'$ satisfying $\mathcal{C}\mathcal{C}' = \iota_M$ do not necessarily satisfy $\mathcal{C}'\mathcal{C} = \iota_M$, as the following simple example with $M = S^1$ shows [figure: 2 cylinders glued together; moreover, a disk of the left part belongs to the right one].

The set $\mathrm{Mor}(M, M)$ of cobordism classes from M to itself will be henceforth denoted by H_M and that of *invertible* cobordism classes by G_M. Example 2.2.2 provides a map $\mathrm{Diffeo}(M) \to H_M$ via $h \mapsto \mathcal{C}_h := [([0,1] \times M, \{0\} \times M, \{1\} \times M; p_2, h \circ p_2)]$. It is elementary to show that $\mathcal{C}_h \mathcal{C}_{h'} = \mathcal{C}_{h' \circ h}$, thus $h \mapsto \mathcal{C}_h$ is a group-antihomomorphism $\mathrm{Diffeo}(M) \to G_M$. It is however not injective in general. In order to describe its kernel, we need the notion of pseudo-isotopy.

Definition 2.6 *Two diffeomorphisms $h_0, h_1 : M \longrightarrow M'$ are called* pseudo-isotopic *if and only if there exists a diffeomorphism $[0,1] \times M \xrightarrow{g} [0,1] \times M'$ such that $g(i, \cdot) = (i, h_i(\cdot))$ for both $i = 0, 1$.*

Recall that two diffeomorphisms $h_0, h_1 : M \longrightarrow M'$ are called *isotopic* if and only if they are (smoothly) homotopic through diffeomorphisms, i.e., iff there exists a smooth map $H : [0,1] \times M \longrightarrow M'$ with $H(i, \cdot) = h_i$ for both $i = 0, 1$ and $H(t, \cdot) : M \longrightarrow M'$ is a diffeomorphism for all $t \in [0,1]$. Any isotopy is obviously a pseudo-isotopy (just set $g(t, x) := (t, H(t, x))$ for all $(t, x) \in [0,1] \times M$), the converse being wrong in general.

Lemma 2.7 *Given any two closed manifolds M and M', isotopy and pseudo-isotopy define equivalence relations on* $\mathrm{Diffeo}(M, M')$.

We can now describe the kernel of the above map $h \mapsto \mathcal{C}_h$.

Proposition 2.8 *With the above notations, two diffeomorphisms $h_0, h_1 : M \longrightarrow M'$ satisfy $\mathcal{C}_h = \mathcal{C}_{h'}$ if and only if they are pseudo-isotopic.*

2.2 Oriented cobordisms

There is an oriented version of cobordisms. Given an oriented manifold M, we shall denote by $-M$ the manifold with the same smooth structure and *opposite* orientation.

Definition 2.9

i) *An* oriented manifold triad *is manifold triad* $(W; V_0, V_1)$*, where* W *is* oriented*.*

ii) *An* oriented cobordism *from an* oriented *manifold* M_0 *to an* oriented *manifold* M_1 *is a* 5*-tuple* $(W; V_0, V_1; h_0, h_1)$*, where* $(W; V_0, V_1)$ *is an oriented manifold triad and* $h_0 : V_0 \longrightarrow M_0$*,* $h_1 : V_1 \longrightarrow -M_1$ *are (smooth) orientation-preserving diffeomorphisms.*

Beware that, if W is oriented, then ∂W carries an induced orientation as follows: a basis $(X_2, \ldots, X_{n+1})$ of $T_x\partial W$ is oriented iff $(X_1, X_2, \ldots, X_{n+1})$ is an oriented basis of T_xW for an (hence all) *outward-pointing* vector $X_1 \in T_xW$ (and all $x \in \partial W$). In case W is 1-dimensional (hence a union of finitely many compact intervals), we define the orientation at a point $x \in \partial W$ to be 1 or -1 according to that point standing to the right or left end of the corresponding interval respectively.

2.3 Cobordism groups

Definition 2.10 *Two (closed) manifolds M_0 and M_1 are called*

i) cobordant *if and only if there exists a cobordism from M_0 to M_1.*

ii) oriented cobordant *if and only if they are oriented and there exists an oriented cobordism from M_0 to M_1.*

Both define equivalence relations on the set of all n-dimensional closed manifolds (resp. all oriented n-dimensional closed manifolds): each such manifold is bordant to itself (via ι_M, which also respects orientations in case M is oriented), symmetry is clear (change the orientation of W in the oriented case) and transitivity follows from Lemma 2.4, which also adapts to the oriented case and to the case where $M_1 = \varnothing$ (then just consider $W_0 \coprod W_2$, where $\partial W_0 = V_0$ and $\partial W_2 = V_2$).

Definition 2.11 *Let $n \in \mathbb{N}$.*

i) The n^{th} cobordism group *is defined as*

$$\Omega_n^{\mathrm{O}} := \{\textit{closed n-dimensional manifolds}\}/_{\sim},$$

where $M_0 \sim M_1$ iff there exists a cobordism from M_0 to M_1.

ii) The n^{th} oriented cobordism group *is defined as*

$$\Omega_n^{\mathrm{SO}} := \{\textit{oriented closed n-dimensional manifolds}\}/_{\sim_{\mathrm{or}}},$$

where $M_0 \sim_{\mathrm{or}} M_1$ iff there exists an oriented cobordism from M_0 to M_1.

Both Ω_n^{O} and Ω_n^{SO} are abelian groups in a very natural way: define the additive law via $[M]+[M'] := [M \coprod M']$ in both cases. This is obviously well-defined, commutative and associative, with neutral element $[\varnothing]$, and the inverse of $[M]$ is $[M]$ in the unoriented case and $[-M]$ in the oriented one. Note in particular that $[M] = 0$ if and only if M bounds a compact manifold (and an oriented one in the oriented case). If one lets n runs over $\mathbb{N}$, then one actually obtains a (graded) ring structure on $\Omega^{\mathrm{O}} := \bigoplus_{n\in\mathbb{N}} \Omega_n^{\mathrm{O}}$ and $\Omega^{\mathrm{SO}} := \bigoplus_{n\in\mathbb{N}} \Omega_n^{\mathrm{SO}}$ via $[M] \cdot [N] := [M \times N]$.

Examples 2.12

1. For $n = 0$ it is easy to see that $\Omega_0^{\mathrm{O}} \cong \mathbb{Z}_2$ since any two points can be joined by a segment. Moreover, $\Omega_0^{\mathrm{SO}} \cong \mathbb{Z}$, where the isomorphism is given by the sum of the signs of the (finitely many) points.

2. For $n = 1$ both $\Omega_1^{\mathrm{O}} = \Omega_1^{\mathrm{SO}} = 0$ since S^1 obviously bounds an oriented manifold.

3. For $n = 2$ the oriented cobordism group Ω_2^{SO} also vanishes since any orientable closed surface bounds a compact manifold (called "handle-body" in higher genus). It is a bit of work to show that $\Omega_2^{\mathrm{O}} \cong \mathbb{Z}_2$, with the class of the real projective plane as a generator.

4. It is however not trivial to show that $\Omega_3^{\mathrm{SO}} = 0$ (Rokhlin's theorem).

All cobordism groups turn out to be finitely generated, see Corollary 3.10.

3 Thom spaces, their homology and homotopy

3.1 The Thom space of a vector bundle

Definition 3.1 *Let $E \longrightarrow B$ be a Riemannian (real) vector bundle over a topological space B. The* Thom space *of E is the topological quotient $T(E) := E/_A$, where $A := \{X \in E, |X| \geq 1\}$.*

Notes 3.2

1. In particular there exists a preferred point $t_0 := [A] \in T(E)$.

2. The map $X \mapsto \frac{X}{\sqrt{1-|X|^2}}$ defines a homeomorphism $E \setminus A \to E$. In particular, if the base B is compact, then the Thom space of E is homeomorphic to the Alexandrov 1-point compactification $\hat{E} := E \bigsqcup \{\infty\}$ of E (map t_0 to ∞).

3. As another consequence of Note 3.2.2, Thom spaces associated to different Riemannian metrics on E are homeomorphic. Therefore, we do not need any longer to specify any metric on E.

If B is a CW-complex, so is its Thom space:

Proposition 3.3 *Let $E \longrightarrow B$ be an n-ranked real vector bundle over a CW-complex B. Then $T(E)$ is an $(n-1)$-connected CW-complex with exactly one 0-cell and one $(n+k)$-cell for each k-cell in B.*

Proof: If $e_\alpha \subset B$ is an open k-dimensional cell (that is, e_α is homeomorphic to $\overset{\circ}{D}{}^k$), then $\pi^{-1}(e_\alpha) \cap (E \setminus A)$ is homeomorphic to $\overset{\circ}{D}{}^n \times \overset{\circ}{D}{}^k \cong \overset{\circ}{D}{}^{n+k}$, so that $\pi^{-1}(e_\alpha) \cap (E \setminus A)$ is an open $(n+k)$-cell. Together with the 0-cell $\{t_0\}$, their disjoint union is $T(E)$. The characteristic maps gluing the cells together can be constructed as follows: if $\phi_\alpha : D^k \to B$ is a characteristic map for e_α (that is, ϕ_α is continuous, maps $\overset{\circ}{D}{}^k$ homeomorphically onto e_α and $\phi_\alpha(\partial D^k)$ is contained in the union of finitely many cells of lower dimension), then the pull-back bundle $\phi_\alpha^* E \to D^k$ is trivial because of D^k being contractible (see e.g. [4, Sec. 11.3] for the lifting property of homotopies in C^0 bundles), hence there exists a disk-bundle isomorphism $F_\alpha : D^k \times D^n \to \phi_\alpha^*(\overline{E \setminus A})$. Composing with the canonical projection $\overline{E \setminus A} \to T(E)$, we obtain a continuous map $\Phi_\alpha : D^k \times D^n \to T(E)$ which turns out to be a characteristic map for the open cell $\pi^{-1}(e_\alpha) \cap (E \setminus A)$. □

3.2 Homology groups of the Thom spaces

Proposition 3.4 *Let $E \longrightarrow B$ be an oriented n-ranked real vector bundle over a topological space B. Then there exists a canonical group isomorphism $H_{n+k}(T(E), t_0; \mathbb{Z}) \to H_k(B; \mathbb{Z})$ for all $k \in \mathbb{Z}$.*

Proof: Set $T_0 := T(E) \setminus \{0 - \text{section}\}$. Since S^{n-1} is a deformation retract of $D^n \setminus \{0\}$, there exists a deformation retract from T_0 onto $\{t_0\}$. In particular $H_l(T_0, \{t_0\}; \mathbb{Z}) = 0$ for all $l \in \mathbb{Z}$. Thus, the long exact homology sequence for the triple $(T, T_0, \{t_0\})$ yields $H_l(T(E), \{t_0\}; \mathbb{Z}) \cong H_l(T(E), T_0; \mathbb{Z})$ for all $l \in \mathbb{Z}$. Since $\{t_0\}$ is closed and contained in the interior T_0, an excision argument provides $H_l(T(E), T_0; \mathbb{Z}) \cong H_l(T(E) \setminus \{t_0\}, T_0 \setminus \{t_0\}; \mathbb{Z})$, which is by construction of $T(E)$ just $H_l(E, E_0; \mathbb{Z})$, where $E_0 := E \setminus \{0-\text{section}\}$. Now we know that there exists a unique cohomology class $u \in H^n(E, E_0; \mathbb{Z})$ such that $u_{|_F} \in H^n(F, F_0)$ is the given orientation class of the fibre F, for every F; moreover, the map $y \mapsto u \cap y$ is an isomorphism $H_{n+k}(E, E_0; \mathbb{Z}) \to H_k(E)$(it is the so-called Thom isomorphism), for every $k \in \mathbb{Z}$. Since B, seen as the zero-section of E, is a deformation retract of E, we obtain by composing the isomorphisms above

$$H_{n+k}(T(E), t_0; \mathbb{Z}) \cong H_{n+k}(T(E), T_0; \mathbb{Z}) \cong H_{n+k}(E, E_0; \mathbb{Z}) \cong H_k(E; \mathbb{Z})) \cong H_k(B; \mathbb{Z}),$$

which was to be shown. □

3.3 Homotopy groups of the Thom spaces

Corollary 3.5 *Let $E \longrightarrow B$ be an oriented $n(\geq 2)$-ranked real vector bundle over a finite CW-complex B. Then for any $m \in \{0, \dots, n-2\}$ there exists a canonical group homomorphism $\pi_{m+n}(T(E)) \to H_m(B; \mathbb{Z})$ which has finite kernel and co-kernel.*

Proof: Since by Proposition 3.3 the Thom space $T(E)$ is $(n-1)$-connected and, by assumption, $m + n < 2n - 1$, the Hurewicz homomorphism $h : \pi_{m+n}(T(E)) \longrightarrow H_{m+n}(T(E); \mathbb{Z})$ has finite kernel and cokernel. Proposition 3.4 yields a canonical group isomorphism $H_{m+n}(T(E); \mathbb{Z}) \to H_m(B; \mathbb{Z})$, which concludes the proof. □

From now on we concentrate on *smooth* vector bundles (over smooth bases). We denote by $s_0 : B \longrightarrow E$ the zero-section of a vector bundle $E \longrightarrow B$.

Theorem 3.6 *Let $E \longrightarrow B$ be a smooth $n(\geq 1)$-ranked real vector bundle over a smooth manifold B and $m \geq 0$ be a non-negative integer. Then for* *any continuous map $S^{m+n} \xrightarrow{f} T(E)$, there exists a continuous map $S^{m+n} \xrightarrow{g} T(E)$ which is homotopic to f, smooth on $g^{-1}(T(E) \setminus \{t_0\})$ and transverse to $B \cong s_0(B)$. Moreover, for any such map g, the cobordism class $[g^{-1}(B)] \in \Omega_m^{\mathrm{O}}$ only depends on the homotopy class of g. In particular, $[g] \mapsto [g^{-1}(B)]$ defines a group homomorphism $\pi_{m+n}(T(E)) \longrightarrow \Omega_m^{\mathrm{O}}$ which induces a group homomorphism $\pi_{m+n}(T(E)) \longrightarrow \Omega_m^{\mathrm{SO}}$ in case E is oriented.*

Proof: The proof of Theorem 3.6 relies on the following lemma:

Lemma 3.7 *Let $m, n \in \mathbb{N}$ and $f : V \longrightarrow \mathbb{R}^n$ be a smooth map from an open subset $V \subset \mathbb{R}^{m+n}$ into $\mathbb{R}^n$. Assume that $0 \in \mathbb{R}^n$ is a regular value of f throughout some closed subset X of V (that is, $d_x f : \mathbb{R}^{m+n} \to \mathbb{R}^n$ is surjective for all $x \in X$). Let $K \subset V$ be a compact subset. Then for every $\varepsilon > 0$, there exists a smooth map $g : V \to \mathbb{R}^n$ such that $g_{|_{V \setminus K'}} = f$ for some compact subset $K' \subset V$, the map g has 0 as a regular value throughout $X \cup K$ and $\|g - f\|_\infty < \varepsilon$.*

Proof: There exist a compact neighbourhood K' of K in V and a smooth function $\chi : V \longrightarrow [0, 1]$ which is 1 on a neighbourhood of K and vanishes outside K'. Given any regular value y of f, consider the map $g_y : V \to \mathbb{R}^n$, $x \mapsto f(x) - \chi(x) \cdot y$. Then g_y is smooth, has 0 as a regular value throughout $(X \setminus K') \cup K$ (because of $\chi_{|K} = 1$ and y being a regular value of f), coincides with f outside K' and satisfies $\|g_y - f\|_\infty \leq |y|$. By the Sard-Brown theorem the set of regular values of a smooth map is dense (in the target manifold), we can achieve $\|g_y - f\|_\infty < \varepsilon$ by choosing y sufficiently close to 0 in $\mathbb{R}^n$. Moreover, since actually $\|g_y - f\|_{C^1} < C(y)|y|$ for some continuous and bounded function C of y, we can by possibly making ε smaller ensure that 0 remains a regular value of g_y throughout $X \cap K'$ as well. On the whole, there is an appropriate choice of y making g_y fulfill all required conditions. $\surd$

First, we have to accept the existence of a map $f_0 : S^{m+n} \to T(E)$, homotopic to f with $f_0^{-1}(\{t_0\}) = f^{-1}(\{t_0\})$ and such that f_0 is smooth outside $f_0^{-1}(\{t_0\})$ (see [4, Sec. 6.7]). Now we prove the result for f_0 instead of f. Since $f_0^{-1}(B)$ is compact, included in $f_0^{-1}(T(E) \setminus \{t_0\})$ and E is locally trivial, there exist finite families $\{K_i\}_{1 \leq i \leq r}$ and $\{V_i\}_{1 \leq i \leq r}$ of compact subsets of $f_0^{-1}(T(E) \setminus \{t_0\})$ and finitely many trivializing open subsets $\{U_i\}_{1 \leq i \leq r}$ for E (i.e., $\pi^{-1}(U_i) \cong U_i \times \mathbb{R}^n$ where $E \xrightarrow{\pi} B$ is the projection map) with $f_0^{-1}(B) \subset \bigcup_{i=1}^r \overset{\circ}{K}_i \subset \bigcup_{i=1}^r K_i \subset \bigcup_{i=1}^r V_i$, and $f_0(V_i) \subset U_i$ for each $1 \leq i \leq r$. We construct inductively on i continuous maps $f_i : S^{m+n} \to T(E)$, $1 \leq i \leq r$, satisfying:

1. The map f_i is homotopic and as close as desired to f_{i-1}, $f_i^{-1}(\{t_0\}) = f_0^{-1}(\{t_0\})$, f_i is smooth outside $f_0^{-1}(\{t_0\})$, $f_{i|_{S^{m+n}\setminus K_i'}} = f_{i-1|_{S^{m+n}\setminus K_i'}}$ for some compact subset K_i' of V_i;

2. The map f_i is a fibrewise deformation of f_{i-1}, that is, $\pi \circ f_i = \pi \circ f_0$ on $S^{m+n} \setminus f_0^{-1}(\{t_0\})$;

3. The map f_i is transverse to B throughout $K_1 \cup \ldots \cup K_i$.

The construction of f_i from f_{i-1} is an elementary application of Lemma 3.7. First we only need to care for f_i one the open subset $S^{m+n} \setminus f_0^{-1}(\{t_0\})$. Let $\rho_i := p_2^{(i)} : \pi^{-1}(U_i) \to \mathbb{R}^n$ be the second projection coming from the trivialization over U_i. To construct f_i on V_i we only need to define $\rho_i \circ f_i$, since the first component of f_i is determined by $\pi \circ f_i = \pi \circ f_0$. By assumption, $\rho_i \circ f_{i-1}$ has 0 as a regular value throughout $V_i \cap (K_1 \cup \ldots K_{i-1})$ (the differential $d\rho_i$ is fibrewise a linear isomorphism). Considering V_i as an open subset of $\mathbb{R}^{m+n} \cong S^{m+n} \setminus \{\mathrm{pt}\}$ (make V_i slightly smaller in case $V_i = S^{m+n}$) and $\rho_i \circ f_{i-1}$ as a smooth function $V \longrightarrow \mathbb{R}^n$ with 0 as a regular value throughout the closed subset $X_i := V_i \cap (K_1 \cup \ldots \cup K_{i-1})$ of V_i, Lemma 3.7 provides a smooth map $r_i : V_i \longrightarrow \mathbb{R}^n$, coinciding with $\rho_i \circ f_{i-1}$ outside some compact subset K_i' of V_i and having 0 as a regular value throughout $X_i \cup K_i = V_i \cap (K_1 \cup \ldots \cup K_i)$. Define f_i to be f_{i-1} on $S^{m+n} \setminus K_i'$ and $f_i := (\pi \circ f_0, r_i)$ on V_i. Then f_i is, by construction of r_i, well-defined as a map $S^{m+n} \to T(E)$, satisfies $f_{i|_{S^{m+n}\setminus K_i'}} = f_{i-1|_{S^{m+n}\setminus K_i'}}$, in particular $f_i^{-1}(\{t_0\}) = f_0^{-1}(\{t_0\})$, f_i is smooth outside $f_0^{-1}(\{t_0\})$ and transverse to B throughout $K_1 \cup \ldots \cup K_i$, as well as $\pi \circ f_i = \pi \circ f_0$ outside $f_0^{-1}(\{t_0\})$. On the whole, the map f_i satisfies all we want. Now start with f_0, take $K_0 := \varnothing$ (keep no other condition in mind that f_0 is smooth outside $f_0^{-1}(\{t_0\})$) and construct sucessively $f_1, \ldots, f_r$ as above. Set $g := f_r$, then g does the job... provided the inclusion $f_r^{-1}(B) \subset K_1 \cup \ldots \cup K_r$ is fulfilled. Note that the homotopy property $f_{i-1} \simeq f_i$ at each step can be achieved by choosing and r_i sufficiently close to $\rho_i \circ f_{i-1}$. Even better: since f_0 is away from B on the compact subset $S^{m+n} \setminus (\bigcup_{i=1}^r \overset{\circ}{K}_i)$, one can choose at each step the corresponding ε_i small enough such that f_i again remains away from B. Therefore, by an appropriate choice of the r_i's at each step, we obtain the required map g.
Next we prove that the oriented cobordism class of the submanifold $g^{-1}(B)$ only depends on the homotopy class of g. Let $g, g' : S^{m+n} \to T(E)$ be continuous maps with $g^{-1}(\{t_0\}) = g'^{-1}(\{t_0\})$, g and g' are smooth outside $g^{-1}(\{t_0\})$, are homotopic to each other and transverse to B. Then there exists a homotopy (actually homotopic to the homotopy from g to g') $h_0 : [0,1] \times S^{m+n} \longrightarrow T(E)$ from g to g' which is smooth outside $h_0^{-1}(\{t_0\})$ and

satisfies $h_0(t,\cdot) = g$ for all $0 \leq t \leq \frac{1}{3}$ as well as $h_0(t,\cdot) = g'$ for all $\frac{2}{3} \leq t \leq 1$. As above one constructs a new continuous map $h : [0,1] \times S^{m+n} \to T(E)$ with $h^{-1}(\{t_0\}) = h_0^{-1}(\{t_0\})$, which is smooth outside that subset, coincides with h_0 outside a compact subset of $]0,1[\times S^{m+n}$ and is transverse to B; beware that the transversality throughout $([0,\frac{1}{3}] \cup [\frac{2}{3},1]) \times S^{m+n}$ has to be required to be preserved at each step. Now $h^{-1}(B)$ is a cobordism[4] from $g^{-1}(B) = h^{-1}(B) \cap (\{0\} \times S^{m+n})$ to $g'^{-1}(B) = h^{-1}(B) \cap (\{1\} \times S^{m+n})$ and induces the desired orientations on both $g^{-1}(B)$ and $g'^{-1}(B)$. The groups laws on $\pi_{m+n}(T(E),t_0)$ and Ω_n^{SO} are respected by the assignment $[g] \mapsto [g^{-1}(B)]$, since by definition of the composition in $\pi_{m+n}(T(E))$ and because of $t_0 \notin B$ the submanifold we obtain from $f \vee f'$ is $g^{-1}(B) \coprod g'^{-1}(B)$.
By forgetting about orientations, the proof of the oriented case obviously adapts to the non-orientable case, giving rise to a group homomorphism $\pi_{m+n}(T(E)) \longrightarrow \Omega_m^{\mathrm{O}}$ for all smooth n-ranked real vector bundles $E \longrightarrow B$ and all $m \in \mathbb{N}$. This concludes the proof of Theorem 3.6. $\Box$

Next we look at the particular case where E is either the universal bundle $\gamma^n := \gamma^n(\mathbb{R}^\infty) \to G_n(\mathbb{R}^\infty)$ or the oriented universal bundle $\widetilde{\gamma^n} := \widetilde{\gamma^n}(\mathbb{R}^\infty) \longrightarrow \widetilde{G_n}(\mathbb{R}^\infty)$. The main theorem of this section is the following:

Theorem 3.8 (R. Thom) *Let $n \in \mathbb{N} \setminus \{0,1\}$. Then for all $m \in \{0,\ldots,n-2\}$, the homomorphisms $\pi_{m+n}(T(\gamma^n)) \to \Omega_m^{\mathrm{O}}$ and $\pi_{m+n}(T(\widetilde{\gamma^n})) \to \Omega_m^{\mathrm{SO}}$ of* Theorem 3.6 *are isomorphisms.*

Proof: As in [3], we only prove the surjectivity, which follows from the

Lemma 3.9 *Let $k,m,n \in \mathbb{N}$ with $m \leq k,n$. Then the homomorphisms $\pi_{m+n}(T(\gamma^n(\mathbb{R}^{n+k}))) \longrightarrow \Omega_m^{\mathrm{O}}$ and $\pi_{m+n}(T(\widetilde{\gamma^n}(\mathbb{R}^{n+k}))) \longrightarrow \Omega_m^{\mathrm{SO}}$ of* Theorem 3.6 *are isomorphisms.*

Proof: Pick $[M^m] \in \Omega_m^{\mathrm{SO}}$. By the Whitney embedding theorem, there exists an embedding $M^m \hookrightarrow \mathbb{R}^{m+n}$ because of $n \geq m$. Since M^m is embedded in $\mathbb{R}^{m+n}$, there exists an open tubular neighbourhood U of M in $\mathbb{R}^{m+n}$ which is diffeomorphic to the normal bundle $T^\perp M \to M$ of M in $\mathbb{R}^{m+n}$. Composing the Gauß map $M \to \widetilde{G_n}(\mathbb{R}^{m+n})$, $x \mapsto T_x^\perp M$ with the canonical embedding by the first coordinates $\widetilde{G_n}(\mathbb{R}^{m+n}) \to \widetilde{G_n}(\mathbb{R}^{k+n})$, we obtain a map $M \to \widetilde{G_n}(\mathbb{R}^{n+k})$

[4] If $f : W \to N$ is a smooth map from a manifold with boundary to another manifold and $y \in N$ is a regular value of f (meaning that $d_x f : T_x W \to T_y N$ and $d_x(f_{|\partial W}) : T_x \partial W \to T_y N$ are surjective for $x \in f^{-1}(\{y\}) \cap (W \setminus \partial W)$ and $x \in f^{-1}(\{y\}) \cap \partial W$ respectively), then $f^{-1}(\{y\})$ is a smooth submanifold with boundary $f^{-1}(\{y\}) \cap \partial W$. Similarly, if B is a boundaryless submanifold of N and f is transverse to B in a sense analogous to the one above, then $f^{-1}(B)$ is a manifold with boundary $f^{-1}(B) \cap \partial W$.

which obviously pulls $\widetilde{\gamma^n}(\mathbb{R}^{n+k}) \to \widetilde{G_n}(\mathbb{R}^{n+k})$ back to $T^\perp M$. Therefore, we obtain a smooth map $U \longrightarrow \widetilde{\gamma^n}(\mathbb{R}^{n+k})$ which is fibrewise a diffeomorphism and hence transverse to the zero-section $B = \widetilde{G_n}(\mathbb{R}^{n+k})$ of $\widetilde{\gamma^n}(\mathbb{R}^{n+k}) \to \widetilde{G_n}(\mathbb{R}^{n+k})$. Composing this map with the inclusion $\widetilde{\gamma^n}(\mathbb{R}^{n+k}) \to T(\widetilde{\gamma^n}(\mathbb{R}^{n+k}))$, we obtain a continuous map $U \longrightarrow T(\widetilde{\gamma^n}(\mathbb{R}^{n+k}))$ which can be extended to a continuous map $S^{m+n} = \mathbb{R}^{m+n} \bigsqcup \{\infty\} \stackrel{g}{\longrightarrow} T(\widetilde{\gamma^n}(\mathbb{R}^{n+k}))$ by sending $\mathbb{R}^{m+n} \setminus U$ onto $\{t_0\}$. By construction, g is smooth outside $g^{-1}(\{t_0\})$, transverse to $B = \widetilde{G_n}(\mathbb{R}^{n+k})$ with $g^{-1}(B) = M$. Moreover, the orientation of M obviously coincides with that induced by g and $\widetilde{\gamma^n}(\mathbb{R}^{n+k}) \to \widetilde{G_n}(\mathbb{R}^{n+k})$. In other words, $[M]$ is the image of g through the homomorphism $\pi_{m+n}(T(\widetilde{\gamma^n}(\mathbb{R}^{n+k}))) \longrightarrow \Omega_m^{\mathrm{SO}}$. Note that, forgetting again about the orientation, the same arguments show that the homomorphism $\pi_{m+n}(T(\gamma^n(\mathbb{R}^{n+k}))) \longrightarrow \Omega_m^{\mathrm{O}}$ is surjective. $\surd$

For sufficiently large k the inclusion $\widetilde{\gamma^n}(\mathbb{R}^{n+k}) \to \widetilde{\gamma^n}(\mathbb{R}^\infty)$ induces an isomorphism $\pi_{m+n}(T(\widetilde{\gamma^n}(\mathbb{R}^{n+k}))) \to \pi_{m+n}(T(\widetilde{\gamma^n}(\mathbb{R}^\infty)))$, therefore the group homomorphism $\pi_{m+n}(T(\widetilde{\gamma^n}(\mathbb{R}^\infty))) \longrightarrow \Omega_m^{\mathrm{SO}}$ is an isomorphism. The non-orientable case is analogous. □

Corollary 3.10 *The oriented cobordism group Ω_m^{SO} is finite for all $m \notin 4\mathbb{Z}$ and finitely generated with rank the number of partitions of $\frac{m}{4}$ for $m \in 4\mathbb{Z}$.*

Proof: By Lemma 3.9, the group Ω_m^{SO} is the image of the homomorphism $\pi_{m+n}(T(\widetilde{\gamma^n}(\mathbb{R}^{n+k}))) \longrightarrow \Omega_m^{\mathrm{SO}}$ as soon as $k, n \geq m$. But since $\widetilde{G_n}(\mathbb{R}^{n+k})$ is a finite CW-complex, Corollary 3.5 states the existence of a group homomorphism $\pi_{m+n}(T(\widetilde{\gamma^n}(\mathbb{R}^{n+k}))) \to H_m(\widetilde{G_n}(\mathbb{R}^{n+k}); \mathbb{Z})$ with finite kernel and cokernel, at least if $n \geq m + 2$. Now $H_m(\widetilde{G_n}(\mathbb{R}^{n+k}); \mathbb{Z})$ is finite if $m \notin 4\mathbb{Z}$ and finitely generated with rank equal to the number p of partitions of $\frac{m}{4}$ if $m \in 4\mathbb{Z}$. Therefore Ω_m^{SO} is finite if $m \notin 4\mathbb{Z}$ and is finitely generated with rank at most the number of partitions of $\frac{m}{4}$ if $m \in 4\mathbb{Z}$. Now an explicit computation shows that the products $\mathbb{C}\mathrm{P}^{2k_1} \times \ldots \times \mathbb{C}\mathrm{P}^{2k_r}$, where $k_1, \ldots, k_r$ run over the set of partitions of $r = \frac{m}{4}$, all have different Pontrjagin numbers (see [3, Sec. 16 & 17]), therefore they give linearly independent elements in Ω_m^{SO}. On the whole the rank of Ω_m^{SO} is at least and hence equal to p in the case where $m \in 4\mathbb{Z}$. □

References

[1] A. Hatcher, *Algebraic topology*, Cambridge University Press, 2002.

[2] J. Milnor, *Lectures on the h-cobordism theorem*, Princeton University Press, Princeton, 1965.

[3] J. Milnor, J.D. Stasheff, *Characteristic classes*, Annals of Mathematics Studies **76**, Princeton University Press, 1974.

[4] N. Steenrod, *The Topology of Fibre Bundles*, Princeton Mathematical Series **14**, Princeton University Press, 1951.