The classification of 3-manifolds admitting positive scalar curvature

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Abstract: Following [5, Sec. IV.6] and Perelman's solution to the geometrisation conjecture, we present the classification of those closed 3-manifolds admitting a Riemannian metric with positive scalar curvature.

The classification is two-step: first we show that a 3-manifold admitting a $K(G,1)$-factor cannot carry any metric with positive scalar curvature (PSC); then we use the geometrisation for 3-manifolds to deduce that, in the connected-sum-decomposition of a 3-manifold carrying PSC into prime factors, only $S^1 \times S^2$'s or quotients of S^3 can appear.

1 Closed 3-manifolds with a $K(G,1)$-factor

Definition 1.1 *Given a group G, a* $K(G,1)$-space *is a topological space X such that $\pi_1(X) = G$ and $\pi_k(X) = 0$ otherwise.*

For instance, the circle S^1 is a $K(\mathbb{Z},1)$-space and more generally, the n-torus $\mathbb{T}^n := (S^1)^n$ is a $K(\mathbb{Z}^n,1)$-space. Any path-connected covering of a $K(G,1)$-space is a $K(G',1)$-space for some subgroup $G' \subset G$. The homotopy- (in particular the homology-)type of a $K(G,1)$-CW-complex is uniquely determined by G, see [4, Thm. 1B.8]. Moreover, a CW-complex X is $K(G,1)$ iff its universal covering is *contractible.*

If an $(n \geq 1)$-dimensional closed manifold M^n is $K(G,1)$, then G is infinite, since otherwise the universal covering $\widetilde{M}^n$ of M^n would be *compact* and contractible, in particular orientable; but then $H_n(\widetilde{M}^n;\mathbb{Z}) \cong \mathbb{Z} \neq 0$, contradiction. Actually a much stronger statement holds: if a (finite-dimensional but non-necessarily compact) manifold M^n is $K(G,1)$, then G has no nontrivial element of finite order: since otherwise there would exist $g \in G$ with $\langle g \rangle \cong \mathbb{Z}_k$ for some $k \in \mathbb{N} \setminus \{0,1\}$ and, corresponding to $\langle g \rangle$, a covering $\widehat{M} \to M$ of M with $\pi_1(\widehat{M}) = \langle g \rangle$, in particular $\widehat{M}$ would be $K(\langle g \rangle, 1) = K(\mathbb{Z}_k, 1)$; but for all $l \in \mathbb{N}$, one has $H^{2l}(K(\mathbb{Z}_k,1);\mathbb{Z}_k) \cong \mathbb{Z}_k$ (see [4, Ex. 1B.4] for a concrete description of a $K(\mathbb{Z}_k,1)$-space), in particular $H^{2l}(\widehat{M};\mathbb{Z}_k) \cong \mathbb{Z}_k \neq 0$ for any l with $2l > \dim(\widehat{M})$, contradiction. Therefore g has to be of infinite order, i.e., $\langle g \rangle \cong \mathbb{Z}$.

Another important feature of $K(G,1)$-spaces is the following "classifying" property [4, Prop. 1B.9]: if X is a connected CW-complex, Y a $K(G,1)$-space and $(x_0, y_0) \in X \times Y$ arbitrary points, then any group homomorphism $\pi_1(X, x_0) \to G = \pi_1(Y, y_0)$ is induced by a continuous map $(X, x_0) \to (Y, y_0)$ which is unique up to homotopy fixing x_0.

The main result of this section if the following theorem [5, Thm. IV.6.18], originally stated and proved in [3] (see [3, Thm. 8.1]).

Theorem 1.2 (Gromov-Lawson [3]) *Any closed smooth 3-manifold M^3 which can be written as the connected sum with a (closed smooth) $K(\pi,1)$-manifold cannot carry any metric with positive scalar curvature. Moreover, any metric with non-negative scalar curvature on M^3 must be flat.*

Proof: W.l.o.g. we may assume that M is connected and, up to taking a two-fold-covering of M, that M is orientable. We first assume that the closed manifold M itself is a $K(\pi,1)$-manifold. Since any orientable 3-manifold is already spin, M is spin. Since $\pi \neq 1$, there exists a loop – which, up to homotopy, may be assumed smooth and embedded – γ such that $1 \neq [\gamma] \in \pi$. Consider the covering $\widehat{M} := \widetilde{M}/_{\langle [\gamma] \rangle} \to M$, where $\widetilde{M} \to M$ is the universal covering of M and $\langle [\gamma] \rangle \subset \pi$ is the subgroup of π generated by $[\gamma]$. Then the loop γ lifts to $\widehat{M}$ as a curve $\widehat{\gamma}$ which, by construction of $\widehat{M}$, is actually a loop in $\widehat{M}$ and whose homotopy class generates $\pi_1(\widehat{M})$; in an equivalent way, $[\widehat{\gamma}] \in \pi_1(\widehat{M})$ is the preimage of $[\gamma]$ via the canonical isomorphism $\pi_1(\widehat{M}) \cong \langle [\gamma] \rangle$ provided by the lifting property for curves and homotopies to coverings. Note that $\langle [\gamma] \rangle \cong \mathbb{Z}$ by the preliminary remarks above, in particular $\widehat{M}$ is a $K(\mathbb{Z};1)$-space and therefore cannot be compact (for $\widehat{M}$ is homotopy-equivalent to the $K(\mathbb{Z},1)$-space S^1, for which $H_3(S^1;\mathbb{Z}) = 0$ holds).
Now if M carried a metric with positive scalar curvature, then the pull-back metric on the spin manifold $\widehat{M}$ would be *complete* (as is the pull-back of any complete metric on a covering), would have *uniformly* positive scalar

curvature and *bounded* Ricci curvature (since M is compact and the ranges of Scal and |Ric| do not change when passing to coverings). Choose a sufficiently small open tubular neighbourhood U of $\widehat{\gamma}$ in $\widehat{M}$ and consider the (non-compact complete) manifold $X := \widehat{M} \setminus U$ with boundary $\partial X = \partial U \cong S^1 \times S^1$. If we show that X is a *bad end* for $\widehat{M}$ in the sense of [5, Def. IV.6.16], that is, if there exists a smooth map $F : X \to Y$ with Y enlargeable[1] and $\deg(F_{|\partial X}) \neq 0$, then [5, Thm. IV.6.17] ("*Any non-compact spin manifold containing a bad end cannot carry any complete metric with uniformly positive scalar curvature and bounded Ricci curvature*") implies a contradiction for $\widehat{M}$.

To prove that X is a bad end, we first notice that the inclusion $\iota : \partial X \to X$ induces an isomorphism $H_1(\partial X; \mathbb{Z}) \to H_1(X; \mathbb{Z})$: indeed since both U and $X = \widehat{M} \setminus U$ have small neighbourhoods which deformation retract onto them, one may write the Mayer-Vietoris long exact homology sequence for $(\widehat{M} = X \cup \overline{U}, X, \overline{U})$ down and obtain (with $\overline{U} \cap X = \partial X$)

$$\ldots \to H_2(\widehat{M}; \mathbb{Z}) \to H_1(\partial X; \mathbb{Z}) \to H_1(X; \mathbb{Z}) \oplus H_1(U; \mathbb{Z}) \stackrel{j}{\to} H_1(\widehat{M}; \mathbb{Z}) \to H_0(\partial X; \mathbb{Z}) \to \ldots,$$

where $H_1(\widehat{M}; \mathbb{Z}) \to H_0(\partial X; \mathbb{Z})$ is the zero-map since both $H_0(\partial X; \mathbb{Z}) \to H_0(X; \mathbb{Z}) \cong \mathbb{Z}$ and $H_0(\partial X; \mathbb{Z}) \to H_0(U; \mathbb{Z}) \cong \mathbb{Z}$ are isomorphisms, and where $H_2(\widehat{M}; \mathbb{Z}) = 0$ since the $K(\mathbb{Z}; 1)$-space $\widehat{M}$ is homotopy equivalent to S^1. Moreover, the inclusion $U \subset \widehat{M}$ is a homotopy equivalence since all induced group homomorphisms $\pi_k(U) \cong \pi_k(\widehat{\gamma}(S^1)) \to \pi_k(\widehat{M})$ are isomorphisms (see e.g. [4, Thm. 4.5] for Whitehead's theorem characterising homotopy equivalences between connected CW-complexes), in particular $H_1(U; \mathbb{Z}) \to H_1(\widehat{M}; \mathbb{Z})$ must be an isomorphism, which implies that the injective homomorphism $H_1(\partial X; \mathbb{Z}) \to H_1(X; \mathbb{Z})$ also has to be surjective: for any $c \in H_1(X; \mathbb{Z})$, there exists a unique $c' \in H_1(U; \mathbb{Z})$ with $\iota_X(c) = \iota_U(c') \in H_1(\widehat{M}; \mathbb{Z})$, so that $(c, c') \in \ker(j) = \mathrm{im}(\iota^X_{\partial X} \oplus \iota^U_{\partial X})$ and thus $c \in \mathrm{im}(\iota^X_{\partial X})$, as claimed. Hence, $H_1(\partial X; \mathbb{Z}) \to H_1(X; \mathbb{Z})$ is an isomorphism and therefore $H_1(X; \mathbb{Z}) \cong H_1(S^1 \times S^1; \mathbb{Z}) = \mathbb{Z}^2$.

The trick is now to reinterpret the Hurewicz group homomorphism $\pi_1(X) \to H_1(X; \mathbb{Z}) \cong \mathbb{Z}^2$ as a group homomorphism $\pi_1(X) \to \pi_1(K(\mathbb{Z}^2, 1))$; but then the "classifying property" of $K(G, 1)$-spaces yields the existence of a continuous (which we can probably assume to be smooth) map $F : X \to S^1 \times S^1 = K(\mathbb{Z}^2, 1)$ inducing that group homomorphism $\pi_1(X) \to \pi_1(K(\mathbb{Z}^2, 1))$. Since

[1] A smooth manifold Y is called *enlargeable* iff, for any $\varepsilon > 0$, there exists an oriented covering $\hat{Y}_\varepsilon \to Y$ and an ε-contracting C^1 map $f_\varepsilon : \hat{Y}_\varepsilon \to S^n$ which is constant at infinity and of non-zero degree [5, Def. IV.5.2].

the diagramme

$$\begin{array}{ccc} \pi_1(\partial X) & \longrightarrow & H_1(\partial X;\mathbb{Z}) \\ \downarrow & & \downarrow \cong \\ \pi_1(X) & \longrightarrow & H_1(X;\mathbb{Z}) \cong \pi_1(S^1 \times S^1) \end{array}$$

commutes and the Hurewicz homomorphism $\pi_1(\partial X) \to H_1(\partial X;\mathbb{Z})$ is an isomorphism (the group $\pi_1(\partial X)$ is abelian), the group homomorphism $\pi_1(F_{|_{\partial X}}) : \pi_1(\partial X) \to \pi_1(S^1 \times S^1)$ must be an isomorphism. Since $\pi_k(F_{|_{\partial X}}) : 0 = \pi_k(\partial X) \to 0 = \pi_k(S^1 \times S^1)$ is anyway an isomorphism for all $k \neq 1$, we deduce that $F_{|_{\partial X}} : \partial X \to S^1 \times S^1$ is a *homotopy equivalence* (see again e.g. [4, Thm. 4.5]), which in turn implies that it must have degree ± 1. Since $S^1 \times S^1$ is enlargeable, the subset X is a bad end for $\widehat{M}$ and therefore we obtain a contradiction.

In the general case, the closed connected oriented manifold M can be written in the form $M = M' \sharp N$, where M' is $K(\pi, 1)$. Take any smooth map $\mathrm{col} : M \to M'$ collapsing the N-factor to a point $p' \in M'$. As before, pick a smooth embedded loop γ in $M' \setminus \{p'\}$ such that $1 \neq [\gamma] \in \pi_1(M')$ (one may assume that γ does not run through p') and let $\widehat{M'} := \widetilde{M'}/\langle[\gamma]\rangle \stackrel{\mathrm{pr}'}{\to} M'$ be a covering of M' with $\pi_1(\widehat{M'}) \cong \langle[\gamma]\rangle \cong \mathbb{Z}$. Let $\widehat{\gamma}$ be the lifted loop in $\widehat{M'} \setminus \mathrm{pr}'^{-1}(\{p'\})$. Pulling the covering $\widehat{M'} \stackrel{\mathrm{pr}'}{\to} M'$ back via the map col provides a covering $\widehat{M} \stackrel{\mathrm{pr}}{\to} M$ making the diagramme

$$\begin{array}{ccc} \widehat{M} & \xrightarrow{\widehat{\mathrm{col}}} & \widehat{M'} \\ \downarrow \mathrm{pr} & & \downarrow \mathrm{pr}' \\ M & \xrightarrow{\mathrm{col}} & M' \end{array}$$

commute. Actually the map $\widehat{\mathrm{col}} : \widehat{M} \to \widehat{M'}$ collapses the copies of N-factors in $\widehat{M}$ to points of $\mathrm{pr}'^{-1}(\{p'\})$ and its restriction to the complement of the N-factors is a diffeomorphism onto $\widehat{M'} \setminus \mathrm{pr}'^{-1}(\{p'\})$. As above, if a metric with positive scalar curvature is given on M, then it can be lifted to the spin manifold $\widehat{M}$ as a complete metric with uniformly positive scalar curvature and bounded Ricci curvature. Now we can choose a sufficiently small tubular neighbourhood U' about the lifted loop $\widehat{\gamma}$ in $\widehat{M'}$ such that $U' \subset \widehat{M'} \setminus \mathrm{pr}'^{-1}(\{p'\})$, take the (relatively compact) preimage $U := \widehat{\mathrm{col}}^{-1}(U') \subset \widehat{M}$ and consider $X := \widehat{M} \setminus U$ and $X' := \widehat{M'} \setminus U'$ respectively. As before, there exists a smooth map $F' : X' \to S^1 \times S^1$ such that

$F'_{|\partial X'} : \partial X' \to S^1 \times S^1$ is of non-zero degree. Composing with $\widehat{\mathrm{col}}$, one obtains a smooth map $F = F' \circ \widehat{\mathrm{col}} : X \to S^1 \times S^1$ such that $F_{|\partial X}$ is of non-zero degree since $\widehat{\mathrm{col}}_{|\partial X} : \partial X \to \partial X'$ is a diffeomorphism. Therefore X is a bad end for $\widehat{M}$ and we obtain again a contradiction.
For the proof of the flatness of any metric with non-negative scalar curvature on M, we refer to [3, Thm. 7.48]. □

2 Classification

We are now ready to state the main result of this talk. Recall that any closed (orientable) 3-manifold M can be written as the connected sum of finitely many irreducible manifolds and of copies of $S^1 \times S^2$'s (Kneser's theorem). More precisely, J. Milnor [6] showed that the irreducible factors of M are either $K(\pi, 1)$-manifolds or closed 3-manifolds Σ_j with finite fundamental group. If M carries PSC, then by Theorem 1.2, there is no $K(\pi, 1)$-factor in the connected sum, so that only $S^1 \times S^2$'s or Σ_j's can appear. But Perelman's solution to the geometrisation conjecture [7, 8, 9] implies that the universal covering of Σ_j, being simply-connected and closed, is diffeomorphic to S^3 and therefore $\Sigma_j \cong S^3/\Gamma_j$, where Γ_j is a finite subgroup of SO_4. Obviously, $S^1 \times S^2$ and each quotient of S^3 by a finite fixed-point-free subgroup of SO_4 carry a metric with PSC (even a homogeneous one); since PSC is preserved by codimension $k \geq 3$-surgery [2, 10], the connected sum of any two closed 3-manifolds with PSC also admits PSC. Therefore, we obtain the following

Theorem 2.1 (Closed orientable 3-manifolds with PSC) *A closed orientable 3-dimensional smooth manifold M^3 admits a metric with positive scalar curvature iff it is diffeomorphic to the connected sum of finitely many copies of $S^1 \times S^2$'s and of quotients of S^3 by (finite) fixed-point-free subgroups of SO_4, that is, iff*

$$M^3 \cong S^3/\Gamma_1 \sharp \ldots \sharp S^3/\Gamma_p \sharp \underbrace{(S^1 \times S^2) \sharp \ldots \sharp (S^1 \times S^2)}_{q \text{ times}}$$

for $p, q \in \mathbb{N}$, where $\Gamma_j \subset \mathrm{SO}_4$ is finite and fixed-point-free, for all $1 \leq j \leq p$.

We refer to [1, Ch. 4] for the classification of the finite subgroups of SO_4.[2]

[2]It would remain to identify those which are fixed-point-free!

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