The intersection form of a smooth 4-manifold

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January 14, 2013

Abstract: We present the proof *à la Seiberg-Witten* given in [8, Sec. 2.4.3] of a theorem due to S.K. Donaldson [2] stating that, if the intersection form of a closed oriented smooth 4-dimensional manifold is (negative or positive) definite, then it is diagonal.

1 Unimodular quadratic forms

Definition 1.1 *A symmetric bilinear map* $q : \mathbb{Z}^n \times \mathbb{Z}^n \longrightarrow \mathbb{Z}$ *is called* unimodular *iff* $\mathbb{Z}^n \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^n, \mathbb{Z})$*,* $x \mapsto q(x, \cdot)$*, is an isomorphism.*

Obviously, a symmetric bilinear form on $\mathbb{Z}^n$ as above is unimodular iff its matrix in any basis of $\mathbb{Z}^n$ has determinant ± 1. Note that every unimodular form is non-degenerate (which, by definition, is equivalent to the induced map $\mathbb{Z}^n \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^n, \mathbb{Z})$ being injective) but that the converse statement is wrong.

From now on, a *unimodular quadratic form* on $\mathbb{Z}^n$ will be the quadratic form $x \mapsto q(x, x)$ associated to a unimodular symmetric bilinear form $q : \mathbb{Z}^n \times \mathbb{Z}^n \longrightarrow \mathbb{Z}$. Since the matrix of q in any basis of $\mathbb{Z}^n$ is symmetric, it is diagonalizable – as a *real* symmetric matrix – in a basis of $\mathbb{R}^n$ and all eigenvalues of that matrix are real (and non-vanishing since the matrix is invertible). In particular, one can define the following:

Definition 1.2 *Let* q *be a unimodular quadratic form on* $\mathbb{Z}^n$*.*

i) The rank *of* q *is* $\mathrm{rk}(q) := n \in \mathbb{N}$ *and the* signature *of* q *is* $\tau(q) := \sharp\{\textit{pos. eigenv. of } q\} - \sharp\{\textit{neg. eigenv. of } q\} \in \mathbb{Z}$*.*

ii) The quadratic form q is called positive definite *(resp.* negative definite*) iff $\tau(q) = \mathrm{rk}(q)$ (resp. $\tau(q) = -\mathrm{rk}(q)$).*

iii) The quadratic form q is called even *iff $q(x,x) \equiv 0$ (2) for all $x \in \mathbb{Z}^n$. Otherwise q is called* odd*.*

iii) A further unimodular quadratic form q' on $\mathbb{Z}^n$ is said to be equivalent *to q iff there is $T \in \mathrm{GL}(n,\mathbb{Z})$ s.t. $q'(Tx,Tx) = q(x,x)$ for all $x \in \mathbb{Z}^n$.*

iv) The quadratic form q is said to be diagonalizable *iff q is equivalent to a diagonal form (i.e., whose matrix in the canonical basis of $\mathbb{Z}^n$ is diagonal – and hence has only ± 1 on the diagonal).*

It is easy to see that a unimodular form q is even iff its matrix in a (hence any) basis of $\mathbb{Z}^n$ has even diagonal entries. By definition, a unimodular form q is positive definite iff $q(x,x) > 0$ for all $x \in \mathbb{Z}^n \setminus \{0\}$. The signature of q is, by Sylvester's invariance theorem, well-defined. Beware that only odd forms can be diagonalizable.

Examples 1.3

1. The form E_8 is the unimodular quadratic form of rank 8 defined in the canonical basis of $\mathbb{Z}^8$ by[1]

$$E_8 := \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 \end{pmatrix}.$$

   The form E_8 is even, positive definite, however not diagonalizable.

2. The form H is the unimodular quadratic form of rank 2 defined in the canonical basis of $\mathbb{Z}^2$ by

$$H := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

   The form H is even, indefinite, has signature 0 and is not diagonalizable.

[1] see [10, Sec. V.1.4] (there may be a mistake in Nicolaescu's book).

Another famous example of unimodular form is provided by the *intersection form* of an oriented closed topological 4-dimensional manifold:

Proposition 1.4 *Given any oriented closed topological* 4*-dimensional manifold* M*, the intersection form* $q_M : H^2(M;\mathbb{Z}) \times H^2(M;\mathbb{Z}) \to H^4(M;\mathbb{Z}) \cong \mathbb{Z}$*,* $(\alpha,\beta) \mapsto \langle \alpha \cup \beta, [M]\rangle =: q_M(\alpha,\beta)$*, of* M *defines a unimodular quadratic form on* ${}^{H^2(M;\mathbb{Z})}/_{\mathrm{Tor}(H^2(M;\mathbb{Z}))} \cong {}^{H_2(M;\mathbb{Z})}/_{\mathrm{Tor}(H_2(M;\mathbb{Z}))}$*. Its rank is* $\mathrm{rk}(q_M) = b_2(M) := \mathrm{rk}(H^2(M;\mathbb{Z}))$ *and its signature is* $\tau(q_M) = \mathrm{sign}(M)$*. In case* M *is smooth, we have* $\mathrm{sign}(M) = b_2^+(M) - b_2^-(M)$*.*[2]

Sketch of proof: By definition of the cup product, q_M is clearly symmetric. Note that, since its values belong to the free group $\mathbb{Z}$, the form $q_M(x,\cdot)$ vanishes as soon as $x \in \mathrm{Tor}(H^2(M;\mathbb{Z})) = \mathrm{Tor}(H_1(M;\mathbb{Z}))$. By the universal coefficient theorem, $H^2(M;\mathbb{Z}) \cong \mathrm{Tor}(H_1(M;\mathbb{Z})) \oplus {}^{H_2(M;\mathbb{Z})}/_{\mathrm{Tor}(H_2(M;\mathbb{Z}))}$, so that the free part of $H^2(M;\mathbb{Z})$ is isomorphic to ${}^{H_2(M;\mathbb{Z})}/_{\mathrm{Tor}(H_2(M;\mathbb{Z}))}$. That q_M is unimodular is the statement of e.g. [1, Thm. VI.9.4]. In case M is smooth, we can identify the singular cohomology groups $H^p(M;\mathbb{R})$ with the de Rham cohomology groups $H^p_{\mathrm{dR}}(M;\mathbb{R})$ and the intersection form with $(\alpha,\beta) \mapsto \int_M \alpha \wedge \beta$. Fixing a Riemannian metric g on M with associated volume form $d\mu_g$, we obtain $q_M(\alpha,\beta) = \int_M \langle \alpha, *\beta\rangle d\mu_g$, where $* : H^2_{\mathrm{dR}}(M;\mathbb{R}) \to H^2_{\mathrm{dR}}(M;\mathbb{R})$ is the Hodge isomorphism induced by g. In particular, we obtain $q_M(\alpha,\alpha) = \pm \int_M |\alpha|^2 d\mu_g$ for all $\alpha \in H^2_\pm(M;\mathbb{R}) := H^2(M;\mathbb{R}) \cap \ker(* \mp \mathrm{Id})$, which implies that q_M has exactly $b_2^+(M) = \dim_{\mathbb{R}}(H^2_+(M;\mathbb{R}))$ positive and $b_2^-(M) = \dim_{\mathbb{R}}(H^2_-(M;\mathbb{R}))$ negative eigenvalues, in particular $\mathrm{sign}(M) = b_2^+(M) - b_2^-(M)$. $\square$

Changing the orientation of the manifold obviously changes its intersection form by a sign. An orientation-preserving homeomorphim between 4-manifolds provides an equivalence of the corresponding intersection forms. Indefinite forms are always equivalent to relatively simple ones, in virtue of the following

Proposition 1.5 (see e.g. Ch. 5 in [10])

i) For any even *unimodular quadratic form* q*, one has* $\tau(q) \equiv 0\ (8)$*.*

ii) Any indefinite even *unimodular quadratic form* q *with* $\tau(q) \geq 0$ *is equivalent to* $\frac{\tau(q)}{8} \cdot E_8 \oplus (\frac{\mathrm{rk}(q)-\tau(q)}{2}) \cdot H$*.*[3]

[2]Note that both $b_2^+(M)$ and $b_2^-(M)$ are *topological* invariants since they coincide with the number of positive and negative eigenvalues of q_M respectively.

[3]In case $\tau(q) < 0$ just replace q by $-q$.

iii) Any indefinite odd *unimodular quadratic form is diagonalizable.*

Positive (and negative) definite unimodular forms are much harder to classify. However, there is an invariant sorting out those (necessarily odd) which are diagonalizable. For this, we have to introduce the following concept:

Definition 1.6 *Given any unimodular quadratic form q on $\mathbb{Z}^n$, a* characteristic *vector for q is a vector $x \in \mathbb{Z}^n$ such that,for all $y \in \mathbb{Z}^n$,*

$$q(x, y) \equiv q(y, y) \quad (2).$$

For instance, q is even iff $0 \in \mathbb{Z}^n$ is a characteristic vector for q. As above, x is a characteristic vector for q iff the matrix A of q in a (hence any) basis of $\mathbb{Z}^n$ satisfies $A_{ii} \equiv x_i$ (2) for all $1 \leq i \leq n$.[4] Note that, since q is unimodular, q always admits a characteristic vector (use the non-degeneracy of the mod 2-reduction of q and the fact that the mod 2-reduction $y \mapsto q(y, y)$ is *linear*).

Example 1.7 Given any 4-dimensional oriented closed smooth manifold M, Wu's formula [13]

$$w_2(TM) \cup y = y \cup y \qquad \forall\, y \in H^2(M; \mathbb{Z}_2)$$

implies that the mod 2-reduction of any characteristic vector of q_M coincides with the second Stiefel-Whitney class $w_2(TM)$ of M. In particular, such a manifold M is spin iff its intersection form is even.

Proposition 1.5.i) generalises to

Proposition 1.8 *For any characteristic vector x of a unimodular quadratic form q, one has*

$$\tau(q) \equiv q(x, x) \quad (8).$$

We can now define the so-called *Elkies invariant* of a negative definite unimodular quadratic form.

Definition 1.9 *Given a negative definite unimodular form q, the* Elkies invariant $\Theta(q)$ *of q is defined by*

$$\Theta(q) := \mathrm{rk}(q) + \max\{q(x, x)\,,\ x \text{ char. vect. for } q\} \in \mathbb{Z}.$$

Note that $\Theta(q)$ is well-defined since q is negative definite, in particular $\Theta(q) \leq \mathrm{rk}(q) = -\tau(q)$. Moreover, Proposition 1.8 yields $\Theta(q) \equiv \mathrm{rk}(q) + \tau(q) \equiv 0$ (8).

[4] Why?

Theorem 1.10 (N.D. Elkies [4]) *For any* negative definite *unimodular quadratic form q, one has $\Theta(q) \geq 0$ with equality iff q is diagonal.*

It is easy to see that, if q is diagonal – hence actually *equal* to the standard form $(x, y) \mapsto -\langle x, y\rangle$ on $\mathbb{Z}^n$ – then a vector $x = (x_1, \ldots, x_n) \in \mathbb{Z}^n$ is characteristic for q iff $x_i \equiv 1$ (2) and then the maximum of $q(x, x)$ is obviously attained for $x = (1, 1, \ldots, 1)$, in which case $\Theta(q) = 0$.

2 Signature and smooth structures

Given $r, s \in \mathbb{N}$, the matrix intersection form of the connected sum $M := r\mathbb{C}\mathrm{P}^2 \sharp s\overline{\mathbb{C}\mathrm{P}^2}$ (where $\overline{\mathbb{C}\mathrm{P}^2}$ denotes $\mathbb{C}\mathrm{P}^2$ with the opposite orientation) is, in suitable basis of $H^2(M; \mathbb{Z}) \cong \mathbb{Z}^{r+s}$, exactly $\mathbb{1}_r \oplus -\mathbb{1}_s$. In particular, Proposition 1.5 implies that any indefinite odd unimodular quadratic form is the intersection form of – at least – one closed smooth 4-manifold (namely one of the connected sums above). Actually a much stronger result holds:

Theorem 2.1 (M.H. Freedman [5]) *Let q be any (equivalence class of) unimodular quadratic form(s) on some $\mathbb{Z}^n$, $n \geq 0$.*

i) If q is even*, then there exists up to orientation-preserving homeomorphism a unique oriented simply-connected closed topological* 4*-dimensional manifold with intersection form q.*

ii) If q is odd*, then there exist up to orientation-preserving homeomorphism exactly two oriented simply-connected closed topological* 4*-dimensional manifolds with intersection form q. Moreover, at most one of them is smoothable, i.e., admits a smooth structure.*

Freedman's theorem 2.1 generalises an earlier result by Whitehead [12], stating that any two simply-connected closed topological 4-dimensional manifolds with the same intersection form must be homotopy equivalent. Note that, as an immediate consequence of Theorem 2.1, two oriented *smooth* simply-connected closed 4-dimensional manifolds are homeomorphic (as oriented manifolds) iff they have equivalent intersection forms.

Theorem 2.1 already yields the existence of *non-smoothable* topological 4-manifolds – actually of infinitely many homeomorphism classes of such manifolds. The following theorem gives another criterion for the existence of a smooth structure on a given 4-manifold:

Theorem 2.2 (V.A. Rokhlin [9]) *Let M be any simply-connected closed* smooth 4-*dimensional manifold with* even *intersection form. Then its signature* $\mathrm{sign}(M) \equiv 0 \ (16)$.

Theorem 2.1 provides the existence of a (unique up to homeomorphism) simply-connected closed topological 4-manifold with intersection form E_8 (see Example 1.3.1). Since its intersection form q_M is positive definite, its signature $\mathrm{sign}(M) = 8$ and as a consequence of Rokhlin's theorem 2.2, the manifold M cannot carry any smooth structure.

Theorem 2.3 (S.K. Donaldson [2]) *Let M be any closed oriented* smooth 4-*dimensional manifold with* negative *(or* positive*)* definite *intersection form q_M. Then q_M is diagonal.*

Proof: Up to changing the orientation of M, we assume that q_M is negative definite. The proof goes by contradiction. Fix a Riemannian metric g on M and assume q_M were not diagonal. We first show this leads to a contradiction in a special case; then we show that, in general, one may reduce the proof to that special case.

• *Case* $b_1(M) = 0$: Since q_M is not diagonal, Elkies' theorem 1.10 implies that $\Theta(q_M) > 0$ – in particular $\Theta(q_M) \geq 8$ –, so that, by definition, there exists at least one characteristic vector $\alpha \in H^2(M;\mathbb{Z})$ (actually one can choose α to lie in the free part $H^2(M;\mathbb{Z})/\mathrm{Tor}(H^2(M;\mathbb{Z}))$) for q_M such that $\Theta(q_M) = \mathrm{rk}(q_M) + q_M(\alpha,\alpha) > 0$. Since the first Chern class $c_1 : H^1(M;\mathbb{U}_1) \longrightarrow H^2(M;\mathbb{Z})$ is an isomorphism, there exists a $\mathbb{U}_1$-bundle $P_{\mathbb{U}_1} = P \to M$ with $c_1(P) = \alpha$, so that $\mathrm{rk}(q_M) + q_M(c_1(P), c_1(P)) > 0$. By Example 1.7, since the mod 2-reduction of $c_1(P)$ has to coincide with the second Stiefel-Whitney class $w_2(TM)$ of M, there exists a spin^c structure $P_{\mathrm{Spin}_4^c}(TM) = \widetilde{P} \to M$ on M with determinant bundle $P \to M$ (see first talk of the seminar). To compute the formal dimension $d(P)$ of the Seiberg-Witten moduli space, we notice that, using Poincaré duality, M connected and $b_1(M) = 0$,

$$\chi(M) = \sum_{i=0}^{4} (-1)^i b_i(M) = 2(b_0(M) - b_1(M)) + b_2(M) = b_2(M) + 2$$

and $\mathrm{sign}(M) = \tau(q_M) = -b_2(M)$ (since q_M is negative definite), so that

$$\begin{aligned} d(P) &= \frac{1}{4}\left(c_1(P)^2[M] - 2\chi(M) - 3\mathrm{sign}(M)\right) \\ &= \frac{1}{4}\left(q_M(c_1(P), c_1(P)) - 2b_2(M) - 4 + 3b_2(M)\right) \\ &= \frac{1}{4}\left(q_M(c_1(P), c_1(P)) + b_2(M)\right) - 1 \\ &= \frac{\Theta(q_M)}{4} - 1. \end{aligned}$$

In particular, $d(P) \in 1 + 2\mathbb{N}$.
Now recall the following facts from Seiberg-Witten theory. Given any $h \in H^{3,2}(\Lambda^2_+ T^*M)$, the *Seiberg-Witten moduli space* parametrised by h is the space $\mathcal{M}(\widetilde{P}, h) := \widetilde{\mathcal{M}}(\widetilde{P}, h)/_{\mathcal{G}(\widetilde{P})}$ (endowed with the quotient topology), where

$$\widetilde{\mathcal{M}}(\widetilde{P}, h) := \left\{(A, \psi) \in \mathcal{A}^{4,2}(P) \times H^{4,2}(\Sigma^+ M)\,,\, F_A^+ = q_\psi + ih \text{ and } D^A\psi = 0\right\}$$

and where the gauge group $\mathcal{G}(\widetilde{P}) := H^{5,2}(M, \mathbb{U}_1)$ acts via $(A, \psi) \cdot \sigma := (R^*_{\sigma^2} A, \sigma^{-1}\psi)$ for all $\sigma \in \mathcal{G}(\widetilde{P})$ (for more details on notations, see notes from Nicolas' second talk). An element $[(A, \psi)] \in \mathcal{M}(\widetilde{P}, h)$ (named *Seiberg-Witten monopole*) is called *irreducible* (resp. *reducible*) iff its stabilizer under the $\mathcal{G}(\widetilde{P})$-action is trivial (resp. non-trivial); actually $[(A, \psi)]$ is reducible iff $\psi = 0$ and in that case its stabilizer is $\mathbb{U}_1$ (the subgroup of $\mathcal{G}(\widetilde{P})$ consisting of constant $\mathbb{U}_1$-valued functions). As in the preceding talks, for any 2-form ω on M, we denote by $[\omega] \in \ker(d) \cap \ker(\delta) = \mathscr{H}^2(M)$ its harmonic component in the Hodge L^2-orthogonal decomposition

$$\Omega^2(M) = \mathscr{H}^2(M) \oplus d\Omega^1(M) \oplus \delta\Omega^3(M)$$

(which hopefully has an analogue for non-smooth differential forms). Recall that $\mathscr{H}^2(M) \longrightarrow H^2_{\mathrm{dR}}(M; \mathbb{R})$, $\omega \longmapsto [\omega]$, is an isomorphism. Note that, if $p_+ = (\cdot)_+ := \frac{\mathrm{Id}+*}{2} : \Omega^2(M) \to \Omega^2_+(M)$ denotes the (pointwise orthogonal) projection onto the space of self-dual 2-forms, then $[h]_+ = [h_+]$ for any $h \in \Omega^2(M)$: for $[h] = [h_+] + [h_-]$ and, using the Hodge decomposition above, $([h_+], \omega)_{L^2(M)} = 0$ for all $\omega \in \mathscr{H}^2_-(M)$, so that $[h_+] \in \mathscr{H}^2_+(M)$ (where $*(\mathscr{H}^2(M)) = \mathscr{H}^2(M)$ is implicitely used) and similarly $[h_-] \in \mathscr{H}^2_-(M)$.

We shall need Lemma 2.5 from Andreas' second talk, in which the following useful lemma from Bernd's first talk plays a central role:

Lemma 2.4 *Let (M^4, g) be any closed oriented smooth Riemannian manifold.*

i) The L^2-orthogonal decomposition $\Omega^2_+(M) = \mathscr{H}^2_+(M) \oplus p_+ \circ d(\Omega^1(M))$ holds. In particular, for any $h \in \Omega^2_+(M)$, there exists (at least one) $\eta \in \Omega^2(M) \cap \ker(d)$ with $\eta_+ = h$.

ii) Given $\alpha \in \Omega^1(M)$, we have $(d\alpha)_+ = 0$ iff $d\alpha = 0$.

Proof of Lemma 2.4: The splitting in i) follows from $*$ exchanging $d\Omega^1(M)$ with $\delta\Omega^3(M)$, so that $d\Omega^1(M) \oplus \delta\Omega^3(M) = p_+(d\Omega^1(M)) \oplus p_-(d\Omega^1(M))$. As a consequence of the splitting $\Omega^2_+(M) = \mathscr{H}^2_+(M) \oplus p_+ \circ d(\Omega^1(M))$, one can write $h = [h] + (d\alpha)_+ = (\underbrace{[h] + d\alpha}_{=:\eta})_+$, where $d\eta = d[h] + d^2\alpha = 0$. Statement ii) follows from $\int_M d\alpha \wedge d\alpha = \int_M d(\alpha \wedge d\alpha) = 0$. $\surd$

Lemma 2.5 *Given any $h \in H^{3,2}(\Lambda^2_+ T^*M)$, the space $\mathcal{M}(\widetilde{P}, h)$ contains at least one reducible monopole iff $[h] = 2\pi[c_1(P)]_+$. In that case, the set of reducible monopoles in $\mathcal{M}(\widetilde{P}, h)$ is an affine space modelled on* $\ker(d) \cap \Omega^1(M)$.

Proof of Lemma 2.5: A reducible monopole is of the form $[(A, 0)]$, where $F_A^+ = ih$. If there exists such a monopole, then taking the harmonic components we obtain $[F_A^+] = [F_A]_+ = i[h]_+ = i[h]$, where $[F_A] = 2i\pi[c_1(P)]$ (property of the first Chern class), so that $[h] = 2\pi[c_1(P)]_+$. Conversely, if $[h] = 2\pi[c_1(P)]_+$, then choosing an arbitrary connection 1-form $\widehat{A}$ on $P \to M$, we have $[F^+_{\widehat{A}}] = i[h]$, so that there exists a real-valued $\alpha \in \Omega^1(M)$ with $F^+_{\widehat{A}} + i(d\alpha)_+ = ih$ by Lemma 2.4. In particular, the connection 1-form $A_0 := \widehat{A} + i\alpha$ on $P \to M$ satisfies $F^+_{A_0} = ih$, i.e., $[(A_0, 0)] \in \mathcal{M}(\widetilde{P}, h)$. Moreover, for any other connection 1-form A on $P \to M$, we can write $A = A_0 + i\beta$ with $\beta \in \Omega^1(M)$ and then $F_A^+ = ih$ iff $(d\beta)_+ = 0$, that is, using Lemma 2.4, iff $d\beta = 0$. This concludes the proof. $\surd$

As a consequence of Lemma 2.5, if $b_2^+(M) = 0$ (which we have assumed here), then the condition $[h] = 2\pi[c_1(P)]_+$ is void, so that, for *any* $h \in H^{3,2}(\Lambda^2_+ T^*M)$, the subset of reducible monopoles in $\mathcal{M}(\widetilde{P}, h)$ is an affine space modelled on $\ker(d) \cap \Omega^1(M)$. Since by assumption $b_1(M) = 0$, we have $\ker(d) \cap \Omega^1(M) = \mathrm{im}(d) \cap \Omega^1(M)$ (we do not worry about the regularity here, though we should), so that any two reducible monopoles differ by a term of the form $2idf$ for a real function f and hence are gauge equivalent[5], i.e., they coincide in the moduli space. Hence we have shown that, for

[5] Explain.

any $h \in H^{3,2}(\Lambda^2_+ T^*M)$, the moduli space $\mathcal{M}(\widetilde{P}, h)$ has only one reducible monopole, which we denote by $[(A_0, 0)]$.

Now Sard-Smale theorem [11] implies that, for "most" $h \in H^{3,2}(\Lambda^2_+ T^*M)$, the obstruction space $\mathcal{H}^2_{[A,\psi]}$ at an *irreducible* monopole $[(A,\psi)] \in \mathcal{M}(\widetilde{P}, h)$ vanishes (cf. Nicolas' second talk). In a similar way, for most $h \in H^{3,2}(\Lambda^2_+ T^*M)$, the obstruction space $\mathcal{H}^2_{[A_0,0]} \cong \mathrm{coker}(p_+ \circ d) \oplus \mathrm{coker}(D_+^{A_0}) = \mathrm{coker}(D_+^{A_0})$ (for $\mathscr{H}^2_+(M) = 0$ by assumption) at the reducible monopole $[(A_0, 0)] \in \mathcal{M}(\widetilde{P}, h)$ vanishes, see e.g. [6][6]. Thus, for most $h \in H^{3,2}(\Lambda^2_+ T^*M)$, the obstruction space at *any* monopole in $\mathcal{M}(\widetilde{P}, h)$ vanishes. Fixing such an h, the (compact) moduli space $\mathcal{M}(\widetilde{P}, h)$ becomes a smooth oriented $d(P)$-dimensional manifold outside $[(A_0, 0)]$ and there exists an open neighbourhood of $[(A_0, 0)]$ which is homeomorphic to the quotient space $\mathcal{H}^1_{[A_0,0]}/\mathbb{U}_1$, where $\mathcal{H}^1_{[A_0,0]} \cong H^1(M;\mathbb{R}) \oplus \ker(D_+^{A_0}) = \ker(D_+^{A_0})$ is the formal (or Zariski) tangent space at $[(A_0, 0)]$. Since $\mathrm{coker}(D_+^{A_0}) = 0$ for that choice of h, we have[7], using the Atiyah-Singer index theorem:

$$\begin{aligned}
\dim_{\mathbb{C}}(\mathcal{H}^1_{[A_0,0]}) &= \dim_{\mathbb{C}}(\ker(D_+^{A_0})) \\
&= \mathrm{ind}_{\mathbb{C}}(D_+^{A_0}) \\
&= \frac{1}{8}(c_1(P)^2[M] - \mathrm{sign}(M)) \\
&= \frac{1}{8}\Theta(q_M) \\
&= \frac{d(P)+1}{2}.
\end{aligned}$$

We deduce that $\mathcal{H}^1_{[A_0,0]}/\mathbb{U}_1$ is homeomorphic to a cone $\mathbb{C}\mathrm{P}^{\frac{d(P)-1}{2}} \times [0,1[$ over $\mathbb{C}\mathrm{P}^{\frac{d(P)-1}{2}}$. Cutting a small neighbourhood W of $[(A_0, 0)]$ out of $\mathcal{M}(\widetilde{P}, h)$, we obtain the smooth compact oriented $d(P)$-dimensional manifold $X := \mathcal{M}(\widetilde{P}, h) \setminus W$ with boundary $\partial X = \mathbb{C}\mathrm{P}^{\frac{d(P)-1}{2}}$. If $d(P) = 1$, then X would be a compact 1-dimensional manifold with boundary consisting of only one point, which is a contradiction. If $d(P) > 1$, then consider the so-called *universal line bundle* $\mathbb{U}_{\widetilde{P}} \to \mathcal{C}^*(\widetilde{P})/\mathcal{G}$ obtained as follows: let $x_0 \in M$ be any fixed point, then the subgroup $\mathcal{G}_0 := \{\sigma \in \mathcal{G}\,,\, \sigma(x_0) = 1\}$ of $\mathcal{G}$ keeps acting freely on $\mathcal{C}^*(\widetilde{P})$, so that $\mathcal{C}^*(\widetilde{P})/\mathcal{G}_0$ becomes a smooth Hilbert manifold on which $\mathcal{G}/\mathcal{G}_0 \cong \mathbb{U}_1$ acts freely from the right (action induced by the free right

[6] Thanks to Bernd Ammann for that reference.

[7] The following computation is probably not necessary since we already know that the dimension of the neighbourhood $\mathcal{H}^1_{[A_0,0]}/\mathbb{U}_1$ has to be $d(P)$.

$\mathcal{G}$-action on $\mathcal{C}^*(\widetilde{P})$). Identifying ${}^{\mathcal{G}}\!/\!_{\mathcal{G}_0}$ with $\mathbb{U}_1$ and taking the standard representation of $\mathbb{U}_1$ onto $\mathbb{C}$, the associated line bundle $\left\{{}^{\mathcal{C}^*(\widetilde{P})}\!/\!_{\mathcal{G}_0} \times \mathbb{C}\right\}\!/\!_{\mathbb{U}_1} \to \left\{{}^{\mathcal{C}^*(\widetilde{P})}\!/\!_{\mathcal{G}_0}\right\}\!/\!_{\mathbb{U}_1} \cong {}^{\mathcal{C}^*(\widetilde{P})}\!/\!_{\mathcal{G}}$ is, by definition, $\mathbb{U}_{\widetilde{P}} \to {}^{\mathcal{C}^*(\widetilde{P})}\!/\!_{\mathcal{G}}$. Observe that its restriction to $\mathcal{M}(\widetilde{P}, h) \setminus \{[(A_0, 0)]\}$ and then to ∂X provides either the tautological bundle $\tau \to \mathbb{C}\mathrm{P}^{\frac{d(P)-1}{2}}$ or its dual $-\tau$. In particular, we obtain $\int_{\partial X} c_1(\tau)^{\frac{d(P)-1}{2}} = \pm 1$. But since τ (or $-\tau$) comes from the line bundle $\mathbb{U}_{\widetilde{P}|X} \to X$, Stokes' formula yields $\int_{\partial X} c_1(\tau)^{\frac{d(P)-1}{2}} = \int_X d\left(c_1(\tau)^{\frac{d(P)-1}{2}}\right) = 0$, which again is a contradiction[8]. This concludes the proof in case $b_1(M) = 0$.

- *Case* $b_1(M) > 0$: The main trick is to reduce that case to the first one by performing surgery on the manifold M. Fixing a basis $\{\mathfrak{c}_1, \ldots, \mathfrak{c}_{b_1}\}$ of ${}^{H_1(M;\mathbb{Z})}\!/\!_{\mathrm{Tor}(H_1(M;\mathbb{Z}))}$ consisting of *smoothly embedded* (oriented) circles S^1, we perform 1-dimensional surgery along each $\mathfrak{c}_j$: we remove a small open neighbourhood $U_j \cong S^1 \times D^3$ of $\mathfrak{c}_j$ and glue in $D^2 \times S^2$ along $\partial U_j \cong S^1 \times S^2$. Next we show that this operation "kills" the homology classes $\mathfrak{c}_1, \ldots, \mathfrak{c}_{b_1}$ – and hence the free part of H_1 – while it modifies neither the free part of the second homology nor the intersection form.

Claim: *The manifold* $\widetilde{M}$ *obtained after those* 1*-dimensional surgeries has* $b_1(\widetilde{M}) = 0$ *while* ${}^{H_2(\widetilde{M};\mathbb{Z})}\!/\!_{\mathrm{Tor}(H_2(\widetilde{M};\mathbb{Z}))} \cong {}^{H_2(M;\mathbb{Z})}\!/\!_{\mathrm{Tor}(H_2(M;\mathbb{Z}))}$ *and* $q_{\widetilde{M}} \simeq q_M$.

Proof of Claim: Denote by $\overline{H}_p(X;\mathbb{Z}) := {}^{H_p(X;\mathbb{Z})}\!/\!_{\mathrm{Tor}(H_p(X;\mathbb{Z}))}$ the free part of the p^{th} homology group of a topological space X. It suffices to show that, after performing surgery along one of the paths $\mathfrak{c}_j$, we obtain a new manifold M' with $\overline{H}_1(M';\mathbb{Z}) \cong {}^{\overline{H}_1(M;\mathbb{Z})}\!/\!_{\langle \mathfrak{c}_j \rangle}$ and $\overline{H}_2(M';\mathbb{Z}) \cong \overline{H}_2(M;\mathbb{Z})$ as well as $q_{M'} \simeq q_M$. Let $M_1 := S^1 \times D^3 \subset M$ and $M_1' := D^2 \times S^2 \subset M'$. The long exact homology sequences associated to the pairs (M, M_1) and (M', M_1') provide, using $H_1(M_1';\mathbb{Z}) = 0$ and the injectivity of $H_1(M_1;\mathbb{Z}) \to H_1(M;\mathbb{Z})$ (by assumption), isomorphisms $H_1(M, M_1;\mathbb{Z}) \cong {}^{H_1(M;\mathbb{Z})}\!/\!_{H_1(M_1;\mathbb{Z})}$ and $H_1(M', M_1';\mathbb{Z}) \cong H_1(M';\mathbb{Z})$, so that, by $H_1(M', M_1';\mathbb{Z}) \cong H_1(M, M_1;\mathbb{Z})$, we obtain $H_1(M';\mathbb{Z}) \cong {}^{H_1(M;\mathbb{Z})}\!/\!_{H_1(M_1;\mathbb{Z})}$. This implies the first statement. The second one should follow in an analogous way.[9] That the intersection form remains unchanged (up to equivalence) remains to be shown. $\checkmark$

Therefore, we obtain after surgery a new closed smooth 4-manifold $\widetilde{M}$ with $b_1 = 0$ and with equivalent intersection form and the first case applies. $\square$

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[8] One could also argue that the Seiberg-Witten invariants of ∂X have to vanish since they are bordim invariants (Bernd's remark).

[9] Unclear.

As a consequence of Theorem 2.3, any closed oriented 4-dimensional manifold with definite but non-diagonalizable intersection form cannot carry any smooth structure. This applies to the example with intersection form E_8 above.

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