Absolute continuous approximations for multifractional processes and fields

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Let \((\Omega, \mathcal{F}, P)\) be a complete probability space.

**Definition**

The fractional Brownian motion (fBm) with Hurst index \(H \in (0, 1)\) is a centered Gaussian process \(B^H = \{B^H_t, t \geq 0\}\) with stationary increments and the covariance function

\[
EB^H_t B^H_s = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}).
\]
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\[
\mathbb{E}B^H_t B^H_s = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}).
\]

**Remark**

Since \(\mathbb{E}(B^H_t - B^H_s)^2 = |t - s|^{2H}\) and \(B^H\) is a Gaussian process, it has a continuous modification, according to the Kolmogorov theorem.
Let $H: [0, +\infty) \rightarrow \left(\frac{1}{2}, 1\right)$ be a function which satisfies Hölder condition: there exist $C_1 > 0$ and $\gamma > \frac{1}{2}$ such that for all $t_1, t_2 \in [0, +\infty)$

$$|H_{t_1} - H_{t_2}| \leq C_1 |t_1 - t_2|^\gamma.$$ 

Let $H_{\text{min}} := \min \left\{ \gamma, \min_{t \in [0, T]} H_t \right\}$. 
**Moving average mBm:** \( Y_t = B^H_t \), where

\[
B^H_t = \frac{1}{\Gamma(H + \frac{1}{2})} \left\{ \int_{-\infty}^{0} \left[ (t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}} \right] dW_s \right. \\
+ \left. \int_{0}^{t} (t-s)^{H-\frac{1}{2}} dW_s \right\},
\]
• Moving average mBm: \( Y_t = B_t^H \), where

\[
B_t^H = \frac{1}{\Gamma \left( H + \frac{1}{2} \right)} \left\{ \int_{-\infty}^{0} \left[ (t - s)^{H - \frac{1}{2}} - (-s)^{H - \frac{1}{2}} \right] dW_s \right. \\
+ \int_{0}^{t} (t - s)^{H - \frac{1}{2}} dW_s \right\},
\]

• Volterra-type mBm: \( Y_t = B_t^H \), where

\[
B_t^H = \int_{0}^{t} K_H(t, s) \, dW_s, \quad t \geq 0,
\]

\[
K_H(t, s) = C_H s^{\frac{1}{2} - H} \int_{s}^{t} (v - s)^{H - \frac{3}{2}} v^{H - \frac{1}{2}} \, dv,
\]

\[
C_H = \left( \frac{H(2H - 1)}{B(2 - 2H, H - \frac{1}{2})} \right)^{\frac{1}{2}}.
\]
Harmonizable mBm

Let $W(\cdot)$ be a complex-valued random measure on $\mathbb{R}$ such that

1) for all $A, B \in \mathcal{B}(\mathbb{R})$

$$EW(A)W(B) = \lambda(A \cap B),$$

where $\lambda$ is Lebesgue measure,

2) for any sequence $\{A_1, A_2, \ldots\} \subset \mathcal{B}(\mathbb{R})$ such that $A_i \cap A_j = \emptyset$, $i \neq j$, $\{W(A_i), i \geq 1\}$ are centered and normal,

$$W\left(\bigcup_{i \geq 1} A_i\right) = \sum_{i \geq 1} W(A_i),$$

3) for all $A \in \mathcal{B}(\mathbb{R})$ $W(A) = \overline{W(-A)}$,

4) for all $\theta \in \mathbb{R}$

$$\{e^{i\theta} W(A), A \in \mathcal{B}(\mathbb{R})\} \overset{d}{=} \{W(A), A \in \mathcal{B}(\mathbb{R})\}.$$

Harmonizable mBm is defined as $Y_t = B_t^H$ where

$$B_t^H = \int_{\mathbb{R}} \frac{e^{itx} - 1}{|x|^{\frac{1}{2}+H}} W(dx).$$
We consider generalizations of fBm of the form $Y_t = B_t^{H_t}$, where 
\( \left\{ B_t^H, t \in [0, T], H \in \left( \frac{1}{2}, 1 \right) \right\} \) is a set of random variables such that 

(i) for each $H \in \left( \frac{1}{2}, 1 \right)$ \( \left\{ B_t^H, t \in [0, T] \right\} \) is fBm with Hurst parameter $H$;

(ii) for all $t \in [0, T]$, $H_1, H_2 \in \left( \frac{1}{2}, 1 \right)$

\[
\mathbb{E} \left( B_t^{H_1} - B_t^{H_2} \right)^2 \leq C_2 (H_1 - H_2)^2.
\]
We consider generalizations of fBm of the form $Y_t = B^H_t$, where
\[ \{B^H_t, t \in [0, T], H \in \left(\frac{1}{2}, 1\right)\} \]
is a set of random variables such that

(i) for each $H \in \left(\frac{1}{2}, 1\right)$ \[ \{B^H_t, t \in [0, T]\} \]
is fBm with Hurst parameter $H$;

(ii) for all $t \in [0, T]$, $H_1, H_2 \in \left(\frac{1}{2}, 1\right)$

\[ E \left( B^{H_1}_t - B^{H_2}_t \right)^2 \leq C_2 (H_1 - H_2)^2. \]

**Remark**

The process $Y_t = B^H_t$ has a continuous modification, according to the Kolmogorov theorem.
Let for \(0 < \beta < 1\)

\[
\varphi_f^\beta(t) := |f(t)| + \int_0^t \frac{|f(t) - f(s)|}{(t - s)^{1+\beta}} ds,
\]

and \(W_0^\beta = W_0^\beta[0, T]\) be the space of measurable functions \(f: [0, T] \rightarrow \mathbb{R}\) with

\[
\|f\|_{0,\beta} := \sup_{t \in [0, T]} \varphi_f^\beta(t) < \infty.
\]
Let for $0 < \beta < 1$

$$\varphi_f^\beta(t) := |f(t)| + \int_0^t \frac{|f(t) - f(s)|}{(t - s)^{1+\beta}} \, ds,$$

and $W_0^\beta = W_0^\beta[0, T]$ be the space of measurable functions $f : [0, T] \to \mathbb{R}$ with

$$\|f\|_{0,\beta} := \sup_{t \in [0, T]} \varphi_f^\beta(t) < \infty.$$

Also let $W_1^\beta = W_1^\beta[0, T]$ be the space of functions $f : [0, T] \to \mathbb{R}$ with

$$\|f\|_{1,\beta} := \sup_{0 \leq s < t \leq T} \left( \frac{|f(t) - f(s)|}{(t - s)^\beta} + \int_s^t \frac{|f(u) - f(s)|}{(u - s)^{1+\beta}} \, du \right) < \infty.$$
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Let us consider such an approximation:

\[
B_{t}^{H_{t},\varepsilon} := \frac{1}{\phi_{t}(\varepsilon)} \int_{t}^{t+\phi_{t}(\varepsilon)} B_{s}^{H_{s}} \, ds = \frac{1}{\phi_{t}(\varepsilon)} \int_{0}^{\phi_{t}(\varepsilon)} B_{u+t}^{H_{u+t}} \, du,
\]
Let us consider such an approximation:

\[ B_{t}^{H_{t},\varepsilon} := \frac{1}{\phi_{t}(\varepsilon)} \int_{t}^{t+\phi_{t}(\varepsilon)} B_{s}^{H_{s}} \, ds = \frac{1}{\phi_{t}(\varepsilon)} \int_{0}^{B_{u+t}} B_{u+t}^{H_{u+t}} \, du, \]

where \( \phi_{t}(\varepsilon) = \phi(t, \varepsilon): [0, T] \times \mathbb{R}_{+} \rightarrow \mathbb{R}_{+} \) is a set of measurable functions such that

1. \( \sup_{t \in [0, T]} \phi_{t}(\varepsilon) \rightarrow 0, \varepsilon \rightarrow 0^{+}; \)
2. for all \( t, s \in [0, T] \) and for all \( \varepsilon > 0 \)

\[ \left| \frac{\phi_{s}(\varepsilon) - \phi_{t}(\varepsilon)}{\phi_{s}(\varepsilon)} \right| \leq C_{3} |t - s|^{H_{\min}}, \quad (1) \]

\( C_{3} \) is a constant which does not depend on \( \varepsilon. \)
Let us consider such an approximation:

\[
B_{t}^{H_{t},\varepsilon} := \frac{1}{\phi_{t}(\varepsilon)} \int_{t}^{t+\phi_{t}(\varepsilon)} B_{s}^{H} \, ds = \frac{1}{\phi_{t}(\varepsilon)} \int_{0}^{\phi_{t}(\varepsilon)} B_{u+t}^{H_{u+t}} \, du,
\]

where \(\phi_{t}(\varepsilon) = \phi(t, \varepsilon): [0, T] \times \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}\) is a set of measurable functions such that

1. \(\sup_{t \in [0, T]} \phi_{t}(\varepsilon) \to 0, \varepsilon \to 0^{+}\);
2. for all \(t, s \in [0, T]\) and for all \(\varepsilon > 0\)

\[
\left| \frac{\phi_{s}(\varepsilon) - \phi_{t}(\varepsilon)}{\phi_{s}(\varepsilon)} \right| \leq C_{3} |t - s|^{H_{\text{min}}}.
\]  

(1)

\(C_{3}\) is a constant which does not depend on \(\varepsilon\).

For example, one can consider functions of the form \(\phi_{t}(\varepsilon) = \psi(t)\varepsilon\), where \(\psi(t)\) satisfies the following conditions:

1. \(\psi(t) > c > 0\),
2. \(|\psi(t) - \psi(s)| \leq C |t - s|^{H_{\text{min}}}\).
Theorem

For any $\beta \in (0, H_{\text{min}})$ one has the convergence in Besov space $W^{\beta}_1$

$$\left\| B^{H, \varepsilon} - B^H \right\|_{1, \beta} \xrightarrow{\mathbb{P}} 0, \quad \varepsilon \to 0^+.$$
Corollary

Let \( \left\{ B^H_t, t \in [0, T] \right\} \) be fBm with Hurst parameter \( H \in \left( \frac{1}{2}, 1 \right) \), functions \( \phi_t(\varepsilon) \) satisfy conditions:

1) \( \sup_{t \in [0, T]} \phi_t(\varepsilon) \to 0, \varepsilon \to 0^+; \)

2) for all \( t, s \in [0, T] \) and for all \( \varepsilon > 0 \)

\[
\left| \frac{\phi_s(\varepsilon) - \phi_t(\varepsilon)}{\phi_s(\varepsilon)} \right| \leq C_3 |t - s|^H.
\]

Then for approximations

\[
B^H_t,\varepsilon = \frac{1}{\phi_t(\varepsilon)} \int_t^{t + \phi_t(\varepsilon)} B^H_s \, ds
\]

one has the convergence

\[
\left\| B^{H,\varepsilon} - B^H \right\|_{1,\beta} \xrightarrow{P} 0, \quad \varepsilon \to 0^+.
\]
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Consider the SDE with mBm:

$$X_t = X_0 + \int_0^t b(s, X_s) \, ds + \int_0^t \sigma(s, X_s) \, dB^H_s, \quad t \in [0, T].$$

The stochastic integral here is understood in the pathwise sense.
Consider the SDE with mBm:

\[ X_t = X_0 + \int_0^t b(s, X_s) \, ds + \int_0^t \sigma(s, X_s) \, dB^H_s, \quad t \in [0, T]. \]

The stochastic integral here is understood in the pathwise sense. We construct approximations for the solution of this equation as solutions of

\[ X_t^\varepsilon = X_0 + \int_0^t b(s, X_s^\varepsilon) \, ds + \int_0^t \sigma(s, X_s^\varepsilon) \, dB^{H_s, \varepsilon}_s, \quad t \in [0, T]. \]
I. $\sigma(t, x)$ is differentiable in $x$, and there exist some constants $1 - H_{\min} < \kappa \leq 1$ and $\frac{1}{H_{\min}} - 1 < \delta \leq 1$, and for every $N > 0$ there exists $M_N > 0$ such that the following properties hold:

(i) $\forall x \in \mathbb{R}, \forall t \in [0, T]$

$$|\sigma(t, x) - \sigma(t, y)| \leq M_0 |x - y|;$$

(ii) $\forall|x|, |y| \leq N, \forall t \in [0, T]$

$$\left| \frac{\partial}{\partial x} \sigma(t, x) - \frac{\partial}{\partial x} \sigma(t, y) \right| \leq M_N |x - y|^{\delta};$$

(iii) $\forall x \in \mathbb{R}, \forall t, s \in [0, T]$

$$|\sigma(t, x) - \sigma(s, x)| + \left| \frac{\partial}{\partial x} \sigma(t, x) - \frac{\partial}{\partial x} \sigma(s, x) \right| \leq M_0 |t - s|^{\kappa}.$$
II. There exists $b_0 \in L^\rho(0, T)$, where $\rho \geq 2$, and for every $N > 0$ there exists $L_N > 0$ such that the following properties hold:

(iv) $\forall |x|, |y| \leq N, \forall t \in [0, T]$ 

$$|b(t, x) - b(t, y)| \leq L_N |x - y|;$$

(v) $\forall x \in \mathbb{R}, \forall t \in [0, T]$ 

$$|b(t, x)| \leq L_0 |x| + b_0(t).$$
Let
\[ \alpha_0 = \min \left\{ \frac{1}{2}, \kappa, \frac{\delta}{1 + \delta} \right\}. \]

**Theorem**

Suppose that \( \alpha \in (1 - H_{\min}, \alpha_0) \), \( X_0 \) is a random variable, the coefficients \( \sigma(t, x) \) and \( b(t, x) \) satisfy assumptions (i)–(v) with \( \rho \geq 1/\alpha \). Then there exists a unique solution \( \{X_t, t \in [0, T]\} \) of the equation

\[ X_t = X_0 + \int_0^t b(s, X_s) \, ds + \int_0^t \sigma(s, X_s) \, dB^H_s, \quad t \in [0, T], \]

\( X \in L^0(\Omega, \mathcal{F}, P, W^\alpha_{0}[0, T]), \) with trajectories from \( C^{1-\alpha}[0, T] \) a.s.
Theorem

Suppose that $\alpha \in (1 - H_{\text{min}}, \alpha_0)$, $X_0$ is a random variable, the coefficients $\sigma(t, x)$ and $b(t, x)$ satisfy assumptions (i)–(v) with $\rho \geq 1/\alpha$. Then one has the uniform convergence in probability

$$\sup_{t \in [0, T]} |X_t - X^\varepsilon_t| \xrightarrow{P} 0, \quad \varepsilon \to 0^+.$$
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Let $T = (T_1, T_2) \in (0, \infty)^2$, $[0, T] = [0, T_1] \times [0, T_2]$.
Let $s, t \in [0, T]$, $s = (s_1, s_2)$, $t = (t_1, t_2)$, $f : [0, T] \to \mathbb{R}$.

$$\Delta_s f(t) := f(s_1, s_2) - f(s_1, t_2) - f(t_1, s_2) + f(t_1, t_2)$$
$$f_{t-}(s) := f(s_1, s_2) - f(s_1, t_2-) - f(t_1-, s_2) + f(t_1-, t_2-)$$

$$s < t \iff \begin{cases} s_1 < t_1 \\ s_2 < t_2 \end{cases}$$

$$s \leq t \iff \begin{cases} s_1 \leq t_1 \\ s_2 \leq t_2 \end{cases}$$
Let $W^{\beta_1,\beta_2}_1 = W^{\beta_1,\beta_2}_1([0, T])$ be a space of measurable functions $f: [0, T] \rightarrow \mathbb{R}$ with

$$
\|f\|_{1,\beta_1,\beta_2} = \sup_{0 \leq s < t \leq T} \left( \frac{\left| \Delta_s f(t) \right|}{(t_1 - s_1)^{\beta_1}(t_2 - s_2)^{\beta_2}} \right)
+ \frac{1}{(t_2 - s_2)^{\beta_2}} \int_{s_1}^{t_1} \frac{\left| f_t(u, s_2) - f_t(s) \right|}{(u - s_1)^{1+\beta_1}} \, du
+ \frac{1}{(t_1 - s_1)^{\beta_1}} \int_{s_2}^{t_2} \frac{\left| f_t(s_1, v) - f_t(s) \right|}{(v - s_2)^{1+\beta_2}} \, dv
+ \int_{[s,t]} \frac{\left| \Delta_s f(r) \right|}{(r_1 - s_1)^{1+\beta_1}(r_2 - s_2)^{1+\beta_2}} \, dr \right) < \infty,
$$
Let \( \{B_t, t \in [0, T]\} \) be a random field which satisfy the following conditions

1) \( B_t \) is Gaussian field;
2) there exists constants \( C > 0 \) and \( \lambda > 1 \) such that for all \( s, t \in [0, T] \)
   \[
   \mathbb{E}(\Delta_s B_t)^2 \leq C(|t_1 - s_1||t_2 - s_2|)^\lambda
   \]
3) the trajectories of \( B_t \) are continuous with probability one.
We consider the following approximation for $B_t$:

$$B_t^\varepsilon = \frac{1}{\varepsilon^2} \int_{t_1}^{t_1+\varepsilon} \int_{t_2}^{t_2+\varepsilon} B_s \, ds = \frac{1}{\varepsilon^2} \int_{[0,\varepsilon]^2} B_{s+t} \, ds.$$ 

**Theorem**

*For all $\beta_1, \beta_2 \in (0, \lambda/2)$*

$$\|B^\varepsilon - B\|_{1, \beta_1, \beta_2} \xrightarrow{P} 0, \quad \varepsilon \to 0 +.$$

Definition

A random field \( \{ B_t^H, t \in \mathbb{R}_+^2 \} \) is called a fractional Brownian field with
Hurst index \( H = (H_1, H_2) \in (0, 1)^2 \), if

1) \( B_t^H \) is a Gaussian field such that \( B_t^H = 0, \ t \in \partial \mathbb{R}_+^2 \),

2) \( \mathbb{E}B_t^H = 0, \ \mathbb{E}B_t^HB_s^H = \frac{1}{4} \prod_{i=1,2} \left( t_i^{2H_i} + s_i^{2H_i} - |t_i - s_i|^{2H_i} \right) \),
Definition

A random field \( \{ B^H_t, t \in \mathbb{R}^2_+ \} \) is called a fractional Brownian field with Hurst index \( H = (H_1, H_2) \in (0, 1)^2 \), if

1) \( B^H_t \) is a Gaussian field such that \( B^H_t = 0, \ t \in \partial \mathbb{R}^2_+ \),

2) \( E B^H_t = 0, \ E B^H_t B^H_s = \frac{1}{4} \prod_{i=1,2} \left( t_i^{2H_i} + s_i^{2H_i} - |t_i - s_i|^{2H_i} \right) \).

The increments of fBf satisfy the following equality

\[
E \left( \Delta_s B^H_t \right)^2 = |t_1 - s_1|^{2H_1} |t_2 - s_2|^{2H_2}.
\]
Corollary

For $\beta_1, \beta_2 \in (0, H_1 \wedge H_2)$ the following convergence holds:

$$\left\| B^{H,\varepsilon} - B^H \right\|_{1, \beta_1, \beta_2} \xrightarrow{P} 0, \quad \varepsilon \to 0^+,$$

where

$$B^H_{t,\varepsilon} = \frac{1}{\varepsilon^2} \int_{t_1}^{t_1 + \varepsilon} \int_{t_2}^{t_2 + \varepsilon} B^H_s \, ds.$$
Let $H(t) = (H_1(t), H_2(t)) : [0, T] \rightarrow (1/2, 1)^2$ be a continuous function such that

$$\frac{1}{2} < \mu < \min_{t \in [0, T]} H_i(t) \leq \max_{t \in [0, T]} H_i(t) < \nu < 1.$$ 

Suppose that for all $t, s \in [0, T]$

(H1) $|H_i(t) - H_i(s)| \leq c_1 (|t_1 - s_1|^\nu + |t_2 - s_2|^\nu)$,

(H2) $|\Delta_s H_i(t)| \leq c_2 (|t_1 - s_1||t_2 - s_2|)^\nu$. 


Let $H(t) = (H_1(t), H_2(t)) : [0, T] \rightarrow (1/2, 1)^2$ be a continuous function such that
\[
\frac{1}{2} < \mu < \min_{t \in [0, T]} H_i(t) \leq \max_{t \in [0, T]} H_i(t) < \nu < 1.
\]

Suppose that for all $t, s \in [0, T]$

(H1) $|H_i(t) - H_i(s)| \leq c_1 (|t_1 - s_1|^{\nu} + |t_2 - s_2|^{\nu})$,

(H2) $|\Delta s H_i(t)| \leq c_2 (|t_1 - s_1| |t_2 - s_2|)^{\nu}$.

Definition ([Meerschaert, Wu, and Xiao (2008)])

Multifractional Brownian sheet with Hurst index $H(t)$ is defined as follows

\[
B_t^{H(t)} := \int_{\mathbb{R}^2} \prod_{i=1,2} \left[ (t_i - u_i)^{H_i(t) - 1/2} - (-u_i)^{H_i(t) - 1/2} \right] dW_u, \quad t \in [0, T],
\]

where $s_+ = \max\{s, 0\}$, $W = \{W_s, s \in \mathbb{R}^2\}$ is a Wiener field.
Theorem

$B^H_t(t)$ has a continuous modification.
Theorem

$B_t^{H(t)}$ has a continuous modification.

Theorem

There exists $C > 0$ such that for all $s, t \in [0, T]$

$$\mathbb{E}(\Delta_s Y_t)^2 \leq C(|t_1 - s_1| |t_2 - s_2|)^{2\mu}.$$
Theorem

For any $\beta_1, \beta_2 \in (0, \mu)$ one has the convergence in Besov space $W^{1, \beta_1, \beta_2}_{1, \beta_1, \beta_2}$

$$
\left\| B_t^{H(t), \epsilon} - B_t^{H(t)} \right\|_{1, \beta_1, \beta_2} \xrightarrow{P} 0, \quad \epsilon \to 0+,
$$

where

$$
B_t^{H(t), \epsilon} = \frac{1}{\epsilon^2} \int_{t_1}^{t_1+\epsilon} \int_{t_2}^{t_2+\epsilon} B_s^{H(s)} \, ds.
$$
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