Drift parameter estimation in models with fractional Brownian motion

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Outline

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2. Models with long-memory
3. Model description and preliminaries
7. Auxiliary results for Gaussian processes related to the fractional Brownian motion.
8. Main results
This is a joint work with Yu. Kozachenko and A. Melnikov.

We consider a stochastic differential equation involving standard and fractional Brownian motion with unknown drift parameter to be estimated. We investigate the standard maximum likelihood estimate of the drift parameter, two non-standard estimates and three estimates for the sequential estimation. Model strong consistency and some other properties are proved. The linear model and Ornstein-Uhlenbeck model are studied in detail. As an auxiliary result, an asymptotic behavior of the fractional derivative of the fractional Brownian motion is established.
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8. Main results
Modern mathematical statistics tends to shift away from the standard statistical schemes based on independent random variables; besides, these days many statistical models are based on continuous time. Therefore, the corresponding statistical problems (e.g., parameter estimation) can be handled by methods of the theory of stochastic processes in addition to the standard statistical methods. Statistics for stochastic processes is well-developed for diffusion processes and even for semimartingales (see, for instance, [LipSh(1978)]) but is still developing for the processes with long-range dependence. The latter is an integral part of stochastic processes, featuring a wide spectrum of applications in economics, physics, finance and other fields.
This work is devoted to the parameter estimation in such models involving fractional Brownian motion (fBm) with Hurst parameter $H > \frac{1}{2}$ which is a well-known long-memory process. We study also a mixed model based on both standard and fractional Brownian motion which turns out to be more flexible. One of the reasons to consider such model comes from the modern mathematical finance where it has become very popular to assume that the underlying random noise consists of two parts: the fundamental part, describing the economical background for the stock price, and the trading part, related to the randomness inherent to the stock market. In our case the fundamental part of the noise has a long memory while the trading part is a white noise.
Statistical aspects of the models with long-memory

Statistical aspects of models involving fractional Brownian motion were studied in many sources. One of the important problems in particular is the drift parameter estimation. In this connection, let us mention papers [HuNu(2010)] and [KlLeBr(2002)], where the fractional Ornstein-Uhlenbeck process with unknown drift parameter originally was studied, books [Bish(2008)], [M.(2008)] and [Prara(2010)] and the references therein, and papers [BTT(2011)], [XZX(2011)], [XZZ(2011)], and [HuXZ(2009)], where the estimate was constructed via discrete observations. We shall also use the results for sequential estimates for semimartingales from [MN(1988)].
Statistical estimates of the drift parameter

We consider stochastic differential equations involving fractional Brownian motion along with equations involving both standard and fractional Brownian motion. We derive the standard maximum likelihood estimate and propose non-standard estimates for the unknown drift parameter. Several non-standard estimates for the drift parameter were proposed in [HuNu(2010)] for the fractional Ornstein-Uhlenbeck process. We go a step ahead and propose non-standard estimates for the drift parameter in a general stochastic differential equation involving fBm. For the models involving only fractional Brownian motion, we compare properties of the estimates. In the mixed models the standard maximum likelihood estimate does not exist but the non-standard estimate works. To formulate the conditions for strong consistency of the non-standard estimates, we need to investigate the asymptotic behavior of the fractional derivative of the fractional Brownian motion using the general growth results for Gaussian processes.
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Model description

Let \((\Omega, \mathcal{F}, \mathcal{F}, P)\) be a complete probability space with filtration \(\mathcal{F} = \{\mathcal{F}_t, t \in R^+\}\) satisfying the standard assumptions. It is assumed that all processes under consideration are adapted to filtration \(\mathcal{F}\).

Definition 1

Fractional Brownian motion (fBm) with Hurst index \(H \in (0, 1)\) is a Gaussian process \(B^H = \{B^H_t, t \in R^+\}\) on \((\Omega, \mathcal{F}, P)\) featuring the properties

(a) \(B^H_0 = 0\);
(b) \(EB^H_t = 0, t \in R^+\);
(c) \(EB^H_t B^H_s = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}), s, t \in R^+\).

We consider the continuous modification of \(B^H\) whose existence is guaranteed by the classical Kolmogorov theorem.
To describe the statistical model, we need to introduce the pathwise integrals w.r.t. fBm. Consider two non-random functions $f$ and $g$ defined on some interval $[a, b] \subset R^+$. Suppose also that the following limits exist: $f(u+) := \lim_{\delta \downarrow 0} f(u + \delta)$ and $g(u-) := \lim_{\delta \downarrow 0} g(u - \delta)$, $a \leq u \leq b$. Let

$$f_{a+}(x) := (f(x) - f(a+))1_{(a,b)}(x), \quad g_{b-}(x) := (g(b-) - g(x))1_{(a,b)}(x).$$

Suppose that $f_{a+} \in L^{\alpha}_{a+}(L^p[a, b]))$, $g_{b-} \in L^{1-\alpha}_{b-}(L^q[a, b]))$ for some $p \geq 1$, $q \geq 1$, $1/p + 1/q \leq 1$, $0 \leq \alpha \leq 1$. (For the standard notation and statements concerning fractional analysis, see [SMK(1993)]).
Description of the integrals w.r.t. fBM

Introduce the fractional derivatives

\[
(D^\alpha_{a+} f_{a+})(x) = \frac{1}{\Gamma(1-\alpha)} \left( \frac{f_{a+}(s)}{(s-a)^{1-\alpha}} + \alpha \int_a^s \frac{f_{a+}(s) - f_{a+}(u)}{(s-u)^{1+\alpha}} \, du \right) 1_{(a,b)}(x)
\]

\[
(D^{1-\alpha}_{b-} g_{b-})(x) = \frac{e^{-i\pi\alpha}}{\Gamma(\alpha)} \left( \frac{g_{b-}(s)}{(b-s)^{1-\alpha}} + (1 - \alpha) \int_s^b \frac{g_{b-}(s) - g_{b-}(u)}{(s-u)^{2-\alpha}} \, du \right) 1_{(a,b)}(x).
\]

It is known that \( D^\alpha_{a+} f_{a+} \in L_p[a, b], \ D^{1-\alpha}_{b-} g_{b-} \in L_q[a, b] \).
Definition 2

([[Zah(1998)], [Zah(1999)]]) Under above assumptions, the generalized (fractional) Lebesgue-Stieltjes integral $\int_a^b f(x)dg(x)$ is defined as

$$\int_a^b f(x)dg(x) := e^{i\pi \alpha} \int_a^b (D^\alpha_{a+} f_{a+})(x)(D^{1-\alpha}_{b-} g_{b-})(x)dx + f(a+)(g(b-) - g(a+)),$$

and for $\alpha \rho < 1$ it can be simplified to

$$\int_a^b f(x)dg(x) := e^{i\pi \alpha} \int_a^b (D^\alpha_{a+} f)(x)(D^{1-\alpha}_{b-} g_{b-})(x)dx.$$
Description of the integrals w.r.t. fBm

As follows from [SMK(1993)], for any $1 - H < \alpha < 1$ there exist fractional derivatives $D_{b-}^{1-\alpha} B^{H}_{b-}$ and $D_{b-}^{1-\alpha} B^{H}_{b-} \in L_{\infty}[a, b]$ for any $0 \leq a < b$. Therefore, for $f \in \mathcal{I}_{a+}(L_{1}[a, b])$ we can define the integral w.r.t. fBm in the following way.

**Definition 3** ([NuaR(2002)], [Zah(1998)], [Zah(1999)]) The integral with respect to fBm is defined as

\[
\int_{a}^{b} f dB^{H} := e^{i\pi\alpha} \int_{a}^{b} (D_{a+}^{\alpha} f)(x)(D_{b-}^{1-\alpha} B^{H}_{b-})(x) dx. \tag{1}
\]

An evident estimate follows immediately from (1):

\[
\left| \int_{a}^{b} f dB^{H} \right| \leq \sup_{a \leq x \leq b} \left| (D_{b-}^{1-\alpha} B^{H}_{b-})(x) \right| \int_{a}^{b} \left| (D_{a+}^{\alpha} f)(x) \right| dx. \tag{2}
\]
Let us take a Wiener process $W = \{W_t, t \in R^+\}$ on probability space $(\Omega, \mathcal{F}, \mathcal{F}, P)$, possibly correlated with $B^H$. Assume that $H > \frac{1}{2}$ and consider a one-dimensional mixed stochastic differential equation involving both the Wiener process and the fractional Brownian motion

$$X_t = x_0 + \theta \int_0^t a(s, X_s) ds + \int_0^t b(s, X_s) dB^H_s + \int_0^t c(s, X_s) dW_s, \quad t \in R^+, \quad (3)$$

where $x_0 \in R$ is the initial value, $\theta$ is the unknown parameter to be estimated, the first integral in the right-hand side of (3) is the Lebesgue-Stieltjes integral, the second integral is the generalized Lebesgue-Stieltjes integral introduced in Definition 3, and the third one is the Itô integral.
The main equation

From now on, we shall assume that the coefficients of equation (3) satisfy the following assumptions on any interval \([0, T]\):

\((A_1)\) *Linear growth of \(a, b\) and \(c:\*)* for any \(s \in [0, T], x \in \mathbb{R}\)

\[|a(s, x)| + |b(s, x)| + |c(s, x)| \leq K(1 + |x|).\]

\((A_2)\) *Lipschitz continuity of \(a, c\) in space:* for any \(t \in [0, T]\) and \(x, y \in \mathbb{R}\)

\[|a(t, x) - a(t, y)| + |c(t, x) - c(t, y)| \leq K|x - y|.\]

\((A_3)\) *Hölder continuity in time:* function \(b(t, x)\) is differentiable in \(x\), for some \(\beta \in (1 - H, 1)\) and any \(s, t \in [0, T], x \in \mathbb{R}\)

\[|a(s, x) - a(t, x)| + |b(s, x) - b(t, x)| + |c(s, x) - c(t, x)|\]
\[+ |\partial_x b(s, x) - \partial_x b(t, x)| \leq K|s - t|^\beta.\]

\((A_4)\) *Lipschitz continuity of \(\partial_x b\) in space:* for any \(t \in [0, T], x, y \in \mathbb{R}\)

\[|\partial_x b(t, x) - \partial_x b(t, y)| \leq K|x - y|.\]

\((A_5)\) *Boundedness of \(\partial_x b\):* for any \(s \in [0, T]\) and \(x \in \mathbb{R}\)

\[|\partial_x b(s, x)| \leq K.\]
Here $K$ is a constant independent of $x$, $y$, $s$ and $t$. For an arbitrary interval $[0, T]$, $\alpha > 0$ and $\kappa = \frac{1}{2} \wedge \beta$ define the following norm:

$$
\|f\|_{\infty, \alpha, [0, T]} = \sup_{s \in [0, T]} \left( |f(s)| + \int_0^s |f(s) - f(z)| (s - z)^{-1-\alpha} \, dz \right).
$$

It was proved in [MiSh(2011)] that under assumptions $(A_1) - (A_5)$ there exists solution $X = \{X_t, \mathcal{F}_t, t \in [0, T]\}$ for equation (3) on any interval $[0, T]$ which satisfies

$$
\|X\|_{\infty, \alpha, [0, T]} < \infty \quad \text{a.s.}
$$

(4)

for any $\alpha \in (1 - H, \kappa)$. This solution is unique in the class of processes satisfying (4) for some $\alpha > 1 - H$. 

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To start with, consider the case $c(t, x) \equiv 0$ which was studied, for instance, in [KlLeBr(2002)] and [M.(2008)]. Recall some facts from the theory of drift parameter estimation in this case. Consider the equation

$$X_t = x_0 + \theta \int_0^t a(s, X_s)ds + \int_0^t b(s, X_s)dB_s^H, \ t \in \mathbb{R}. \quad (5)$$
Let assumptions \((A_1)\) and \((A_3)\) with \(c \equiv 0\) hold on any interval \([0, T]\), together with the following assumptions:

\((A'_2)\) Lipschitz continuity of \(a, b\) in space: for any \(t \in [0, T]\) and \(x, y \in \mathbb{R}\)

\[ |a(t, x) - a(t, y)| + |b(t, x) - b(t, y)| \leq K|x - y|, \]

\((A'_4)\) Hölder continuity of \(\partial_x b(t, x)\) in space: there exists such \(\rho \in (3/2 - H, 1)\) that for any \(t \in [0, T]\) and \(x, y \in \mathbb{R}\)

\[ |\partial_x b(t, x) - \partial_x b(t, y)| \leq D|x - y|^{\rho}, \]
Then, according to [NuaR(2002)], solution for equation (5) exists on any interval $[0, T]$ and is unique in the class of processes satisfying (4) for some $\alpha > 1 - H$.

In addition, suppose that the following assumption holds:

$(B_1) \ b(t, X_t) \neq 0, \ t \in [0, T]$ and $\frac{a(t, X_t)}{b(t, X_t)}$ is a.s. Lebesgue integrable on $[0, T]$ for any $T > 0$. 
Denote $\psi(t,x) = \frac{a(t,x)}{b(t,x)}$, $\varphi(t) := \psi(t,X_t)$. Also, let the kernel

$$l_H(t,s) = c_H s^{\frac{1}{2} - H} (t - s)^{\frac{1}{2} - H} I_{\{0 < s < t\}},$$

with $c_H = \left( \frac{\Gamma(3 - 2H)}{2H \Gamma\left(\frac{3}{2} - H\right)^3 \Gamma(H + \frac{1}{2})} \right)^{\frac{1}{2}}$, and introduce the integral

$$J_t = \int_0^t l_H(t,s) \varphi(s) ds = c_H \int_0^t (t - s)^{\frac{1}{2} - H} s^{\frac{1}{2} - H} \varphi(s) ds. \quad (6)$$

Finally, let $M_t^H = \int_0^t l_H(t,s) dB_s^H$ be Gaussian martingale with square bracket $\langle M \rangle_t^H = t^{2 - 2H}$ (Molchan martingale, see [NVV(1999)]).
Consider two processes:

\[
Y_t = \int_0^t b^{-1}(s, X_s) dX_s = \theta \int_0^t \varphi(s) ds + B_t^H
\]

and

\[
Z_t = \int_0^t l_H(t, s) dY_s = \theta J_t + M_t^H.
\]

Note that we can rewrite process \(Z\) as

\[
Z_t = \int_0^t l_H(t, s) b^{-1}(s, X_s) dX_s,
\]

so \(Z\) is a functional of the observable process \(X\). The following smoothness condition for the function \(\psi\) (Lemma 6.3.2 [M.(2008)]) ensures the semimartingale property of \(Z\).
Lemma 4

Let $\psi(t, x) \in C^1(R^+) \times C^2(R)$. Then for any $t > 0$

$$J'(t) = (2 - 2H)C_H \psi(0, x_0)t^{1-2H}$$

$$+ \int_0^t I(t, s)(\psi'_t(s, X_s) + \theta \psi'_x(s, X_s)a(s, X_s))ds$$

$$- \left(H - \frac{1}{2}\right)c_H \int_0^t s^{-\frac{1}{2} - H}(t - s)^{\frac{1}{2} - H} \int_0^s \left(\psi'_t(u, X_u) + \theta \psi'_x(u, X_u)a(u, X_u)\right)du ds$$

$$+(2 - 2H)c_H t^{1-2H} \int_0^t s^{2H - 3} \int_0^s u^{\frac{3}{2} - H}(s - u)^{\frac{1}{2} - H} \psi'_x(u, X_u)b(u, X_u)dB_u^H ds$$

$$+c_H t^{-1} \int_0^t u^{\frac{3}{2} - H}(t - u)^{\frac{1}{2} - H} \psi'_x(u, X_u)b(u, X_u)dB_u^H,$$

(7) where $C_H = B\left(\frac{3}{2} - H, \frac{3}{2} - H\right)$, $c_H = \left(\frac{\Gamma\left(\frac{3}{2} - H\right)}{2H\Gamma\left(H + \frac{1}{2}\right)\Gamma(3 - 2H)}\right)^{\frac{1}{2}}$, and all of the involved integrals exist a.s.
Remark 1

Suppose that $\psi(t, x) \in C^1(R^+) \times C^2(R)$ and limit $\varsigma(0) = \lim_{s \to 0} \varsigma(s)$ exists a.s., where $\varsigma(s) = s^{\frac{1}{2}-H} \varphi(s)$. In this case $J(t)$ can be presented as

$$J(t) = c_H \int_0^t (t-s)^{\frac{1}{2}-H} \varsigma(s) ds = \frac{c_H t^{\frac{3}{2}-H}}{3/2 - H} \varsigma(0) + c_H \int_0^t \frac{(t-s)^{\frac{3}{2}-H}}{3/2 - H} \varsigma'(s) ds,$$

and $J'(t)$ from (7) can be simplified to

$$J'(t) = c_H t^{\frac{1}{2}-H} \varsigma(0) + \int_0^t l_H(t, s) \left( \left( \frac{1}{2} - H \right) s^{-1} \varphi(s) + \psi'_t(s, X_s) + \theta \psi'_x(s, X_s) a(s, X_s) \right) ds + \int_0^t l_H(t, s) \psi'_x(s, X_s) b(s, X_s) dB^H_s.$$
Same way as $Z$, processes $J$ and $J'$ are functionals of $X$. It is more convenient to consider process $\chi(t) = (2 - 2H)^{-1} J'(t) t^{2H-1}$, so that

$$Z_t = (2 - 2H) \theta \int_0^t \chi(s) s^{1-2H} ds + M_t^H = \theta \int_0^t \chi(s) d\langle M^H \rangle_s + M_t^H.$$
Suppose that the following conditions hold:

\[(B_2) \quad E I_T := E \int_0^T \chi_s^2 d\langle M^H \rangle_s < \infty \text{ for any } T > 0,\]

\[(B_3) \quad I_\infty := \int_0^\infty \chi_s^2 d\langle M^H \rangle_s = \infty \text{ a.s.}\]
Then we can consider the maximum likelihood estimate
\[
\theta^{(1)}_T = \frac{\int_0^T \chi_s dZ_s}{\int_0^T \chi_s^2 d\langle M^H \rangle_s} = \theta + \frac{\int_0^T \chi_s dM^H_s}{\int_0^T \chi_s^2 d\langle M^H \rangle_s}.
\]

Condition \((B_2)\) ensures that process \(\int_0^t \chi_s dM^H_s, t > 0\) is a square integrable martingale, and condition \((B_3)\) alongside with the law of large numbers for martingales ensure that \(\frac{\int_0^T \chi_s dM^H_s}{\int_0^T \chi_s^2 d\langle M^H \rangle_s} \to 0\) a.s. as \(T \to \infty\).

Summarizing, we arrive at the following result ([M.(2008)]).

**Proposition 1**

Let \(\psi(t, x) \in C^1(R^+) \times C^2(R)\) and assumptions \((A_1), (A_3), (A_2'), (A_4')\) and \((B_1)-(B_3)\) hold. Then estimate \(\theta^{(1)}_T\) is strongly consistent as \(T \to \infty\).
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The next results are obtained in [KoMeMi(2011)]. In case when $c = 0$, it is possible to construct another estimate for parameter $\theta$, preserving the structure of the standard maximum likelihood estimate. Similar approach was applied in [HuNu(2010)] to the fractional Ornstein-Uhlenbeck process with constant coefficients. We shall use process $Y$ to define the estimate as

$$\theta_T^{(2)} = \frac{\int_0^T \varphi_s dY_s}{\int_0^T \varphi_s^2 ds} = \theta + \frac{\int_0^T \varphi_s dB_s^H}{\int_0^T \varphi_s^2 ds}. \quad (8)$$
Let us return to general equation (3) with non-zero \( c \) and construct the estimate of parameter \( \theta \). Suppose that the following assumption holds:

\[(C_1)\] \( c(t, X_t) \neq 0, t \in [0, T], \frac{a(t, X_t)}{c(t, X_t)} \) is a.s. Lebesgue integrable on \([0, T]\) for any \( T > 0 \) and there exists generalized Lebesgue–Stieltjes integral \( \int_0^T \frac{b(t, X_t)}{c(t, X_t)} dB_t^H \).

Define functions \( \psi_1(t, x) = \frac{a(t, x)}{c(t, x)} \) and \( \psi_2(t, x) = \frac{b(t, x)}{c(t, x)} \), processes \( \varphi_i(t) = \psi_i(t, X_t), i = 1, 2 \) and process

\[Y_t = \int_0^t b^{-1}(s, X_s) dX_s = \theta \int_0^t \varphi_1(s) ds + \int_0^t \varphi_2(s) dB_t^H + W_t.\]

Evidently, \( Y \) is a functional of \( X \) and is observable. Assume additionally that the generalized Lebesgue–Stieltjes integral \( \int_0^T \varphi_1(t) \varphi_2(t) dB_t^H \) exists and

\[(C_2)\] for any \( T > 0 \) \( E \int_0^T \varphi_1(s)^2 ds < \infty. \)
Denote \( \vartheta(s) = \varphi_1(s) \varphi_2(s) \). We can consider the following estimate of parameter \( \theta \):

\[
\theta_T^{(3)} = \frac{\int_0^T \varphi_1(s) dY_s}{\int_0^T \varphi_1^2(s) ds} = \theta + \frac{\int_0^T \vartheta(s) dB^H_s}{\int_0^T \varphi_1^2(s) ds} + \frac{\int_0^T \varphi_1(s) dW_s}{\int_0^T \varphi_1^2(s) ds}. \tag{9}
\]

Estimate \( \theta_T^{(3)} \) preserves the traditional form of maximum likelihood estimates for diffusion models. The right-hand side of (9) provides a stochastic representation of \( \theta_T^{(3)} \). We shall use it to investigate the strong consistency of this estimate.
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Return to model (5) and suppose that conditions \((B_1) - (B_3)\) hold. For any \(h > 0\) consider the stopping time

\[
\tau(h) = \inf\{t > 0 : \int_0^t \chi_s^2 d\langle M\rangle_s = h\}.
\]

Under conditions \((B_1) - (B_2)\) we have \(\tau(h) < \infty\) a.s. and

\[
\int_0^{\tau(h)} \chi_s^2 d\langle M\rangle_s = h.
\]

The sequential maximum likelihood estimate has a form

\[
\theta^{(1)}_{\tau(h)} = \int_0^{\tau(h)} \frac{\chi_s dZ_s}{h} = \theta + \frac{\int_0^{\tau(h)} \chi_s dM^H_s}{h}. \quad (10)
\]

Sequential versions of estimates \(\theta_T^{(2)}\) and \(\theta_T^{(3)}\) have a form

\[
\theta^{(2)}_{\tau(h)} = \theta + \frac{\int_0^{\tau(h)} \varphi_s dB^H_s}{h}
\]

and

\[
\theta^{(3)}_{\nu(h)} = \theta + \frac{\int_0^{\nu(h)} \nu(s) dB^H_s}{h} + \frac{\int_0^{\nu(h)} \varphi_1(s) dW_s}{h},
\]
where

$$v(h) = \inf\{ t > 0 : \int_0^t \varphi_1^2(s) ds = h \}.$$ 

To provide an exhaustive study of the introduced estimates, we will need a number of auxiliary facts about Gaussian processes.
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Theorem 5

Let $0 < H < 1, 1 - H < \alpha < 1$, $T = \{ t = (t_1, t_2), 0 \leq t_2 < t_1 \}$,

$$X(t) = \frac{B_{t_1}^H - B_{t_2}^H}{(t_1 - t_2)^{1-\alpha}} + \int_{t_2}^{t_1} \frac{B_u^H - B_{t_2}^H}{(u - t_2)^{2-\alpha}} du.$$

Then for any $p > 1$ there exists random variable $\xi = \xi(p)$ such that for any $t \in T$

$$|X(t)| \leq ((t_1^{H+\alpha-1}(\log(t_1))^p) \lor 1)\xi(p),$$

where $\xi(p)$ satisfies assumption $(D_4)$ with some constants $B_1$ and $C_1$. 
Remark 2

We can consider the fractional Brownian motion $B_t^H$ itself and apply the same reasoning to it. This case is much simpler and we immediately obtain that $\sup_{0 \leq s \leq t} |B_s^H| \leq ((t^H(\log(t))^p) \vee 1)\xi(p)$ for any $p > 1$. 
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General results on strong consistency

We shall establish conditions for strong consistency of $\theta_T^{(2)}$ and $\theta_T^{(3)}$.

**Theorem 6**

Let assumptions $(A_1), (A_3), (A'_2), (A'_4), (B_1)$ and $(B_2)$ hold and let function $\varphi$ satisfy the following assumption:

[(B_4)] There exists such $\alpha > 1 - H$ and $p > 1$ that

$$
T^{H+\alpha-1}(\log T)^p \frac{\int_0^T |(D^\alpha_0+\varphi)(s)|ds}{\int_0^T \varphi^2_s ds} \to 0 \quad \text{a.s. as } T \to \infty. \quad (11)
$$

Then estimate $\theta_T^{(2)}$ is correctly defined and strongly consistent as $T \to \infty$. 
Relation (11) ensures convergence $\frac{\int_0^T \varphi_s dB^H_s}{\int_0^T \varphi^2_s ds} \to 0$ a.s. in the general case.

In a particular case when function $\varphi$ is non-random and integral $\int_0^T \varphi_s dB^H_s$ is a Wiener integral w.r.t. the fractional Brownian motion, conditions for existence of this integral are simpler since assumption (11) can be simplified.

**Theorem 7**

Let assumptions $(A_1), (A_3), (A'_2), (A'_4) (B_1)$ and $(B_2)$ hold and let function $\varphi$ be non-random and satisfy the following assumption:

$(B_5)$ There exists such $p > 0$ that

$$\lim_{T \to \infty} \sup_{T} \frac{T^{2H-1+p}}{\int_0^T \varphi^2(t)dt} < \infty.$$ 

Then estimate $\theta_T^{(2)}$ is strongly consistent as $T \to \infty$. 
Theorem 8

Let assumptions (C₁) and (C₂) hold, and, in addition,

(C₃) \( \int_0^T \varphi_1^2(s) ds = \infty \) a.s.

(C₄) There exist such \( \alpha > 1 - H \) and \( p > 1 \) that

\[
\frac{T^{H+\alpha-1}(\log T)^p \int_0^T |(\mathcal{D}_{0+}^{\alpha,y})(s)| ds}{\int_0^T \varphi_1^2(s) ds} \to 0 \quad \text{a.s. as} \quad T \to \infty. \tag{12}
\]

Then estimate \( \theta^{(3)}_T \) is strongly consistent as \( T \to \infty \).
Similarly to Theorem 7, conditions stated in Theorem 8 can be simplified in case when function \( \vartheta \) is non-random.

**Theorem 9**

Let assumptions \((C_1)\) and \((C_2)\) hold. Then, if functions \( \varphi_1 \) and \( \varphi_2 \) are non-random, function \( \varphi_1 \) satisfies condition \((B_5)\), function \( \varphi_2 \) is bounded, then estimate \( \hat{\theta}_T^{(3)} \) is strongly consistent as \( T \to \infty \).
Now we shall take a look at the properties of sequential estimates.

**Theorem 10**

(a) Let assumptions \((B_1) - (B_3)\) hold. Then estimate \(\theta^{(1)}_{\tau(h)}\) is unbiased, efficient, strongly consistent, \(E(\theta^{(1)}_{\tau(h)} - \theta)^2 = \frac{1}{h}\), and for any estimate of the form

\[
\theta_{\tau} = \frac{\int_0^{\tau} \chi_s dZ_s}{\int_0^{\tau} \chi_s^2 d\langle M^H \rangle_s} = \theta + \frac{\int_0^{\tau} \chi_s dM^H_s}{\int_0^{\tau} \chi_s^2 d\langle M^H \rangle_s}
\]

with \(\tau < \infty\) a.s. and \(E \int_0^{\tau} \chi_s^2 d\langle M^H \rangle_s \leq h\) we have that

\[
E(\theta^{(1)}_{\tau(h)} - \theta)^2 \leq E(\theta_{\tau} - \theta)^2.
\]
Theorem 11

(To continue)

(b) Let function $\varphi$ be separated from zero, $|\varphi(s)| \geq c > 0$ a.s. and satisfy the assumption: for some $1 - H < \alpha < 1$ and $p > 0$

$$\int_0^{\tau(h)} \frac{|(D_{0+}^\alpha \varphi)(s)| ds}{(\tau(h))^{2-\alpha-H-p}} \rightarrow 0 \quad \text{a.s.} \quad (13)$$

as $h \rightarrow \infty$. Then estimate $\theta_{\tau(h)}^{(2)}$ is strongly consistent.
Theorem 12

(To continue)

(c) Let function \( \varphi_1 \) be separated from zero, \(| \varphi(s) | \geq c > 0 \) a.s. and let function \( \vartheta \) satisfy the assumption: for some \( 1 - H < \alpha < 1 \) and \( p > 0 \)

\[
\int_0^{\nu(h)} \frac{|(\mathcal{D}_0^\alpha \vartheta)(s)| ds}{(\nu(h))^{2-\alpha-H-p}} \rightarrow 0 \quad \text{a.s.} \tag{14}
\]

as \( h \rightarrow \infty \). Then estimate \( \theta^{(3)}_{\nu(h)} \) is strongly consistent.

(d) Let function \( \vartheta \) be non-random, bounded and positive, \( \varphi_1 \) be separated from zero. Then estimate \( \theta^{(3)}_{\nu(h)} \) is consistent in the following sense: for any \( p > 0 \),

\[
E \left| \theta - \theta^{(3)}_{\nu(h)} \right|^p \rightarrow 0 \quad \text{as} \quad h \rightarrow \infty.
\]

Remark 3

Another proof of statement (a) is contained in [Prara(2010)]. Assumptions (13) and (14) hold, for example, for bounded and Lipschitz functions \( \varphi \) and \( \vartheta \) correspondingly.
Linear models and strong consistency.

1. Consider the linear version of model (5):

\[ dX_t = \theta a(t)X_t \, dt + b(t)X_t \, dB^H_t, \]

where \( a \) and \( b \) are locally bounded non-random measurable functions. In this case solution \( X \) exists, is unique and can be presented in the integral form

\[ X_t = x_0 + \theta \int_0^t a(s)X_s \, ds + \int_0^t b(s)X_s \, dB^H_s \]

\[ = x_0 \exp \left\{ \theta \int_0^t a(s) \, ds + \int_0^t b(s) \, dB^H_s \right\}. \]

Suppose that function \( b \) is non-zero and note that in this model

\[ \varphi(t) = \frac{a(t)}{b(t)}. \]

Suppose that \( \varphi(t) \) is also locally bounded and consider maximum likelihood estimate \( \theta_T^{(1)} \).
According to (6), to guarantee existence of process $J'$, we have to assume that the fractional derivative of order $\frac{3}{2} - H$ for function $\varsigma(s) := \varphi(s)s^{\frac{1}{2} - H}$ exists and is integrable. The sufficient conditions for the existence of fractional derivatives can be found in [SMK(1993)]. One of these conditions states:
(B₆) Functions \( \varphi \) and \( \varsigma \) are differentiable and their derivatives are locally integrable.

So, the maximum likelihood estimate does not exist for an arbitrary locally bounded function \( \varphi \). Suppose that condition \((B₆)\) holds and limit 
\( \varsigma₀ = \lim_{s \to 0} \varsigma(s) \) exists. In this case, according to Lemma 4 and Remark 1, process \( J' \) admits both of the following representations:

\[
J'(t) = (2 - 2H)C₇\varphi(0)t^{1-2H} + \int_0^t l_H(t, s)\varphi'(s)ds \\
- \left(H - \frac{1}{2}\right)c_H \int_0^t s^{-\frac{1}{2}-H}(t - s)^{\frac{1}{2}-H} \int_0^s \varphi'(u)duds \\
= c_H\varsigma₀t^{\frac{1}{2}-H} + c_H \int_0^t (t - s)^{\frac{1}{2}-H}\varsigma'(s)ds,
\]

and assuming \((B₃)\) also holds true, the estimate \( \theta^{(1)}_T \) is strongly consistent.
Let us formulate some simple conditions sufficient for the strong consistency.

**Lemma 13**

*If function $\varphi$ is non-random, locally bounded, satisfies $(B_6)$, limit $\varsigma(0)$ exists and one of the following assumptions hold:*

(a) *function $\varphi$ is not identically zero and $\varphi'$ is non-negative and non-decreasing;*

(b) *derivative $\varsigma'$ preserves the sign and is separated from zero;*

(c) *derivative $\varsigma'$ is non-decreasing and has a non-zero limit,*

*then the estimate $\theta_T^{(1)}$ is strongly consistent as $T \to \infty$.***
Example 14

Let the coefficients are constant, \( a(s) = a \neq 0 \) and \( b(s) = b \neq 0 \), then the estimate has a form

\[
\theta_T^{(1)} = \theta + \frac{bM_H^H}{aC_H T^{2-2H}}
\]

and is strongly consistent. In this case assumption (a) holds. In addition, power functions \( \varphi(s) = s^\rho \) are appropriate for \( \rho > H - 1 \): this can be verified directly from (6).
Let us now apply estimate $\theta_T^{(2)}$ to the same model. It has a form (8). We can use Theorem 7 directly and under assumption $(B_5)$ estimate $\theta_T^{(2)}$ is strongly consistent. Note that we do not need any assumptions on the smoothness of $\varphi$, which is a clear advantage of $\theta_T^{(2)}$. We shall consider two more examples.

**Example 15**

If the coefficients are constant, $a(s) = a \neq 0$ and $b(s) = b \neq 0$, then the estimate has a form $\theta_T^{(2)} = \theta + \frac{b B_H^T}{a T}$. We can refer to Theorem 7 and conclude that $\theta_T^{(2)}$ is strongly consistent. Alternatively, we can use Remark 2 which states that $|B_H^T| \leq \xi T^H (\log T)^p$ for any $p > 1$ and some random variable $\xi$, therefore $\frac{B_H^T}{T} \to 0$ a.s. as $T \to \infty$. In this case both estimates $\theta_T^{(2)}$ and $\theta_T^{(2)}$ are strongly consistent and $E(\theta - \theta_T^{(1)})^2 = \frac{\gamma^2 T^{2H-2}}{a^2 C_H^2}$ has the same asymptotic behavior as $E(\theta - \theta_T^{(2)})^2 = \frac{\gamma^2 T^{2H-2}}{a^2}$. 
Example 16

If non-random functions $\varphi$ and $\varsigma$ are bounded on some fixed interval $[0, t_0]$ but $\varsigma$ is sufficiently irregular on this interval and has no fractional derivative of order $\frac{3}{2} - H$ or higher then we can not even calculate $J'(t)$ on this interval and the maximum likelihood estimate does not exist. However, if we assume that $\varphi(t) \sim t^{H-1+\rho}$ at infinity with some $\rho > 0$, then assumption $(B_5)$ holds and estimate $\theta_T^{(2)}$ is strongly consistent as $T \to \infty$. In this sense estimate $\theta_T^{(2)}$ is more flexible.
The mixed model

II. Consider a mixed linear model of the form

\[ dX_t = X_t(\theta a(t)dt + b(t)dB_t^H + c(t)dW_t), \]

where \( a, b \) and \( c \) are non-random measurable functions. Assume that they are locally bounded. In this case solution \( X \) for equation (15) exists, is unique and can be presented in the integral form

\[ X_t = x_0 \exp \left\{ \theta \int_0^t a(s)ds + \int_0^t b(s)dB_s^H + \int_0^t c(s)dW_s - \frac{1}{2} \int_0^t c^2(s)ds \right\}. \]

In what follows assume that \( c(s) \neq 0 \). We have that \( \varphi_1(t) = \frac{a(t)}{c(t)} \) and \( \varphi_2(t) = \frac{b(t)}{c(t)} \). Estimate \( \theta^{(3)}_T \) has a form

\[ \theta^{(3)}_T = \frac{\int_0^T \varphi_1(s)dY_s}{\int_0^T \varphi_1^2(s)ds} = \theta + \frac{\int_0^T \varphi_1(s)\varphi_2(s)dB_s^H}{\int_0^T \varphi_1^2(s)ds} + \frac{\int_0^T \varphi_1(s)dW_s}{\int_0^T \varphi_1^2(s)ds}. \]

In accordance with Theorem 9, assume that function \( \varphi_1 \) satisfies \((B_5)\) and \( \varphi_2 \) is bounded. Then estimate \( \theta^{(3)}_T \) is strongly consistent. Evidently, these assumptions hold for the constant coefficients.
The fractional Ornstein-Uhlenbeck model and strong consistency.

I. Consider the fractional Ornstein-Uhlenbeck, or Vasicek, model with non-constant coefficients. It has a form

\[ dX_t = \theta(a(t)X_t + b(t))dt + \gamma(t)dB_t^H, \quad t \geq 0, \]

where \( a, b \) and \( \gamma \) are non-random measurable functions. Suppose they are locally bounded and \( \gamma = \gamma(t) > 0 \). The solution for this equation is a Gaussian process and has a form

\[ X_t = e^{\theta A(t)} \left( x_0 + \theta \int_0^t b(s)e^{-\theta A(s)}ds + \int_0^t \gamma(s)e^{-\theta A(s)}dB_s^H \right) := E(t) + G(t), \]

where \( A(t) = \int_0^t a(s)ds \), \( E(t) = e^{\theta A(t)} \left( x_0 + \theta \int_0^t b(s)e^{-\theta A(s)}ds \right) \) is a non-random function, \( G(t) = e^{\theta A(t)} \int_0^t \gamma(s)e^{-\theta A(s)}dB_s^H \) is a Gaussian process with zero mean.
Denote \( c(t) = \frac{a(t)}{\gamma(t)} \), \( d(t) = \frac{b(t)}{\gamma(t)} \). Now we shall state the conditions for strong consistency of the maximum likelihood estimate.

**Theorem 17**

Let functions \( a, c, d \) and \( \gamma \) satisfy the following assumptions:

\[(B_7) \quad -a_1 \leq a(s) \leq -a_2 < 0, \quad -c_1 \leq c(s) \leq -c_2 < 0, \quad 0 < \gamma_1 \leq \gamma(s) \leq \gamma_2, \]

functions \( c \) and \( d \) are continuously differentiable, \( c' \) is bounded, \( c'(s) \geq 0 \) and \( c'(s) \to 0 \) as \( s \to \infty \).

Then estimate \( \theta_T^{(1)} \) is strongly consistent as \( T \to \infty \).
Remark 4

The assumptions of the theorem are fulfilled, for example, if \( a(s) = -1 \), \( b(s) = b \in \mathbb{R} \) and \( \gamma(s) = \gamma > 0 \). In this case we deal with a standard Ornstein-Uhlenbeck process \( X \) with constant coefficients that satisfies the equation

\[
dX_t = \theta(b - X_t) dt + \gamma dB_t^H, \quad t \geq 0.
\]

This model with constant coefficients was studied in [KlLeBr(2002)] where the Laplace transform \( \Theta_T(\lambda) \) was calculated explicitly and strong consistency of \( \theta_T^{(1)} \) was established. Therefore, our results generalize the statement of strong consistency to the case of variable coefficients.
II. Consider a simple version of the Ornstein-Uhlenbeck model where $a = \gamma = 1$, $b = x_0 = 0$. The SDE has a form $dX_t = \theta X_t dt + dH_t^H$, $t \geq 0$ with evident solution $X_t = e^{\theta t} \int_0^t e^{-\theta s} dH_s^H$. Let us construct an estimate which is a modification of $\theta^{(2)}_T$:

$$
\tilde{\theta}^{(2)}_T = \frac{\int_0^T e^{-2\theta s} X_s dX_s}{\int_0^T e^{-2\theta s} X_s^2 ds} = \theta + \frac{\left( \int_0^T e^{-\theta s} dH_s^H \right)^2}{\int_0^T \left( \int_0^s e^{-\theta u} dH_u^H \right)^2 ds}.
$$

**Theorem 18**

Let $\theta > 0$. Then estimate $\tilde{\theta}^{(2)}_T$ is strongly consistent as $T \to \infty$. 
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