Counting patterns in digital expansions

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Outline

Motivation

Number systems and additive functions
Distributional properties
Normal numbers

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Transfer of the problems
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Division of the expansion

Results

Arithmetic sum
Asymptotic distribution
Construction of normal numbers
Number systems

Let \( \mathcal{R} \) be an integral domain, \( b \in \mathcal{R} \), and \( \mathcal{N} = \{n_1, \ldots, n_m\} \subset \mathcal{R} \). Then we call the pair \((b, \mathcal{N})\) a number system in \( \mathcal{R} \) if every \( g \in \mathcal{R} \) admits a unique and finite representation of the form

\[
g = \sum_{j=0}^{h} a_j(g)b^j \quad \text{with} \quad a_i(g) \in \mathcal{N} \quad \text{for} \quad i = 0, \ldots, h \quad (1)
\]

and \( a_h(g) \neq 0 \) if \( h \neq 0 \). We call \( b \) the base and \( \mathcal{N} \) the set of digits.
Examples for number systems

- $b \in \mathbb{Z}$, $b \leq -2$ and $\mathcal{N} := \{0, 1, \ldots, |b| - 1\}$, then $(b, \mathcal{N})$ is a number system in $\mathbb{Z}$.

- $B \in \mathbb{F}_q[X]$ a polynomial, $\deg B > 1$, $\mathcal{N} := \{P \in \mathbb{F}_q[X] : \deg P < \deg B\}$. Then $(B, \mathcal{N})$ is a number system in $\mathbb{F}_q[X]$.

- Let $\beta$ be an algebraic integer over $\mathbb{Z}$. Furthermore let $b \in \mathbb{Z}[\beta]$ and $\mathcal{N} := \{0, 1, \ldots, N(b) - 1\}$. Then under some mild conditions the pair $(b, \mathcal{N})$ is a number system in $\mathbb{Z}[\beta]$. 

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  \( b \in \mathbb{Z}[\beta] \) and \( \mathcal{N} := \{0, 1, \ldots, \mathcal{N}(b) - 1\} \). Then under some mild conditions the pair \((b, \mathcal{N})\) is a number system in \( \mathbb{Z}[\beta] \).
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Additive functions

Let $\mathcal{R}$ be an integral domain and $(b, \mathcal{N})$ be a number system in this domain. Then we call a function $f : \mathcal{R} \rightarrow \mathbb{R}$ $b$-additive, if for $g$ as in (1) we have that

$$f(g) = \sum_{k=0}^{h} f(a_k b^k).$$

Moreover we call it strictly $b$-additive, if for $g$ as in (1) we have that

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The sum-of-digits function

A very simple example of a strictly $b$-additive function is the sum-of-digits function $s_b$, which is defined by

$$s_b(g) = \sum_{k=0}^{h} a_k$$

for $g$ as in (1).
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Theorem  Delange (1975)

\[ \sum_{n \leq N} s_q(n) = \frac{q-1}{2} N \log_q N + NF (\log_q N), \]

where \( \log_q \) is the logarithm to base \( q \) and \( F \) is a 1-periodic, continuous and nowhere differentiable function.
Distribution result for integers

Let $f$ be a $q$-additive function in $\mathbb{N}$ such that $f(aq^k) = O(1)$ as $k \to \infty$ and $a \in \mathcal{N}$. Furthermore let

$$m_{k,q} := \frac{1}{q} \sum_{a \in \mathcal{N}} f(aq^k), \quad \sigma^2_{k,q} := \frac{1}{q} \sum_{a \in \mathcal{N}} f^2(aq^k) - m^2_{k,q},$$

and

$$M_q(x) := \sum_{k=0}^N m_{k,q}, \quad D^2_q(x) = \sum_{k=0}^N \sigma^2_{k,q}$$

with $N = \lfloor \log_q x \rfloor$. l
Distribution result for integers

Theorem  Bassily and Katai (1995)

Assume that \( \frac{D_q(x)}{(\log x)^{1/3}} \to \infty \) as \( x \to \infty \) and let \( p(x) \) be a polynomial with integer coefficients, degree \( d \) and positive leading term. Then, as \( x \to \infty \),

\[
\frac{1}{x} \# \left\{ n < x \left| \frac{f(p(n)) - M_q(x^d)}{D_q(x^d)} < y \right. \right\} \to \Phi(y),
\]

where \( \Phi \) is the normal distribution function.
Distribution result for Gaussian integers

Let $f$ be a $b$-additive function in $\mathbb{Z}[i]$ such that $f(ab^k) = \mathcal{O}(1)$ as $k \to \infty$ and $a \in \mathcal{N}$. Furthermore let

$$m_{k,b} := \frac{1}{N(b)} \sum_{a \in \mathcal{N}} f(ab^k), \quad \sigma_{k,b}^2 := \frac{1}{N(b)} \sum_{a \in \mathcal{N}} f^2(ab^k) - m_{k,b}^2,$$

and

$$M_b(x) := \sum_{k=0}^{L} m_{k,q}, \quad D_b^2(x) = \sum_{k=0}^{L} \sigma_{k,q}^2$$

with $L = \lfloor \log_{N(b)} x \rfloor$. 
Distribution result for Gaussian integers

Theorem  Gittenberger and Thuswaldner (2000)

Assume that $D_b(x)/(\log x)^{1/3} \to \infty$ as $x \to \infty$ and let $p \in \mathbb{Z}[i][X]$ be a polynomial of degree $d$. Then, as $T \to \infty$,

$$
\frac{1}{\# \{z | N(z) \leq N\} \# \left\{ N(z) \leq N \left| \frac{f(p(z)) - M_b(N^d)}{D_b(N^d)} < y \right\}} \to \Phi(y),
$$

where $\Phi$ is the normal distribution function.
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We extend our number system onto $\mathcal{K}_\infty$ the completion of the field of quotients $\mathcal{K}$ of $\mathcal{R}$. Then we get that every $\gamma \in \mathcal{K}_\infty$ has a (not necessarily unique) representation of the shape

$$\gamma = \sum_{j=-\infty}^{\ell(\gamma)} a_j(\gamma) b^j \quad (a_j(\gamma) \in \mathcal{N}).$$
Fundamental domain

In this context the fundamental domain $\mathcal{F}$ indicates the properties of this extension. It is defined as all numbers with zero in the integer part of their $b$-ary representation, i.e.,

$$\mathcal{F} := \left\{ \gamma \in \mathcal{K}_\infty \mid \gamma = \sum_{j \geq 1} a_j b^{-j}, a_j \in \mathcal{N} \right\}.$$
Block count

Let $\theta \in \mathcal{K}_\infty$ be such that

$$\theta = \sum_{j \geq 1} a_j b^{-j} \in \mathcal{F}.$$ 

Then for $d_1 \ldots d_r \in \mathcal{N}^r$ being a block of digits of length $r$ we denote by $\mathcal{N}(\theta; d_1 \ldots d_r; N)$ the number of occurrences of this block in the first $N$ digits of $\theta$. Thus

$$\mathcal{N}(\theta; d_1 \ldots d_r; N) := \#\{0 \leq n < N : d_1 = a_{n+1}, \ldots, d_r = a_{n+r}\}.$$
Normal number

Now we call $\theta$ normal in $(b, \mathcal{N})$ if for every $r \geq 1$ we have that

$$R_N(\theta) = R_{N,r}(\theta) := \sup_{d_1 \ldots d_r} \left| \frac{1}{N} \mathcal{N}(\theta; d_1 \ldots d_r; N) - \frac{1}{|\mathcal{N}|^r} \right| = o(1)$$

where the supremum is taken over all possible blocks $d_1 \ldots d_r \in \mathcal{N}^r$ of length $r$. 
Construction of normal numbers

In order to construct a normal number in base $b$ we often take a strictly increasing sequence $(s_n)_{n \geq 1}$ of real numbers and concatenate its values written in base $b$. Thus we define

$$\theta((s_n)_{n \geq 1}) := 0.s_1s_2s_3s_4s_5 \ldots$$
Constructions of normal numbers

Theorem Champernowne (1933)

$$\theta((n)_{n \geq 1}) = 0.1234567891011121314$$

is normal to base 10.

Theorem Copeland and Erdős (1946)

Let $$s_n \in \mathbb{N}$$. If

$$\forall \delta > 0 \exists N \in \mathbb{N} : \# \{ s_n : s_n \leq N \} \geq N^\delta,$$

then $$\theta((s_n)_{n \geq 1})$$ is normal.
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Our goals

Let $p$ be an arbitrary function from the reals into the reals and $f$ be an $q$-additive function. Then we want to estimate the following

\[ \sum_{n \leq N} f(p(n)) \]

\[ \# \left\{ n \leq N : \frac{f(p(n)) - \mu_f}{\sigma_f} < y \right\} \]

\[ \mathcal{N}(\theta (p(n))_{n \geq 1}, d_1 \ldots d_r, N) \]
Indicator function

\[
I_{\ell,a}(x) = \frac{1}{q} + \psi \left( \frac{x}{q^{\ell+1}} - \frac{a + 1}{q} \right) - \psi \left( \frac{x}{q^{\ell+1}} - \frac{a}{q} \right)
\]

\[
= \begin{cases} 
1 & \text{if } a_\ell(x) = a, \\
0 & \text{else,}
\end{cases}
\]

where

\[
\psi(x) = x - \lfloor x \rfloor - \frac{1}{2} = \{x\} - \frac{1}{2}.
\]
Rewriting our goals

For the arithmetic sum we get, that

\[
\sum_{n \leq N} f(p(n)) = \sum_{n \leq N} \sum_{\ell=0}^{q-1} \sum_{a=0}^{\lfloor \log_q p(n) \rfloor} f(aq^\ell) \mathcal{I}_{\ell,a}(p(n))
\]

\[
= \sum_{a=0}^{q-1} \sum_{\ell=0}^{\lfloor \log_q p(n) \rfloor} \sum_{n \leq N} f(aq^\ell) \mathcal{I}_{\ell,a}(p(n)).
\]
Transferation of the problem

For the asymptotic distribution result there is more work to be done.

1. Truncate the range of the function $f$.
2. Consideration of the moments.
3. Getting rid of the polynomial.
**Transferation: Truncation**

Let $0 \leq A \leq B \leq L$ be well chosen. Then define the truncated function $f'$ to be

$$f'(p(n)) = \sum_{j=A}^{B} f(a_j(p(n))b^j).$$

We easily get that

$$\max_{n \leq N} \left| \frac{f(p(n)) - M(T^d)}{D(T^d)} - \frac{f'(p(n)) - M'(T^d)}{D'(T^d)} \right| \to 0.$$
Transferation: Moments

Thus it suffices to consider

$$\frac{1}{N} \# \left\{ n \leq N \left| \frac{f'(p(n)) - M'(T^d)}{D'(T^d)} < y \right. \right\} \rightarrow \Phi(y).$$

and by the Frechet-Shohat Theorem this holds if and only if the moments

$$\xi_k(T) := \frac{1}{N} \sum_{n \leq N} \left( \frac{f'(p(n)) - M'(T^d)}{D'(T^d)} \right)^k$$

converge to the moments of the normal law.
Finally we want to get rid of the polynomial $p$ and this is done by instead of considering the moments

$$
\xi_k(T) := \frac{1}{N} \sum_{n \leq N} \left( \frac{f'(p(n)) - M'(T^d)}{D'(T^d)} \right)^k
$$

considering the following moments

$$
\eta_k(T) := \frac{1}{N} \sum_{n \leq N} \left( \frac{f'(n) - M'(T^d)}{D'(T^d)} \right)^k.
$$
Transferation: The central tool

Now we are left with showing that

\[ \xi_k(T) - \eta_k(T) \to 0 \quad \text{for} \quad T \to \infty. \]

In particular we are left with estimating

\[ \# \{ n \leq N | a_{\ell_j}(p(n)) = a_j, j = 1, \ldots, h \} = \sum_{n \leq N} \prod_{j=1}^{h} I_{\ell_j,a_j}(p(n)), \]

where \( A \leq l_1 < l_2 < \cdots < l_h \leq B. \)
Block counting

For proving that one of the constructions above really yields a normal number one counts the number of occurrences of a pattern within the expansion and ignores the number occurring between two expansions. Thus we need to estimate

\[ \sum_{n \leq N} \sum_{\ell=0}^{r-1} \prod_{j=0}^{r-1} \mathcal{I}_{\ell+j,a_j}(p(n)). \]
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\[ \sum_{n \leq N} \sum_{\ell=0}^{r-1} \prod_{j=0}^{r-1} I_{\ell+j,a_j}(p(n)). \]
Counting the patterns

Thus we have to count the following "patterns":

1. 
\[
\sum_{a=0}^{q-1} \sum_{\ell=0}^{\lfloor \log_q p(n) \rfloor} \sum_{n \leq N} f(aq^\ell) I_{\ell,a}(p(n)),
\]

2. 
\[
\sum_{n \leq N} \prod_{j=1}^{h} I_{\ell,j,a}(p(n)),
\]

3. 
\[
\sum_{n \leq N} \sum_{\ell=0}^{\lfloor \log_q p(n) \rfloor - r + 1} \prod_{j=0}^{r-1} I_{\ell+j,a}(p(n)).
\]
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Fourier transform of the indicator function

In the one dimensional case we can easily do Fourier transform.

\[
\sum_{n \leq N} \left( I_{\ell,a} (p(n)) - \frac{1}{q} \right)
\ll \frac{N}{\delta} + \sum_{\nu=1}^{\infty} \min \left( \frac{\delta}{\nu^2}, \frac{1}{\nu} \right) \left| \sum_{n \leq N} e \left( \frac{\nu}{q^{\ell+1}} p(n) \right) \right|
\]
Fourier transform of $\psi$

In the same manner we get for the function $\psi$

$$
\sum_{n \leq N} \psi(g(n)) \ll \frac{N}{\delta} + \sum_{\nu=1}^{\infty} \min \left( \frac{\delta}{\nu^2}, \frac{1}{\nu} \right) \left| \sum_{n \leq N} e(\nu g(n)) \right|.
$$

Recall that

$$
I_{\ell,a}(x) = \frac{1}{q} + \psi \left( \frac{x}{q^{\ell+1}} - \frac{a+1}{q} \right) - \psi \left( \frac{x}{q^{\ell+1}} - \frac{a}{q} \right)
$$
Higher dimension: Embedding

Since it is more easy to consider the properties of number systems in $\mathbb{R}^n$, we define an embedding of $\mathcal{K}$ in $\mathbb{R}^n$. Thus let $\phi$ be defined by

$$\phi : \left\{ \begin{array}{ccc} \mathbb{C} & \rightarrow & \mathbb{R}^n, \\ \alpha_0 + \alpha_1 b + \cdots + \alpha_{n-1} b^{n-1} & \mapsto & (\alpha_0, \ldots, \alpha_{n-1}) \end{array} \right.$$
Higher dimension: Matrix number system

Let \( m_b(x) = a_0 + a_1x + \cdots + a_{n-1}x^{n-1} \) be the minimal polynomial of \( b \), then we define the corresponding matrix \( B \) by

\[
B = \begin{pmatrix}
0 & 0 & \cdots & \cdots & \cdots & -a_0 \\
1 & 0 & \cdots & \cdots & 0 & \\
0 & 1 & \ddots & \ddots & \vdots & \\
& \ddots & \ddots & \ddots & \ddots & \\
& & \ddots & \ddots & 1 & 0 \\
0 & 0 & \cdots & 0 & 1 & -a_{n-1}
\end{pmatrix}.
\]

One easily checks that

\[
\phi(b \cdot z) = B \cdot \phi(z).
\]
Higher dimension: Fundamental domain

By this we define the embedding of the fundamental domain by

\[ \mathcal{F}' = \phi(\mathcal{F}) = \left\{ z \in \mathbb{R}^n \mid z = \sum_{k \geq 1} B^{-k} a_k, a_k \in \phi(\mathcal{N}) \right\} . \]
Higher dimension: Twin dragon

Figure: Fundamental domain for $b = -1 + i$
Higher dimension: Urysohn function

Let $\mathcal{F}_a$ be the domain consisting of all elements of $\mathcal{F}'$ starting with $a$, i.e.,

$$\mathcal{F}_a = B^{-1} (\mathcal{F}' + a)$$

In order to count the number of hits in $\mathcal{F}_a$ we define

$$\psi_a(x_1, \ldots, x_n) = \begin{cases} 1 & \text{if } (x_1, \ldots, x_n) \in I_{k,a} \\ \frac{1}{2} & \text{if } (x_1, \ldots, x_n) \in \Pi_{k,a} \\ 0 & \text{otherwise.} \end{cases}$$
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0 & \text{otherwise.}
\end{cases}
$$
Higher dimension: Urysohn function

Now we smooth the function $\psi$ in order to get

$$u_a(x_1, \ldots, x_n) = \frac{1}{\Delta^n} \int_{-\Delta/2}^{\Delta/2} \cdots \int_{-\Delta/2}^{\Delta/2} \psi_a(x_1 + y_1, \ldots, x_n + y_n) \, dy_1 \cdots dy_n,$$

where

$$\Delta := 2c_u |b|^{-k}$$
Higher dimension: Fourier transform

Finally we do Fourier transformation of the Urysohn function. Thus

\[ u_a(x_1, \ldots, x_n) = \sum_{(m_1, \ldots, m_n) \in \mathbb{Z}^n} c_{m_1, \ldots, m_n} e^{i(m_1 x_1 + \cdots + m_n x_n)} \]
Higher dimension: Estimation of $\Theta$

Now we get that

$$\left| \Theta - \sum_{z \in P(T)} t(\phi(p(z))) \right| \leq \sum_{i=1}^{h} F_{\ell_i},$$

where $t$ is defined by

$$t(\nu) = \sum_{M \in M} T_M e \left( \sum_{j=1}^{h} \mu_j B^{-\ell_j-1} \nu \right)$$

with $T_M = \prod_{j=1}^{h} c_{m_{j_1},\ldots,m_{j_n}}$. 
Higher dimension: The border

We collect all points near the border of the fundamental domain in the following set.

\[ F_\ell := \# \left\{ z \in P(T) \mid \phi \left( \frac{p(z)}{b^{\ell+1}} \right) \in \bigcup_{a \in \mathcal{N}} P_{k,a} \mod B^{-1}\mathbb{Z}^n \right\}. \]
Higher dimension: Erdős-Turan-Koksma

Lemma

Let $x_1, \ldots, x_L$ be points in the $n$-dimensional real vector space $\mathbb{R}^n$ and $H$ an arbitrary positive integer. Then the discrepancy $D_L(x_1, \ldots, x_L)$ fulfils the inequality

$$D_L(x_1, \ldots, x_L) \ll \frac{2}{H + 1} + \sum_{0 \leq \|h\|_\infty \leq H} \frac{1}{r(h)} \left| \frac{1}{L} \sum_{\ell=1}^{L} e(h \cdot x_\ell) \right|,$$

where $h \in \mathbb{Z}^n$ and $r(h) = \prod_{i=1}^{n} \max(1, |h_i|).$
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Weyl Sums

Since in most of the examples above we used polynomials we write

\[ p(x) = \alpha_d x^d + \cdots + \alpha_1 x + \alpha_0. \]

Then we are interested in the estimation of sums of the following kind.

\[ \sum_{n \leq N} e \left( \frac{\nu}{q^{\ell+1}} \sum_{k=0}^{d} \alpha_k x^k \right) \]
Higher Dimension: Weyl sums

In both cases, the Fourier-Transform of the Urysohn-function and the Erdős-Turan-Koksma inequality, we end up with sums of the form

$$\sum_{\ell \leq L} e(m_1x_1 + \cdots + m_nx_n).$$

A typical Weyl sum in this field looks like

$$\sum_{z \in P(T)} e(\text{Tr}(g(z))).$$
Higher Dimension: Weyl sums

Our sums are of the form

$$\sum_{z \in P(T)} e \left( h \cdot \phi \left( \frac{p(z)}{b^{\ell+1}} \right) \right).$$

We can rewrite these sums and get the following

$$\sum_{z \in P(T)} e \left( \text{Tr} \left( \frac{q(z)}{b^{\ell}} \right) \right).$$

In order to estimate these Weyl sums we consider them according to the size of $\ell$. This is the reason why we do not get an estimate for the whole range of $\ell$s.
Higher Dimension: Siegel’s Lemma

Lemma Siegel

Let \( N > D^{\frac{1}{n}} \). Then, corresponding to any \( \xi \in K \), there exist \( q \in \mathcal{O}_K \) and \( a \in \delta^{-1} \) such that

\[
\max_{1 \leq i \leq n} \left| q(i) \xi(i) - a(i) \right| < \frac{1}{N}, \quad 0 < \max_{1 \leq i \leq n} |q(i)| \leq N, \\
\max(N \left| q(i) \xi(i) - a(i) \right|, |q(i)|) \geq D^{-\frac{1}{2}}, \quad 1 \leq i \leq n,
\]

and

\[
N((q, a\delta)) \leq D^{\frac{1}{2}}.
\]
Higher Dimension: Weyl’s Lemma

Lemma

Let \( g(z) = \alpha_k z^k + \cdots + \alpha_1 z \) with

\[
\max_{1 \leq i \leq n} \left| \alpha_k^{(i)} q_k^{(i)} - a_k^{(i)} \right| \leq T^{-k} (\log T)^{\sigma_1}
\]

and

\[
(\log T)^{\sigma_1} \leq \max_{1 \leq i \leq n} \left| q_k^{(i)} \right| \leq T^k (\log T)^{-\sigma_1}.
\]

Then

\[
\sum_{z \in P(T)} e(Tr(g(z))) \ll T^n (\log T)^{-\sigma_0}
\]

with \( \sigma_1 \geq 2^{k-1} (\sigma_0 + r 2^{2k}) \).
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We will now focus on Delange’s problem:

\[
\sum_{n \leq N} f(aq^\ell) I_{\ell,d}(p(n)) \\
= f(aq^\ell) \left( \frac{N}{q} + \sum_{n \leq N} \psi \left( \frac{p(n)}{q^{\ell+1}} - \frac{a + 1}{q} \right) - \psi \left( \frac{p(n)}{q^{\ell+1}} - \frac{a}{q} \right) \right).
\]
Division of the expansion

Since in our case the leading coefficient looks like

\[ \nu q^{\ell+1} \alpha_i. \]

The Diophantine approximation leads us to a division of the expansion according to the position of the digit within the expansion.

- Least significant digits \((0 \leq \ell < C_i \log \log N)\).
- Middle digits \((C_i \log \log N \leq \ell < C_u \log \log N)\).
- Most significant digits \((C_u \log \log N \leq \ell < \log N)\).
Treating the middle digits

By an application of Weyl’s Lemma we get for $C_l \log \log N \leq \ell < C_u \log \log N$ that

$$\sum_{n \leq N} e \left( \frac{\nu}{q^{\ell+1} p(n)} \right) \ll \frac{N}{(\log N)^\sigma}.$$

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Treating the least significant digits

This treatment depends on the shape of the coefficients of the polynomials. Under the assumption, that at least one of them is of finite type we again get that

$$\sum_{\ell=0}^{C_l \log \log N} \left| \sum_{n \leq N} |I_{\ell,a}(p(n)) - \frac{1}{q}| \right| \ll \frac{N}{(\log N)^\sigma}.$$
Treating the most significant digits

Here we want to extract the periodic error part in the shape of Delange (periodic error term). Therefore we rewrite the sum as follows.

$$\sum_{N < n \leq 2N} \psi \left( \frac{p(n)}{q^{\ell+1}} - \frac{a}{q} \right)$$

$$= \frac{1}{d} \alpha_0 \frac{1}{d} q^{r+1} \int_{q^{-r-1}p(N)} q^{-r-1}p(2N) \psi \left( t - \frac{a}{q} \right) t^{\frac{1}{d}-1} dt + O \left( N^{1-\varepsilon} \right),$$

where $C_u \log \log N \leq \ell < \log N$. 

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Peter’s Result

**Theorem** Peter (2002)

There are $c \in \mathbb{R}$ and $\varepsilon > 0$ such that

$$\sum_{n \leq N} s_q(\lfloor \alpha n^k \rfloor) = \frac{q-1}{2} N \log_q(\alpha N^k) + cN$$

$$+ NF \left( \log_q(\alpha N^k) \right) + \mathcal{O}(N^{1-\varepsilon})$$

where $\lfloor x \rfloor$ is the greatest integer less than $x$, $F$ a 1-periodic function and $\alpha = 1$ or $\alpha > 0$ an irrational of finite type.
Pseudo polynomial

Let $\alpha_0, \beta_0, \ldots, \alpha_d, \beta_d \in \mathbb{R}$, $\alpha_0 > 0$ and $\beta_0 > \beta_1 > \cdots > \beta_d \geq 1$, where at least one $\beta_i \notin \mathbb{Z}$. Then we define a \textit{pseudo} polynomial $p$ as

$$p(x) := \alpha_0 x^{\beta_0} + \cdots + \alpha_d x^{\beta_d}.$$
Over a pseudo-polynomial sequence

**Theorem**   Nakai and Shiokawa (1990)

Let $p$ be a pseudo polynomial. Then

$$\sum_{n \leq N} s_q(\lfloor p(n) \rfloor) = \frac{q - 1}{2} N \log_q p(N) + O(N)$$

where $\log_q$ denotes the logarithm to base $q$. 
Arbitrary additive functions

**Theorem M (201?)**

Let \( q \in \mathbb{N} \setminus \{1\} \) and \( f \) be a strictly \( q \)-additive function with \( f(0) = 0 \). If \( p \) is a pseudo polynomial, then there exists \( \varepsilon > 0 \) such that

\[
\sum_{n \leq N} f \left( \lfloor p(n) \rfloor \right) = \mu_f N \log_q (p(N))
\]

\[
+ NF \left( \log_q (p(N)) \right) + \mathcal{O} \left( N^{1-\varepsilon} \right).
\]
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Length of expansion

We note that with one expansion for $\gamma$ you also have one for all the conjugates, by

$$\gamma^{(i)} = \sum_{k=0}^{h} a_k (b^{(i)})^k \quad (1 \leq i \leq n).$$

**Theorem** Kovacs and Pethő (1992)

Let $\ell(\gamma)$ be the length of the expansion of $\gamma$ to the base $b$. Then

$$\left| \ell(\gamma) - \max_{1 \leq i \leq n} \frac{\log |\gamma^{(i)}|}{\log |b^{(i)}|} \right| \leq C.$$
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Area of interest

We fix a \( T \) and set \( T_i \) for \( 1 \leq i \leq n \) such that

\[
\log T_i = \log T \frac{\log |b(i)|^n}{\log |N(b)|}.
\]

Furthermore we will write

\[
N(T) = N(T_1, \ldots, T_n) := \{ \lambda \in R : |\lambda^{(i)}| \leq T_i, 1 \leq i \leq n \}.
\]
Asymptotic distribution in $\mathbb{Z}[\beta]$

**Theorem M (2009)**

Assume that there exists an $\varepsilon > 0$ such that $D_b(x)/(\log x)^{\varepsilon} \to \infty$ as $x \to \infty$ and let $p$ be a polynomial of degree $d$. Then, as $T \to \infty$,

$$\frac{1}{\#N(T)} \# \left\{ z \in N(T) \left| \frac{f(\lfloor p(z) \rfloor) - M_b(T^d)}{D_b(T^d)} < y \right. \right\} \to \Phi(y),$$

where $\Phi$ is the normal distribution function.
Some remarks

- It should suffice that

\[ D_b(x) \to \infty \quad \text{for} \quad x \to \infty. \]

- One can shift the “decimal” dot.
Asymptotic distribution in function fields

Let \((B, \mathcal{N})\) be a number system in \(S := \mathbb{F}_q[X, Y]/p\mathbb{F}_q[X, Y]\) with \(d(B) = \frac{a}{b}\) and let \(f : S \to \mathbb{R}\) be a strict \(B\)-additive function. Set

\[
\mu_f := \frac{1}{\# \mathcal{N}} \sum_{D \in \mathcal{N}} f(A) \quad \text{and} \quad \sigma^2_f := \frac{1}{\# \mathcal{N}} \sum_{D \in \mathcal{N}} f(A)^2 - \mu_f^2.
\]
Asymptotic distribution in function fields

**Theorem**  M and Thuswaldner (2009)

Let \( h \in \mathbb{L}_\infty [\mathbb{Z}] \) be a polynomial of degree \( d \). Suppose that \( \sigma_f > 0 \) and \( S \) is the ring of integers of \( \mathbb{L} \), then for \( n \to \infty \)

\[
\# \left\{ A \in S(n) : \frac{f(h(A)) - \frac{ndb}{a} \mu_f}{\sqrt{\frac{ndb}{a} \sigma_f}} \leq x \right\} \to \Phi(x),
\]

where \( \Phi \) denotes the standard normal distribution function.
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**Theorem**  Nakai and Shiokawa (1992)

Let $f$ be a polynomial with real coefficients. Then $\theta((f(n))_{n \geq 1})$ is normal.

**Theorem**  M, Thuswaldner and Tichy (2007)

Let $f$ be an entire function of bounded logarithmic order. Then $\theta((f(n))_{n \geq 1})$ and $\theta((f(p))_{p \in \mathbb{P}})$ are normal.
Further projects

- Extend the existing results to the more general case of canonical number systems in algebraic number fields and function fields.
- Build some connections to ergodic, measure theoretic and/or analytic methods.
- Consider different number systems such as Cantor and Zeckendorf.