UNIFORM STABILIZATION IN WEIGHTED SOBOLEV SPACES FOR THE KDV EQUATIONPOSED ON THE HALF-LINE

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Abstract. Studied here is the large-time behavior of solutions of the Korteweg-de Vries equation posed on the right half-line under the effect of a localized damping. Assuming as in [19] that the damping is active on a set \((a_0, +\infty)\) with \(a_0 > 0\), we establish the exponential decay of the solutions in the weighted spaces \(L^2((x+1)^m dx)\) for \(m \in \mathbb{N}^*\) and \(L^2(e^{2bx} dx)\) for \(b > 0\) by a Lyapunov approach. The decay of the spatial derivatives of the solution is also derived.

1. Introduction. The Korteweg-de Vries (KdV) equation was first derived as a model for the propagation of small amplitude long water waves along a channel [8, 14, 15]. It has been intensively studied from various aspects for both mathematics and physics since the 1960s when solitons were discovered through solving the KdV equation, and the inverse scattering method, a so-called nonlinear Fourier transform, was invented to seek solitons [12, 21]. It is now well known that the KdV equation is not only a good model for water waves but also a very useful approximation model in nonlinear studies whenever one wishes to include and balance weak nonlinear and dispersive effects.

The initial boundary value problems (IBVP) arise naturally in modeling small-amplitude long waves in a channel with a wavemaker mounted at one end [1, 2, 3, 28]. Such mathematical formulations have received considerable attention in the past, and a satisfactory theory of global well-posedness is available for initial and boundary conditions satisfying physically relevant smoothness and consistency assumptions (see e.g. [1, 4, 6, 7, 9, 10, 11] and the references therein).

The analysis of the long-time behavior of IBVP on the quarter-plane for KdV has also received considerable attention over recent years, and a review of some of the results related to the issues we address here can be found in [5, 7, 18]. For stabilization and controllability issues on the half line, we refer the reader to [19] and [26, 27, 28], respectively.

In this work, we are concerned with the asymptotic behavior of the solutions of the IBVP for the KdV equation posed on the positive half line under the presence

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of a localized damping represented by the function $a$; that is,
\[
\begin{aligned}
    u_t + u_x + u_{xxx} + uu_x + a(x)u &= 0, \quad x, t \in \mathbb{R}^+,
    \\
    u(0, t) &= 0, \quad t > 0,
    \\
    u(x, 0) &= u_0(x), \quad x > 0.
\end{aligned}
\]

Assuming $a(x) \geq 0$ a.e. and that $u(\cdot, t) \in H^3(\mathbb{R}^+)$, it follows from a simple computation that
\[
\frac{dE}{dt} = -\int_0^\infty a(x)|u(x, t)|^2\,dx - \frac{1}{2}|u_x(0, t)|^2
\]
where
\[
E(t) = \frac{1}{2} \int_0^\infty |u(x, t)|^2\,dx
\]
is the total energy associated with (1). Then, we see that the term $a(x)u$ plays the role of a feedback damping mechanism and, consequently, it is natural to wonder whether the solutions of (1) tend to zero as $t \to \infty$ and under what rate they decay. When $a(x) > a_0 > 0$ almost everywhere in $\mathbb{R}^+$, it is very simple to prove that $E(t)$ converges to zero as $t$ tends to infinity. The problem of stabilization when the damping is effective only in a subset of the domain is much more subtle. The following result was obtained in [19].

**Theorem 1.1.** Assume that the function $a = a(x)$ satisfies the following property
\[
a \in L^\infty(\mathbb{R}^+), \quad a \geq 0 \text{ a.e. in } \mathbb{R}^+ \text{ and } a(x) \geq a_0 > 0 \text{ a.e. in } (x_0, +\infty)
\]
for some numbers $a_0, x_0 > 0$. Then for all $R > 0$ there exist two numbers $C > 0$ and $\nu > 0$ such that for all $u_0 \in L^2(\mathbb{R}^+)$ with $\|u_0\|_{L^2(\mathbb{R}^+)} \leq R$, the solution $u$ of (1) satisfies
\[
\|u(t)\|_{L^2(\mathbb{R}^+)} \leq Ce^{-\nu t}\|u_0\|_{L^2(\mathbb{R}^+)}.
\]

Actually, Theorem 1.1 was proved in [19] under the additional hypothesis that
\[
a(x) \geq a_0 \text{ a.e. in } (0, \delta)
\]
for some $\delta > 0$, but (6) may be dropped by replacing the unique continuation property [19, Lemma 2.4] by [29, Theorem 1.6]. The exponential decay of $E(t)$ is obtained following the methods in [22, 24, 25] which combine multiplier techniques and compactness arguments to reduce the problem to some unique continuation property for weak solutions of KdV. For the stabilization of the Korteweg-de Vries equation on the torus (i.e. with periodic boundary conditions), we refer the reader to [17, 30] and the references therein.

Along this work we assume that the real-valued function $a = a(x)$ satisfies the condition (4) for some given positive numbers $a_0, x_0$. In this paper we investigate the stability properties of (1) in the weighted spaces introduced by Kato in [13]. More precisely, for $b > 0$ and $m \in \mathbb{N}$, we prove that the solution $u$ exponentially decays to 0 in $L^2$ and $L^2_{(x+1)^m}$ (if $u(0)$ belongs to one of these spaces), where
\[
L^2 = \{u : \mathbb{R}^+ \to \mathbb{R} : \int_0^\infty |u(x)|^2e^{2bx}\,dx < \infty\},
\]
\[
L^2_{(x+1)^m} = \{u : \mathbb{R}^+ \to \mathbb{R} : \int_0^\infty |u(x)|^2(x+1)^m\,dx < \infty\}.
\]
The following weighted Sobolev spaces
\[ H^s_b = \{ u : \mathbb{R}^+ \to \mathbb{R}; \partial^i_x u \in L^2_b \text{ for } 0 \leq i \leq s; \ u(0) = 0 \text{ if } s \geq 1 \} \]
and
\[ H^s_{(x+1)^m dx} = \{ u : \mathbb{R}^+ \to \mathbb{R}; \partial^i_x u \in L^2_{(x+1)^m dx} \text{ for } 0 \leq i \leq s; \ u(0) = 0 \text{ if } s \geq 1 \}, \]
endowed with their usual inner products, will be used thereafter. Note that \( H^0_b = L^2_b \) and that \( H^0_{(x+1)^m dx} = L^2_{(x+1)^m dx} \).

The exponential decay in \( L^2_{(x+1)^m dx} \) is obtained by constructing a convenient Lyapunov function (which actually decreases strictly on the sequence of times \( \{kt\}_{k \geq 0} \)) by induction on \( m \). For \( u_0 \in L^2_{(x+1)^m dx} \), we also prove the following estimate
\[ \|u(t)\|_{H^s_{(x+1)^m dx}} \leq C \frac{e^{-\mu t}}{\sqrt{t}} \|u_0\|_{L^2_{(x+1)^m dx}} \quad (7) \]
in two situations: (i) \( m = 1 \) and \( \|u_0\|_{L^2_{(x+1)^m dx}} \) is arbitrarily large; (ii) \( m \geq 2 \) and \( \|u_0\|_{L^2_{(x+1)^m dx}} \) is small enough. In the situation (ii), we first establish a similar estimate for the linearized system and next apply the contraction mapping principle in a space of functions fulfilling the exponential decay. Note that (7) combines the (global) Kato smoothing effect to the exponential decay.

The exponential decay in \( L^2_b \) is established for any initial data \( u_0 \in L^2_b \) under the additional assumption that \( 4b^3 + b < a_0 \). Next, we can derive estimates of the form
\[ \|u(t)\|_{H^s_b} \leq C \frac{e^{-\mu t}}{\sqrt{b^{s/2}}} \|u_0\|_{L^2_b} \]
for any \( s \geq 1 \), revealing that \( u(t) \) decays exponentially to 0 in strong norms.

It would be interesting to see if such results are still true when the function \( a \) has a smaller support. It seems reasonable to conjecture that similar positive results can be derived when the support of \( a \) contains a set of the form \( \cup_{k \geq 1} [ka_0, ka_0 + b_0] \) where \( 0 < b_0 < a_0 \), while a negative result probably holds when the support of \( a \) is a finite interval, as the \( L^2 \) norm of a soliton-like initial data may not be sufficiently dissipated over time. Such issues will be discussed elsewhere.

The plan of this paper is as follows. Section 2 is devoted to global well-posedness results in the weighted spaces \( L^2_b \) and \( L^2_{(x+1)^2 dx} \). In section 3, we prove the exponential decay in \( L^2_{(x+1)^m dx} \) and \( L^2_b \), and establish the exponential decay of the derivatives as well.

2. Global well-posedness.

2.1. Global well-posedness in \( L^2_b \). Fix any \( b > 0 \). To begin with, we apply the classical semigroup theory to the linearized system
\[ \begin{cases}
    u_t + u_x + u_{xxx} + a(x)u = 0, & x, t \in \mathbb{R}^+, \\
    u(0, t) = 0, & t > 0, \\
    u(x, 0) = u_0(x), & x > 0.
\end{cases} \quad (8) \]

Let us consider the operator
\[ A : D(A) \subset L^2_b \to L^2_b \]
with domain
\[ D(A) = \{ u \in L^2_b; \partial^i_x u \in L^2_b \text{ for } 1 \leq i \leq 3 \text{ and } u(0) = 0 \} \]
defined by

\[ Au = -u_{xx} - u_x - a(x)u. \]

Then, the following result holds.

**Lemma 2.1.** The operator \( A \) defined above generates a continuous semigroup of operators \( (S(t))_{t \geq 0} \) in \( L_2^x \).

**Proof.** We first introduce the new variable \( v = e^{bx}u \) and consider the following (IBVP)

\[
\begin{cases}
  v_t + (\partial_x - b)v + (\partial_x - b)^3v + a(x)v = 0, & x, t \in \mathbb{R}^+, \\
  v(0, t) = 0, & t > 0, \\
  v(x, 0) = v_0(x), & x > 0.
\end{cases}
\]

(9)

Clearly, the operator \( B : D(B) \subset L^2(\mathbb{R}^+) \rightarrow L^2(\mathbb{R}^+) \) with domain

\[ D(B) = \{ u \in H^2(\mathbb{R}^+); \ u(0) = 0 \} \]

defined by

\[ Bv = -(\partial_x - b)v - (\partial_x - b)^3v - a(x)v \]

is densely defined and closed. So, we are done if we prove that for some real number \( \lambda \) the operator \( B - \lambda \) and its adjoint \( B^* - \lambda \) are both dissipative in \( L^2(\mathbb{R}^+) \). It is readily seen that \( B^* : D(B^*) \subset L^2(\mathbb{R}^+) \rightarrow L^2(\mathbb{R}^+) \) is given by \( B^*v = (\partial_x + b)v + (\partial_x + b)^3v - a(x)v \) with domain

\[ D(B^*) = \{ v \in H^3(\mathbb{R}^+); \ v(0) = v'(0) = 0 \}. \]

Pick any \( v \in D(B) \). After some integration by parts, we obtain that

\[ (Bv, v)_{L^2} = -\frac{1}{2}v_x^2(0) - 3b \int_0^\infty v_x^2 dx + (b + b^3) \int_0^\infty v^2 dx - \int_0^\infty a(x)v^2 dx, \]

that is,

\[ ([B - (b^3 + b)]v, v)_{L^2} \leq 0. \]

Analogously, we deduce that for any \( v \in D(B^*) \)

\[ (v, [B^* - (b^3 + b)]v)_{L^2} \leq 0 \]

which completes the proof. \( \square \)

The following linear estimates will be needed.

**Lemma 2.2.** Let \( u_0 \in L_2^x \) and \( u = S(\cdot)u_0 \). Then, for any \( T > 0 \)

\[
\begin{align*}
  \frac{1}{2} \int_0^\infty |u(x, T)|^2 dx \\
  -\frac{1}{2} \int_0^\infty |u_0(x)|^2 dx + \int_0^T \int_0^\infty a(x)|u|^2 dx dt + \frac{1}{2} \int_0^T u_x^2(0, t) dt = 0
\end{align*}
\]

(10)

\[
\begin{align*}
  \frac{1}{2} \int_0^\infty |u(x, T)|^2 e^{2bx} dx \\
  -\frac{1}{2} \int_0^\infty |u_0(x)|^2 e^{2bx} dx + \frac{1}{2} \int_0^\infty |u_0(x)|^2 e^{2bx} dx \\
  + 3b \int_0^T \int_0^\infty u_x^2 e^{2bx} dx dt - (4b^3 + b) \int_0^T \int_0^\infty u^2 e^{2bx} dx dt \\
  + \int_0^T \int_0^\infty a(x)|u|^2 e^{2bx} dx dt + \frac{1}{2} \int_0^T u_x^2(0, t) dt = 0.
\end{align*}
\]

(11)

As a consequence,

\[ ||u||_{L^\infty(0,T;L_2^x)} + ||u_x||_{L^2(0,T;L_2^x)} \leq C ||u_0||_{L_2^x}, \]

(12)
where $C = C(T)$ is a positive constant.

Proof. Pick any $u_0 \in D(A)$. Multiplying the equation in (1) by $u$ and integrating over $(0, +\infty) \times (0, T)$, we obtain (10). Then, the identity may be extended to any initial state $u_0 \in L^2_b$ by a density argument. To derive (11) we first multiply the equation by $(e^{2bx} - 1)u$ and integrate by parts over $(0, +\infty) \times (0, T)$ to deduce that

\[
\begin{align*}
\frac{1}{2} \int_0^\infty |u(x, T)|^2 (e^{2bx} - 1)dx - \frac{1}{2} \int_0^\infty |u_0(x)|^2 (e^{2bx} - 1)dx + \\
+3b \int_0^T \int_0^\infty u_x^2 e^{2bx} dxdt - (4b^3 + b) \int_0^T \int_0^\infty u^2 e^{2bx} dxdt + \\
+ \int_0^T \int_0^\infty a(x)|u|^2 (e^{2bx} - 1)dxdt = 0.
\end{align*}
\]

Adding the above equality and (10) hand to hand, we obtain (11) using the same density argument. Then, Gronwall inequality, (4) and (11) imply that

\[||u||_{L^\infty(0;T;L^2_b)} \leq C ||u_0||_{L^2_b},\]

with $C = C(T) > 0$. This estimate together with (11) gives us

\[||u_x||_{L^2(0;T;L^2_b)} \leq C ||u_0||_{L^2_b},\]

where $C = C(T)$ is a positive constant.

The global well-posedness result reads as follows:

**Theorem 2.3.** For any $u_0 \in L^2_b$ and any $T > 0$, there exists a unique solution $u \in C([0, T]; L^2_b) \cap L^2(0, T; H^1_b)$ of (1).

Proof. By computations similar to those performed in the proof of Lemma 2.2, we obtain that for any $f \in C^1([0, T]; L^2_b)$ and any $u_0 \in D(A)$, the solution $u$ of the system

\[
\begin{align*}
\left\{ 
\begin{array}{ll}
  u_t + u_x + u_{xxx} + a(x)u = f, & x \in \mathbb{R}^+, t \in (0, T), \\
  u(0, t) = 0, & t \in (0, T), \\
  u(x, 0) = u_0(x), & x \in \mathbb{R}^+, 
\end{array}
\right.
\end{align*}
\]

fulfills

\[\sup_{0 \leq t \leq T} ||u(t)||_{L^2_b} + \left( \int_0^T \int_0^\infty |u_x|^2 e^{2bx} dxdt \right)^{\frac{1}{2}} \leq C \left( ||u_0||_{L^2_b} + \int_0^T ||f||_{L^2_b} dt \right) \]

(13)

for some constant $C = C(T)$ nondecreasing in $T$. A density argument yields that $u \in C([0, T]; L^2_b)$ when $f \in L^1(0, T; L^2_b)$ and $u_0 \in L^2_b$.

Let $u_0 \in L^2_b$ be given. To prove the existence of a solution of (1) we introduce the map $\Gamma$ defined by

\[(\Gamma u)(t) = S(t)u_0 + \int_0^t S(t-s)N(u(s)) ds\]

where $N(u) = -uu_x$, and the space

\[F = C([0, T]; L^2_b) \cap L^2(0, T; H^1_b)\]

endowed with its natural norm. We shall prove that $\Gamma$ has a fixed-point in some ball $B_R(0)$ of $F$. We need the following
CLAIM 1. If \( u \in H^1_0 \) then
\[
\|u^2 e^{2bx}\|_{L^\infty(\mathbb{R}^+)} \leq (2 + 2b) \|u\|_{L^2_0} \|u\|_{H^1_0}.
\]

From Cauchy-Schwarz inequality, we get for any \( \bar{F} \in \mathbb{R}^+ \)
\[
u^2(\bar{F})e^{2bx} = \int_0^\infty \nu^2 e^{2bx} dx = \int_0^\infty [2uu_x e^{2bx} + 2bu^2 e^{2bx}] dx \
\leq 2\left( \int_0^\infty \nu^2 e^{2bx} dx \right)^{\frac{1}{2}} \left( \int_0^\infty u_x^2 e^{2bx} dx \right)^{\frac{1}{2}} + 2b \int_0^\infty \nu^2 e^{2bx} dx \
\leq (2 + 2b)\|u\|_{L^2_0} \|u\|_{H^1_0}
\]
which guarantees that Claim 1 holds.

CLAIM 2. There exists a constant \( K > 0 \) such that for \( 0 < T \leq 1 \)
\[
\|\Gamma(u) - \Gamma(v)\|_F \leq KT^{\frac{1}{4}}(\|u\|_F + \|v\|_F)\|u - v\|_F, \quad \forall u, v \in F.
\]
According to the previous analysis,
\[
\|\Gamma(u) - \Gamma(v)\|_F \leq C\|uu_x - vv_x\|_{L^1(0,T;L^2_D)}.
\]
So, applying triangular inequality and Hölder inequality, we have
\[
\|\Gamma(u) - \Gamma(v)\|_F \leq C\{\|u - v\|_{L^2(0,T;L^\infty(0,\infty))}\|u\|_{L^2(0,T;H^1_0)} + \|v\|_{L^2(0,T;L^\infty(0,\infty))}\|u - v\|_{L^2(0,T;H^1_0)}\}. \tag{14}
\]
Now, by Claim 1, we have
\[
\|u\|_{L^2(0,T;L^\infty(0,\infty))} \leq CT^{\frac{1}{4}}\|u\|_{L^2(0,T;L_0^2)}\|u\|_{H^1_0}. \tag{15}
\]
Then, combining (14) and (15), we deduce that
\[
\|\Gamma(u) - \Gamma(v)\|_F \leq CT^{\frac{1}{4}}\{\|u\|_F + \|v\|_F\}\|u - v\|_F. \tag{16}
\]
Let \( T > 0 \), \( R > 0 \) be numbers whose values will be specified later, and let \( u \in B_R(0) \subset F \) be given. Then, by Claim 2 and Lemma 2.2, \( \Gamma u \in F \) and
\[
\|\Gamma u\|_F \leq C (\|u_0\|_{L^2_D} + T^{\frac{1}{4}}\|u\|_F^2).
\]
Consequently, for \( R = 2C\|u_0\|_{L^2_D} \) and \( T > 0 \) small enough, \( \Gamma \) maps \( B_R(0) \) into itself. Moreover, we infer from (16) that this mapping contracts if \( T \) is small enough. Then, by the contraction mapping theorem, there exists a unique solution \( u \in B_R(0) \subset F \) to the problem (1) for \( T \) small enough.

In order to prove that this solution is global, we need some a priori estimates. So, we proceed as in the proof of Lemma 2.2 to obtain for the solution \( u \) of (1)
\[
\frac{1}{2} \int_0^\infty |u(x,T)|^2 dx \\
-\frac{1}{2} \int_0^\infty |u_0(x)|^2 dx + \int_0^T \int_0^\infty a(x)|u|^2 dx dt + \frac{1}{2} \int_0^T u_x^2(0,t) dt = 0 \tag{17}
\]
and
\[
\frac{1}{2} \int_0^\infty |u(x,T)|^2 e^{2bx} dx - \frac{1}{2} \int_0^\infty |u_0(x)|^2 e^{2bx} dx + \frac{1}{2} \int_0^T u_x^2(0,t) dt \\
+ 3b \int_0^T \int_0^\infty u_x^2 e^{2bx} dx dt - (4b^3 + b) \int_0^T \int_0^\infty u_x^2 e^{2bx} dx dt \\
+ \int_0^T \int_0^\infty a(x)|u|^2 e^{2bx} dx dt - \frac{2b}{3} \int_0^T \int_0^\infty u^3 e^{2bx} dx dt = 0. 
\] (18)

First, observe that
\[
|\int_0^\infty u^2 e^{2bx} dx| = - \frac{1}{b} \int_0^\infty uu_x e^{2bx} dx \leq \frac{1}{b}(\int_0^\infty u^2 e^{2bx} dx) \frac{1}{2}(\int_0^\infty u_x^2 e^{2bx} dx) \frac{1}{2},
\]
therefore,
\[
\int_0^\infty u^2 e^{2bx} dx \leq \frac{1}{b^2} \int_0^\infty u_x^2 e^{2bx} dx.
\]
Combined to Claim 1, this yields
\[
||u(x)e^{bx}||_{L^\infty(R^+)} \leq C||u_x||_{L_x^2}.
\]

On the other hand, it follows from (17) that
\[
||u(t)||_{L^2(R^+)} \leq ||u_0||_{L^2(R^+)},
\]

hence
\[
\int_0^T \int_0^\infty |u|^2 e^{2bx} dx dt \leq \int_0^T ||ue^{bx}||_{L^\infty(R^+)}(\int_0^\infty |u|^2 e^{2bx} dx) dt \\
\leq C \int_0^T ||u_x||_{L_x^2} ||u||_{L_x^2} ||u||_{L^2} dt \\
\leq \delta ||u_x||_{L_x^2} ||u||_{L_x^2} ||u||_{L^2} + C_0 ||u||_{L^2(0,T;L_x^2)}^2,
\]
where \( \delta > 0 \) is arbitrarily chosen and \( C = C(b, \delta, ||u_0||_{L^2(R^+)}) \) is a positive constant.

Combining this inequality (with \( \delta < 9/2 \)) to (18) results in
\[
||u(T)||_{L_x^2}^2 \leq ||u_0||_{L_x^2}^2 + C \int_0^T ||u||_{L_x^2}^2 dt
\]
where \( C = C(b, ||u_0||_{L^2(R^+)}) \) does not depend on \( T \). It follows from Gronwall lemma that
\[
||u(T)||_{L_x^2}^2 \leq ||u_0||_{L_x^2}^2 e^{CT}
\]
for all \( T > 0 \), which gives the global well-posedness.

\[ \square \]

2.2. Global well-posedness in \( L^2_{(x+1) dx} \).

**Definition 2.4.** For \( u_0 \in L^2_{(x+1) dx} \) and \( T > 0 \), we denote by a mild solution of (1) any function \( u \in C([0,T];L^2_{(x+1) dx}) \cap L^2(0,T;H^1_{(x+1) dx}) \) which solves (1), and such that for some \( b > 0 \) and some sequence \( \{u_n,0\} \subset L^2_0 \) we have
\[
u_n,0 \to u_0 \text{ strongly in } L^2_{(x+1) dx},
\]
\[
u_n \rightharpoonup u \text{ weakly } \ast \text{ in } L^\infty(0,T;L^2_{(x+1) dx}),
\]
\[
u_n \to u \text{ weakly in } L^2(0,T;H^1_{(x+1) dx}),
\]
u_n denoting the solution of (1) emanating from \( u_n,0 \) at \( t = 0 \).
\textbf{Theorem 2.5.} For any \( u_0 \in L^2_{(x+1)^2dx} \) and any \( T > 0 \), there exists a unique mild solution \( u \in C([0,T]; L^2_{(x+1)^2dx}) \cap L^2(0,T; H^1_{(x+1)^2dx}) \) of (1).

\textit{Proof.} We prove the existence and the uniqueness in two steps.

\textbf{Step 1. Existence}

Since the embedding \( L^2_0 \subset L^2_{(x+1)^2dx} \) is dense, for any given \( u_0 \in L^2_{(x+1)^2dx} \) we may construct a sequence \( \{u_{n,0}\} \subset L^2_0 \) such that \( u_{n,0} \to u_0 \) in \( L^2_{(x+1)^2dx} \) as \( n \to \infty \). For each \( n \), let \( u_n \) denote the solution of (1) emanating from \( u_{n,0} \) at \( t = 0 \), which is given by Theorem 2.3. Then \( u_n \in C([0,T]; L^2_0) \cap L^2(0,T; H^1_0) \) and it solves

\[
\begin{align*}
  u_{n,t} + u_{n,x} + u_{n,xxx} + u_n u_{n,x} + a(x)u_n &= 0, \quad (19) \\
  u_n(0, t) &= 0, \quad (20) \\
  u_n(x, 0) &= u_{n,0}(x). \quad (21)
\end{align*}
\]

Multiplying (19) by \((x + 1)^2u_n\) and integrating by parts, we obtain

\[
\begin{align*}
  \frac{1}{2} \int_0^\infty (x + 1)^2 |u_{n,0}(x)|^2 dx & = \frac{1}{2} \int_0^\infty (x + 1)^2 |u_n(x, T)|^2 dx + 3 \int_0^T \int_0^\infty (x + 1)|u_{n,x}|^2 dx dt \\
  & + \frac{1}{2} \int_0^\infty |u_n(x, 0)|^2 dx - \int_0^T \int_0^\infty (x + 1)|u_n|^2 dx dt \\
  & - \frac{2}{3} \int_0^T \int_0^\infty (x + 1)u_n^3 dx dt + \int_0^T \int_0^\infty (x + 1)^2 u_n^2 a(x) dx.
\end{align*}
\]

Scaling in (19) by \( u_n \) gives

\[
\begin{align*}
  \frac{1}{2} \int_0^\infty |u_{n,0}(x)|^2 dx & = \frac{1}{2} \int_0^\infty |u_n(x, T)|^2 dx + \frac{1}{2} \int_0^T |u_{n,x}(0, t)|^2 dt + \int_0^T \int_0^\infty a(x)|u_n(x, t)|^2 dx dt,
\end{align*}
\]

hence

\[
\|u_n\|_{L^2(\mathbb{R}^+)} \leq \|u_{n,0}\|_{L^2(\mathbb{R}^+)} \leq C \quad (23)
\]

where \( C = C(||u_0||_{L^2(\mathbb{R}^+)}) \). It follows that

\[
\begin{align*}
  \frac{2}{3} \int_0^\infty (x + 1)|u_n|^3 dx & \leq \frac{2\sqrt{2}}{3} \|u_{n,x}\|_{L^2(\mathbb{R}^+)}\|u_n\|_{L^2(\mathbb{R}^+)}\|((x + 1)u_n)_{L^2(\mathbb{R}^+)}\| (x + 1)u_n ||_{L^2(\mathbb{R}^+)} \\
  & \leq \int_0^\infty (x + 1)|u_{n,x}|^2 dx + C \int_0^\infty (x + 1)^2 |u_n|^2 dx
\end{align*}
\]

which, combined to (22), gives

\[
\begin{align*}
  \frac{1}{2} \int_0^\infty (x + 1)^2 |u_n(x, T)|^2 dx + 2 \int_0^T \int_0^\infty (x + 1)|u_{n,x}|^2 dx dt + \frac{1}{2} \int_0^T |u_{n,x}(0, t)|^2 dt & \leq \frac{1}{2} \int_0^\infty (x + 1)^2 |u_{n,0}(x)|^2 dx + C \int_0^T \int_0^\infty (x + 1)^2 |u_n(x, t)|^2 dx dt.
\end{align*}
\]
An application of Gronwall’s lemma yields
\[
\|u_n\|_{L^\infty(0,T;L^2(x+1)2dx)} \leq C(T, \|u_{n,0}\|_{L^2(x+1)2dx}),
\]
\[
\|u_{n,x}\|_{L^2(0,T;H^1(x+1)2dx)} \leq C(T, \|u_{n,0}\|_{L^2(x+1)2dx}),
\]
\[
\|u_{n,x}(0,\cdot)\|_{L^2(0,T)} \leq C(T, \|u_{n,0}\|_{L^2(x+1)2dx}).
\]

Therefore, there exists a subsequence of \(\{u_n\}\), still denoted by \(\{u_n\}\), such that
\[
\begin{align*}
&u_n \to u \text{ weakly * in } L^\infty(0,T;L^2(x+1)2dx), \\
&u_n \to u \text{ weakly in } L^2(0,T;H^1(x+1)2dx), \\
&u_{n,x}(0,\cdot) \to u_{x}(0,\cdot) \text{ weakly in } L^2(0,T).
\end{align*}
\]

Note that, for all \(L > 0\), \(\{u_n\}\) is bounded in \(L^2(0,T;H^1(0,L)) \cap H^1(0,T;H^{-2}(0,L))\), hence by Aubin’s lemma, we have (after extracting a subsequence if needed)
\[
u_n \to u \text{ strongly in } L^2(0,T;L^2(0,L)) \text{ for all } L > 0.
\]

This gives that \(u_n u_{n,x} \to u_{x} u\) in the sense of distributions, hence the limit \(u \in L^\infty(0,T;L^2(x+1)2dx)) \cap L^2(0,T;H^1(x+1)2dx)\) is a solution of (1). Let us check that \(u \in C([0,T];L^2(x+1)2dx)\). Since \(u \in C([0,T];H^2(R^+)) \cap L^\infty(0,T;L^2(x+1)2dx)\), we have that \(u \in C_w([0,T]; L^2(x+1)2dx)\) (see e.g. [20]), where \(C_w([0,T]; L^2(x+1)2dx)\) denotes the space of sequentially weakly continuous functions from \([0,T]\) into \(L^2(x+1)2dx\).

We claim that \(u \in L^3(0,T;L^3(R^+))\). Indeed, from Moser estimate (see [31])
\[
\|u\|_{L^\infty(R^+)} \leq \sqrt{2}\|u_x\|_{L^2(R^+)}^{\frac{1}{2}}\|u\|_{L^2(R^+)}^{\frac{1}{2}} \tag{25}
\]
and Young inequality we get
\[
\int_0^\infty |u|^3dx \leq \|u\|_{L^\infty} \|u\|_{L^2}^2 \leq \sqrt{2}\|u_x\|_{L^2}^{\frac{1}{2}}\|u\|_{L^2}^{\frac{3}{2}} \leq \varepsilon \|u_x\|_{L^2}^2 + c_\varepsilon \|u\|_{L^2}^3 \tag{26}
\]
where \(\varepsilon > 0\) is arbitrarily chosen and \(c_\varepsilon\) denotes some positive constant. Since \(u \in C_w([0,T];L^2(x+1)2dx) \cap L^2(0,T;H^1(x+1)2dx)\), it follows that \(u \in L^3(0,T;L^3(R^+))\). On the other hand, \(u(0,t) = 0\) for \(t \in (0,T)\) and \(u_x(0,\cdot) \in L^2(0,T)\). Scaling in (1) by \((x+1)^2u\) yields for all \(t_1, t_2 \in (0,T)\)
\[
\begin{align*}
\frac{1}{2} \int_0^{t_2} (x+1)^2|u(x,t_2)|^2dx - \frac{1}{2} \int_0^{t_2} (x+1)|u(x,t_1)|^2dx \\
-\frac{1}{2} \int_0^{t_2} (x+1)|u_x|^2dx - \frac{1}{2} \int_0^{t_2} |u_x(0,t)|^2dt \\
+ \int_{t_1}^{t_2} \int_0^{\infty} (x+1)|u|^2dxdt + \frac{2}{3} \int_{t_1}^{t_2} \int_0^{\infty} (x+1)u^3dxdt \\
- \int_{t_1}^{t_2} \int_0^{\infty} (x+1)^2a(x)|u|^2dxdt.
\end{align*}
\]

Therefore \(\lim_{t_1 \to t_2} \left| \|u(t_2)\|_{L^2(x+1)2dx}^2 - \|u(t_1)\|_{L^2(x+1)2dx}^2 \right| = 0\). Combined to the fact that \(u \in C_w([0,T];L^2(x+1)2dx)\), this yields \(u \in C([0,T];L^2(x+1)2dx)\).

**STEP 2. UNIQUENESS**

Here, \(C\) will denote a universal constant which may vary from line to line. Pick
$u_0 \in L^2_{(x+1)^2}dx$, and let $u, v \in C([0, T]; L^2_{(x+1)^2} \cap L^2(0, T; H^1_{(x+1)^2}dx)$ be two mild solutions of (1). Pick two sequences $\{u_{n,0}\}, \{v_{n,0}\}$ in $L^2_b$ for some $b > 0$ such that
\begin{align*}
    u_{n,0} & \to u_0 \text{ strongly in } L^2_{(x+1)^2}dx, \\
    u_n & \to u \text{ weakly * in } L^\infty(0, T; L^2_{(x+1)^2}dx), \\
    u_n & \to u \text{ weakly in } L^2(0, T; H^1_{(x+1)^2}dx)
\end{align*}
and also
\begin{align*}
    v_{n,0} & \to u_0 \text{ strongly in } L^2_{(x+1)^2}dx, \\
    v_n & \to v \text{ weakly * in } L^\infty(0, T; L^2_{(x+1)^2}dx), \\
    v_n & \to v \text{ weakly in } L^2(0, T; H^1_{(x+1)^2}dx).
\end{align*}

We shall prove that $w = u - v$ vanishes on $\mathbb{R}^+ \times [0, T]$ by providing some estimate for $w_n = u_n - v_n$. Note first that $w_n$ solves the system
\begin{align*}
    w_n, t + w_n, x + w_n, xx + aw_n = f_n = v_n v_n, x - u_n u_n, x, \\
    w_n(0, t) = 0, \\
    w_n(x, 0) = w_{n,0}(x) = u_{n,0}(x) - v_{n,0}(x).
\end{align*}

Scaling in (34) by $(x+1)w_n$ yields
\begin{align*}
    &\frac{1}{2} \int_0^\infty (x+1)|w_n(x, t)|^2dx + \frac{3}{2} \int_0^t \int_0^\infty |w_n, x|^2dxdt - \frac{1}{2} \int_0^t \int_0^\infty |w_n|^2dxdt \\
    &\leq \frac{1}{2} \int_0^\infty (x+1)|w_n, 0|^2dx + \int_0^t \int_0^\infty (x+1)|w_n|^2dx + \int_0^t \int_0^\infty (x+1)|f_n|^2dx d\tau \\
    &\leq \frac{1}{2} \int_0^\infty (x+1)|w_n, 0|^2dx + \frac{1}{4} \sup_{0 < \tau < t} \int_0^\infty (x+1)|w_n(x, \tau)|^2dx + \\
    &\quad + \left[ \int_0^T \int_0^\infty (x+1)|f_n|^2dx d\tau \right]^2.
\end{align*}

Since $||w_n(t)||_{L^2(\mathbb{R}^+)} \leq ||w_n(t)||_{L^2_{(x+1)^2}dx}$, this yields for $T < 1/10$
\begin{align*}
    &\sup_{0 < t < T} \int_0^\infty (x+1)|w_n(x, t)|^2dx + \int_0^T \int_0^\infty |w_n, x|^2dxdt \\
    &\leq C \left[ \int_0^\infty (x+1)|w_n, 0(x)|^2dx + \left( \int_0^T \int_0^\infty (x+1)|f_n|^2dx d\tau \right)^2 \right].
\end{align*}

It remains to estimate $\int_0^T \int_0^\infty (x+1)|f_n|^2dx d\tau$. We split $f_n$ into
\begin{align*}
    f_n = (v_n - u_n)v_n, x + u_n(v_n, x - u_n, x) = f_n^1 + f_n^2.
\end{align*}

We have that
\begin{align*}
    \int_0^T \left( \int_0^\infty (x+1)|f_n|^2dx \right)^{1/2}d\tau &= \int_0^T \left( \int_0^\infty (x+1)|w_n|^2|v_n, x|^2dx \right)^{1/2}d\tau \\
    &\leq \int_0^T ||w_n||_{L^\infty(\mathbb{R}^+)} \left( \int_0^\infty (x+1)|v_n, x|^2dx \right)^{1/2}d\tau \\
    &\leq \left( \int_0^T ||w_n||_{L^\infty(\mathbb{R}^+)}^2dt \right)^{1/2} \left( \int_0^\infty (x+1)|v_n, x|^2dxdt \right)^{1/2}.
\end{align*}
By Sobolev embedding, we have that
\[
\left( \int_0^T \| w_n \|_{L^\infty(\mathbb{R}^+)}^2 \right)^{\frac{1}{2}} \leq \left( \int_0^T \| w_n \|_{H^1(\mathbb{R}^+)}^2 \right)^{\frac{1}{2}} \leq \sqrt{T} \sup_{0 < t < T} \| w_n \|_{L^2(\mathbb{R}^+)} + \| w_{n,x} \|_{L^2(0,T;L^2(\mathbb{R}^+))}.
\]

Thus
\[
\int_0^T \left( \int_0^\infty (x + 1)|f_n|^2 dx \right)^{\frac{1}{2}} dt \leq \| v_{n,x} \|_{L^2(0,T,L^2(x+1)dx)} \left( \sqrt{T} \sup_{0 < t < T} \| w_n \|_{L^2(\mathbb{R}^+)} + \| w_{n,x} \|_{L^2(0,T;L^2(\mathbb{R}^+))} \right). \tag{38}
\]

On the other hand, we have that
\[
\int_0^T \left( \int_0^\infty (x + 1)|f_n|^2 dx \right)^{\frac{1}{2}} dt = \int_0^T \left( \int_0^\infty (x + 1)|u_n|^2 |w_{n,x}|^2 dx \right)^{\frac{1}{2}} dt
\leq \int_0^T \|(x + 1)^{\frac{1}{2}} u_n\|_{L^\infty(\mathbb{R}^+)} \| w_{n,x} \|_{L^2(\mathbb{R}^+)} dt
\leq C \int_0^T \left( \|(x + 1)^{\frac{1}{2}} u_n\|_{L^2(\mathbb{R}^+)} + \|(x + 1)^{\frac{1}{2}} u_{n,x}\|_{L^2(\mathbb{R}^+)} \right) \| w_{n,x} \|_{L^2(\mathbb{R}^+)} dt \tag{39}
\leq C \left( \sqrt{T} \|(x + 1)u_n\|_{L^\infty(0,T;L^2(\mathbb{R}^+))} \right)
+ \| (x + 1)^{\frac{1}{2}} u_{n,x} \|_{L^2(0,T,L^2(\mathbb{R}^+))} \right) \| w_{n,x} \|_{L^2(0,T;L^2(\mathbb{R}^+))}.
\]

Gathering together (37), (38) and (39), we conclude that for \( T < 1/10 \)
\[
h_n(T) \leq K_n(T) h_n(T) + C \| w_{n,0} \|_{L^2_{L^2(x+1)dx}}^2
\]
where
\[
h_n(t) := \sup_{0 < \tau < T} \int_0^\infty (x + 1)|w_n(x, \tau)|^2 dx + \int_0^T \int_0^\infty |w_{n,x}|^2 dx dt \tag{40}
\]
\[
K_n(T) \leq C \left( \int_0^T \int_0^\infty |v_{n,x}|^2 dx dt + T \| (x + 1)u_n \|_{L^\infty(0,T;L^2(\mathbb{R}^+))} \right) + \int_0^T \int_0^\infty (x + 1)^{\frac{1}{2}} u_{n,x}^2 dx dt \tag{41}
\]
and \( C \) denotes a universal constant. The following claim is needed.

**Claim 3.**
\[
\lim_{T \to 0} \limsup_{n \to \infty} \int_0^T \int_0^\infty (x + 1)^{\frac{1}{2}} |u_{n,x}|^2 dx dt = 0, \quad \lim_{T \to 0} \limsup_{n \to \infty} \int_0^T \int_0^\infty (x + 1)^{\frac{1}{2}} |v_{n,x}|^2 dx dt = 0.
\]

Clearly, it is sufficient to prove the claim for the sequence \( \{ u_n \} \) only. From (27) applied with \( u = u_n \) on \([0,T]\), we obtain
\[
\frac{1}{2} \int_0^\infty (x + 1)^2 |u_n(x,T)|^2 dx + \frac{3}{2} \int_0^\infty (x + 1)|u_{n,x}|^2 dx dt \\
\leq \frac{1}{2} \int_0^\infty (x + 1)^2 |u_{n,0}|^2 dx + \int_0^T \int_0^\infty (x + 1)|u_{n,0}|^2 dx dt \\
+ \frac{2}{3} \int_0^T \int_0^\infty (x + 1)^3 |u_n|^3 dx dt.
\]
Combined to (23)-(24), this gives
\[ ||u_n(T)||_{L^2_{(x+1)^2dx}}^2 + \int_0^T \int_0^\infty (x+1)|u_{n,x}|^2 dx dt \leq ||u_{n,0}||_{L^2_{(x+1)^2dx}}^2 + C \int_0^T ||u_n||_{L^2_{(x+1)^2dx}}^2 dt. \] (42)

It follows from Gronwall lemma that
\[ ||u_n(t)||_{L^2_{(x+1)^2dx}}^2 \leq ||u_{n,0}||_{L^2_{(x+1)^2dx}}^2 e^{Ct} \] (43)

Using (43) in (42) and taking the limit sup as \( n \to \infty \) gives for a.e. \( T \)
\[ ||u(T)||_{L^2_{(x+1)^2dx}}^2 + \limsup_{n \to \infty} \int_0^T \int_0^\infty |u_{n,x}|^2 dx dt \leq e^{Ct} ||u_0||_{L^2_{(x+1)^2dx}}^2. \]

As \( u \) is continuous from \( \mathbb{R}^+ \) to \( L^2_{(x+1)^2dx} \), we infer that
\[ \lim_{T \to 0} \limsup_{n \to \infty} \int_0^T \int_0^{\infty} |u_{n,x}|^2 dx dt = 0. \]

The claim is proved. Therefore, we have that for \( T > 0 \) small enough and \( n \) large enough, \( K_n(T) < \frac{1}{T} \), and hence
\[ h_n(T) \leq 2C ||w_n(0)||_{L^2_{(x+1)^2dx}}^2. \]

This yields
\[ ||u - v||_{L^2_{(0,T;L^2_{(x+1)^2dx})}}^2 = \liminf_{n \to \infty} h_n(T) \leq 2C \liminf_{n \to \infty} ||w_n(0)||_{L^2_{(x+1)^2dx}}^2 = 0 \]
and \( u = v \) for \( 0 < t < T \). This proves the uniqueness for \( T \) small enough. The general case follows by a classical argument.

**Remark 1.**
1. If we assume only that \( u_0 \in L^2_{(x+1)^2dx} \), then a proof similar to Step 1 gives the existence of a mild solution \( u \in C([0,T]; L^2_{(x+1)^2dx}) \cap L^2(0,T; H^1_{(x+1)^2dx}) \) of (1). The uniqueness of such a solution is open. The existence and uniqueness of a solution issuing from \( u_0 \in L^2_{(x+1)^2dx} \) in a class of functions involving a Bourgain norm has been given in [11].
2. If \( u_0 \in L^2_{(x+1)^m dx} \) with \( m \geq 3 \), then \( u \in C([0,T]; L^2_{(x+1)^m dx}) \cap L^2(0,T; H^1_{(x+1)^m dx}) \) for all \( T > 0 \) (see below Theorem 3.1).

3. **Asymptotic behavior.**

3.1. **Decay in \( L^2_{(x+1)^m dx} \).**

**Theorem 3.1.** Assume that the function \( a = a(x) \) satisfies (4). Then, for all \( R > 0 \) and \( m \geq 1 \), there exist numbers \( C > 0 \) and \( \nu > 0 \) such that
\[ ||u(t)||_{L^2_{(x+1)^m dx}} \leq C e^{-\nu t} ||u_0||_{L^2_{(x+1)^m dx}} \]
for any solution given by Theorem 2.5, whenever \( ||u_0||_{L^2_{(x+1)^m dx}} \leq R \).

**Proof.** The proof will be done by induction in \( m \). We set
\[ V_0(u) = E(u) = \frac{1}{2} \int_0^\infty u^2 dx \]
and define the Lyapunov function \( V_m \) for \( m \geq 1 \) in an inductive way
\[ V_m(u) = \frac{1}{2} \int_0^\infty (x + 1)^m u^2 dx + d_{m-1}V_{m-1}(u), \]
\[ d_{m-1} = \frac{\nu}{m + 1} \]
and
\[ K_m(T) = \sup_{t \in [0,T]} \frac{d_{m-1} ||w_m(t)||_{L^2_{(x+1)^m dx}}^2}{||w_{m-1}(t)||_{L^2_{(x+1)^{m-1} dx}}^2}. \]

Using (43) in (42) for \( m \geq 1 \), we have
\[ \sup_{t \in [0,T]} \frac{d_{m-1} ||w_m(t)||_{L^2_{(x+1)^m dx}}^2}{||w_{m-1}(t)||_{L^2_{(x+1)^{m-1} dx}}^2} \leq e^{Ct} \] (44)

and define the Lyapunov function \( V_m \) for \( m \geq 1 \) in an inductive way
\[ V_m(u) = \frac{1}{2} \int_0^\infty (x + 1)^m u^2 dx + d_{m-1}V_{m-1}(u), \] (45)
where $d_{m-1} > 0$ is chosen sufficiently large (see below).

Suppose first that $m = 1$ and put $V = V_1$. Multiplying the first equation in (1) by $u$ and integrating by parts over $\mathbb{R}^+ \times (0, T)$, we obtain

\[
\frac{1}{2} \int_0^\infty |u(x, T)|^2 \, dx = \frac{1}{2} \int_0^\infty |u_0(x)|^2 \, dx - \int_0^T \int_0^\infty a(x)|u|^2 \, dx \, dt - \frac{1}{2} \int_0^T u_x^2(0, t) \, dt.
\]  

(46)

Now, multiplying the equation by $xu$, we deduce that

\[
\frac{1}{2} \int_0^\infty x|u(x, T)|^2 \, dx - \frac{1}{2} \int_0^\infty x|u_0(x)|^2 \, dx + \frac{3}{2} \int_0^T \int_0^\infty u_x^2 \, dx \, dt
\]

\[-\frac{1}{2} \int_0^T \int_0^\infty u^2 \, dx \, dt - \frac{1}{3} \int_0^T \int_0^\infty u^3 \, dx \, dt + \int_0^T \int_0^\infty xa(x)|u|^2 \, dx \, dt = 0.
\]  

(47)

Combining (46) and (47) it follows that

\[
V(u) - V(u_0) + (d_0 + 1) \left( \frac{1}{2} \int_0^T u_x^2(0, t) \, dt + \int_0^T \int_0^\infty a(x)|u|^2 \, dx \, dt \right)
\]

\[+ \frac{3}{2} \int_0^T \int_0^\infty u_x^2 \, dx \, dt - \frac{1}{2} \int_0^T \int_0^\infty u^2 \, dx \, dt - \frac{1}{3} \int_0^T \int_0^\infty u^3 \, dx \, dt
\]

\[+ \int_0^T \int_0^\infty xa(x)|u|^2 \, dx \, dt = 0.
\]  

(48)

The next step is devoted to estimate the nonlinear term in the left hand side of (48). To do that, we first assume that $||u_0||_{L^2} \leq 1$.

By (26) we have that

\[
\int_0^\infty |u|^3 \, dx \leq \varepsilon ||u_x||^2_{L^2} + c_\varepsilon ||u||^\frac{10}{3}_{L^2}
\]

for any $\varepsilon > 0$ and some constant $c_\varepsilon > 0$. Thus, if $||u_0||_{L^2} \leq 1$, we have $||u||^\frac{10}{3}_{L^2} \leq ||u||^2_{L^2}$ and

\[
\int_0^T \int_0^\infty |u|^3 \, dx \, dt \leq \varepsilon \int_0^T \int_0^\infty u_x^2 \, dx \, dt + c_\varepsilon \int_0^T \int_0^\infty u^2 \, dx \, dt.
\]  

(49)

Moreover, according to [19], there exists $c_1 > 0$, satisfying

\[
\int_0^T \int_0^\infty u^2 \, dx \, dt \leq c_1 \left( \frac{1}{2} \int_0^T u_x^2(0, t) \, dt + \int_0^T \int_0^\infty a(x)u^2 \, dx \, dt \right).
\]  

(50)

Now, combining (48)-(50) and taking $\varepsilon < \frac{1}{2}$ and $d_0 := 2c_1 \left( \frac{1}{2} + \frac{\varepsilon}{3} \right)$ we obtain

\[
V(u(T)) - V(u_0) + \frac{d_0}{2} + 1 \left( \frac{1}{2} \int_0^T u_x^2(0, t) \, dt + \int_0^T \int_0^\infty a(x)|u|^2 \, dx \, dt \right)
\]

\[+ \left( \frac{3}{2} - \frac{5}{3} \right) \int_0^T \int_0^\infty u_x^2 \, dx \, dt + \int_0^T \int_0^\infty xa(x)|u|^2 \, dx \, dt \leq 0
\]  

(51)

or

\[
V(u(T)) - V(u_0) \leq -\tilde{\varepsilon} \left\{ \int_0^T u_x^2(0, t) \, dt + \int_0^T \int_0^\infty (x + 1)a(x)|u|^2 \, dx \, dt + \int_0^T \int_0^\infty u_x^2 \, dx \, dt \right\}
\]  

(52)
where \( \bar{c} > 0 \). We aim to prove the existence of a constant \( c > 0 \) satisfying
\[
V(u(T)) - V(u_0) \leq -c V(u_0)
\]
(53)
Indeed, such an inequality gives at once the decay \( V(u(t)) \leq ce^{-\nu t}V(u_0) \). To this end, we need to establish two claims.

**Claim 4.** There exists \( c > 0 \) such that
\[
\int_0^T V(u) dt \leq c \left\{ \int_0^T u_x^2(0,t) dt + \int_0^T \int_0^\infty (x+1)a(x)u^2 dx dt \right\}
\]
Since \( u_0 \in L^2_{(x+1)dx} \subset L^2 \), from (4) and (50) we get
\[
\int_0^T V(u) dt = \frac{1}{2} \int_0^T \int_0^\infty (x+1)u^2 dx dt + \frac{d_0}{2} \int_0^T \int_0^\infty u^2 dx dt
\]
\[
\leq \frac{c_1 d_0}{2} \left( \frac{1}{2} \int_0^T u_x^2(0,t) dt + \int_0^T \int_0^\infty u(x)u^2 dx dt \right)
\]
\[
+ \frac{1}{2} \int_0^T \int_0^{x_0} (x+1)u^2 dx dt + \frac{1}{2} \int_0^T \int_{x_0}^\infty (x+1)u^2 dx dt
\]
\[
\leq \frac{c_1 d_0}{2} \left( \frac{1}{2} \int_0^T u_x^2(0,t) dt + \int_0^T \int_0^\infty u(x)u^2 dx dt \right)
\]
\[
+ \frac{1}{2} (x_0 + 1) \int_0^T \int_0^{x_0} u^2 dx dt + \frac{1}{2} \int_0^T \int_{x_0}^\infty (x+1) \frac{a(x)}{a_0} u^2 dx dt
\]
\[
\leq c \left\{ \int_0^T u_x^2(0,t) dt + \int_0^T \int_0^\infty (x+1)a(x)u^2 dx dt \right\}
\]

**Claim 5.**
\[
V(u_0) \leq C \left( \int_0^T u_x^2(0,t) dt + \int_0^T \int_0^\infty (x+1)a(x)u^2 dx dt + \int_0^T \int_0^\infty u_x^2 dx dt \right)
\]
(54)
where \( C > 0 \).

Multiplying the first equation in (1) by \((T-t)u\) and integrating by parts in \((0,\infty) \times (0, T)\), we obtain
\[
\frac{T}{2} \int_0^\infty |u_0(x)|^2 dx = \frac{1}{2} \int_0^T \int_0^\infty |u|^2 dx dt
\]
\[
+ \int_0^T \int_0^\infty (T-t)a(x)|u|^2 dx dt + \frac{1}{2} \int_0^{T} (T-t)u_x^2(0,t) dt,
\]
and therefore, using (50)
\[
\int_0^\infty |u_0(x)|^2 dx \leq C \left( \int_0^T \int_0^\infty a(x)|u|^2 dx dt + \int_0^T u_x^2(0,t) dt \right).
\]
(56)
Now, multiplying by \((T-t)xu\), it follows that
\[
-\frac{T}{2} \int_0^\infty u^2 dx + \frac{1}{2} \int_0^T \int_0^\infty x|u|^2 dx dt + \frac{3}{2} \int_0^T \int_0^\infty (T-t)u_x^2 dx dt
\]
\[
- \frac{1}{2} \int_0^T \int_0^{x_0} (T-t)a(x)|u|^2 dx dt + \int_0^T \int_0^\infty (T-t)xa(x)|u|^2 dx dt
\]
\[
- \frac{1}{3} \int_0^T \int_0^\infty (T-t)u^3 dx dt = 0.
\]
The identity above and (49) allow us to conclude that
\[
\int_0^\infty x |u_0(x)|^2 \, dx \\
\leq C \left\{ \int_0^T \int_0^\infty (x+1)^{m-1} u_x^2 \, dx \, dt + \int_0^T \int_0^\infty |u|^2 \, dx \, dt \\
+ \int_0^T \int_0^\infty x a(x) u_x^2 \, dx \, dt + \int_0^T \int_0^\infty |u|^3 \, dx \, dt \right\} \\
\leq C \left\{ \int_0^T V(u(t)) \, dt + \int_0^T \int_0^\infty x a(x) u^2 \, dx \, dt + \int_0^T \int_0^\infty u_x^2 \, dx \, dt \right\} \\
\leq C \left\{ \int_0^T V(u(T)) \, dt + \int_0^T \int_0^\infty x a(x) u^2 \, dx \, dt + \int_0^T \int_0^\infty u_x^2 \, dx \, dt \right\} \\
= C \left\{ \int_0^T V(u(t)) \, dt + \int_0^T \int_0^\infty x a(x) u^2 \, dx \, dt + \int_0^T \int_0^\infty u_x^2 \, dx \, dt \right\}
\]
for some $C > 0$. Claim 5 follows from Claim 4 and (56)-(57).

The previous computations give us (53) (and the exponential decay) when $\|u_0\|_{L^2} \leq 1$. The general case is proved as follows. Let $u_0 \in L^2_{(x+1)dx} \subseteq L^2$ be such that $\|u_0\|_{L^2} \leq R$. Since $u \in C(\mathbb{R}^+; L^2(\mathbb{R}^+))$ and $\|u(t)\|_{L^2} \leq \alpha e^{-\beta t} \|u_0\|_{L^2}$, where $\alpha = \alpha(R)$ and $\beta = \beta(R)$ are positive constants, $\|u(T)\|_{L^2} \leq 1$ if we pick $T$ satisfying $\alpha e^{-\beta T} R < 1$. Then, it follows from (48)-(26) and (53) that for some constants $\nu > 0$, $c > 0$, $C > 0$

\[
V(u(T)) \leq ce^{-\nu T} V(u(T)) \leq c(T\|u_0\|_{L^2}^2 + T\|u_0\|_{L^2}^4 + V(u_0))e^{-\nu t},
\]
hence

\[
V(u(t)) \leq Ce^{-\nu t} V(u_0),
\]
where $C = C(R)$, which concludes the proof when $m = 1$.

**Induction Hypothesis:** There exist $c > 0$ and $\rho > 0$ such that if $V_{m-1}(u_0) \leq \rho$, we have

\[
V_m(u) - V_m(u_0) \\
\leq -c \left\{ \int_0^T u_x^2(0,t) \, dt + \int_0^T \int_0^\infty (x+1)^{m-1} u_x^2 \, dx \, dt + \int_0^T \int_0^\infty (x+1)^{m-1} a(x) u_x^2 \, dx \, dt \right\}
\]

\[
V_m(u_0) \\
\leq c \left\{ \int_0^T u_x^2(0,t) \, dt + \int_0^T \int_0^\infty (x+1)^{m-1} u_x^2 \, dx \, dt + \int_0^T \int_0^\infty (x+1)^{m-1} a(x) u_x^2 \, dx \, dt \right\}.
\]

By (52)-(54), the induction hypothesis is true for $m = 1$. Pick now an index $m \geq 2$ and assume that $d_0, ..., d_{m-2}$ have been constructed so that (*)$_k$-(**)$_k$ are fulfilled for $1 \leq k \leq m-1$. We aim to prove that for a convenient choice of the constant $d_{m-1}$ in (45), the properties (*)$_m$ - (**)$_m$ hold true.

Let us investigate first (*)$_m$. We multiply the first equation in (1) by $(x+1)^m u$ to obtain

\[
V_m(u) - V_m(u_0) - d_{m-1}(V_{m-1}(u) - V_{m-1}(u_0)) \\
- \frac{m(m-1)(m-2)}{2} \int_0^T \int_0^\infty (x+1)^{m-3} u_x^2 \, dx \, dt + \frac{1}{2} \int_0^T u_x^2(0,t) \, dt \\
+ \frac{3m}{2} \int_0^T \int_0^\infty (x+1)^{m-1} u_x^2 \, dx \, dt - \frac{m}{2} \int_0^T \int_0^\infty (x+1)^{m-1} u_x^2 \, dx \, dt \\
- \frac{m}{3} \int_0^T \int_0^\infty (x+1)^{m-1} u^3 \, dx \, dt + \int_0^T \int_0^\infty (x+1)^{m} a(x) u_x^2 \, dx \, dt = 0.
\]
Indeed, the existence of a positive constant \( c > 0 \) such that
\[
\int_0^T \int_0^\infty (x + 1)^{m-1} u^2 \, dx \, dt \\
= \int_0^T \int_0^{x_0} (x + 1)^{m-1} u^2 \, dx \, dt + \int_0^T \int_{x_0}^\infty (x + 1)^{m-1} u^2 \, dx \, dt \\
\leq (x_0 + 1)^{m-1} \int_0^T \int_0^\infty u^2 \, dx \, dt + \frac{1}{a_0} \int_0^T \int_0^\infty a(x)(x + 1)^{m-1} u^2 \, dx \, dt \\
\leq c \left\{ \int_0^T u_x^2(0, t) \, dt + \int_0^T \int_0^\infty (x + 1)^{m-1} a(x) u^2 \, dx \, dt \right\} \\
\leq -c \{ V_{m-1}(u) - V_{m-1}(u_0) \}
\] (59)
where we used \((*)_{m-1}\). In the same way
\[
\int_0^T \int_0^\infty (x + 1)^{m-3} u^2 \, dx \, dt \\
\leq \int_0^T \int_0^\infty (x + 1)^{m-1} u^2 \, dx \, dt \leq -c \{ V_{m-1}(u) - V_{m-1}(u_0) \}
\] (60)
where \( c > 0 \) is a positive constant. Moreover, assuming \( V_{m-1}(u_0) \leq \rho \) with \( \rho > 0 \) small enough (so that by exponential decay of \( V_{m-1}(u(t)) \) we have \( \int_0^\infty (x + 1)^{m-1} |u(x, t)|^2 \, dx \leq 1 \) for all \( t \geq 0 \) and proceeding as in the case \( m = 1 \), we obtain the existence of \( \varepsilon > 0 \) and \( c_\varepsilon > 0 \) satisfying
\[
\int_0^T \int_0^\infty (x + 1)^{m-1} |u|^3 \, dx \, dt \\
\leq \varepsilon \int_0^T \int_0^\infty (x + 1)^{m-1} u_x^2 \, dx \, dt + c_\varepsilon \int_0^T \int_0^\infty (x + 1)^{m-1} u^2 \, dx \, dt.
\] (61)
Indeed,
\[
\int_0^\infty (x + 1)^{m-1} |u|^3 \, dx \leq ||u||_{L^\infty} \int_0^\infty (x + 1)^{m-1} u^2 \, dx \\
\leq \sqrt{2} ||u||_{L^\infty} \frac{1}{2} ||u||_{L^2}^{\frac{1}{2}} \int_0^\infty (x + 1)^{m-1} u^2 \, dx \leq \varepsilon \int_0^\infty (x + 1)^{m-1} u_x^2 \, dx \\
+ c_\varepsilon \int_0^\infty u^2 \, dx + c_\varepsilon \left( \int_0^\infty (x + 1)^{m-1} u^2 \, dx \right)^2.
\] (62)
Then, if we return to (58) and take \( \varepsilon < 9/2 \) and \( d_{m-1} > 0 \) large enough, from (59)-(61) if follows that
\[
V_m(u) - V_m(u_0) \\
\leq -c \left\{ \int_0^T u_x^2(0, t) \, dt + \int_0^T \int_0^\infty (x + 1)^{m-1} u_x^2 \, dx \, dt \right\} \\
+ \int_0^T \int_0^\infty a(x)(x + 1)^{m-1} u_x^2 \, dx \, dt + \frac{d_{m-1}}{2} \{ V_{m-1}(u) - V_{m-1}(u_0) \}.
\] (63)
This yields \((*)_m\), by \((*)_{m-1}\). Let us now check \((**)_m\). It remains to estimate the terms in the right hand side of (63). We multiply the first equation in (1) by
\[(T - t)(x + 1)^m u \text{ to obtain}
\]
\[
\frac{T}{2} \int_0^\infty (x + 1)^m u_0^2 \, dx = \frac{1}{2} \int_0^T \int_0^\infty (x + 1)^m u^2 \, dx \, dt
\]
\[
-m(m - 1)(m - 2) \int_0^T \int_0^\infty (T - t)(x + 1)^{-3} u^2 \, dx \, dt + \frac{1}{2} \int_0^T (T - t) u_x^2(0, t) \, dt
\]
\[
+ \frac{3m}{2} \int_0^T \int_0^\infty (T - t)(x + 1)^{m-1} u_x^2 \, dx \, dt - m \int_0^T \int_0^\infty (T - t)(x + 1)^{-1} u^2 \, dx \, dt
\]
\[
- \frac{m}{3} \int_0^T \int_0^\infty (T - t)(x + 1)^{-1} u^3 \, dx \, dt + \int_0^T \int_0^\infty (T - t)(x + 1)^m a(x) u^2 \, dx \, dt.
\]

Then, proceeding as above, we deduce that
\[
\int_0^T (x + 1)^m u_0^2 \, dx
\]
\[
\leq c \left\{ \int_0^T \int_0^\infty (x + 1)^{m-1} u^2 \, dx \, dt + \int_0^T u_x^2(0, t) \, dt + \int_0^T \int_0^\infty (x + 1)^{m-1} u_x^2 \, dx \, dt
\]
\[
+ \int_0^T \int_0^\infty (x + 1)^m a(x) u^2 \, dx \, dt \right\}
\]
\[
\leq c \left\{ \int_0^T u_x^2(0, t) \, dt + \int_0^T \int_0^\infty (x + 1)^{m-1} u_x^2 \, dx \, dt + \int_0^T \int_0^\infty (x + 1)^m a(x) u^2 \, dx \, dt \right\}.
\]

Combined to \((**)_m\), this yields \((**)_m\). This completes the construction of the sequence \(\{V_m\}_{m \geq 1}\) by induction.

Let us now check the exponential decay of \(V_m\) for \(m \geq 2\). It follows from \((*)_m - (**)_m\) that
\[
V_m(u) - V_m(u_0) \leq -c V_m(u_0)
\]
where \(c > 0\), which completes the proof when \(V_{m-1}(u_0) \leq \rho\). The global result \((V_{m-1}(0) \leq R)\) is obtained as above for \(m = 1\).

**Corollary 1.** Let \(a = a(x)\) fulfilling (4) and \(a \in W^{2, \infty}(0, \infty)\). Then for any \(R > 0\), there exist positive constants \(c = c(R)\) and \(\mu = \mu(R)\) such that
\[
\|u(x,t)\|_{L^2(R_+)} \leq \frac{e^{-\mu t}}{\sqrt[4]{t}} \|u_0\|_{L^2_{x+1}(\mathbb{R})} \tag{64}
\]
for all \(t > 0\) and all \(u_0 \in L^2_{x+1}(\mathbb{R})\) satisfying \(\|u_0\|_{L^2_{x+1}(\mathbb{R})} \leq R\).

**Proof.** Pick any \(R > 0\) and any \(u_0 \in L^2_{x+1}(\mathbb{R})\) with \(\|u_0\|_{L^2_{x+1}(\mathbb{R})} \leq R\). By Theorem 3.1 there are some constants \(C = C(R)\) and \(\nu = \nu(R)\) such that
\[
\|u(t)\|_{L^2_{x+1}(\mathbb{R})} \leq Ce^{-\nu t} \|u_0\|_{L^2_{x+1}(\mathbb{R})} \tag{65}
\]
Using the multiplier \(t(u^2 + 2u_{xx})\) we obtain after some integrations by parts that for all \(0 < t_1 < t_2\)
\[
t_2 \int_0^\infty u_x^2(x, t_2) \, dx + \int_t^{t_2} tu_x^2(0, t) \, dt + 2 \int_{t_1}^{t_2} \int_0^\infty ta(x) u_x^2 \, dx \, dt + \int_t^{t_2} tu_x^2(0, t) \, dt
\]
\[
= -\frac{1}{3} \int_{t_1}^{t_2} \int_0^\infty u^3 \, dx \, dt + \frac{t_2}{3} \int_0^\infty u^3(x, t_2) \, dx + \int_{t_1}^{t_2} \int_0^\infty tu^3a(x) \, dx \, dt
\]
\[
+ \int_{t_1}^{t_2} \int_0^\infty u^2 \, dx \, dt + \int_{t_1}^{t_2} \int_0^\infty ta''(x) u^2 \, dx \, dt. \tag{66}
\]
1. Let us assume first that $T > 1$. Applying (66) on the time interval $[T - 1, T]$, we infer that
\begin{equation}
\int_0^\infty |u_x(x, T)|^2 dx 
\leq c \left( \int_{T - 1}^T \int_0^\infty |u|^3 dx dt + ||u(T)||_{L^2(\mathbb{R}^+)}^3 + \int_{T - 1}^T ||u||_{H^1(\mathbb{R}^+)}^2 dt \right).
\end{equation}

To estimate the cubic terms in (67), we use (26) to obtain
\begin{equation}
\int_0^\infty |u_x(x, T)|^2 dx 
\leq \varepsilon \int_0^\infty |u_x(x, T)|^2 dx 
+ c_\varepsilon \left( ||u(T)||_{L^2(\mathbb{R}^+)}^3 + \int_{T - 1}^T (||u||_{L^2(\mathbb{R}^+)}^2 + ||u||_{H^1(\mathbb{R}^+)}^2) dt \right).
\end{equation}

Note that by (65)
\begin{equation}
||u(T)||_{L^2(\mathbb{R}^+)}^3 \leq (Ce^{-\nu T} ||u_0||_{L^2_{(x+1)dx}})^\frac{2}{3} \leq C \frac{2}{3} R^4 e^{-\nu T} ||u_0||_{L^2_{(x+1)dx}}^2.
\end{equation}

It follows from (48), (26), and (65) that
\begin{equation}
\int_{T - 1}^T (||u||_{H^1(\mathbb{R}^+)}^2 + ||u||_{L^2(\mathbb{R}^+)}^2) dt 
\leq C \left( \int_{T - 1}^T (||u||_{L^2(\mathbb{R}^+)}^2 + ||u||_{H^1(\mathbb{R}^+)}^2) dt \right)
\end{equation}
where $C = C(R, \nu)$. (64) for $T \geq 1$ follows from (68) and (69) by choosing $\varepsilon < 1$ and $\mu < \nu$.

2. Assume now that $T \leq 1$. Estimating again the cubic terms in (66) (with $[t_1, t_2] = [0, T]$) by using (26), we obtain
\begin{equation}
T \int_0^\infty u_x^2(x, T) dx 
\leq \frac{T}{3} \left( \varepsilon ||u_x(T)||_{L^2(\mathbb{R}^+)}^2 + C_\varepsilon ||u(T)||_{L^2(\mathbb{R}^+)}^3 \right) 
+ C_\varepsilon \int_0^T (||u||_{H^1(\mathbb{R}^+)}^2 + ||u||_{L^2(\mathbb{R}^+)}^2) dt.
\end{equation}

By (48), (26) and (65), we have that
\begin{equation}
\int_0^1 \int_0^\infty |u_x|^2 dx dt \leq C(R) ||u_0||_{L^2_{(x+1)dx}}^2
\end{equation}
which, combined to (70) with $\varepsilon = 1$ and (65), gives
\begin{equation}
||u_x(T)||_{L^2(\mathbb{R}^+)}^2 \leq C(R) T^{-1} ||u_0||_{L^2_{(x+1)dx}}^2
\end{equation}
for all $T < 1$. This gives (64) for $T < 1$.

Corollary 1 may be extended (locally) to the weighted space $L^2_{(x+1)m dx}$ ($m \geq 2$) in following the method of proof of [23, Theorem 1.1].

**Corollary 2.** Let $a = a(x)$ fulfilling (4) and $m \geq 2$. Then there exist some constants $\rho > 0$, $C > 0$ and $\mu > 0$ such that
\begin{equation}
||u(t)||_{H^{1_{(x+1)m}dx}} \leq C \frac{e^{-\mu t}}{\sqrt{t}} ||u_0||_{L^2_{(x+1)m dx}}.
\end{equation}
for all $t > 0$ and all $u_0 \in L^2_{(x+1)^m dx}$ satisfying $||u_0||_{L^2_{(x+1)^m dx}}^2 \leq \rho$.

**Proof.** We first prove estimates for the linearized problem

\[ u_t + u_x + u_{xxx} + au = 0 \]  
\[ u(0, t) = 0 \]  
\[ u(x, 0) = u_0(x) \]

and next apply a perturbation argument to extend them to the nonlinear problem (1). Let us denote by $W(t)u_0 = u(t)$ the solution of (72)-(74). By computations similar to those performed in the proof of Theorem 3.1, we have that

\[ ||W(t)u_0||_{L^2_{(x+1)^m dx}} \leq C_0 e^{-\nu t} ||u_0||_{L^2_{(x+1)^m dx}}. \]

We need the

**Claim 6.** Let $k \in \{0, ..., 3\}$. Then there exists a constant $C_k > 0$ such that for any $u_0 \in H^k_{(x+1)^m dx}$,

\[ ||W(t)u_0||_{H^k_{(x+1)^m dx}} \leq C_k e^{-\nu t} ||u_0||_{H^k_{(x+1)^m dx}}. \]

Indeed, if $u_0 \in H^3_{(x+1)^m dx}$, then $u_t(., 0) \in L^2_{(x+1)^{m-3} dx}$, and since $v = u_t$ solves (72)-(73), we also have that

\[ ||u_t(., t)||_{L^2_{(x+1)^{m-3} dx}} \leq C_0 e^{-\nu t} ||u_t(., 0)||_{L^2_{(x+1)^{m-3} dx}}. \]

Using (72), this gives

\[ ||W(t)u_0||_{H^3_{(x+1)^m dx}} \leq C_3 e^{-\nu t} ||u_0||_{H^3_{(x+1)^m dx}}. \]

This proves (75) for $k = 3$. The fact that (75) is valid for $k = 1, 2$ follows from a standard interpolation argument, for $H^k_{(x+1)^m dx} = [H^0_{(x+1)^m dx}, H^3_{(x+1)^m dx}]^{\frac{k}{3}}$.

**Lemma 3.2.** Pick any number $\mu \in (0, \nu)$. Then there exists some constant $C = C(\mu) > 0$ such that for any $u_0 \in L^2_{(x+1)^m dx}$

\[ ||W(t)u_0||_{H^1_{(x+1)^m dx}} \leq C \frac{e^{-\mu t}}{\sqrt{t}} ||u_0||_{L^2_{(x+1)^m dx}}. \]

**Proof.** Let $u_0 \in L^2_{(x+1)^m dx}$, and set $u(t) = W(t)u_0$ for all $t > 0$. By scaling in (72) by $(x+1)^m u$, we see that for some constant $C_K = C_K(T)$

\[ ||u||_{L^2(0, T; H^1_{(x+1)^m dx})} \leq C_K ||u_0||_{L^2_{(x+1)^m dx}}. \]

This implies that $u(t) \in H^1_{(x+1)^m dx}$ for a.e. $t \in (0, 1)$ which, combined to (75), gives that $u(t) \in H^1_{(x+1)^m dx}$ for all $t > 0$. Pick any $T \in (0, 1]$. Note that, by (75),

\[ ||u(t)||_{H^1_{(x+1)^m dx}} \leq C_1 e^{-\nu(T-t)} ||u(t)||_{H^1_{(x+1)^m dx}}, \quad \forall t \in (0, T). \]

Integrating with respect to $t$ in (77) yields

\[ [C_1^{-1} ||u(T)||_{H^1_{(x+1)^m dx}}^2 \int_0^T e^{2\nu(T-t)} dt \leq \int_0^T ||u(t)||_{H^1_{(x+1)^m dx}}^2 dt, \]

and hence

\[ ||u(T)||_{H^1_{(x+1)^m dx}} \leq C_K C_1 \sqrt{\frac{2\nu}{e^{2\nu T} - 1}} ||u_0||_{L^2_{(x+1)^m dx}} \]

\[ \leq \frac{C_K C_1}{\sqrt{T}} ||u_0||_{L^2_{(x+1)^m dx}} \]
for $0 < T \leq 1$. Therefore
\[
\|u(t)\|_{H^1_t L^2_x} \leq C K C_1 e^{\nu t} \frac{\mu t}{\sqrt{t}} \|u_0\|_{L^2_t H^1_x} \quad \forall t \in (0, 1).
\] (78)

(76) follows from (78) and (75), since $\mu < \nu$. \hfill \square

Let us return to the proof of Corollary 2. Fix a number $\mu \in (0, \nu)$, where $\nu$ is as in (75), and let us introduce the space
\[
F = \{ u \in C(\mathbb{R}^+; H^1_{t+1}) : \|e^{\mu t} u(t)\|_{L^\infty(\mathbb{R}^+; H^1_{t+1})} < \infty \}
\]
endowed with its natural norm. Note that (1) may be recast in the following integral form
\[
u(t) = W(t)u_0 + \int_0^t W(t-s)N(u(s)) \, ds \quad (79)
\]
where $N(u) = -uu_x$. We first show that (79) has a solution in $F$ provided that $u_0 \in H^1_{t+1}$ small enough. Let $u_0 \in H^1_{t+1}$ and $u \in F$ with $\|u_0\|_{H^1_{t+1}} \leq r_0$ and $\|u\|_F \leq R$, $r_0$ and $R$ being chosen later. We introduce the map $\Gamma$ defined by
\[
(\Gamma u)(t) = W(t)u_0 + \int_0^t W(t-s)N(u(s)) \, ds \quad \forall t \geq 0.
\] (80)

We shall prove that $\Gamma$ has a fixed point in the closed ball $B_R(0) \subset F$ provided that $r_0 > 0$ is small enough.

For the forcing problem
\[
\begin{cases}
  u_t + u_x + u_{xx} + au = f \\
  u(0, t) = 0 \\
  u(x, 0) = u_0(x)
\end{cases}
\]
we have the following estimate
\[
\sup_{0 \leq t \leq T} \|u(t)\|_{L^2_t H^1_x}^2 + \int_0^T \int_0^\infty (x + 1)^{m-1} u_x^2 \, dx \, dt 
\]
\[
\leq C \left( \|u_0\|_{L^2_x}^2 + \|f\|_{L^2(0, T; L^2_x)}^2 \right).
\]

Let us take $f = N(u) = -uu_x$. Observe that for all $x > 0$
\[
(x + 1)u^2(x) = \left| \int_0^\infty \frac{d}{dx}[(x + 1)u^2(x)] \, dx \right|
\leq C \left( \int_0^\infty (x + 1)^m |u|^2 \, dx + \int_0^\infty (x + 1)^{m-1} |u_x|^2 \, dx \right)
\]
whenever $m \geq 2$. It follows that for some constant $K > 0$
\[
\|uu_x\|_{L^2_t H^1_x} \leq \|(x + 1)u^2\|_{L^\infty(\mathbb{R}^+)} \int_0^\infty (x + 1)^{m-1} |u_x|^2 \, dx 
\leq K\|u\|_{H^1_{t+1}}^4.
\]
Therefore, for any $T > 0$,
\[
\sup_{0 \leq t \leq T} \left\| (\Gamma u)(t) \right\|_{L^2(x+1)^m_{dx}}^2 + \int_0^T \int_0^\infty (x+1)^{m-1} |(\Gamma u)_x|^2 dxdt \\
\leq C \left( \left\| u_0 \right\|_{L^2(x+1)^m_{dx}}^2 + \left( \int_0^T \left\| u(t) \right\|_{H^1(x+1)^m_{dx}}^2 dt \right)^2 \right) < \infty.
\]
Thus $\Gamma u \in C(\mathbb{R}^+, L^2(x+1)^m_{dx}) \cap L^2_{loc}(\mathbb{R}^+; H^1(x+1)^m_{dx})$ with $(\Gamma u)(0) = u_0$. We claim that $\Gamma u \in F$. Indeed, by (75),
\[
\left\| e^{\mu t} W(t) u_0 \right\|_{H^1(x+1)^m_{dx}} \leq C \left\| u_0 \right\|_{H^1(x+1)^m_{dx}}
\]
and for all $t \geq 0$
\[
\left\| e^{\mu t} \int_0^t W(t-s) N(u(s)) ds \right\|_{H^1(x+1)^m_{dx}} \leq C e^{\mu t} \int_0^t \frac{e^{-\mu(t-s)}}{\sqrt{1-s}} \left\| N(u(s)) \right\|_{L^2(x+1)^m_{dx}} ds \\
\leq C \int_0^t \frac{e^{\mu s}}{\sqrt{1-s}} K(e^{-\mu s}) \left\| u \right\|_{H^1}^2 ds \\
\leq CK \left\| u \right\|_{H^1}^2 \int_0^t \frac{e^{-\mu(t-s)}}{\sqrt{1-s}} ds \\
\leq CK(2 + \mu^{-1}) \left\| u \right\|_{H^1}^2
\]
where we used Lemma 3.2. Pick $R > 0$ such that $CK(2 + \mu^{-1})R \leq \frac{1}{2}$, and $r_0$ such that $C_1 r_0 = \frac{R}{4}$. Then, for $\left\| u_0 \right\|_{H^1(x+1)^m_{dx}} \leq r_0$ and $\left\| u \right\|_{H^1} \leq R$, we obtain that
\[
\left\| e^{\mu t} (\Gamma u)(t) \right\|_{H^1(x+1)^m_{dx}} \leq C_1 r_0 + CK(2 + \mu^{-1})R^2 \leq R, \quad t \geq 0.
\]
Hence $\Gamma$ maps the ball $B_R(0) \subset F$ into itself. Similar computations show that $\Gamma$ contracts. By the contraction mapping theorem, $\Gamma$ has a unique fixed point $u$ in $B_R(0)$. Thus $\left\| u(t) \right\|_{H^1(x+1)^m_{dx}} \leq C e^{-\mu t} \left\| u_0 \right\|_{H^1(x+1)^m_{dx}}$, provided that $\left\| u_0 \right\|_{H^1(x+1)^m_{dx}} \leq r_0$ with $r_0$ small enough. Proceeding as in the proof of Lemma 3.2, we have that
\[
\left\| u(t) \right\|_{H^1(x+1)^m_{dx}} \leq C e^{-\mu t} \left\| u_0 \right\|_{L^2(x+1)^m_{dx}} \quad \text{for } 0 < t < 1,
\]
provided that $\left\| u_0 \right\|_{L^2(x+1)^m_{dx}} \leq \rho_0$ with $\rho_0 < 1$ small enough. The proof is complete with a decay rate $\mu' \leq \mu$.

**Corollary 3.** Assume that $a(x)$ satisfies (4) and that $\partial_x^k a \in L^\infty(\mathbb{R}^+)$ for all $k \geq 0$. Pick any $u_0 \in L^2(x+1)^m_{dx}$. Then for all $\varepsilon > 0$, all $T > \varepsilon$, and all $k \in \{1, ..., m\}$, there exists a constant $C = C(\varepsilon, T, k) > 0$ such that
\[
\int_{\varepsilon}^T (x+1)^{m-k} |\partial_x^k u(t,x)|^2 dx \leq C \left\| u_0 \right\|_{L^2(x+1)^m_{dx}} \quad \forall t \in [\varepsilon, T]. \tag{81}
\]

**Proof.** The proof is very similar to the one in [16, Lemma 5.1] and so we only point out the small changes. First, it should be noticed that the presence in the KdV equation of the extra terms $u_x$ and $a(x)u$ does not cause any serious trouble. On the other hand, choosing a cut-off function in $x$ of the form $\eta(x) = \psi_0(x/\varepsilon)$ (instead of $\eta(x) = \psi_0(x-x_0 + 2)$ as in [16]) where $\psi_0 \in C^\infty(\mathbb{R}, [0,1])$ satisfies $\psi_0(x) = 0$ for $x \leq 1/2$ and $\psi_0(x) = 1$ for $x \geq 1$, allows to overcome the fact that $u$ is a solution of (1) on the half-line only. \qed
3.2. Decay in $L^2_b$. This section is devoted to the exponential decay in $L^2_b$. Our result reads as follows:

**Theorem 3.3.** Assume that the function $a = a(x)$ satisfies (4) with $4b^3 + b < a_0$. Then, for all $R > 0$, there exist $C > 0$ and $\nu > 0$, such that

$$
||u(t)||_{L^2_b} \leq C e^{-\nu t}||u_0||_{L^2_b} \quad t \geq 0
$$

for any solution $u$ given by Theorem 2.3.

**Proof.** We introduce the Lyapunov function

$$
V(u) = \frac{1}{2} \int_0^\infty u^2 e^{2bx}dx + c_b \int_0^\infty u^2dx,
$$

(82)

where $c_b$ is a positive constant that will be chosen later. Then, adding (17) and (18) hand by hand we obtain

$$
V(u) - V(u_0) = (4b^3 + b) \int_0^T \int_{x_0}^{\infty} u^2 e^{2bx}dxdt + (4b^3 + b) \int_0^T \int_0^{x_0} u^2 e^{2bx}dxdt
$$

$$
- 3b \int_0^\infty \int_0^{\infty} u^2 e^{2bx}dxdt + \frac{2b}{3} \int_0^T \int_0^{\infty} u^3 e^{2bx}dxdt
$$

$$
- (c_b + \frac{1}{2}) \int_0^T u^2(0,t)dt - \int_0^T \int_0^{\infty} a(x)|u|^2(e^{2bx} + 2c_b)dxdt,
$$

(83)

where $x_0$ is the number introduced in (4). On the other hand, since $L^2_b \subset L^2_{(x+1)dx}$, $||u(t)||_{L^2(0,\infty)}$ and $||u_x(t)||_{L^2(0,\infty)}$ decays to zero exponentially. Consequently, from Moser estimate we deduce that $||u(t)||_{L^\infty(0,\infty)} \to 0$. We may assume that $(2b/3)||u(t)||_{L^\infty} < \varepsilon = a_0 - (4b^3 + b)$ for all $t \geq 0$, by changing $u_0$ into $u(t_0)$ for $t_0$ large enough. Therefore

$$
\frac{2b}{3} \int_0^T \int_0^{\infty} |u|^3 e^{2bx}dxdt \leq \frac{2b}{3} \int_0^T \int_0^{\infty} |u(t)||_{L^\infty(0,\infty)} \left( \int_0^{\infty} |u|^2 e^{2bx}dx \right) dt \leq \varepsilon \int_0^T \int_0^{\infty} u^2 e^{2bx}dxdt.
$$

(84)

So, returning to (83), the following holds

$$
V(u) - V(u_0) = (4b^3 + b + \varepsilon) \int_0^T \int_0^{x_0} u^2 e^{2bx}dxdt + 3b \int_0^T \int_0^{\infty} u^2 e^{2bx}dxdt
$$

$$
+(c_b + \frac{1}{2}) \int_0^T u^2(0,t)dt + 2c_b \int_0^T \int_0^{\infty} a(x)|u|^2dxdt \leq 0.
$$

(85)

Moreover, according to [19] there exists $C > 0$ satisfying

$$
\int_0^T \int_0^{x_0} u^2 e^{2bx}dxdt \leq e^{2b} \int_0^T \int_0^{x_0} u^2dxdt \leq C \left\{ \int_0^T u^2_2(0,t)dt + \int_0^T \int_0^{\infty} a(x)|u|^2dxdt \right\}
$$
since \( L^2_b \subset L^2(\mathbb{R}^+) \). Then, choosing \( c_b \) sufficiently large, the above estimate and (85) give us that

\[
V(u) - V(u_0) \leq -C \left\{ \int_0^T u^2_x(0,t)dt + \int_0^T \int_0^\infty a(x)u^2 dx dt 
+ \int_0^T \int_0^\infty u^2_2 e^{2bx} dx dt \right\} \leq -C V(u_0),
\]

(86)

which allows to conclude that \( V(u) \) decays exponentially. The last inequality is a consequence of the following results:

**Claim 7.** There exists a positive constant \( C > 0 \), such that

\[
\int_0^T V(u(t)) dt \leq C \int_0^T \int_0^\infty u^2_2 e^{2bx} dx dt.
\]

First, observe that

\[
| \int_0^\infty u^2 e^{2bx} dx | = \frac{1}{b} \int_0^\infty u u e^{2bx} dx \leq \frac{1}{b} \left( \int_0^\infty u^2 e^{2bx} dx \right)^2 \left( \int_0^\infty u^2 e^{2bx} dx \right)^2,
\]

therefore,

\[
\int_0^\infty u^2 e^{2bx} dx \leq \frac{1}{b^2} \int_0^\infty u^2 e^{2bx} dx.
\]

(87)

Then, from (4) and (87) we have

\[
V(u(t)) \leq \frac{1}{2} + c_b \int_0^\infty u^2 e^{2bx} dx \leq \frac{1}{2} + c_b b^{-2} \int_0^\infty u^2 e^{2bx} dx
\]

which gives us Claim 7.

**Claim 8.**

\[
V(u_0) \leq C \left\{ \int_0^T u^2_x(0,t)dt + \int_0^T \int_0^\infty u^2_2 e^{2bx} dx dt + \int_0^T V(u(t)) dt \right\},
\]

where \( C \) is a positive constant.

Multiplying the first equation in (1) by \((T-t)u^2 e^{2bx}\) and integrating by parts in \((0,\infty) \times (0,T)\), we obtain

\[
-\frac{T}{2} \int_0^\infty |u_0(x)|^2 e^{2bx} dx + \frac{1}{2} \int_0^T \int_0^\infty |u|^2 e^{2bx} dx dt
+ 3b \int_0^T \int_0^\infty (T-t)u^2 e^{2bx} dx dt + \frac{1}{2} \int_0^T (T-t)u^2_x(0,t)dt
- (4b^3 + b) \int_0^T \int_0^\infty (T-t)u^2 e^{2bx} dx dt + \int_0^T \int_0^\infty (T-t)a(x)|u|^2 e^{2bx} dx dt
- \frac{2b}{3} \int_0^T \int_0^\infty (T-t)u^3 e^{2bx} dx dt = 0
\]

(88)

and therefore,

\[
\int_0^\infty |u_0(x)|^2 e^{2bx} dx \leq C \left\{ \int_0^T u^2_x(0,t)dt + \int_0^T \int_0^\infty u^2 e^{2bx} dx dt
+ \int_0^T \int_0^\infty u^2_2 e^{2bx} dx dt + \int_0^T \int_0^\infty |u|^3 e^{2bx} dx dt \right\}.
\]

(89)

Then, combining (87) and (84), we derive Claim 8. (86) follows at once. This proves the exponential decay when \( ||u(t)||_{L^\infty} \leq 3\varepsilon/(2b) \). The general case is obtained as in Theorem 3.1.
Assume that the function \( a = a(x) \) satisfies (4) with \( 4b^3 + b < a_0 \). Then for any \( R > 0 \), there exist positive constants \( c = c(R) \) and \( \mu = \mu(R) \) such that

\[
\|u_x(t)\|_{L^2_t} \leq \frac{e^{-\mu t}}{\sqrt{t}} \|u_0\|_{L^2_t} \tag{90}
\]

for all \( t > 0 \) and all \( u_0 \in L^2_t \) satisfying \( \|u_0\|_{L^2_t} \leq R \).

Corollary 5. Assume that the function \( a = a(x) \) satisfies (4) with \( 4b^3 + b < a_0 \), and let \( s \geq 2 \). Then there exist some constants \( \rho > 0 \), \( C > 0 \) and \( \mu > 0 \) such that

\[
\|u(t)\|_{H^s} \leq Ce^{-\mu t} \|u_0\|_{L^2_t}
\]

for all \( t > 0 \) and all \( u_0 \in L^2_t \) satisfying \( \|u_0\|_{L^2_t} \leq \rho \).

The proof of Corollary 4 (resp. 5) is very similar to the proof of Corollary 1 (resp. 2), so it is omitted.

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REFERENCES


