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# FINITE-TIME STABILIZATION OF $2 \times 2$ HYPERBOLIC SYSTEMS ON TREE-SHAPED NETWORKS*

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**Abstract.** We investigate the finite-time boundary stabilization of a one-dimensional first order quasilinear hyperbolic system of diagonal form on [0,1]. The dynamics of both boundary controls are governed by a finite-time stable ODE. The solutions of the closed-loop system issuing from small initial data in $\mathrm{Lip}([0,1])$ are shown to exist for all times and to reach the null equilibrium state in finite time. When only one boundary feedback law is available, a finite-time stabilization is shown to occur roughly in a twice longer time. The above feedback strategy is then applied to the Saint-Venant system for the regulation of water flows in a network of canals.

**Key words.** finite-time stability, stabilization, hyperbolic systems, shallow water equations, water management, network

**AMS subject classifications.** 35L50, 35L60, 76B75, 93D15

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## 1. Introduction.

Solutions of certain asymptotically stable ODEs may reach the equilibrium state in finite time. This phenomenon, which is common when using feedback laws that are not Lipschitz continuous, was called *finite-time stability* in [4] and investigated in that paper.

A *finite-time stabilizer* is a feedback control for which the closed-loop system is finite-time stable around some equilibrium. In some sense, it satisfies a controllability objective with a control in feedback form. On the other hand, a finite-time stabilizer may be seen as an exponential stabilizer yielding an arbitrarily large decay rate for the solutions to the closed-loop system. This explains why some efforts were made in the last decade to construct finite-time stabilizers for controllable systems, including the linear ones. See [25] for some recent developments and up-to-date references, and [2] for some connections with Lyapunov theory.

For PDEs, the relationship between exact controllability and rapid stabilization was investigated in [18, 19, 30]. (See also [20] for the rapid semiglobal stabilization of the Korteweg–de Vries equation using a time-varying feedback law.)

To the best knowledge of the authors, analysis of the finite-time stabilization of PDE has not yet been developed. However, the phenomenon of finite-time extinction exists naturally for certain nonlinear evolution equations (see [5, 11, 29]). On the other hand, it has been well known since [24] that solutions of the wave equation on a bounded domain may disappear when using "transparent" boundary conditions. For instance, the solution of the one-dimensional wave equation

$$\partial_t^2 y - \partial_x^2 y = 0 \quad \text{in } (0,T)\times(0,1), \tag{1.1}$$

$$\partial_x y(t,1) = -\partial_t y(t,1) \quad \text{in } (0,T), \tag{1.2}$$

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$$\partial_x y(t,0) = \partial_t y(t,0) \quad \text{in } (0,T), \tag{1.3}$$

$$(y(0,.), \partial_t y(0,.)) = (y_0, z_0) \quad \text{in } (0,1) \tag{1.4}$$

is finite-time stable in $\{(y,z) \in H^1(0,1) \times L^2(0,1);\ y(0) + y(1) + \int_0^1 z(x)dx = 0\}$, with $T = 1$ as extinction time (see, e.g., [19, Theorem 0.5] for details). The condition (1.2) is transparent in the sense that a wave $y(t,x) = f(x-t)$ traveling to the right satisfies (1.2) and leaves the domain at $x = 1$ without generating any reflected wave. Note that we can replace (1.3) by the boundary condition $y(t,0) = 0$ (or $\partial_x y(t,0) = 0$). Then a finite-time extinction still occurs (despite the fact that waves bounce at $x = 0$) with an extinction time $T = 2$. We refer the reader to [8] for an analysis of the finite-time extinction property for a nonhomogeneous string with a viscous damping at one extremity, and to [1] for an investigation of the finite-time stabilization of a network of strings.

The finite-time stability of (1.1)–(1.4) is easily established when writing (1.1) as a first order hyperbolic system

$$\partial_t \begin{pmatrix} r \\ s \end{pmatrix} - \partial_x \begin{pmatrix} s \\ r \end{pmatrix} = 0$$

with $(r,s) = (\partial_x y, \partial_t y)$, and next introducing the Riemann invariants $u = r - s$, $v = r + s$ that solve the system of the two transport equations

$$\begin{aligned} \partial_t u + \partial_x u &= 0, \\ \partial_t v - \partial_x v &= 0. \end{aligned}$$

The boundary conditions (1.2) and (1.3) yield $u(t,0) = v(t,1) = 0$ (and hence $u(t,.) = v(t,.) = 0$ for $t \geq 1$), while the boundary conditions (1.2) and $y(t,0) = 0$ yield $v(t,1) = 0$ and $u(t,0) = v(t,0)$ (and hence $v(t,.) = 0$ for $t \geq 1$ and $u(t,.) = 0$ for $t \geq 2$).

The goal of this paper is to show that the finite-time extinction property can be realized for a system of conservation laws

$$\partial_t Y + \partial_x F(Y) = 0, \tag{1.5}$$

which can be put in diagonal form, i.e., for which there is a smooth change of (dependent) variables that transforms (1.5) into a system of two nonlinear transport equations of the form

$$\partial_t u + \lambda(u,v)\partial_x u = 0, \tag{1.6}$$

$$\partial_t v + \mu(u,v)\partial_x v = 0, \tag{1.7}$$

where $\mu(u,v) \leq -c < c \leq \lambda(u,v)$ are smooth functions and $c > 0$ is some constant. In practice, the functions $u$ and $v$ are Riemann invariants of (1.5) (see, e.g., [12, 31]).

The generalization of the finite-time extinction property of the wave equation to systems of the form (1.6)–(1.7) on general networks is the main aim of this paper. It should be noticed that this extension was done in [21] for the Saint-Venant system on an interval (or a star-shaped tree) with the homogeneous Dirichlet conditions

$$u(t,0) = v(t,1) = 0. \tag{1.8}$$

We stress that the use of the boundary conditions (1.8) forces us to restrict ourselves to initial data $(u_0, v_0)$ fulfilling the compatibility conditions

$$u_0(0) = v_0(1) = 0. \tag{1.9}$$

Indeed, if the initial conditions are such that $u_0(0) \neq 0$ and $v_0(1) \neq 0$, then shocks, appearing at $t = 0$ at both extremities, propagate inside the domain along the characteristics and meet at some positive time, as explained in [16]. The corresponding solution, whose mathematical analysis should be done with some care, fails to be regular, and this may be a serious concern in applications.

In order to provide a result for a class of initial data *without any ($C^0$ or $C^1$) compatibility condition*, we consider boundary conditions whose dynamics obey a finite-time stable ODE, namely,

$$\frac{d}{dt}u(t,0) = -K\text{sgn}(u(t,0))|u(t,0)|^\gamma, \tag{1.10}$$

$$\frac{d}{dt}v(t,1) = -K\text{sgn}(v(t,1))|v(t,1)|^\gamma, \tag{1.11}$$

$(K, \gamma) \in (0, +\infty) \times (0, 1)$ being some arbitrary constants. From the control viewpoint, it means that we added to the state $(u, v)$ a variable in $\mathbb{R}^2$ whose dynamics is given by (1.10)–(1.11). Finally, we supplement the system (1.6)–(1.7), (1.10)–(1.11) with the initial condition

$$u(0, x) = u_0(x), \ v(0, x) = v_0(x). \tag{1.12}$$

The idea is that, after some transient period, the boundary conditions (1.8) are indeed effective without assuming that (1.9) holds. Note that any boundary condition involving a finite-time stable ODE could be considered, but we limit ourselves here to the simplest one. Of course, the existence/uniqueness of a (regular) solution to (1.6)–(1.7) and (1.10)–(1.12) is our first concern.

The first main result in this paper (Theorem 1) asserts that for any pair $(u_0, v_0)$ of Lipschitz continuous initial data, which are small enough in the $W^{1,\infty}(0, 1)$ norm, system (1.6)–(1.7) and (1.10)–(1.12) admits a unique solution in some class of Lipschitz continuous functions, and that this solution is defined for all times $t \geq 0$, is stable in the $W^{1,\infty}(0, 1)$ norm, and vanishes for roughly $t \geq 1/c$. Theorem 1 is proved by using a fixed-point argument (Schauder theorem) and energy estimates.

Sometimes the boundary condition at one extremity of the domain (say 0) is imposed by the context, so that we cannot choose the condition $u(t, 0) = 0$ (or its generalization (1.10)) for the Riemann invariant $u$. Then we have to replace (1.10) by a boundary condition of the form

$$u(t, 0) = h(v(t, 0), t) \tag{1.13}$$

for some (smooth) function $h = h(v, t)$. The second main result in this paper (Theorem 2) asserts that the system (1.6)–(1.7) and (1.11)–(1.13) is still locally well-posed with roughly an extinction time $T = 2/c$. The result is obtained for small initial data and for $||\partial_t h||_\infty$ small enough.

The results obtained in this paper can be applied to any $2 \times 2$ hyperbolic system that can be put in the form (1.6)–(1.7) with $\mu(u, v) \leq -c < c \leq \lambda(u, v)$ (e.g., the $p$-system, the shallow water equations, and Euler's equations for barotropic compressible gas; see [12]).

For the sake of brevity, we will limit ourselves to the stabilization of Saint-Venant equations, and will give an extension of the above finite-time stabilization results to a tree-shaped network of canals. The obtained extinction time will be roughly $d/c$, where $d$ denotes the depth of the tree (Theorem 3).

There is a substantial literature on the controllability and stabilization of first order hyperbolic equations (see, e.g., [13, 14, 15, 22, 23, 26]). In particular, the control of Saint-Venant equations has attracted the attention of the control community because of its relevance to the regulation of water flows in networks of canals or rivers. We refer the reader to, e.g., [3, 7, 6, 16, 17, 10, 21, 32], where Riemann invariants often played a great role in the design of the controls. Our main contribution here is to notice that a *finite-time* stabilization can be achieved as well, i.e., that bounces of waves at the two ends of the domain can be *avoided*. Some numerical experiments for the finite-time stabilization of water flows in a canal can be found in [28]. When source terms are incorporated into (1.6)–(1.7), then the finite-time stability of (1.6)–(1.8) may be lost. However, in a work in progress, we aim to prove that an exponential stability still occurs in the $W^{1,\infty}(0,1)$ norm with a "large" decay rate for a "small" source term.

The paper is outlined as follows. In section 2, after recalling classical properties of linear transport equations, we consider a $2\times 2$ hyperbolic system with two boundary controls whose dynamics are governed by a finite-time stable ODE. We prove the well-posedness and the finite-time stability of the closed-loop system. In section 3, we address the same issue for a system with only one boundary control. In the last section, we apply the results in sections 2 and 3 to the regulation of water flows in a tree-shaped network of canals.

## 2. Finite-time boundary stabilization of a system of two conservation laws.

**2.1. Notation.** $\mathcal{C}^0([0,T]\times[0,1])$ denotes the space of continuous functions $u : [0,T]\times[0,1]\to\mathbb{R}$. It is endowed with the norm

$$||u||_{\mathcal{C}^0([0,T]\times[0,1])} = \sup_{(t,x)\in[0,T]\times[0,1]} |u(t,x)|.$$

The norm of the space $L^p(0,1)$ is denoted $||\cdot||_p$ for $1\le p\le\infty$. Lip$([0,1])$ denotes the space of Lipschitz continuous functions $u:[0,1]\to\mathbb{R}$. It may be identified with the Sobolev space $W^{1,\infty}(0,1)$. Lip$([0,1])$ is endowed with the $W^{1,\infty}(0,1)$ norm; that is,

$$||u||_{\mathrm{Lip}([0,1])} = ||u||_{W^{1,\infty}(0,1)} = ||u||_\infty + ||u'||_\infty .$$

We use similar norms for Lip$(\mathbb{R})$, Lip$([0,T]\times[0,1])$, etc.

**2.2. Linear transport equations.** In this section, we recall without proof two results about linear transport equations that will be used thereafter (see [26, Appendix] or [27] for details). Given $T>0$, assume $y_0\in$ Lip$([0,1])$, $y_l\in$ Lip $([0,T])$, and $a=a(t,x)$ with

$$a\in\mathcal{C}^0([0,T]\times[0,1])\cap L^\infty(0,T;\mathrm{Lip}([0,1])), \tag{2.1}$$

$$a(t,x)\ge c>0 \qquad \forall (t,x)\in[0,T]\times[0,1], \tag{2.2}$$

where $c$ denotes some constant. We consider the following boundary initial value problem:

$$\partial_t y + a(t,x)\partial_x y = 0,\quad (t,x)\in(0,T)\times(0,1), \tag{2.3}$$

$$y(t,0)=y_l(t),\quad t\in(0,T), \tag{2.4}$$

$$y(0,x)=y_0(x),\quad x\in(0,1). \tag{2.5}$$

By (2.1), $a$ is uniformly Lipschitz continuous in the variable $x$. Since we intend to use the method of characteristics to solve (2.3)–(2.5), we need to study the flow associated with $a$.

For $(t,x) \in [0,T] \times [0,1]$, let $\phi(.,t,x)$ denote the $\mathcal{C}^1$ maximal solution to the Cauchy problem

$$\begin{cases} \partial_s \phi(s,t,x) = a(s,\phi(s,t,x)), \\ \phi(t,t,x) = x, \end{cases} \tag{2.6}$$

which is defined on a certain subinterval $[e(t,x), f(t,x)]$ of $[0,T]$ (which is closed since $[0,1]$ is compact), and with possibly $e(t,x)$ and/or $f(t,x) = t$. Let

$$\operatorname{Dom}\phi = \{(s,t,x);\ (t,x) \in [0,T]\times[0,1],\ s \in [e(t,x), f(t,x)]\}$$

denote the *domain* of $\phi$. Note that

$$e(t,x) > 0 \Rightarrow \phi(e(t,x),t,x) = 0. \tag{2.7}$$

We take into account the influence of the boundaries by introducing the sets

$$\begin{aligned} P &:= \{(s,\phi(s,0,0));\quad s \in [0, f(0,0)]\}, \\ I &:= \{(t,x) \in [0,T]\times[0,1] \setminus P;\ e(t,x) = 0\}, \\ J &:= \{(t,x) \in [0,T]\times[0,1] \setminus P;\ \phi(e(t,x),t,x) = 0\}. \end{aligned}$$

(See Figure 2.1.) Note that both $I$ and $J$ are open in $[0,T]\times[0,1]$.

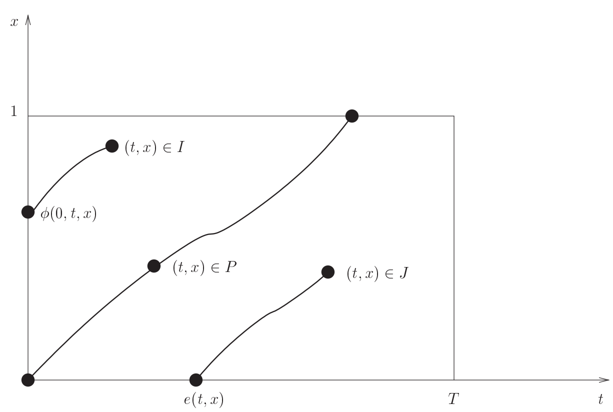


FIG. 2.1. *Partition of* $[0,T]\times[0,1]$ *into* $I \cup P \cup J$.

The first result is concerned with the regularity of $e$.

PROPOSITION 2.1. *Let $a$ be as above, let $(t,x) \in [0,T]\times[0,1]$, let $\{a_n\} \subset \mathcal{C}^0([0,T]\times[0,1]) \cap L^\infty(0,T;\mathrm{Lip}([0,1]))$ be a sequence such that $||a_n||_{L^\infty(0,T;\mathrm{Lip}([0,1]))}$ is bounded and*

$$||a_n - a||_{\mathcal{C}^0([0,T]\times[0,1])} \to 0 \qquad \text{as } n \to +\infty,$$

*and let $\{(t_n,x_n)\} \subset [0,T]\times[0,1]$ be a sequence such that $(t_n,x_n) \to (t,x)$. Then*

$$e_n(t_n,x_n) \to e(t,x). \tag{2.8}$$

Introduce the space

$$\mathcal{T} = \{\psi \in \mathcal{C}^1([0,T]\times[0,1]);\ \psi(t,1) = \psi(T,x) = 0 \quad \forall (t,x)\in[0,T]\times[0,1]\}. \tag{2.9}$$

We say that a function $y \in L^1((0,T)\times(0,1))$ is a *weak solution* of (2.3)–(2.5) if for any $\psi \in \mathcal{T}$ we have

$$\iint_{(0,T)\times(0,1)} y(t,x)(\psi_t(t,x) + a(t,x)\psi_x(t,x) + a_x(t,x)\psi(t,x))dtdx \\ + \int_0^T \psi(t,0)y_l(t)a(t,0)dt + \int_0^1 \psi(0,x)y_0(x)dx = 0. \tag{2.10}$$

The second result asserts the existence and uniqueness of a weak solution to (2.3)–(2.5).

Proposition 2.2. *Let us assume that $a$, $y_l$, and $y_0$ are uniformly Lipschitz continuous with Lipschitz constants $L$, $L_l$, and $L_0$, respectively, and that $y_l(0) = y_0(0)$. Then the function $y$ defined by*

$$y(t,x) = \begin{cases} y_l(e(t,x)) & \text{if } (t,x)\in J, \\ y_0(\phi(0,t,x)) & \text{if } (t,x)\in I\cup P \end{cases} \tag{2.11}$$

*is a weak solution of* (2.3)–(2.5)*. Furthermore, $y$ is $M$-Lipschitz continuous on $[0,T]\times[0,1]$ with $M$ defined by*

$$M := \max\left(\frac{L_l}{c}, L_0\right)\max\left(1, ||a||_{\mathcal{C}^0([0,T]\times[0,1])}\right)e^{LT}. \tag{2.12}$$

*Finally, $y$ is the unique solution in the class* $\mathrm{Lip}([0,T]\times[0,1])$ *of system* (2.3)–(2.5)*, with* (2.3) *understood in the distributional sense, and* (2.4)–(2.5) *pointwise.*

**2.3. Finite-time stabilization with two boundary controls.** Let $\lambda$ and $\mu$ be given functions with

$$\lambda,\mu \in \mathcal{C}^\infty(\mathbb{R}^2,\mathbb{R}), \tag{2.13}$$

$$\mu(u,v) \le -c < 0 < c \le \lambda(u,v) \qquad \forall (u,v)\in\mathbb{R}^2 \tag{2.14}$$

for some constant $c > 0$. Let $C_1 > 0$ and $C_2 > 0$ be given constants, and pick any $u_0, v_0 \in \mathrm{Lip}([0,1])$ with

$$\max(||u_0||_\infty, ||v_0||_\infty) \le C_1, \tag{2.15}$$

$$\max(||u_0'||_\infty, ||v_0'||_\infty) \le C_2. \tag{2.16}$$

We define $u_l$ and $v_r$ as the solutions of the ODEs

$$\frac{\mathrm{d}}{\mathrm{d}t}u_l(t) = -K\mathrm{sgn}(u_l(t))|u_l(t)|^\gamma, \quad u_l(0) = u_0(0), \tag{2.17}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}v_r(t) = -K\mathrm{sgn}(v_r(t))|v_r(t)|^\gamma, \quad v_r(0) = v_0(1), \tag{2.18}$$

with $(K,\gamma)\in(0,+\infty)\times(0,1)$ arbitrarily chosen. An obvious calculation gives

$$u_l(t) = \begin{cases} \mathrm{sgn}(u_0(0))\left(|u_0(0)|^{1-\gamma} - (1-\gamma)Kt\right)^{\frac{1}{1-\gamma}} & \text{if } 0\le t\le \frac{|u_0(0)|^{1-\gamma}}{(1-\gamma)K}, \\ 0 & \text{if } t \ge \frac{|u_0(0)|^{1-\gamma}}{(1-\gamma)K}, \end{cases}$$

and a similar expression for $v_r(t)$. Let

$$T := \frac{1}{c} + \frac{C_1^{1-\gamma}}{(1-\gamma)K}\cdot$$

Clearly

$$\forall t \geq T - \frac{1}{c}, \quad v_r(t) = u_l(t) = 0, \tag{2.19}$$

$$\max(||u_l||_\infty, ||v_r||_\infty) \leq C_1, \tag{2.20}$$

$$\max(||u_l'||_\infty, ||v_r'||_\infty) \leq KC_1^\gamma. \tag{2.21}$$

We are concerned with the well-posedness of the boundary initial-value problem

$$\partial_t u + \lambda(u,v)\partial_x u = 0, \qquad (t,x) \in (0,+\infty) \times (0,1), \tag{2.22}$$

$$\partial_t v + \mu(u,v)\partial_x v = 0, \qquad (t,x) \in (0,+\infty) \times (0,1), \tag{2.23}$$

$$u(t,0) = u_l(t), \quad v(t,1) = v_r(t), \tag{2.24}$$

$$u(0,x) = u_0(x), \quad v(0,x) = v_0(x). \tag{2.25}$$

We aim to prove the existence and uniqueness of a solution $(u,v)$ of (2.22)–(2.25) in some class of Lipschitz continuous functions. By *solution*, we mean that (2.22)–(2.23) is satisfied in the distributional sense, and that (2.24)–(2.25) are satisfied pointwise.

Let us introduce

$$M_1 := \max\Big(||\lambda||_{\mathcal{C}^0([-C_1,C_1]^2)}, ||\mu||_{\mathcal{C}^0([-C_1,C_1]^2)}\Big), \tag{2.26}$$

$$M_2 := \max\Big(||\partial_u\mu||_{\mathcal{C}^0([-C_1,C_1]^2)}, ||\partial_v\mu||_{\mathcal{C}^0([-C_1,C_1]^2)}, ||\partial_u\lambda||_{\mathcal{C}^0([-C_1,C_1]^2)}, ||\partial_v\lambda||_{\mathcal{C}^0([-C_1,C_1]^2)}\Big). \tag{2.27}$$

Pick any positive number $C_3$. Let $\mathcal{D}$ denote the domain

$$\mathcal{D} := \Big\{(u,v) \in \mathrm{Lip}([0,T]\times[0,1])^2;\ \max(||u||_\infty, ||v||_\infty) \leq C_1 \text{ and } u \text{ and } v \text{ are } C_3\text{-Lipschitz}\Big\}. \tag{2.28}$$

Let us equip the domain $\mathcal{D}$ with the topology of the uniform convergence. Then, by the Ascoli–Arzelà theorem, $\mathcal{D}$ is a compact set in $\mathcal{C}^0([0,T]\times[0,1])^2$.

The main result in this section is the following.

THEOREM 1. *Assume that* $C_1 > 0$ *and* $C_2 > 0$ *are such that*

$$TM_2 \max\Big(1, M_1\Big) \max\Big(\frac{KC_1^\gamma}{c}, C_2\Big) \leq \frac{1}{2e} \tag{2.29}$$

*and let* $C_3 = (2TM_2)^{-1}$. *Pick any pair* $(u_0,v_0) \in \mathrm{Lip}([0,1])^2$ *satisfying* (2.15)–(2.16). *Then there exists a unique solution* $(u,v)$ *of* (2.22)–(2.25) *in the class* $\mathcal{D}$. *Furthermore, the solution is global in time with* $u(t,.) = v(t,.) = 0$ *for* $t \geq T$. *Finally, the equilibrium state* $(0,0)$ *is stable in Lip*$([0,1])^2$ *for* (2.22)–(2.25)*; that is,*

$$||(u,v)||_{L^\infty(\mathbb{R}^+;\mathrm{Lip}([0,1])^2)} \to 0 \quad \text{as} \quad ||(u_0,v_0)||_{\mathrm{Lip}([0,1])^2} \to 0. \tag{2.30}$$

The first task consists in constructing a solution of the closed loop system as a fixed point of a certain operator.

**2.4. Definition of the operator.** If $(\tilde{u},\tilde{v}) \in \mathcal{D}$ are given, we define $(u,v) = \mathcal{F}(\tilde{u},\tilde{v})$ as follows: the function $u$ is the weak solution of the linear transport equation

$$\begin{cases} \partial_t u + \lambda(\tilde{u},\tilde{v})\partial_x u = 0, \\ u(t,0) = u_l(t), \qquad u(0,x) = u_0(x), \end{cases} \quad \forall (t,x) \in [0,T]\times[0,1], \tag{2.31}$$

and the function $v$ is the weak solution of the linear transport equation

$$\begin{cases} \partial_t v + \mu(\tilde{u},\tilde{v})\partial_x v = 0, \\ v(t,1) = v_r(t), \qquad v(0,x) = v_0(x), \end{cases} \quad \forall (t,x) \in [0,T]\times[0,1]. \tag{2.32}$$

**2.5. Stability of the domain.** In this part, we show that for a certain choice of $C_1,\ C_2,\ C_3$, we have $\mathcal{F}(\mathcal{D}) \subset \mathcal{D}$. We first apply the results of subsection 2.2 to get the following.

LEMMA 1. *Let $C_1, C_2, C_3$ be any positive numbers, and let $u_0, v_0 \in \mathrm{Lip}([0,1])$ satisfy* (2.15)–(2.16). *For given $(\tilde{u},\tilde{v}) \in \mathcal{D}$, let $(u,v) = \mathcal{F}(\tilde{u},\tilde{v})$. Then the functions $u$ and $v$ are Lipschitz continuous on $[0,T]\times[0,1]$ and satisfy the following estimates:*

$$\max(||u||_{\mathcal{C}^0([0,T]\times[0,1])}, ||v||_{\mathcal{C}^0([0,T]\times[0,1])}) \le C_1, \tag{2.33}$$

$$\max(||\partial_x u||_{L^\infty((0,T)\times(0,1))}, ||\partial_x v||_{L^\infty((0,T)\times(0,1))}, ||\partial_t u||_{L^\infty((0,T)\times(0,1))}, ||\partial_t v||_{L^\infty((0,T)\times(0,1))})$$
$$\le \max\Big(\frac{KC_1^\gamma}{c}, C_2\Big)\max\Big(1, M_1\Big)\exp\Big(2TM_2C_3\Big). \tag{2.34}$$

*Proof.* Estimate (2.33) follows directly from (2.11), (2.15), (2.20), (2.31), and (2.32).

Estimate (2.34) can be deduced applying (2.12) for (2.31) and (2.32), and using (2.16), (2.21), (2.26), (2.27), and (2.28). □

Thanks to Lemma 1, we see that the domain $\mathcal{D}$ is stable by $\mathcal{F}$ as soon as

$$\max\Big(\frac{KC_1^\gamma}{c}, C_2\Big)\max\Big(1, M_1\Big)\exp\Big(2TM_2C_3\Big) \le C_3. \tag{2.35}$$

If $M_2 = 0$ for some $C_1 > 0$ (linear case), then (2.35) holds for $C_3$ large enough. Otherwise, $M_2 > 0$ for any $C_1 > 0$ small enough, and (2.35) can be written

$$\max\Big(\frac{KC_1^\gamma}{c}, C_2\Big) \le \frac{C_3\exp\Big(-2TM_2C_3\Big)}{\max\Big(1, M_1\Big)}. \tag{2.36}$$

For given $C_1$ and $C_2$, $T, M_1$, and $M_2$ are fixed. Note that $T$, $M_1$, and $M_2$ are independent of $C_2$, and that they are nondecreasing in $C_1$. On the other hand, as a function of $C_3$ the supremum of the right-hand side of (2.36) is attained for $C_3 = (2TM_2)^{-1}$, and for this value of $C_3$ (2.35) then reads

$$TM_2\max\Big(1, M_1\Big)\max\Big(\frac{KC_1^\gamma}{c}, C_2\Big) \le \frac{1}{2e}. \tag{2.37}$$

But the term on the left-hand side of (2.37) tends to 0 when $C_1$ and $C_2$ tend to 0, so that for $C_1, C_2$ small enough the condition (2.37) is satisfied and $\mathcal{D}$ is stable by $\mathcal{F}$.

**2.6. Continuity of the operator.** In this part we consider a sequence $\{(\tilde{u}_n, \tilde{v}_n)\} \subset \mathcal{D}$ and a pair $(\tilde{u}, \tilde{v}) \in \mathcal{D}$ such that

$$\max\Big(||\tilde{u}_n - \tilde{u}||_{\mathcal{C}^0([0,T]\times[0,1])}, ||\tilde{v}_n - \tilde{v}||_{\mathcal{C}^0([0,T]\times[0,1])}\Big) \underset{n\to+\infty}{\to} 0. \tag{2.38}$$

Let us now define

$$(u_n, v_n) = \mathcal{F}(\tilde{u}_n, \tilde{v}_n) \quad \text{for } n \geq 0, \quad \text{and} \quad (u, v) = \mathcal{F}(\tilde{u}, \tilde{v}). \tag{2.39}$$

Our goal in this subsection is to show that

$$\max\Big(||u_n - u||_{\mathcal{C}^0([0,T]\times[0,1])}, ||v_n - v||_{\mathcal{C}^0([0,T]\times[0,1])}\Big) \underset{n\to+\infty}{\to} 0. \tag{2.40}$$

We need the following.

LEMMA 2. *For almost all* $(t, x) \in [0, T] \times [0, 1]$*, we have*

$$(u_n(t, x), v_n(t, x)) \underset{n\to+\infty}{\to} (u(t, x), v(t, x)). \tag{2.41}$$

*Proof.* Let us show that $u_n(t, x) \to u(t, x)$, the convergence $v_n(t, x) \to v(t, x)$ being similar.

The fact that $(\tilde{u}_n, \tilde{v}_n)$ converges uniformly toward $(\tilde{u}, \tilde{v})$ on $[0, T] \times [0, 1]$ implies that $\lambda(\tilde{u}_n, \tilde{v}_n)$ converges uniformly toward $\lambda(\tilde{u}, \tilde{v})$ on $[0, T] \times [0, 1]$. Furthermore, since $(\tilde{u}_n, \tilde{v}_n) \in \mathcal{D}$ for all $n$, we see that the functions $\lambda(\tilde{u}_n, \tilde{v}_n)$ are uniformly Lipschitz continuous for $n \geq 0$. This will allow us to use Proposition 2.1. To this end, we consider the flow $\phi_n$ (resp., $\phi$) of $\lambda(\tilde{u}_n, \tilde{v}_n)$ (resp., $\lambda(\tilde{u}, \tilde{v})$). In the same way, we define $e_n$ and $e$, $I_n$ and $I$, $J_n$ and $J$, and $P_n$ and $P$. Using (2.11) we have that

$$u_n(t, x) = \begin{cases} u_l(e_n(t, x)) & \text{if } (t, x) \in J_n, \\ u_0(\phi_n(0, t, x)) & \text{if } (t, x) \in I_n \cup P_n, \end{cases} \tag{2.42}$$

and also

$$u(t, x) = \begin{cases} u_l(e(t, x)) & \text{if } (t, x) \in J, \\ u_0(\phi(0, t, x)) & \text{if } (t, x) \in I \cup P. \end{cases} \tag{2.43}$$

We infer from Proposition 2.1 that

$$e_n(t, x) \underset{n\to+\infty}{\to} e(t, x) \quad \forall (t, x) \in [0, T] \times [0, 1]. \tag{2.44}$$

This shows in particular that if $(t, x) \in J$, then $e(t, x) > 0$ and hence $e_n(t, x) > 0$ for $n$ large enough, i.e., $(t, x) \in J_n$ for $n$ large enough. Therefore

$$u_n(t, x) \underset{n\to+\infty}{\to} u(t, x) \qquad \forall (t, x) \in J.$$

Now if $(t, x) \in I$, then $e(t, x) = 0$ and $\phi(0, t, x) > 0$. Since $\lambda \geq c > 0$, this implies the existence of $\epsilon > 0$ such that

$$\epsilon < \phi(s, t, x) \qquad \forall s \in [0, t]. \tag{2.45}$$

By a classical result about the convergence of the flows (see, e.g., [27]), we see that for $n$ large enough, $e_n(t, x) = 0$ and $\phi_n(0, t, x) \to \phi(0, t, x)$, so that

$$u_n(t, x) = u_0(\phi_n(0, t, x)) \underset{n\to+\infty}{\to} u_0(\phi(0, t, x)) = u(t, x). \tag{2.46}$$

Finally, $P$ is clearly negligible and $I \cup P \cup L = [0, T] \times [0, 1]$. ∎

To strengthen this convergence, we just need to recall that for every $n \geq 0$, we have $(u_n, v_n) \in \mathcal{D}$, with $\mathcal{D}$ compact in $\mathcal{C}^0([0,T] \times [0,1])$. According to Lemma 2, the only possible limit point is $(u,v)$, and therefore we get the convergence of the whole sequence in $\mathcal{D}$; that is,

$$\max\Big(||u_n - u||_{\mathcal{C}^0([0,T]\times[0,1])}, ||v_n - v||_{\mathcal{C}^0([0,T]\times[0,1])}\Big) \underset{n\to+\infty}{\to} 0. \tag{2.47}$$

This shows that the operator $\mathcal{F}$ is continuous on the domain $\mathcal{D}$, which is a convex compact set in $\mathcal{C}^0([0,T] \times [0,1])^2$. It follows then from Schauder fixed-point theorem that $\mathcal{F}$ has a fixed point. This proves the existence of solutions on the time interval $[0,T]$.

**2.7. Uniqueness of the solution.** Let $u_0, v_0 \in \mathrm{Lip}([0,1])$ be as in (2.15)–(2.16). Assume two pairs $(u^1, v^1), (u^2, v^2) \in \mathcal{D}$ of solutions of (2.22)–(2.25) are given; that is, if $u_l$ and $v_r$ are defined as in (2.17)–(2.18), then $u^i$, $i = 1, 2$, is a (weak) solution of

$$\begin{cases} \partial_t u^i + \lambda(u^i, v^i)\partial_x u^i = 0, \\ u^i(t,0) = u_l(t), \quad u^i(0,x) = u_0(x), \end{cases} \qquad (t,x) \in (0,T) \times (0,1),$$

while $v^i$, $i = 1, 2$, is a (weak) solution of

$$\begin{cases} \partial_t v^i + \mu(u^i, v^i)\partial_x v^i = 0, \\ v^i(t,1) = v_r(t), \quad v^i(0,x) = v_0(x), \end{cases} \qquad (t,x) \in (0,T) \times (0,1).$$

Let $\hat{u} = u^1 - u^2$ and $\hat{v} = v^1 - v^2$. Note that $\hat{u}, \hat{v} \in \mathrm{Lip}([0,T] \times [0,1]) = W^{1,\infty}((0,T) \times (0,1))$ and that $\hat{u}, \hat{v}$ fulfill

$$\partial_t \hat{u} + \lambda^1 \partial_x \hat{u} + \hat{\lambda}\partial_x u^2 = 0, \qquad (t,x) \in (0,T) \times (0,1), \tag{2.48}$$

$$\partial_t \hat{v} + \mu^1 \partial_x \hat{v} + \hat{\mu}\partial_x v^2 = 0, \qquad (t,x) \in (0,T) \times (0,1), \tag{2.49}$$

$$\hat{u}(t,0) = \hat{v}(t,1) = 0, \qquad \hat{u}(0,x) = \hat{v}(0,x) = 0, \tag{2.50}$$

where $\lambda^i = \lambda(u^i, v^i)$, $\mu^i = \mu(u^i, v^i)$, and $\hat{\lambda} = \lambda^1 - \lambda^2$, $\hat{\mu} = \mu^1 - \mu^2$.

Multiplying by $2\hat{u}$ in (2.48) and by $2\hat{v}$ in (2.49), integrating over $(0,t) \times (0,1)$, and adding the two equations gives

$$||\hat{u}(t)||_2^2 + ||\hat{v}(t)||_2^2 + 2\int_0^t\int_0^1 (\lambda^1 \hat{u}\partial_x \hat{u} + \mu^1 \hat{v}\partial_x \hat{v})\, dxds + 2\int_0^t\int_0^1 (\hat{\lambda}\hat{u}\partial_x u^2 + \hat{\mu}\hat{v}\partial_x v^2)\, dxds = 0.$$

Using (2.50) and an integration by parts, we obtain

$$\begin{aligned} 2\int_0^t\int_0^1 &(\lambda^1 \hat{u}\partial_x \hat{u} + \mu^1 \hat{v}\partial_x \hat{v}) \\ &= -\int_0^t\int_0^1 [(\partial_x \lambda^1)|\hat{u}|^2 + (\partial_x \mu^1)|\hat{v}|^2]\, dxds + \int_0^t [\lambda^1 |\hat{u}(s,1)|^2 - \mu^1 |\hat{v}(s,0)|^2] ds \\ &\geq -\int_0^t\int_0^1 [(\partial_x \lambda^1)|\hat{u}|^2 + (\partial_x \mu^1)|\hat{v}|^2]\, dxds, \end{aligned}$$

where we used (2.14). On the other hand, since $\lambda$ and $\mu$ are $M_2$-Lipschitz continuous on $[-C_1, C_1]^2$, we infer that $\lambda^i$ and $\mu^i$ are $2M_2C_3$-Lipschitz continuous on $[0,T] \times [0,1]$. In particular,

$$||\partial_x \lambda^1||_\infty \leq 2M_2C_3, \quad ||\partial_x \mu^1||_\infty \leq 2M_2C_3$$

and

$$|\hat{\lambda}| \le M_2(|\hat{u}| + |\hat{v}|),$$
$$|\hat{\mu}| \le M_2(|\hat{u}| + |\hat{v}|).$$

This yields

$$\left| 2\int_0^t \int_0^1 (\hat{\lambda}\hat{u}\partial_x u^2 + \hat{\mu}\hat{v}\partial_x v^2)\, dxds \right| \le 2M_2C_3 \int_0^t \int_0^1 (|\hat{u}| + |\hat{v}|)^2 dxds.$$

We conclude that for all $t \in (0, T)$

$$||\hat{u}(t)||_2^2 + ||\hat{v}(t)||_2^2 \le 6M_2C_3 \int_0^t (||\hat{u}||_2^2 + ||\hat{v}||_2^2) ds.$$

This yields $\hat{u} = \hat{v} \equiv 0$ by Gronwall's lemma.

**2.8. Finite-time extinction of the maximal solutions.** In this section, $(u, v)$ denotes the only solution of (2.22)–(2.25) in the class $\mathcal{D}$.

LEMMA 3. *At time $t = T$ we have*

$$u(T, x) = v(T, x) = 0 \quad \forall x \in [0, 1]. \tag{2.51}$$

*Proof of Lemma* 3. We infer from (2.22)–(2.23) that

$$u(t, 0) = v(t, 1) = 0 \quad \forall t \ge T - \frac{1}{c}. \tag{2.52}$$

Thanks to (2.13)–(2.14), we have that

$$\lambda(u(t, x), v(t, x)) \ge c > 0 > -c > \mu(u(t, x), v(t, x)) \qquad \forall (t, x) \in [0, T] \times [0, 1].$$

Let $\phi^\lambda$ (resp., $\phi^\mu$) denote the flow of $\lambda(u, v)$ (resp., $\mu(u, v)$), and let $e^\lambda$ (resp., $e^\mu$) denote the corresponding entrance times. (Note that $e^\mu > 0$ implies $\phi^\mu(e^\mu(t, x), t, x) = 1$.) Then the following holds:

$$e^\mu(T, x) \ge T - \frac{1}{c} \quad \text{ and } \quad e^\lambda(T, x) \ge T - \frac{1}{c} \quad \forall x \in [0, 1].$$

Combining this with (2.11) and (2.52), we obtain (2.51). ∎

Finally, it is sufficient to extend $u$ and $v$ by 0 for $t \ge T$ to get a global-in-time solution. The stability property (2.30) follows at once from (2.33)–(2.34), as the right-hand sides of (2.33) and (2.34) tend to 0 as $(C_1, C_2) \to (0, 0)$. The proof of Theorem 1 is complete. ∎

## 3. Finite-time stabilization with a control from one side.

In this section, we consider the system (2.22)–(2.25), with (2.24) replaced by

$$u(t, 0) = h(v(t, 0), t), \quad v(t, 0) = v_r(t), \tag{3.1}$$

where $v_r$ is still defined by (2.18).

In (3.1), $h$ denotes some function in $\mathcal{C}^1([-\overline{C_1}, \overline{C_1}] \times \mathbb{R}^+) \cap W^{1,\infty}((-\overline{C_1}, \overline{C_1}) \times (0, +\infty))$ for some number $\overline{C_1} > 0$ such that, for some time $T_h > 0$,

$$h(0, t) = 0 \qquad \forall t \ge T_h. \tag{3.2}$$

Pick any $C_1 \in (0, \overline{C_1}]$ and introduce the numbers

$$\begin{aligned}
C_1' &:= \max(C_1, ||h||_{L^\infty((-C_1,C_1)\times(0,+\infty))}),\\
T &:= \frac{1}{c} + \max\left(T_h, \frac{1}{c} + \frac{C_1^{1-\gamma}}{(1-\gamma)K}\right),\\
D_1 &:= ||\partial_v h||_{L^\infty((-C_1,C_1)\times(0,+\infty))}, \quad D_2 := ||\partial_t h||_{L^\infty((-C_1,C_1)\times(0,+\infty))},\\
M_1 &:= \max\Big(||\lambda||_{\mathcal{C}^0([-C_1',C_1']\times[-C_1,C_1])}, ||\mu||_{\mathcal{C}^0([-C_1',C_1']\times[-C_1,C_1])}\Big),\\
M_2 &:= \max\Big(1, ||\partial_u \mu||_{\mathcal{C}^0([-C_1',C_1']\times[-C_1,C_1])}, ||\partial_v \mu||_{\mathcal{C}^0([-C_1',C_1']\times[-C_1,C_1])},\\
&\qquad ||\partial_u \lambda||_{\mathcal{C}^0([-C_1',C_1']\times[-C_1,C_1])}, ||\partial_v \lambda||_{\mathcal{C}^0([-C_1',C_1']\times[-C_1,C_1])}\Big),\\
C_3 &:= \frac{1}{2TM_2}, \quad C_3' := \max\left(\frac{KC_1^\gamma}{c}, C_2\right)\max(1, M_1)\exp(2TM_2C_3).
\end{aligned}$$

Note that if $||v||_{\mathcal{C}^0([0,T]\times[0,1])} \le C_1$, then for all $t \in (0,T)$

$$|u(t,0)| \le C_1' \quad \text{and} \quad |\partial_t u(t,0)| \le D_1|\partial_t v(t,0)| + D_2.$$

We shall consider the following conditions:

$$C_3' \le C_3, \tag{3.3}$$

$$C_3'' := \max\left(\frac{1}{c}(D_1C_3' + D_2), C_2\right)\max(1, M_1)\exp(2TM_2C_3) \le C_3. \tag{3.4}$$

Note that (3.3) and (3.4) are satisfied if $C_1$, $C_2$, and $D_2$ are small enough.

We introduce the set

$$\mathcal{D} := \big\{(u,v) \in \mathrm{Lip}([0,T]\times[0,1])^2;\ ||u||_{\mathcal{C}^0([0,T]\times[0,1])} \le C_1',\ ||v||_{\mathcal{C}^0([0,T]\times[0,1])} \le C_1,\\ u \text{ is } C_3\text{-Lipschitz},\ v \text{ is } C_3'\text{-Lipschitz}\big\}.$$

We pick a pair $(u_0, v_0) \in \mathrm{Lip}([0,1])^2$ fulfilling (2.15)–(2.16) and the following compatibility condition:

$$u_0(0) = h(v_0(0), 0). \tag{3.5}$$

Let us make some comments about the boundary condition (3.1). For a system of conservation laws on the interval $(0,1)$, a very general boundary condition at $x = 0$ takes the form $f(u(t,0), v(t,0)) = 0$ with $f(0,0) = 0$. If $\partial_u f(0,0) \ne 0$, then around $(0,0)$ an application of the implicit function theorem gives a relation of the form $u(t,0) = h(v(t,0))$ with $h$ a smooth function of $v$ in a neighborhood of 0. Assume now that the interval represents an edge in a network, and that the left endpoint is a multiple node (i.e., it belongs to at least two edges). The contributions of the other edges at this multiple node can be taken into account in $h$ through its dependence in $t$ in (3.1).

We are in a position to state the main result of this section.

THEOREM 2. *Assume that $C_1, C_2$, and $D_2$ are such that the conditions* (3.3) *and* (3.4) *are satisfied. Then for any pair $(u_0, v_0) \in \mathrm{Lip}([0,1])^2$ fulfilling* (2.15), (2.16), *and* (3.5), *there exists a unique solution $(u,v)$ of* (2.22)–(2.23), (2.25), *and* (3.1) *in the class $\mathcal{D}$. Furthermore, the solution is global in time with $u(t,.) = v(t,.) = 0$ for*

*$t \geq T$. Finally, if $h = h(v)$, then the equilibrium state* $(0,0)$ *is stable in* $\mathrm{Lip}([0,1])^2$ *for* (2.22)–(2.23), (2.25), *and* (3.1)*; that is,*

$$||(u,v)||_{L^\infty(\mathbb{R}^+;\mathrm{Lip}([0,1])^2)} \to 0 \quad as \quad ||(u_0,v_0)||_{\mathrm{Lip}([0,1])^2} \to 0. \tag{3.6}$$

*Proof.* It is very similar to the proof of Theorem 1.

**3.1. Existence.** If $(\tilde{u},\tilde{v}) \in \mathcal{D}$ is given, we define $(u,v) = \mathcal{F}(\tilde{u},\tilde{v})$ as follows: $u$ is the weak solution of the system

$$\begin{cases} \partial_t u + \lambda(\tilde{u},\tilde{v})\partial_x u = 0, \\ u(t,0) = h(\tilde{v}(t,0),t), \qquad u(0,x) = u_0(x), \end{cases} \quad \forall (t,x) \in [0,T]\times[0,1],$$

and $v$ is the weak solution of the system

$$\begin{cases} \partial_t v + \mu(\tilde{u},\tilde{v})\partial_x v = 0, \\ v(t,1) = v_r(t), \qquad v(0,x) = v_0(x), \end{cases} \quad \forall (t,x) \in [0,T]\times[0,1].$$

Then, using Proposition 2.2 and (3.3)–(3.4), one readily sees that

$$||u||_{\mathcal{C}^0([0,T]\times[0,1])} \leq C_1', \ ||v||_{\mathcal{C}^0([0,T]\times[0,1])} \leq C_1,$$

$$u \text{ is } C_3''\text{-Lipschitz, and hence } u \text{ is } C_3\text{-Lipschitz}, \tag{3.7}$$

$$v \text{ is } C_3'\text{-Lipschitz}, \tag{3.8}$$

so that $\mathcal{F}$ maps $\mathcal{D}$ into itself. Let us prove that $\mathcal{F}$ is continuous, $\mathcal{D}$ being equipped with the topology of the uniform convergence. Consider a sequence $\{(\tilde{u}_n,\tilde{v}_n)\} \subset \mathcal{D}$ and a pair $(\tilde{u},\tilde{v}) \in \mathcal{D}$ such that

$$\max\Big(||\tilde{u}_n - \tilde{u}||_{\mathcal{C}^0([0,T]\times[0,1])}, ||\tilde{v}_n - \tilde{v}||_{\mathcal{C}^0([0,T]\times[0,1])}\Big) \underset{n\to+\infty}{\to} 0. \tag{3.9}$$

Let

$$(u_n,v_n) = \mathcal{F}(\tilde{u}_n,\tilde{v}_n) \quad \text{for } n \geq 0, \quad \text{and} \quad (u,v) = \mathcal{F}(\tilde{u},\tilde{v}). \tag{3.10}$$

We aim to prove that $u_n \to u$ and $v_n \to v$ uniformly on $[0,T]\times[0,1]$ as $n \to \infty$. We focus on $u_n$, the argument for $v_n$ being the same as those given in Lemma 2. We consider the same $\phi_n, \phi, e_n, e, I_n, I, J_n, J, P_n$, and $P$ as in the proof of Lemma 2. Then

$$u_n(t,x) = \begin{cases} h(\tilde{v}_n(e_n(t,x),0), e_n(t,x)) & \text{if } (t,x) \in J_n, \\ u_0(\phi_n(0,t,x)) & \text{if } (t,x) \in I_n \cup P_n \end{cases}$$

and

$$u(t,x) = \begin{cases} h(\tilde{v}(e(t,x),0), e(t,x)) & \text{if } (t,x) \in J, \\ u_0(\phi(0,t,x)) & \text{if } (t,x) \in I \cup P. \end{cases}$$

Assume first that $(t,x) \in J$. Then $e(t,x) > 0$ and $e_n(t,x) > 0$ for $n$ large enough, by Proposition 2.1. Since $\tilde{v}_n \to \tilde{v}$ uniformly on $[0,T]\times[0,1]$ and $e_n(t,x) \to e(t,x)$, we infer that

$$u_n(t,x) = h(\tilde{v}_n(e_n(t,x),0), e_n(t,x)) \to h(\tilde{v}(e(t,x),0), e(t,x)) = u(t,x).$$

If now $(t,x)\in I$, one can repeat the argument in Lemma 2 to conclude that

$$u_n(t,x) = u_0(\phi_n(0,t,x)) \to u_0(\phi(0,t,x)) = u(t,x).$$

Thus, $u_n(t,x)\to u(t,x)$ for $(t,x)\in I\cup J$, and hence for a.e. $(t,x)\in[0,T]\times[0,1]$. We have also that $v_n(t,x)\to v(t,x)$ for a.e. $(t,x)\in[0,T]\times[0,1]$. We infer from the compactness of $\mathcal{D}$ in $\mathcal{C}^0([0,T]\times[0,1])^2$ that $(u_n,v_n)\to(u,v)$ in $\mathcal{C}^0([0,T]\times[0,1])^2$. We conclude with the Schauder fixed-point theorem that $\mathcal{F}$ has a fixed point $(u,v)\in\mathcal{D}$, which is a solution of (2.22)–(2.23), (2.25), and (3.1) on $[0,T]\times[0,1]$.

**3.2. Uniqueness.** Let us now establish the uniqueness of the solution of (2.22)–(2.23), (2.25), and (3.1) in the class $\mathcal{D}$. Assume two pairs $(u^1,v^1),(u^2,v^2)\in\mathcal{D}$ of solutions of (2.22)–(2.23), (2.25), and (3.1) are given; that is, with $v_r$ defined as in (2.18), $v^i$, $i=1,2$, is a (weak) solution of

$$\begin{cases} \partial_t v^i + \mu(u^i,v^i)\partial_x v^i = 0, \\ v^i(t,1) = v_r(t), \quad v^i(0,x) = v_0(x), \end{cases} \qquad (t,x)\in(0,T)\times(0,1),$$

while $u^i$, $i=1,2$, is a (weak) solution of

$$\begin{cases} \partial_t u^i + \lambda(u^i,v^i)\partial_x u^i = 0, \\ u^i(t,0) = h(v^i(t,0),t), \quad u^i(0,x) = u_0(x), \end{cases} \qquad (t,x)\in(0,T)\times(0,1).$$

Let $\hat u = u^1-u^2$ and $\hat v = v^1 - v^2$. Note that $\hat u,\hat v\in W^{1,\infty}((0,T)\times(0,1))$ and that $\hat u,\hat v$ satisfy

$$\partial_t\hat u + \lambda^1\partial_x\hat u + \hat\lambda\partial_x u^2 = 0, \tag{3.11}$$

$$\partial_t\hat v + \mu^1\partial_x\hat v + \hat\mu\partial_x v^2 = 0, \tag{3.12}$$

$$\hat u(t,0) = h(v^1(t,0),t) - h(v^2(t,0),t), \tag{3.13}$$

$$\hat v(t,1) = 0, \tag{3.14}$$

$$\hat u(0,x) = \hat v(0,x) = 0, \tag{3.15}$$

where $\lambda^i = \lambda(u^i,v^i)$, $\mu^i=\mu(u^i,v^i)$, and $\hat\lambda = \lambda^1-\lambda^2$, $\hat\mu = \mu^1-\mu^2$.

Multiplying by $2\hat u$ in (3.11) and integrating over $(0,t)\times(0,1)$ gives

$$\begin{aligned} ||\hat u(t)||^2 &= \int_0^t\int_0^1 [(\partial_x\lambda^1)|\hat u|^2 - 2\hat\lambda\hat u\partial_x u^2]dxds - \int_0^t \lambda^1|\hat u|^2\Big|_0^1 ds \\ &\le 2M_2C_3\int_0^t\int_0^1[|\hat u|^2 + |\hat u|(|\hat u|+|\hat v|)]dxds \\ &\quad + ||\lambda||_{\mathcal{C}^0([-C_1',C_1']\times[-C_1,C_1])}D_1^2\int_0^t|\hat v(s,0)|^2ds, \end{aligned} \tag{3.16}$$

where we used (2.14). Multiplying by $2\hat v$ in (3.12) and integrating over $(0,t)\times(0,1)$ gives

$$\begin{aligned} ||\hat v(t)||^2 &= \int_0^t\int_0^1[(\partial_x\mu^1)|\hat v|^2 - 2\hat\mu\hat v\partial_x v^2]dxds + \int_0^t \mu^1|\hat v(s,0)|^2ds \\ &\le 2M_2C_3\int_0^t\int_0^1[|\hat v|^2 + |\hat v|(|\hat u|+|\hat v|)]dxds - c\int_0^t|\hat v(s,0)|^2ds, \end{aligned} \tag{3.17}$$

where we used (2.14) again. Let us introduce the energy

$$E(t) := ||\hat{u}(t)||^2 + ||\lambda||_{\mathcal{C}^0([-C_1',C_1']\times[-C_1,C_1])} \frac{D_1^2}{c} ||\hat{v}(t)||^2.$$

Combining (3.16) with (3.17) yields

$$E(t) \le C \int_0^t E(s)ds$$

for some $C$ depending only on $\mathcal{D}$, so that $E \equiv 0$ by Gronwall's lemma. This proves the uniqueness.

**3.3. Finite-time stability.** For the extinction time, we notice that from the proof of Theorem 1

$$v(t,x) = 0 \qquad \text{for } \frac{1}{c} + \frac{C_1^{1-\gamma}}{(1-\gamma)K} \le t \le T,\ 0 \le x \le 1.$$

Combined with (3.2), this yields

$$u(t,0) = h(v(t,0),t) = 0 \qquad \text{for } \max\left(T_h, \frac{1}{c} + \frac{C_1^{1-\gamma}}{(1-\gamma)K}\right) \le t \le T.$$

Using (2.14), we conclude that

$$u(T,x) = 0 \qquad \forall x \in [0,1].$$

Assume now that $h = h(v)$, i.e., $D_2 = 0$. The stability property (3.6) follows at once from (3.2) and (3.3)–(3.4), as $C_1' \le \max(1, D_1)C_1$ and $(C_3', C_3'') \to (0,0)$ as $(C_1, C_2) \to (0,0)$. The proof of Theorem 2 is complete. □

**4. Application to the regulation of water flow in channels.** In this section, we investigate the regulation of water flow in a network of open horizontal channels. We assume that the channels have a rectangular cross section and that the friction on the walls can be neglected. In this context, the flow of the fluid can be described in a satisfactory way by the shallow water equations (also called Saint-Venant equations) (see [10]). The control in feedback form is applied at the vertices of the network, which is assumed to be a tree.

We introduce some notation needed in what follows (we follow closely [9]). Let $\mathcal{T}$ be a tree whose vertices (or nodes) are numbered by the index $n \in \mathcal{N} = \{1, \ldots, N\}$, and whose edges are numbered by the index $i \in \mathcal{I} = \{1, \ldots, I\}$ with $I = N - 1$. We choose a simple vertex, called the *root* of $\mathcal{T}$ and denoted by $\mathcal{R}$, and which corresponds to the index $n = N$. We choose an orientation of the edges in the tree such that $\mathcal{R}$ is the "last" encountered vertex. It is similar to those of a fluvial network in which each edge stands for a river, and $\mathcal{R}$ indicates the place where the last river enters into the sea.

We denote by $l_i$ the length of the edge with index $i$. Once the orientation is chosen, each point of the $i$th edge is identified with a real number $x \in [0, l_i]$. The points $x = 0$ and $x = l_i$ are termed the *initial point* and the *final point* of the $i$th edge, respectively.

Renumbering the edges if needed, we may assume that the edge with index $i$ has as initial point the vertex with the (same) index $n = i$ for all $i \in \mathcal{I}$. (See Figure 4.1.)

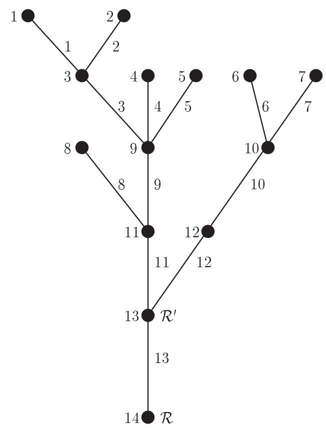


FIG. 4.1. *A tree with* 14 *nodes, a depth equal to* 5, *with simple nodes* $\mathcal{N}_S = \{1, 2, 4, 5, 6, 7, 8, 14\}$ *and multiple nodes* $\mathcal{N}_M = \{3, 9, 10, 11, 12, 13\}$.

We denote by $\mathcal{I}_n \subset \mathcal{I}$, $n = 1, \ldots, N$, the set of indices of those edges having the vertex of index $n$ as one of their ends. Let

$$\varepsilon_{i,n} = \begin{cases} 0 & \text{if the vertex with index } n \text{ is the initial point of the edge with index } i; \\ 1 & \text{if the vertex with index } n \text{ is the final point of the edge with index } i. \end{cases}$$

Note that $\varepsilon_{i,i} = 0$ for all $i \in \mathcal{I}$, and that $\varepsilon_{N-1,N} = 1$. A node with index $n$ is said to be *simple* (resp., *multiple*) if $\#(\mathcal{I}_n) = 1$ (resp., $\#(\mathcal{I}_n) \geq 2$). The sets of indices of simple and multiple nodes are denoted by $\mathcal{N}_S$ and $\mathcal{N}_M$, respectively. The *depth* of the tree is the greater number of edges in a path from one simple node to $\mathcal{R}$.

Pick any channel represented by (say) the $i$th edge of the tree, which is identified with the segment $[0, l_i]$. Then the shallow water equations read

$$\partial_t H_i + \partial_x (H_i V_i) = 0, \quad t > 0,\ 0 < x < l_i, \tag{4.1}$$

$$\partial_t V_i + \partial_x \left( \frac{V_i^2}{2} + g H_i \right) = 0, \quad t > 0,\ 0 < x < l_i. \tag{4.2}$$

Let us introduce

$$Q_i(t, x) := H_i(t, x) V_i(t, x).$$

Then $H_i(t, x)$, $V_i(t, x)$, and $Q_i(t, x)$ denote, respectively, the *water depth*, the *flow velocity*, and the *flow rate* along the $i$th channel, and $g$ is the gravitation constant. Equations (4.1)–(4.2) have to be supplemented with some initial conditions

$$H_i(0, x) = H_{i,0}(x), \quad V_i(0, x) = V_{i,0}(x), \qquad 0 < x < l_i, \tag{4.3}$$

and with two boundary conditions. In general, there are at the two ends of the channel (i.e., at $x = 0$ and at $x = l_i$) some hydraulic devices to assign the values of the flow rate.

At any multiple node $n \in \mathcal{N}_M$, the equation of conservation of the flow

$$\sum_{i \in \mathcal{I}_n} (-1)^{\varepsilon_{i,n}} Q_i(t, \varepsilon_{i,n} l_i) = 0 \tag{4.4}$$

has to be taken into consideration. It yields a boundary condition (coming from the physics) in which no control applies. Note that (4.4) can be rewritten

$$Q_n(t,0) = \sum_{i\in\mathcal{I}_n, i\neq n} Q_i(t,l_i). \tag{4.5}$$

Thus, the flow rate can be controlled at the final points of the edges of indices $i \neq n$, while it is prescribed by (4.5) at the initial point of the edge of index $n$.

We aim to stabilize the system around some equilibrium state, represented by a sequence $\{(H_i^*, V_i^*)\}_{1\le i\le I}$ of pairs of positive numbers. Let $Q_i^* = H_i^* V_i^*$. For (4.4) to be valid as $t \to \infty$, we impose that

$$\sum_{i\in\mathcal{I}_n} (-1)^{\varepsilon_{i,n}} Q_i^* = 0 \qquad \forall n \in \mathcal{N}_M. \tag{4.6}$$

Introduce the characteristic velocities

$$\mu_i = V_i - \sqrt{gH_i}, \tag{4.7}$$

$$\lambda_i = V_i + \sqrt{gH_i} \tag{4.8}$$

and the Riemann invariants (see [12, 10])

$$u_i = V_i + 2\sqrt{gH_i} - (V_i^* + 2\sqrt{gH_i^*}), \tag{4.9}$$

$$v_i = V_i - 2\sqrt{gH_i} - (V_i^* - 2\sqrt{gH_i^*}). \tag{4.10}$$

We shall assume thereafter that the flow is *subcritical* or *fluvial*; that is, the characteristic velocities are of opposite sign

$$\mu_i < 0 < \lambda_i.$$

Clearly, this holds if

$$0 < V_i^* < \sqrt{gH_i^*} \tag{4.11}$$

and $\max(|H_i - H_i^*|, |V_i - V_i^*|)$ is small enough. From now on, we assume that (4.11) holds for all $i \in \mathcal{I}$, and we pick a number $c > 0$ such that

$$\sqrt{gH_i^*} - V_i^* > 2c \quad \forall i \in \mathcal{I}. \tag{4.12}$$

Inverting (4.9)–(4.10) and substituting the values of $H_i, V_i$ in (4.7)–(4.8) yields

$$\mu_i = V_i^* - \sqrt{gH_i^*} + \frac{1}{4}(u_i + 3v_i), \tag{4.13}$$

$$\lambda_i = V_i^* + \sqrt{gH_i^*} + \frac{1}{4}(3u_i + v_i). \tag{4.14}$$

Combined with (4.12), this shows that

$$\max(|u_i|, |v_i|) \le c \quad \Rightarrow \quad \mu_i < -c < c < \lambda_i.$$

The shallow water equations (4.1)–(4.2), when expressed in terms of the Riemann invariants $u_i$ and $v_i$, read

$$\partial_t u_i + \lambda_i(u_i, v_i)\partial_x u_i = 0, \quad t > 0,\ 0 < x < l_i, \tag{4.15}$$

$$\partial_t v_i + \mu_i(u_i, v_i)\partial_x v_i = 0, \quad t > 0,\ 0 < x < l_i. \tag{4.16}$$

Let us now turn our attention to the boundary conditions. Consider first a boundary condition associated with an active control, e.g.,

$$\frac{\mathrm{d}}{\mathrm{dt}} v_i(t, l_i) = -K\mathrm{sgn}(v_i(t, l_i))|v_i(t, l_i)|^\gamma. \tag{4.17}$$

In practice, one would like to assign the value of $Q_i(t, l_i) = H_i(t, l_i)V_i(t, l_i)$ by using the output $H_i(t, l_i)$ only. Using (4.10), it is sufficient to set

$$Q_i(t, l_i) = H_i(t, l_i)\left(v_i(t, l_i) + 2\sqrt{gH_i(t, l_i)} + V_i^* - 2\sqrt{gH_i^*}\right), \tag{4.18}$$

where $v_i$ solves (4.17) together with the initial condition

$$v_i(0, l_i) = V_i(0, l_i) - 2\sqrt{gH_i(0, l_i)} - V_i^* + 2\sqrt{gH_i^*}. \tag{4.19}$$

For a control applied to the initial point of the $i$th edge (a simple node), we set

$$Q_i(t, 0) = H_i(t, 0)\big(u_i(t, 0) - 2\sqrt{gH_i(t, 0)} + V_i^* + 2\sqrt{gH_i^*}\big), \tag{4.20}$$

where $u_i(., 0)$ solves

$$\frac{d}{dt} u_i(t, 0) = -K\mathrm{sgn}(u_i(t, 0))|u_i(t, 0)|^\gamma, \tag{4.21}$$

$$u_i(0, 0) = V_i(0, 0) + 2\sqrt{gH_i(0, 0)} - V_i^* - 2\sqrt{gH_i^*}. \tag{4.22}$$

Finally, for a multiple node $n \in \mathcal{N}_M$, if $i_0 \in \mathcal{I}_n$ is the only index such that $\varepsilon_{i_0,n} = 0$ (i.e., $i_0 = n$), then (4.5) may be written

$$F_{i_0}(U_{i_0}(t, 0), v_{i_0}(t, 0), U(t), V(t)) = 0,$$

where $U(t) = (u_i(t, l_i))_{i\in\mathcal{I}_n, i\neq i_0}$, $V(t) = (v_i(t, l_i))_{i\in\mathcal{I}_n, i\neq i_0}$, and

$$\begin{aligned} F_{i_0}(u_{i_0}, v_{i_0}, U, V) = &\left(\sqrt{H_{i_0}^*} + \frac{1}{4\sqrt{g}}(u_{i_0} - v_{i_0})\right)^2 \left(V_{i_0}^* + \frac{1}{2}(u_{i_0} + v_{i_0})\right) \\ &- \sum_{i\in\mathcal{I}_n, i\neq i_0} \left(\sqrt{H_i^*} + \frac{1}{4\sqrt{g}}(u_i - v_i)\right)^2 \left(V_i^* + \frac{1}{2}(u_i + v_i)\right). \end{aligned}$$

Note that, by (4.6), $F_{i_0}(0, 0, 0, 0) = 0$ and

$$\frac{\partial F_{i_0}}{\partial u_{i_0}}(0, 0, 0, 0) = \frac{1}{2}\sqrt{H_{i_0}^*}\Big(\sqrt{H_{i_0}^*} + \frac{V_{i_0}^*}{\sqrt{g}}\Big) > 0.$$

By the implicit function theorem, we can pick a number $\delta_{i_0} > 0$ and a function $H_{i_0}$ of class $\mathcal{C}^1$ around 0 such that if $|u_i| < \delta_{i_0}$ and $|v_i| < \delta_{i_0}$ for all $i \in \mathcal{I}_n$, we have

$$F_{i_0}(u_{i_0}, v_{i_0}, U, V) = 0 \quad \Longleftrightarrow \quad u_{i_0} = H_{i_0}(v_{i_0}, U, V).$$

Replacing $(u_i, v_i)$ by $(u_i(t, l_i), v(t, l_i))$ for $i \neq i_0$ in $U, V$, we see that (4.5) may be written, at least locally, in the form

$$u_{i_0}(t, 0) = h_{i_0}(v_{i_0}(t, 0), t), \tag{4.23}$$

where $h_{i_0} \in \mathcal{C}^1(\mathbb{R}^2)$ and $h_{i_0}(0, t) = 0$ if $u_i(t, l_i) = v_i(t, l_i) = 0$ for all $i \in \mathcal{I}_n \setminus \{i_0\}$.

We are in a position to state our results for the regulation of water flow in a network of channels. We assume that the incoming flows can be controlled at each multiple node (the outgoing flow being uncontrolled and deduced from the conservation of the flows). In terms of Riemann invariants, for the edge with index $i$, the function $v_i$ is controlled at $x = l_i$ according to (4.17), while the function $u_i$ is controlled at $x = 0$ only if the initial point of the edge is a simple node (otherwise, $u_i(t, 0)$ is given by (4.23)).

THEOREM 3. *Consider a tree with $N$ nodes and $I = N - 1$ edges. Assume that* (4.6) *holds and that* (4.11) *holds for $i = 1, \dots, I$, and pick any $c > 0$ as in* (4.12). *Then there exists a number $\delta > 0$ such that for all $(H_{1,0}, V_{1,0}, \dots, H_{I,0}, V_{I,0}) \in \mathrm{Lip}([0, l_1])^2 \times \cdots \times \mathrm{Lip}([0, l_I])^2$ with*

$$\max(||H_{i,0} - H_i^*||_{W^{1,\infty}(0,l_i)}, ||V_{i,0} - V_i^*||_{W^{1,\infty}(0,l_i)}) < \delta, \qquad i = 1, \dots, I, \tag{4.24}$$

$$H_{n,0}(0, 0)V_{n,0}(0, 0) = \sum_{i \in \mathcal{I}_n, i \neq n} H_{i,0}(0, l_i)V_{i,0}(0, l_i) \qquad \forall n \in \mathcal{N}_M, \tag{4.25}$$

*there exists for any $T > 0$ a unique function $(H_1, V_1, \dots, H_I, V_I) \in \mathrm{Lip}([0, T] \times [0, l_1])^2 \times \cdots \times \mathrm{Lip}([0, T] \times [0, l_I])^2$ such that, for all $i = 1, \dots, I$,* (4.1)–(4.3) *and* (4.17)–(4.19) *hold, and* (4.20)–(4.22) *hold if the initial point of the $i$th edge is simple, while* (4.5) *holds if the initial point of the $i$th edge is multiple. Furthermore, there exists a function $t^*(H_1^*, V_1^*, \dots, H_I^*, V_I^*, \delta, c, K, \gamma)$ with $\lim_{\delta \to 0} t^* = 0$ such that*

(4.26)

$$H_i(t, x) = H_i^*, \quad V_i(t, x) = V_i^*, \qquad t \geq \frac{d \max_{1 \leq i \leq I} l_i}{c} + t^*, \ x \in (0, l_i), \ i = 1, \dots, I,$$

*where $d$ denotes the depth of the tree. Finally, the equilibrium state $(H_i^*, V_i^*)_{1 \leq i \leq I}$ is stable in $\mathrm{Lip}([0, l_1])^2 \times \cdots \times \mathrm{Lip}([0, l_I])^2$ for the system.*

*Proof.* The proof is done by induction on the number of edges $I \geq 1$.

**4.1. Step 1: $I = 1$.** Noticing that the map $\Theta : (H_1, V_1) \to (u_1, v_1)$ defined along (4.9)–(4.10) is locally around $(H_1^*, V_1^*)$ a diffeomorphism of class $\mathcal{C}^\infty$, the condition (4.24) implies (2.15)–(2.16) for $C_1$ and $C_2$ as in Theorem 1 (applied actually on the interval $(0, l_1)$ rather than $(0, 1)$), provided that $\delta < \delta_0$ is small enough. We modify the functions $\mu_1(u, v)$ and $\lambda_1(u, v)$ outside $[-c, c]^2$ so that

$$\mu_1(u, v) \leq -c < c \leq \lambda_1(u, v) \qquad \forall (u, v) \in \mathbb{R}^2.$$

Let $(u_1, v_1)$ be the solution given by Theorem 1, and let $(H_1, V_1) := \Theta^{-1}(u_1, v_1)$. If $C_1$ is chosen sufficiently small, then we infer from (2.33) that

$$\max(|u_1(t, x)|, |v_1(t, x)|) \leq C_1 < c, \quad t \geq 0, \ 0 < x < l_1,$$
$$\max(|H_1(t, x) - H_1^*|, |V_1(t, x) - V_1^*|) < \delta, \quad t \geq 0, \ 0 < x < l_1.$$

It follows that for all $T > 0$, $(H_1, V_1) \in \mathrm{Lip}([0, T] \times [0, l_1])^2$ is a solution of (4.1)–(4.3) and (4.17)–(4.22) such that (4.26) holds with $d = 1$ and $t^* = C_1^{1-\gamma}/((1 - \gamma)K)$. Note that the range of $C_1$ in Theorem 1 depends on $H_1^*$, $V_1^*$, and $c$ through the constants $M_1$ and $M_2$, and that $(C_1, C_2) \to (0, 0)$ as $\delta \to 0$. Thus $t^* \to 0$ as $\delta \to 0$ with $K$ and $\gamma$ kept constant. The uniqueness of $(H_1, V_1)$ in the class $\mathrm{Lip}([0, T] \times [0, l_1])^2$ for all $T > 0$ follows at once from those of $(u_1, v_1)$ in the same class, as stated in Theorem 1. The stability property follows from (2.30). Note that the norm $||(u_1, v_1)||_{L^\infty(\mathbb{R}^+; \mathrm{Lip}([0,l_1])^2)}$ is as small as desired if $\delta$ is small enough.

**4.2. Step 2.** Let $I \geq 2$, and assume the result true for any tree with at most $I-1$ edges, with the norms $||(u_i, v_i)||_{L^\infty(\mathbb{R}^+;\mathrm{Lip}([0,l_i])^2)}$ in the edges of the tree as small as desired if $\delta$ is small enough. Pick any tree with $I$ edges. Recall that the root $\mathcal{R}$ is the node with index $N$, and that it is the final point of the edge of index $I = N-1$. Denote by $\mathcal{R}'$ the node of index $N-1$. Let $k = \#(\mathcal{I}_{N-1})$, and let us denote by $\mathcal{T}_1, \ldots, \mathcal{T}_{k-1}$ the subtrees of $\mathcal{T}$ with $\mathcal{R}'$ as root. ($\mathcal{R}$ does not belong to any of them.) Note that the subsystem associated with any subtree $\mathcal{T}_i$ is decoupled from the other subtrees and from the last edge of index $I$. An application of the induction hypothesis on each subtree $\mathcal{T}_i$, $1 \leq i \leq k-1$, yields the existence (and uniqueness) of the functions $(H_i, V_i)$ for $i = 1, \ldots, I-1$. Next, the existence and uniqueness of $(H_I, V_I)$ follows at once from Theorem 2. Indeed, the constant $D_2$ in Theorem 2 may be taken as small as we want if $\delta$ is sufficiently small, for the quantities $||\partial_t u_i(., l_i)||_\infty$ and $||\partial_t v_i(., l_i)||_\infty$ for $i \in \mathcal{I}_{N-1} \backslash \{N-1\}$ can be taken arbitrarily small by the induction assumption. Furthermore, the norm $||(u_I, v_I)||_{L^\infty(\mathbb{R}^+;\mathrm{Lip}([0,l_I])^2)}$ tends to 0 with $\delta$, by (3.7)–(3.8). The condition (4.26) is obtained by an obvious induction on the depth of the tree. ◻

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