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Determination of the Calcium channel distribution in the olfactory system

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Abstract. In this paper we study a linear inverse problem with a biological interpretation, which is modeled by a Fredholm integral equation of the first kind. When the kernel in the Fredholm equation is represented by step functions, we obtain identifiability, stability and reconstruction results. Furthermore, we provide a numerical reconstruction algorithm for the kernel, whose main feature is that a non-regular mesh has to be used to ensure the invertibility of the matrix representing the numerical discretization of the system. Finally, a second identifiability result for a polynomial approximation of degree less than nine of the kernel is also established.

Keywords. Inverse Problems, Olfactory System, Kernel Determination, Partial Differential Equations, Numerical Reconstruction.

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1 Introduction

In this work we study an integral inverse problem coming from the biology of the olfactory system. The transduction of an odor into an electrical signal is accomplished by a depolarizing influx of ions through cyclic-nucleotide-gated (CNG) channels in the membrane. Those channels, that form the lateral surface of the cilium, are activated by adenosine 3', 5'-cyclic monophosphate (cAMP).

The distribution of the channels should be crucial in determining the kinetics of the neuronal response.

Experimental procedures developed by Steven Kleene and Rick Flannery in the College of Medicine (University of Cincinnati) have produced data from which the distributions of CNG channels can be inferred using mathematical and computational procedures developed by Donald French et al. (see [5]). The techniques for the procedure have been developed in [7–10].

We explore the hypothesis that CNG channel distributions can be derived from the experimental current data and known properties of the cilia (a biological inverse problem). To accomplish this, we consider a mathematical model of this experiment to proposed by D.A. French & C.W. Groetsch [6].

D.A. French et al. [5] proposed a mathematical model for the dynamics of cAMP concentration in this experiment, consisting of two nonlinear differential equations and a constrained Fredholm integral equation of first kind. The unknowns of the system of differential equations proposed by French are the concentration of cAMP, the membrane potential and the distribution ρ of CNG channels along the length of a cilium. A very natural issue is whether it is possible to recover the distribution of CNG channels along the length of a cilium by only measuring the electrical activity produced by the diffusion of cAMP into cilia. A simple numerical method to obtain estimates of channels distribution was proposed in [5]. Certain computations indicated that this mathematical problem was ill-conditioned.

Later, D.A. French & D.A. Edwards [4] studied the above inverse problem by using perturbation techniques. A simple perturbation approximation was derived and used to solve the inverse problem, and to obtain estimates of the spatial distribution of CNG ion channels. A one-dimensional computer minimization and a special delay iteration were used with the perturbation formulas to obtain approximate channel distributions in the cases of simulated and experimental data. On the other hand, D.A. French & C.W. Groetsch [6] introduced some simplifications and approximations in the problem, leading to an analytical solution for the inverse problem. A numerical procedure was proposed for a class of integral equations suggested by this simplified model and numerical results were compared to laboratory data.

In this paper we consider the linear problem proposed in [6], with an improved approximation of the kernel, along with studying the identifiability, stability and numerical reconstruction for the corresponding inverse problem.

The inverse problem we are interesting in this work consists in determining a positive function $\rho = \rho(x) > 0$ from the measurement of

$$I_m[\rho](t) = J_0 \int_0^L \rho(x) K_m(t,x)dx, \tag{1.1}$$

for $t \in \mathcal{I}$, where $\mathcal{I}$ is a time interval, ρ is the channel distribution, J_0 is a positive constant and the kernel $K_m(t,x)$ is defined by

$$K_m(t,x) = F_m(w(t,x)), \tag{1.2}$$

where $w(t,x)$, defined in (2.14), represents an approximation of the concentration of cAMP $c(t,x)$ defined in (2.3), while F_m, defined in (2.7), is a step function approximation of the Hill function F, given by

$$F(x) = \frac{x^n}{x^n + K_{1/2}^n}\cdot \tag{1.3}$$

In (1.3), the exponent n is an experimentally determined parameter and $K_{1/2} > 0$ is a constant which corresponds to the half-bulk concentration.

Under a strong assumption about the regularity of ρ (namely, ρ is analytic), we obtain in Theorem 3.4 an identifiability result for (1.1) with a single measurement of $I_m[\rho]$ on an *arbitrary small* interval around zero. The second identifiability result, Theorem 3.5, requires weaker regularity assumptions about ρ (namely, $\rho \in L^2(0, L)$), but it requires the measurement of $I_m[\rho]$ on a large time interval.

Furthermore, in Theorem 3.9, using appropriate weighted norms and Mellin transform (see [12]), we obtain a general stability result for the operator $I_m[\rho]$ for $\rho \in L^2(0, L)$. Using a *non-regular* mesh for the approximation of F_m, we develop a reconstruction procedure in Theorem 3.10 to recover ρ from I_m. Additionally, for this non-regular mesh, a general stability result for a large class of norms is rigorously established in Theorem 3.11.

Finally, we also investigate the same inverse problem with another approximation of the kernel obtained by replacing Hill's function by its Taylor expansion of degree m around $c_0 > 0$.

More precisely, the polynomial kernel approximation is defined as

$$PK_m(t, x) = P_m(c(t, x) - c_0), \tag{1.4}$$

where $P_m \in \mathbb{R}[x]$ with deg $(P_m) \leq m$ is such that

$$F(x) = P_m(x - c_0) + O(|x - c_0|^{m+1}),$$

and $c(t, x)$, the concentration of cAMP, is defined as the solution of the diffusion problem (2.3). Thus, the total current with polynomial approximation is given by

$$PI_m[\rho](t) = J_0 \int_0^L \rho(x) PK_m(t, x) dx \quad \forall t > 0. \tag{1.5}$$

In Theorem 8.1 we derive an identifiability result for the operator PI_m, when the degree of P_m is less than nine.

The paper is organized as follows. In Section 2, we set the problem, introduce the principal assumptions and some operator Φ_m that we use to derive the main results regarding the operator I_m. Those results are presented in Section 3. Section 4 is devoted to prove the identifiability theorems. Section 5 contains the proof of Theorem 3.9 concerning the stability of I_m. The proof of the results involving the reconstruction procedure are developed in Section 6, while the numerical algorithm and some examples are shown in Section 7. Finally, in Section 8, we prove an identifiability result for PI_m (Theorem 8.1).

2 Setting the problem

In this section we set the mathematical model related to the inverse problem arising in olfaction experimentation.

The starting point is the linear model introduced in [6]. As already mentioned, a nonlinear integral equation model was developed in [5] to determine the spatial distribution of ion channels along the length of frog olfactory cilia. The essential nonlinearity in the model arises from the binding of the channel activating ligand to the cyclic-nucleotide-gated ion channels as the ligand diffuses along the length of the cilium. We investigate a linear model for this process, in which the binding mechanism is neglected, leading to a particular type of linear Fredholm integral equation of the first kind with a diffusive kernel. The linear inverse problem consists in determining $\rho = \rho(x) > 0$ from the measurement of

$$I[\rho](t) = J_0 \int_0^L \rho(x) K(t,x)\,dx, \quad t \geq 0, \tag{2.1}$$

where the kernel is

$$K(t,x) = F(c(t,x)), \tag{2.2}$$

F being given by (1.3) and c denoting the concentration of cAMP, which is governed by the following initial boundary value problem:

$$\begin{aligned} \frac{\partial c}{\partial t} - D\frac{\partial^2 c}{\partial x^2} &= 0, \quad t>0,\ x \in (0,L), \\ c(t,0) &= c_0, \quad t>0, \\ \frac{\partial c}{\partial x}(t,L) &= 0, \quad t>0, \\ c(0,x) &= 0, \quad x \in (0,L). \end{aligned} \tag{2.3}$$

The (unknown) function ρ is the ion channel density function, and c is the concentration of a channel activating ligand that is diffusing from left-to-right in a thin cylinder (the interior of the cilium) of length L with diffusivity constant D. $I[\rho](t)$ is a given total transmembrane current, the constant J_0 has units of current/length, and c_0 is the maintained concentration of cAMP at the open end of the cylinder (while $x = L$ is considered as the closed end). We note that (2.1) is a Fredholm integral equation of the first kind.

The associated inverse problem is in general ill-posed. For instance, if K is sufficiently smooth, then the operator defined above is compact from $L^p(0,L)$ to $L^p(0,T)$ for $1 < p < \infty$. Even if the operator I is injective, its inverse will not be continuous. Indeed, if I is compact and I^{-1} is continuous, then it follows that the identity map in $L^p(0,L)$ is compact, a property which is clearly false.

In what follows, we consider a simplified version of the above problem under more general assumptions than those in [6].

Let us consider the constants J_0, c_0 and D introduced above and a fixed integer $m \in \mathbb{N}$. Then we introduce the approximate total current

$$I_m[\rho](t) = J_0 \int_0^L \rho(x) K_m(t,x)dx, \quad t \geq 0, \tag{2.4}$$

where the kernel K_m is defined as

$$K_m(t,x) = F_m\big(c_0 \,\mathrm{erfc}\big(\frac{x}{2\sqrt{Dt}}\big)\big). \tag{2.5}$$

In (2.5), "erfc" denotes the complementary error function:

$$\mathrm{erfc}(z) = 1 - \frac{2}{\sqrt{\pi}} \int_0^z \exp(-\tau^2)d\tau. \tag{2.6}$$

We note that when L is large, $c_0\,\mathrm{erfc}(x/(2\sqrt{Dt}))$ provides an approximation of the solution of (2.3). The function F_m is a step function defined by

$$F_m(x) = F(c_0) \sum_{j=1}^m a_j H(x-\alpha_j) \quad \forall x \in [0,c_0], \tag{2.7}$$

with F as in (1.3). H is the Heaviside unit step function; that is,

$$H(u) = \begin{cases} 1 & \text{if } u \geq 0, \\ 0 & \text{if } u < 0. \end{cases} \tag{2.8}$$

Finally, the positive constants $\{a_j\}_{j=1}^m$ and $\{\alpha_j\}_{j=1}^m$ satisfy

$$\sum_{j=1}^m a_j = 1, \quad 0 < \alpha_1 < \alpha_2 < \cdots < \alpha_m < c_0, \tag{2.9}$$

and hence $\{\alpha_j\}_{j=1}^m$ defines a partition of the interval $(0, c_0)$.

If we choose $\{a_j\}_{j=1}^m$ such that F_m is an approximation of Hill's function F on the interval $[0, c_0]$, i.e.

$$F(x) \simeq F_m(x) = F(c_0) \sum_{j=1}^m a_j H(x-\alpha_j) \quad \forall x \in [0,c_0], \tag{2.10}$$

then

$$K_m \simeq K.$$

Therefore, we can view the functional I_m in (2.4) as an approximation of the functional I in (2.1).

Now, we introduce the operator (used thereafter)

$$\Phi_m[\varphi](t) = \sum_{j=1}^{m} a_j \varphi\left(h_j(t)\right) \quad \forall t \geq 0, \tag{2.11}$$

where $h_j(s) = \min\{L, \beta_j s\}$ with

$$\beta_j = 2\sqrt{D}\,\mathrm{erfc}^{-1}(\alpha_j/c_0) \quad \text{for } j = 1, ..., m. \tag{2.12}$$

Thus, we have the following useful relation

$$I_m[\rho](t) = J_0 F(c_0)\Phi_m[\varphi](\sqrt{t}) \quad t \geq 0, \tag{2.13}$$

with

$$\varphi(x) = \int_0^x \rho(\tau)d\tau.$$

Indeed, if we define

$$w(t,x) = c_0\,\mathrm{erfc}\left(\frac{x}{2\sqrt{Dt}}\right), \tag{2.14}$$

and put together (2.4), (2.5) and (2.7), we obtain

$$\begin{aligned} I_m[\rho](t) &= J_0 \int_0^L \rho(x) K_m(t,x)dx \\ &= J_0 F(c_0) \sum_{j=1}^{m} a_j \int_0^L \rho(x) H(w(t,x) - \alpha_j)dx \\ &= J_0 F(c_0) \sum_{j=1}^{m} a_j \int\limits_{G_j(t)\cap(0,L)} \rho(x)dx, \end{aligned} \tag{2.15}$$

with $G_j(t) := \{x \in \mathbb{R} : \quad w(t,x) \geq \alpha_j\}$. Since the "erfc" function is decreasing, we see that

$$G_j(t) = \left[0, \beta_j\sqrt{t}\right], \tag{2.16}$$

with $\{\beta_j\}_{j=1}^m$ as in (2.12). (Note that $\beta_1 > \beta_2 > \cdots > \beta_m$.) Thus, we have

$$I_m[\rho](t) = J_0 F(c_0)\Bigg(\sum_{j=1}^{m} a_j \int\limits_0^{h_j(\sqrt{t})} \rho(x)dx\Bigg), \tag{2.17}$$

and using the definition of Φ_m in (2.11), we obtain (2.13).

Clearly, Φ_m is linear, and it follows from (2.9) that $\Phi_m(1) = 1$, and that for any $f \in L^\infty(0,L)$ it holds

$$\|\Phi_m[f]\|_{L^\infty(0,L/\beta_m)} \leq \|f\|_{L^\infty(0,L)}.$$

Furthermore, for any $f \in C^0([0,L])$ with $f(L) = 0$, we have

$$\|\Phi_m[f]\|_{L^p(0,L/\beta_m)} \leq \left(\sum_{j=1}^m a_j \beta_j^{-1/p}\right) \|f\|_{L^p(0,L)}, \qquad 1 \leq p < \infty. \tag{2.18}$$

Note that the operator Φ_m is well defined on $C^0([0,L])$. Therefore, using (2.18) and the fact that the set $\{f \in C^0([0,L] \ : \ f(L) = 0\}$ is dense in $L^p(0,L)$, we can extend the operator Φ_m to $L^p(0,L)$, for all $1 \leq p < +\infty$.

Finally, we introduce some notations. We set

$$L_k = L/\beta_k \ \text{ for } k = 1, ..., m, \quad \text{and } L_0 = 0, \tag{2.19}$$

and for any $\gamma > 0$, we introduce the following weighted norms

$$\begin{aligned} \|f\|_{0,\gamma,b} &= \|\sigma_\gamma f\|_{L^2(0,b)}, \\ \|f\|_{1,\gamma,b} &= \|\sigma_\gamma f\|_{H^1(0,b)}, \\ \|f\|_{-1,\gamma,b} &= \|\sigma_\gamma f\|_{H^{-1}(0,b)} \end{aligned}$$

with $\sigma_\gamma(x) = x^\gamma$.

3 Main results

In this section we present the main results in this paper. We begin by studying the functional Φ_m defined in (2.11). It is worth noticing with (2.13) that the identifiability for Φ_m is equivalent to the identifiability for I_m.

Firstly, we discuss some identifiability results for the operator Φ_m. We begin with the analytic case.

Theorem 3.1 (Identifiability for analytic functions). *Let $\varphi : (-\varepsilon, L+\varepsilon) \to \mathbb{R}$ be an analytic function satisfying*

$$\Phi_m[\varphi](t) = 0 \quad \forall t \in (0,\delta), \tag{3.1}$$

where Φ_m is defined in (2.11)*, and ε and δ are some positive numbers. Then $\varphi \equiv 0$ in $[0,L]$.*

The second identifiability result requires less regularity for φ, provided that a measurement on a sufficiently large time interval is available.

Theorem 3.2. *Let $\varphi : [0, L] \to \mathbb{R}$ be a given continuous function satisfying*

$$\Phi_m[\varphi](t) = 0 \quad \forall t \in [0, L_m], \tag{3.2}$$

where Φ_m is defined in (2.11)*. Then $\varphi \equiv 0$ in $[0, L]$.*

Remark 3.3. Theorem 3.2 is actually true for *any* function $\varphi : [0, L] \to \mathbb{R}$ satisfying (3.2).

The proof of Theorem 3.2, which is based only upon algebraic arguments, gives us an idea on how the kernel could be reconstructed and how one can envision a numerical algorithm.

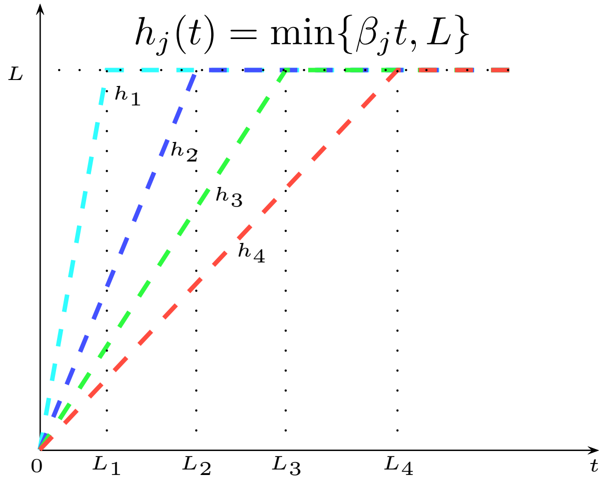

Figure 1. With $m = 4$, we plot the functions h_j on the interval $[0, L_m]$.

Let us give the main ideas in the proof of Theorem 3.2. Recall that the h_j's (see Figure 1) are the functions involved in the definition of Φ_m in (2.11). Note first that $\Phi_m[\varphi](0) = \varphi(0)$ and $\Phi_m[\varphi](t) = \varphi(L)$ for all $t \geq L_m$. Thus, using (3.2), we see that φ vanishes on $\{0, L\}$. Next, we observe that for $t \in [L_{m-1}, L_m)$, we have $\Phi_m[\varphi](t) = a_m\varphi(\beta_m t) + C\varphi(L)$, where $C = \sum_{j=1}^{m-1} a_j$. It follows that φ

vanishes in $[\lambda L, L] \cup \{0\}$, where $\lambda = \frac{\beta_m}{\beta_{m-1}} < 1$. Applying the same argument in $[L_{m-1}, L_m)$, $[L_{m-2}, L_{m-1})$, etc. we can "increase" the set where φ is known to be zero. Note that, in general, it cannot be done directly on $[L_{m-2}, L_{m-1})$.

Indeed, let us consider the case when $m = 4$ (see again Figure 1), and assume that φ vanishes on $[\lambda L, L] \cup \{0\}$, where $\lambda = \frac{\beta_4}{\beta_3} < 1$. For $t \in [L_2, L_3)$, we have

$$\Phi_4[\varphi](t) = a_4\varphi(\beta_4 t) + a_3\varphi(\beta_3 t) + C\varphi(L),$$

where $C = \sum_{j=1}^{2} a_j$, so that, using (2.11) and $\varphi(L) = 0$, we obtain

$$0 = a_4\varphi(\beta_4 t) + a_3\varphi(\beta_3 t), \quad \forall t \in [L_2, L_3),$$

i.e.

$$0 = a_4\varphi(\lambda\tau) + a_3\varphi(\tau), \quad \forall \tau \in [\beta_3 L_2, L).$$

Therefore, if $\lambda_1 = \beta_3/\beta_2 \geq \lambda$, the set $[\lambda_1 L, L)$ is contained in $[\lambda L, L] \cup \{0\}$, and we infer that φ vanishes in $[\beta_4 L_2, \lambda L) \cup [\lambda L, L] \cup \{0\}$. The same argument can be applied in the following interval, namely $[L_1, L_2)$. The above procedure suggests how the reconstruction process could be carried out, but under the condition

$$\frac{\beta_4}{\beta_3} \leq \frac{\beta_3}{\beta_2},$$

which is a restriction on the mesh defined in (2.9).

The corresponding identifiability results for the operator I_m are as follows.

Corollary 3.4 (Identifiability for analytic functions). *Let $\rho : (-\varepsilon, L + \varepsilon) \to \mathbb{R}$ be an analytic function satisfying*

$$I_m[\rho](t) = 0 \quad \forall t \in (0, \delta), \tag{3.3}$$

where I_m is defined in (2.4)*, and ε and δ are some positive numbers. Then $\rho \equiv 0$ in $[0, L]$.*

Corollary 3.5. *Let $\rho : [0, L] \to \mathbb{R}$ be a given function in $L^2(0, L)$ such that*

$$I_m[\rho](t) = 0 \quad \forall t \in [0, L_m^2]. \tag{3.4}$$

where I_m is defined in (2.4)*. Then $\rho \equiv 0$ in $[0, L]$.*

Corollaries 3.4 and 3.5 follow at once from Theorems 3.1 and 3.2 by letting

$$\varphi(x) = \int_0^x \rho(\tau) d\tau.$$

Let us now proceed to the continuity and stability results.

Theorem 3.6. *Let $\varphi \in H^1(0,L)$ be a given function. Then there exists a constant $\tilde{C}_1 > 0$ such that*

$$\|\Phi_m[\varphi]\|_{H^1(0,L_m)} \leq \tilde{C}_1 \|\varphi\|_{H^1(0,L)}, \tag{3.5}$$

where $\tilde{C}_1$ depends only on L, β_1, β_m and Φ_m given by (2.11).

We are now in a position to state our first main result. Firstly, we define the function

$$\Lambda_m^\gamma(s) = \left|\sum_{j=1}^{m} a_j \beta_j^{-(\frac{1}{2}+\gamma-is)}\right|, \tag{3.6}$$

where $i = \sqrt{-1}$ is the imaginary unit.

Theorem 3.7. *Let $\varphi \in C([0,L])$ be a given function. Then there exists a constant $\gamma_0 \in \mathbb{R}$ such that for any $\gamma > \gamma_0$,*

$$C_\gamma \|\varphi(\cdot) - \varphi(L)\|_{0,\gamma,L} \leq \|\Phi_m[\varphi](\cdot) - \Phi_m[\varphi](L_m)\|_{0,\gamma,L_m}, \tag{3.7}$$

with

$$C_\gamma := \inf_{s\in\mathbb{R}} \Lambda_m^\gamma(s) > 0,$$

and Φ_m is given by (2.11).

It is worth noting that (3.7) can be viewed as an inverse inequality of (2.18) for $p = 2$ and for functions $\varphi \in \{f \in C([0,L]);\ f(L) = 0\}$, and it can also be regarded as a stability estimate for the functional Φ_m. Its proof involves some properties of Mellin transform. Hereafter, we refer to γ_0 as the smallest number such that

$$C_\gamma > 0, \quad \forall \gamma > \gamma_0.$$

Next, we present a continuity result for the operator I_m.

Corollary 3.8. *Let $\rho : [0,L] \to \mathbb{R}$ be a function in $L^2(0,L)$. Then, for $\gamma \geq \frac{3}{4}$, there exists a positive constant $C_1 > 0$, such that*

$$\|I_m[\rho]\|_{1,\gamma,L_m^2} \leq C_1 \|\rho\|_{L^2(0,L)}, \tag{3.8}$$

where C_1 depends only on $L, \alpha_1, \alpha_{m-1}, \alpha_m, a_m$ and γ.

Besides, we present a stability result for the operator I_m.

Corollary 3.9. *Let $\rho : [0, L] \to \mathbb{R}$ be a function in $L^2(0, L)$. Then, for any $\gamma > \max\{\gamma_0, 3/4\}$, there exists a positive constant $C_2 > 0$ such that*

$$\|\rho\|_{-1,\gamma+1,L} \leq C_2 \|I_m[\rho]\|_{1,\frac{\gamma}{2}-\frac{1}{4},L_m^2}, \tag{3.9}$$

where C_2 depends only on $L, C_\gamma > 0$ and γ.

Corollaries 3.8 and 3.9 are consequences of Theorems 3.6 and 3.7, respectively.

Even if the proof of Theorem 3.2 is provided for any choice of the partition $\{\alpha_j\}_{j=1}^m$ of $[0, c_0]$, its proof can be considerably simplified in the special case when

$$\alpha_j = c_0 \text{erfc}\left(\frac{\beta_0 \beta^j}{2\sqrt{D}}\right) \quad j = 1, ..., m, \tag{3.10}$$

with $\beta \in (0, 1)$ and $\beta_0 > 0$ constants. Note that the corresponding mesh is non-regular.

In what follows, I_m and Φ_m are denoted by $\tilde{I}_m$ and $\tilde{\Phi}_m$, respectively, when α_j is given by (3.10).

For the reconstruction, we introduce the function

$$g(t) = \frac{\tilde{I}_m[\rho](t^2/\beta_0^2) - \tilde{I}_m[\rho](L_m^2)}{J_0 F(c_0)} \quad \forall t \in [0, \beta_0 L_m). \tag{3.11}$$

As mentioned in the Introduction, we look for a reconstruction algorithm and a numerical scheme to recover function ρ from the measurement of $\tilde{I}_m[\rho]$. We begin by recovering $\tilde{\varphi} : [0, L] \to \mathbb{R}$, which satisfies

$$\tilde{\Phi}_m[\tilde{\varphi}](t/\beta_0) = g(t), \quad \forall t \in [0, \beta_0 L_m). \tag{3.12}$$

Next, we define functions $\varphi_1, \varphi_2, ..., \varphi_m$ by means of the following induction formulae:

$$\varphi_1(x) = \begin{cases} \dfrac{1}{a_m} g\left(\dfrac{x}{\beta^m}\right), & \text{if } x \in [\beta L, L), \\ \\ 0, & \text{otherwise}, \end{cases} \tag{3.13}$$

and

$$\varphi_{k+1}(x) = \begin{cases} \dfrac{1}{a_m}\left(g\left(\dfrac{x}{\beta^m}\right) - \displaystyle\sum_{j=1}^{k} a_{m-k-1+j}\varphi_j\left(\dfrac{\beta^j x}{\beta^{k+1}}\right)\right), & x \in [\beta^{k+1}L, \beta^k L), \\ \\ 0, & \text{otherwise}, \end{cases} \tag{3.14}$$

for $k = 1, .., m-1$. Furthermore for $k \geq m$, we define

$$\varphi_{k+1}(x) = \begin{cases} \dfrac{1}{a_m}\left(g\left(\dfrac{x}{\beta^m}\right) - \displaystyle\sum_{j=1}^{m-1} a_j \varphi_{j+k-m+1}\left(\dfrac{\beta^j x}{\beta^m}\right)\right), x \in [\beta^{k+1}L, \beta^k L), \\ 0, \qquad\qquad \text{otherwise.} \end{cases} \tag{3.15}$$

With the above definitions we have the following reconstruction result:

Theorem 3.10. *Let ρ be a function in $C^0([0,L])$, let g be defined as in* (3.11)*, and let $\{\varphi_j\}_{j\geq 1}$ be given by* (3.13)-(3.15)*. Then the function $\widetilde{\varphi}$ defined by*

$$\tilde{\varphi}(x) = \begin{cases} \displaystyle\sum_{j=1}^{+\infty} \varphi_j(x), & \text{if } x \in (0, L], \\ g(0), & \text{if } x = 0, \end{cases} \tag{3.16}$$

is well defined and satisfies

$$\tilde{\Phi}_m[\tilde{\varphi}](t/\beta_0) = g(t) \quad \forall t \in [0, \beta_0 L_m]. \tag{3.17}$$

Furthermore, ρ satisfies

$$\int_0^x \rho(z)dz = \tilde{\varphi}(x) + \frac{\tilde{I}_m[\rho](L_m^2)}{J_0 F(c_0)} \quad \forall x \in [0, L]. \tag{3.18}$$

Theorem 3.10 provides an *explicit* reconstruction procedure for both operators $\tilde{\Phi}_m$ and $\tilde{I}_m$ and therefore a numerical algorithm for the reconstruction.

The previous reconstruction procedure gives us the possibility to obtain a sharper stability result. We shall provide a stability result for $\tilde{\Phi}_m$ in terms of a quite general norm.

We consider a family of norms $\|\cdot\|_{[a,b)}$ for functions $f : [a,b) \to \mathbb{R}$, where $0 \leq a < b < \infty$, which enjoys the following properties:

(i) $\|f\|_{[a,b)} < \infty$ for any $f \in W^{1,1}(a,b)$;

(ii) If $[a_1, b_1) \subset [a,b)$, then

$$\|f\|_{[a_1,b_1)} \leq \|f\|_{[a,b)}\,; \tag{3.19}$$

(iii) For any $\lambda > 0$, there exists a positive constant $C(\lambda)$ such that

$$\|g_\lambda\|_{[\lambda a,\lambda b)} \leq C(\lambda) \, \|f\|_{[a,b)} \, , \tag{3.20}$$

where $g_\lambda(x) = f(x/\lambda)$, and $C(\cdot)$ is a nondecreasing function with $C(1) = 1$.

A natural family of norms fulfilling (i), (ii), and (iii), is those of L^p norms, where $1 \leq p \leq +\infty$. Indeed, (i) and (ii) are obvious, and (iii) holds with

$$C(\lambda) = \begin{cases} \lambda^{\frac{1}{p}} & \text{if } p \in [1, +\infty), \\ 1 & \text{if } p = \infty. \end{cases}$$

Another family of norms fulfilling (i), (ii), and (iii), is the family of BV-norms:

$$\|f\|_{BV(a,b)} = \|f\|_{L^\infty(a,b)} + \sup_{a \leq x_1 < \cdots < x_k < b} \sum_{j=1}^{k} |f(x_k) - f(x_{k-1})| \, . \tag{3.21}$$

Here, we can pick $C(\lambda) = 1$. (Note that $W^{1,1}(a,b) \subset BV(a,b)$, see e.g. [2].) These kinds of norms are adapted to functions with low regularity, as e.g. step functions. The second main result in this paper is the following stability result.

Theorem 3.11. *Let $\rho \in C^0([0,L])$ be a function and let a family of norms satisfying conditions (i), (ii) and (iii). Then, we have for all $k \geq 0$*

$$\|\varphi(\cdot) - \varphi(L)\|_{[\beta^{k+1}L,\beta^k L)} \leq C(\beta_0) \frac{C(\beta^m)}{a_m^{k+1}} \left\|\tilde{\Phi}_m[\varphi](\cdot) - \tilde{\Phi}_m[\varphi](L_m)\right\|_{[\beta^{k+1}L_m,L_m)} \, , \tag{3.22}$$

where $\varphi(x) = \int_0^x \rho(\tau) d\tau$.

Theorem 3.11 shows in particular that the value of φ in the interval $[\beta^{k+1}L, \beta^k L)$ depends on the value of $\tilde{\Phi}_m[\varphi]$ in the interval $[\beta^{k+1}L_m, L_m)$, a property which is closely related to the nature of the reconstruction procedure.

4 Proof of identifiability results

This section is devoted to proving the identifiability results for the operator Φ_m.

Proof of Theorem 3.1. Let φ be an analytic function such that

$$\Phi_m[\varphi](t) = \sum_{j=1}^{m} a_j \varphi(h_j(t)) = 0 \quad \forall t \in (0, \delta).$$

Then, taking $t \in (0, \min\{\delta, L_1\})$ and using the fact that

$$L_0 < L_1 < \cdots < L_m, \tag{4.1}$$

we see that $h_j(t) = \beta_j t$, $j = 1, ..., m$. Then, we have

$$\sum_{j=1}^{m} a_j \varphi(\beta_j t) = 0, \qquad t \in (0, \min\{\delta, L_1\}).$$

If we derive the above expression and evaluate it at zero, we obtain

$$\varphi^{(k)}(0) \left(\sum_{j=1}^{m} a_j (\beta_j)^k \right) = 0 \quad \forall k \geq 0,$$

where $\varphi^{(k)}(0)$ denotes the $k-$th derivative of φ at zero. Since a_j, β_j are positive, we have that $\sum_{j=1}^{m} a_j (\beta_j)^k > 0$; therefore $\varphi^{(k)}(0) = 0$ for all $k \geq 0$, and hence $\varphi \equiv 0$. This proves the identifiability for Φ_m in the case of analytic functions. □

To prove Theorem 3.2, we need some technical lemmas.

Lemma 4.1. *Let $f, g : [0, L] \to \mathbb{R}$ be functions, and let $s, \alpha_0 \in [0, 1)$ and $\lambda \in (0, 1)$ be numbers such that*

$$f(\tau) + g(\lambda\tau) = 0 \quad \forall \tau \in [sL, L), \tag{4.2}$$

and

$$f(\tau) = 0 \quad \forall \tau \in [\alpha_0 L, L). \tag{4.3}$$

Then

$$g(\tau) = 0 \quad \forall \tau \in [\alpha_1 L, \lambda L), \tag{4.4}$$

where $\alpha_1 = \lambda \max\{s, \alpha_0\}$.

Lemma 4.1 is a direct consequence of (4.2) and (4.3).

Lemma 4.2. *Let $f : [0, L] \to \mathbb{R}$ be a function, and let $s, \alpha_0 \in [0, 1)$ and $\lambda \in (0, 1)$ be some numbers such that*

$$f(\tau) = 0 \quad \forall \tau \in [\tilde{\alpha}_k L, L) \quad \forall k \geq 1, \tag{4.5}$$

where

$$\tilde{\alpha}_k = \lambda \max\{s, \tilde{\alpha}_{k-1}\} \quad \forall k \geq 1, \tag{4.6}$$

with $\tilde{\alpha}_0 = \alpha_0$.

Then, if $s > 0$,

$$f(\tau) = 0 \quad \forall \tau \in [s\lambda L, L),$$

and if $s = 0$,

$$f(\tau) = 0 \quad \forall \tau \in (0, L).$$

Proof. To prove the above lemma, we need to consider two cases: $s = 0$ and $s > 0$.

If $s > 0$, we claim that there exists k_0 such that $\tilde{\alpha}_{k_0} < s$. Otherwise, if $\tilde{\alpha}_k \geq s \; \forall k \geq 0$, replacing in (4.6), we have

$$\tilde{\alpha}_{k+1} = \lambda \tilde{\alpha}_k,$$

and hence $\tilde{\alpha}_k = \tilde{\alpha}_0 \lambda^k \to 0$, which is impossible, for $s > 0$.

Using (4.6), the desired result follows, since

$$\tilde{\alpha}_k = \lambda s \quad \forall k > k_0.$$

Now, if $s = 0$, replacing it in (4.6) we obtain

$$\tilde{\alpha}_k = \alpha_0 \lambda^k.$$

Then, using (4.5) we have

$$f(\tau) = 0, \quad \forall \tau \in (0, L),$$

which completes the proof. □

Lemma 4.3. *Let* $f : [0, L] \to \mathbb{R}$ *be a function, and let* $s, \alpha_0 \in [0, 1)$, $\lambda_1, ..., \lambda_n \in (0, 1)$ *and* $a_k > 0$, $k = 0, ..., n$ *be some numbers such that* $\lambda_1 > \lambda_2 > \cdots > \lambda_n \geq \alpha_0$, *and*

$$a_0 f(t) + \sum_{j=1}^{n} a_j f(\lambda_j t) = 0 \quad \forall t \in [sL, L), \tag{4.7}$$

and

$$f(\tau) = 0 \quad \forall \tau \in [\alpha_0 L, L). \tag{4.8}$$

Then

$$f(\tau) = 0 \quad \forall \tau \in [\overline{\alpha} L, L), \tag{4.9}$$

where $\overline{\alpha} = \lambda_n s$.

Proof. We prove this result by induction on n.

Case $n = 1$. In this case, from (4.7) we have the following equations

$$a_0 f(t) + a_1 f(\lambda_1 t) = 0 \quad \forall t \in [sL, L), \tag{4.10}$$

$$f(\tau) = 0 \quad \forall \tau \in [\alpha_0 L, L), \tag{4.11}$$

and $\alpha_0 \leq \lambda_1$. Then, applying Lemma 4.1 with $g = f$, we get

$$f(\tau) = 0 \quad \forall \tau \in [\alpha_1 L, \lambda_1 L),$$

where $\alpha_1 = \lambda_1 \max\{s, \alpha_0\}$, and thus

$$f(\tau) = 0 \quad \forall \tau \in [\alpha_1 L, L),$$

for $\alpha_0 \leq \lambda_1$.

If $\alpha_0 = 0$, we obtain the desired result:

$$f(\tau) = 0 \quad \forall \tau \in [\lambda_1 s L, L).$$

On the other hand, when $\alpha_0 > 0$, we can apply Lemma 4.1 again with α_0 replaced by α_1, since we have

$$a_0 f(t) + a_1 f(\lambda_1 t) = 0 \quad \forall t \in [sL, L),$$

$$f(\tau) = 0 \quad \forall \tau \in [\alpha_1 L, L),$$

and $\alpha_1 \leq \lambda_1$. Thus, we get by induction on $k \geq 0$

$$f(\tau) = 0 \quad \forall \tau \in [\alpha_k L, L), \quad \forall k \geq 1, \tag{4.12}$$

where

$$\alpha_k = \lambda_1 \max\{s, \alpha_{k-1}\} \quad \forall k \geq 1. \tag{4.13}$$

Note that, if $s = 0$, letting $t = 0$ in (4.10) yields $f(0) = 0$. Using Lemma 4.2 with (4.12)-(4.13), we conclude that

$$f(\tau) = 0 \quad \forall \tau \in [\lambda_1 s L, L),$$

which completes the case $n = 1$.

Case $n + 1$. Assume the lemma proved up to the value n, and let us prove it for the value $n + 1$.

Assume given a function $f:[0,L]\to\mathbb{R}$ and some numbers $s,\alpha_0\in[0,1)$, $a_k>0$ for $0\le k\le n+1$, $\lambda_1,...,\lambda_{n+1}\in(0,1)$ with $1>\lambda_1>\lambda_2>\cdots>\lambda_{n+1}\ge\alpha_0$, and such that

$$a_0f(t)+\sum_{j=1}^{n+1}a_jf(\lambda_jt)=0\quad\forall t\in[sL,L),\tag{4.14}$$

and

$$f(\tau)\equiv 0\quad\forall\tau\in[\alpha_0L,L).\tag{4.15}$$

Then we aim to prove that

$$f(\tau)=0\quad\forall\tau\in[\lambda_{n+1}sL,L).$$

We introduce the function

$$\psi(\tau)=\sum_{j=1}^{n+1}a_jf(\frac{\lambda_j}{\lambda_1}\tau)=a_1f(\tau)+\sum_{j=2}^{n+1}a_jf(\tilde\lambda_j\tau),$$

where $\tilde\lambda_j=\dfrac{\lambda_j}{\lambda_1},\ j=2,..,n+1.$

Then, using (4.15), we have

$$\psi(\tau)=0\quad\forall\tau\in[\lambda_1\frac{\alpha_0}{\lambda_{n+1}}L,L).\tag{4.16}$$

On the other hand, from (4.14), we have

$$a_0f(\tau)+\psi(\lambda_1\tau)=0\quad\forall\tau\in[sL,L).$$

Then, from (4.15) and Lemma 4.1 with $g=\psi$, we conclude

$$\psi(\tau)=0\quad\forall\tau\in[\lambda_1\max\{\alpha_0,s\}L,\lambda_1L).$$

Next, we set $s_1=\lambda_1\max\{\alpha_0,s\}\in[0,1)$. Using (4.16), we have $\psi\equiv 0$ on $[s_1L,\lambda_1L)\cup[\lambda_1\frac{\alpha_0}{\lambda_{n+1}}L,L)$. Therefore, with $\frac{\alpha_0}{\lambda_{n+1}}\le 1$,

$$\psi(\tau)=a_1f(\tau)+\sum_{i=2}^{n+1}a_if(\tilde\lambda_i\tau)=0\quad\forall\tau\in[s_1L,L).\tag{4.17}$$

Note that $1>\tilde\lambda_2>\tilde\lambda_3>\cdots>\tilde\lambda_{n+1}$, and that $\alpha_0\le\lambda_{n+1}<\frac{\lambda_{n+1}}{\lambda_1}=\tilde\lambda_{n+1}$. Then, by using the induction hypothesis with (4.17) and (4.15), we obtain

$$f(\tau)=0\quad\forall\tau\in[\alpha_1L,L),$$

where $\tilde{\alpha}_1 = s_1\tilde{\lambda}_{n+1} = \lambda_{n+1}\max\{s,\alpha_0\} < \lambda_{n+1}$. Then we can repeat the latter argument replacing α_0 by $\tilde{\alpha}_1$, and we obtain

$$f(\tau) = 0 \quad \forall \tau \in [\tilde{\alpha}_k L, L) \quad \forall k \geq 1,$$

where

$$\tilde{\alpha}_k = \lambda_{n+1}\max\{s,\tilde{\alpha}_{k-1}\} \quad \forall k \geq 1, \tag{4.18}$$

with $\tilde{\alpha}_0 = \alpha_0$ given. If $s = 0$, letting $t = 0$ in (4.14) yields $f(0) = 0$. Using Lemma 4.2 we infer that

$$f(\tau) = 0, \quad \forall \tau \in [\overline{\alpha}L, L),$$

where $\overline{\alpha} = \lambda_{n+1}s$, which completes the proof. □

Proof of Theorem 3.2. Let $\varphi : [0, L] \to \mathbb{R}$ be a function such that

$$\Phi_m[\varphi](t) = \sum_{j=1}^{m} a_j\varphi(h_j(t)) = 0 \quad \forall t \in [0, L_m].$$

Then, if $t = L_m$, we obtain

$$h_j(L_m) = L \quad \forall j = 1, ..., m,$$

and hence

$$0 = \Phi_m[\varphi](L_m) = \varphi(L). \tag{4.19}$$

Next, for any $k \in \{1, ..., m\}$, we have

$$\sum_{j=k}^{m} a_j\varphi(\beta_j t) = 0 \quad \forall t \in [L_{k-1}, L_k],$$

which is equivalent to

$$a_k\varphi(t) + \sum_{j=k+1}^{m} a_j\varphi\left(\frac{\beta_j}{\beta_k}t\right) = 0 \quad \forall t \in [\beta_k L_{k-1}, \beta_k L_k] = [\beta_k L_{k-1}, L], \tag{4.20}$$

for $k = 1, 2, ..., m$. We aim to prove that

$$\varphi(\tau) = 0 \quad \forall \tau \in [\beta_m L_{k-1}, L],$$

for $k = 1, ..., m$. We proceed by induction on $i = m - k \in \{0, ..., m-1\}$.

Case $i = 0$. Letting $k = m$ in (4.20) yields

$$a_m\varphi(t) = 0 \quad \forall t \in [\beta_m L_{m-1}, L],$$

which implies

$$\varphi(\tau) = 0 \quad \forall \tau \in [\beta_m L_{m-1}, L], \tag{4.21}$$

which completes the case $i = 0$.

Case $i = 1$**.** Letting $k = m - 1$ in (4.20), we obtain

$$a_{m-1}\varphi(t) + a_m \varphi\left(\frac{\beta_m}{\beta_{m-1}} t\right) = 0 \quad \forall t \in [\beta_{m-1} L_{m-2}, L]. \tag{4.22}$$

We infer from Lemma 4.3 (applied with $\lambda_1 = \frac{\beta_m}{\beta_{m-1}}$, $s = \frac{\beta_{m-1}}{\beta_{m-2}}$ and $\alpha_0 = \frac{\beta_m}{\beta_{m-1}}$) that

$$\varphi(\tau) = 0 \quad \forall \tau \in [\beta_m L_{m-2}, L].$$

Case i. Assume the property satisfied for $i - 1$, i.e.,

$$\varphi(\tau) = 0 \quad \forall \tau \in [\beta_m L_{m-i}, L]. \tag{4.23}$$

Replacing $k = m - i$ in (4.20), we obtain

$$a_{m-i}\varphi(t) + \sum_{j=m-i+1}^{m} a_j \varphi\left(\frac{\beta_j}{\beta_{m-i}} t\right) = 0 \quad \forall t \in [\beta_{m-i} L_{m-i-1}, L]. \tag{4.24}$$

Then, if we set $\lambda_j = \frac{\beta_j}{\beta_{m-i}} < 1$, for $j = m - i + 1, ..., m$,

$$s = \beta_{m-i} \frac{L_{m-i-1}}{L},$$

and $\alpha_0 = \dfrac{\beta_m}{\beta_{m-i}} = \lambda_m$, then we infer from Lemma 4.3 that

$$\varphi(\tau) = 0 \quad \forall \tau \in [\beta_m L_{m-i-1}, L].$$

Thus

$$\varphi(\tau) = 0 \quad \forall \tau \in [\beta_m L_{k-1}, L],$$

and for $k = 1, ..., m$. This implies (with $k = 1$ and $L_0 = 0$)

$$\varphi(\tau) = 0 \quad \forall \tau \in [0, L].$$

The proof of Theorem 3.2 is complete. □

5 Proofs of the stability results

We first prove Theorem 3.6.

Proof of Theorem 3.6. First, some estimates are established.

$$\begin{aligned}
\|\varphi \circ h_j\|^2_{L^2(0,L_m)} &= \int_0^{L_m} \varphi^2(h_j(t))dt \\
&= \int_0^{L_j} \varphi^2(\beta_j t)dt + \varphi^2(L)L\left(\frac{1}{\beta_m} - \frac{1}{\beta_j}\right) \\
&\leq \frac{1}{\beta_j}\int_0^{L} \varphi^2(t)dt + \varphi^2(L)\frac{L}{\beta_m} \\
&\leq \frac{1}{\beta_m}\left\{\|\varphi\|^2_{L^2(0,L)} + \varphi^2(L)L\right\} \\
&\leq \frac{1}{\beta_m}\left(1 + ||T_L||^2 L\right)\|\varphi\|^2_{H^1(0,L)}, \qquad (5.1)
\end{aligned}$$

where $T_L(u) = u(L)$ is the trace operator in $H^1(0,L)$.

Now, if we set

$$c_1 = \frac{1}{\sqrt{\beta_m}}\left(1 + ||T_L||^2 L\right)^{\frac{1}{2}},$$

then using (5.1), we obtain

$$\|\Phi_m[\varphi]\|_{L^2(0,L_m)} \leq \sum_{j=1}^{m} a_j \|\varphi \circ h_j\|_{L^2(0,L_m)} \leq c_1 \|\varphi\|_{H^1(0,L)}. \qquad (5.2)$$

On the other hand, let ψ be any test function with compact support in $(0, L_m)$. Then

$$\begin{aligned}
\int_0^{L_m} \Phi_m[\varphi](t)\psi'(t)dt &= \sum_{j=1}^{m} a_j\left\{\int_0^{L_j} \varphi(\beta_j t)\psi'(t)dt + \varphi(L)\int_{L_j}^{L_m} \psi'(t)dt\right\} \\
&= -\sum_{j=1}^{m} a_j\beta_j\int_0^{L_j} \varphi'(\beta_j t)\psi(t)dt \qquad (5.3) \\
&= -\sum_{j=1}^{m} a_j\beta_j\int_0^{L_m} \varphi'(\beta_j t)\psi(t)(1 - H(\beta_j t - L))dt,
\end{aligned}$$

where H denotes Heaviside's function. Thus

$$(\Phi_m[\varphi])'(t) = \sum_{j=1}^{m} a_j \beta_j \varphi'(\beta_j t)(1 - H(\beta_j t - L)) \quad \forall t \in (0, L_m). \tag{5.4}$$

Therefore, for any $\varphi \in H^1(0,L)$, the function $\Phi_m[\varphi]$ belongs to $H^1(0,L_m)$. This, along with (5.4) yields

$$\left\|(\Phi_m[\varphi])'\right\|_{L^2(0,L_m)} \leq \sum_{j=1}^{m} a_j \sqrt{\beta_j} \left(\int_0^L (\varphi')^2(t)dt \right)^{1/2} \leq \sqrt{\beta_1} \left\|\varphi'\right\|_{L^2(0,L)}. \tag{5.5}$$

Combining (5.5) with equation (5.2), we obtain

$$\|\Phi_m[\varphi]\|_{1,0,L_m} \leq \tilde{C}_1 \|\varphi\|_{1,0,L},$$

where $\tilde{C}_1 = \sqrt{(c_1)^2 + \beta_1}$. The proof of Theorem 3.6 is therefore complete. □

Now we proceed to the proof of Theorem 3.7. Before establishing this stability result, we need recall well-known facts about Mellin Transform (the reader is referred to [12] Chapter VIII, for details).

For any real numbers $\alpha < \beta$, let $<\alpha, \beta>$ denote the open strip of complex numbers $s = \sigma + it$ ($\sigma, t \in \mathbb{R}$) such that $\alpha < \sigma < \beta$.

Definition 5.1 (Mellin transform). Let f be locally Lebesgue integrable over $(0, +\infty)$. The Mellin transform of f is defined by

$$\mathcal{M}[f](s) = \int_0^{+\infty} f(x)x^{s-1}dx \quad \forall s \in <\alpha, \beta>,$$

where $<\alpha, \beta>$ is the largest open strip in which the integral converges (it is called the fundamental strip).

Lemma 5.2. *Let f be locally Lebesgue integrable over $(0, +\infty)$. Then the following properties hold true:*

(i) *Let $s_0 \in \mathbb{R}$. Then for all s such that $s + s_0 \in <\alpha, \beta>$, we have*

$$\mathcal{M}[f(x)](s + s_0) = \mathcal{M}[x^{s_0} f(x)](s).$$

(ii) *For any* $\beta \in \mathbb{R}$*, if* $g(x) = f(\beta x)$*, then*

$$\mathcal{M}[g](s) = \beta^{-s}\mathcal{M}[f](s) \qquad \forall s \in <\alpha, \beta> .$$

Definition 5.3 (Mellin transform as operator in L^2). For functions in $L^2(0, +\infty)$ we define a linear operator $\tilde{\mathcal{M}}$ as

$$\begin{aligned} \tilde{\mathcal{M}} : L^2(0, +\infty) &\longrightarrow L^2(-\infty, +\infty), \\ f &\longrightarrow \tilde{\mathcal{M}}[f](s) := \tfrac{1}{\sqrt{2\pi}}\mathcal{M}[f](\tfrac{1}{2} - is). \end{aligned}$$

Theorem 5.4 (Mellin inversion theorem). *The operator* $\tilde{\mathcal{M}}$ *is invertible with inverse*

$$\begin{aligned} \tilde{\mathcal{M}}^{-1} : L^2(-\infty, +\infty) &\longrightarrow L^2(0, +\infty), \\ \varphi &\longrightarrow \tilde{\mathcal{M}}^{-1}[\varphi](x) := \tfrac{1}{\sqrt{2\pi}} \textstyle\int_{-\infty}^{+\infty} x^{-\frac{1}{2}-is}\varphi(s)ds. \end{aligned}$$

Furthermore, this operator is an isometry; that is,

$$\left\|\tilde{\mathcal{M}}[f]\right\|_{L^2(-\infty,\infty)} = \|f\|_{L^2(0,\infty)} \quad \forall f \in L^2(0, +\infty).$$

Proof of the Theorem 3.7. We note that for any function $f : [0, +\infty[\to \mathbb{R}$ such that $\text{supp}(f) \subset [0, L)$, we have

$$f(h_j(t)) = f(\beta_j t).$$

Thus, we obtain

$$\Phi_m[f](t) = \sum_{j=1}^{m} a_j f(\beta_j t) \quad \forall t \geq 0, \tag{5.6}$$

where $\{\beta_j\}_{j=1}^m$ has been defined in (2.12).

Pick any $\varphi \in C([0, L])$ and let $g : [0, L_m] \to \mathbb{R}$ be such that

$$\Phi_m[\varphi](t) = g(t) \quad \forall t \in [0, L_m]. \tag{5.7}$$

Define the functions

$$\tilde{g}(t)=\begin{cases} g(t)-g(L_m) & 0\le t\le L_m, \\ \\ 0 & t\ge L_m, \end{cases} \quad ,\tilde{\varphi}(t)=\begin{cases} \varphi(t)-\varphi(L) & 0\le t\le L, \\ \\ 0 & t\ge L. \end{cases} \tag{5.8}$$

If we replace t by L_m in (5.7), we have the following compatibility condition

$$\varphi(L)=g(L_m).$$

Since $\Phi_m[1]=1$, we infer that

$$\Phi_m[\tilde{\varphi}](t)=\tilde{g}(t) \quad \forall t\ge 0. \tag{5.9}$$

Letting $f=\tilde{\varphi}$ in (5.6) yields

$$\Phi_m[\tilde{\varphi}](t)=\sum_{j=1}^{m} a_j\tilde{\varphi}(\beta_j t) \quad \forall t\ge 0.$$

It follows from Lemma 5.2 that

$$\mathcal{M}\left[\Phi_m[\tilde{\varphi}]\right](s)=\left(\sum_{j=1}^{m} a_j\beta_j^{-s}\right)\mathcal{M}[\tilde{\varphi}](s) \quad \forall s\in <\alpha,\beta>, \tag{5.10}$$

where $<\alpha,\beta>$ is the fundamental strip associated with $\tilde{\varphi}$.

Let $\gamma>0$ be a fixed constant. Using (5.10) and Lemma 5.2, we obtain

$$\Lambda_m^{\gamma}(s)\left|\tilde{\mathcal{M}}[x^{\gamma}\tilde{\varphi}(x)](s)\right|=\left|\tilde{\mathcal{M}}\left[x^{\gamma}\Phi_m[\tilde{\varphi}](x)\right](s)\right| \quad \forall s\in\mathbb{R}, \tag{5.11}$$

where Λ_m^{γ} has been defined in (3.6). On the other hand,

$$\begin{aligned} \Lambda_m^{\gamma}(s) &\ge a_m\beta_m^{-\gamma-\frac{1}{2}}-\left|\sum_{j=1}^{m-1} a_j\beta_j^{-(\gamma+\frac{1}{2}-is)}\right| \\ &\ge a_m\beta_m^{-\gamma-\frac{1}{2}}-\sum_{j=1}^{m-1} a_j\beta_j^{-(\gamma+\frac{1}{2})} \\ &\ge a_m\beta_m^{-\gamma-\frac{1}{2}}-\beta_{m-1}^{-(\gamma+\frac{1}{2})} \\ &= \beta_m^{-\gamma-\frac{1}{2}}\left(a_m-\left(\frac{\beta_{m-1}}{\beta_m}\right)^{-(\gamma+\frac{1}{2})}\right). \end{aligned} \tag{5.12}$$

Therefore, if we choose

$$\gamma > \frac{\ln(a_m)}{\ln(\frac{\beta_m}{\beta_{m-1}})} - \frac{1}{2},$$

then

$$\Lambda_m^\gamma(s) \geq \beta_m^{-\gamma-\frac{1}{2}} \left(a_m - \left(\frac{\beta_{m-1}}{\beta_m} \right)^{-(\gamma+\frac{1}{2})} \right) > 0 \quad \forall s \in \mathbb{R}.$$

Thus, there exists γ_0 such that

$$C_\gamma = \inf_{s\in\mathbb{R}} \Lambda_m^\gamma(s) > 0 \quad \forall \gamma > \gamma_0.$$

Therefore, using the fact that $\tilde{\mathcal{M}}$ is an isometry and (5.11), we obtain

$$C_\gamma \left\| \tilde{\varphi} \right\|_{0,\gamma,L} \leq \left\| \Phi_m[\tilde{\varphi}] \right\|_{0,\gamma,L_m}, \tag{5.13}$$

which completes the proof of Theorem 3.7. □

We are now in a position to prove Theorems 3.8 and 3.9.

Proof of Theorem 3.8. Let us fix any $\gamma > 0$ and let $\rho : [0, L] \to \mathbb{R}$ be a function in $L^2(0, L)$. From (2.13) we have

$$\begin{aligned} (x^\gamma I_m[\rho](x))' &= \gamma x^{\gamma-1} I_m[\rho](x) + x^\gamma (I_m[\rho](x))' \\ &= \gamma x^{\gamma-1} I_m[\rho](x) + \frac{x^{\gamma-\frac{1}{2}} J_0 F(c_0)}{2} (\Phi_m[\varphi])'(\sqrt{x}), \end{aligned} \tag{5.14}$$

where $\varphi(x) = \int_0^x \rho(\tau)d\tau$. (Note that $\varphi \in H^1(0, L)$.) Since

$$\begin{aligned} \int_0^{L_m^2} x^{2\gamma-1} \left((\Phi_m[\varphi])'(\sqrt{x}) \right)^2 dx &= 2 \int_0^{L_m} \tau^{4\gamma-1} \left((\Phi_m[\varphi])'(\tau) \right)^2 d\tau \\ &= 2 \left\| (\Phi_m[\varphi])' \right\|_{0,2\gamma-\frac{1}{2},L_m}^2, \end{aligned} \tag{5.15}$$

we have

$$\|I_m[\rho]\|^2_{1,\gamma,L^2_m} \le \|I_m[\rho]\|^2_{0,\gamma,L^2_m} + \left(\gamma \|I_m[\rho]\|_{0,\gamma-1,L^2_m} + \frac{|J_0F(c_0)|}{\sqrt{2}} \left\|(\Phi_m[\varphi])'\right\|_{0,2\gamma-\frac{1}{2},L_m}\right)^2$$
$$\le \|I_m[\rho]\|^2_{0,\gamma,L^2_m} + 2\gamma^2 \|I_m[\rho]\|^2_{0,\gamma-1,L^2_m} + (J_0F(c_0))^2 \left\|(\Phi_m[\varphi])'\right\|^2_{0,2\gamma-\frac{1}{2},L_m} \le (L^2+2\gamma^2) \|I_m[\rho]\|^2_{0,\gamma-1,L^2_m} + (J_0F(c_0))^2 \left\|(\Phi_m[\varphi])'\right\|^2_{0,2\gamma-\frac{1}{2},L_m}. \quad (5.16)$$

On other hand, using (2.13) and the change of variable $\tau = x^2$, we have

$$\|\Phi_m[\varphi]\|^2_{0,2\gamma-\frac{3}{2},L_m} = \frac{1}{(F(c_0)J_0)^2}\int_0^{L_m} x^{4\gamma-3}\left(I_m[\rho](x^2)\right)^2 dx = \frac{1}{2\,(F(c_0)J_0)^2} \|I_m[\rho]\|^2_{0,\gamma-1,L^2_m}. \quad (5.17)$$

By replacing (5.17) in (5.16), we obtain

$$\|I_m[\rho]\|^2_{1,\gamma,L^2_m} \le (L^2+2\gamma^2)2\,(F(c_0)J_0)^2 \|\Phi_m[\varphi]\|^2_{0,2\gamma-\frac{3}{2},L_m} + (F(c_0)J_0)^2 \left\|(\Phi_m[\varphi])'\right\|^2_{0,2\gamma-\frac{1}{2},L_m}, \quad (5.18)$$

and assuming that $\gamma \ge \frac{3}{4}$, from Theorem 3.6, we have

$$\|I_m[\rho]\|_{1,\gamma,L^2_m} \le \sqrt{3L^2+4\gamma^2} J_0F(c_0)L^{2\gamma-\frac{3}{2}} \|\Phi_m[\varphi]\|_{H^1(0,L_m)} \le \sqrt{3L^2+4\gamma^2} J_0F(c_0)L^{2\gamma-\frac{3}{2}} \tilde{C}_1 \|\varphi\|_{H^1(0,L)}. \quad (5.19)$$

But, from Cauchy-Schwarz inequality we have $|\varphi(x)| \le \sqrt{L}\,\|\rho\|_{L^2(0,L)}$, and hence

$$\|\varphi\|^2_{H^1(0,L)} = \|\varphi\|^2_{L^2(0,L)} + \|\rho\|^2_{L^2(0,L)} \le (L^2+1)\,\|\rho\|^2_{L^2(0,L)}.$$

Therefore, for any $\gamma \ge \frac{3}{4}$, we have

$$\|I_m[\rho]\|_{1,\gamma,L^2_m} \le C_1 \|\rho\|_{L^2(0,L)},$$

where

$$C_1 = \sqrt{3L^2+4\gamma^2} J_0F(c_0)L^{2\gamma-\frac{3}{2}}\tilde{C}_1(L^2+1)^{1/2},$$

and the proof of Theorem 3.8 is therefore finished. □

Proof of Theorem 3.9. Let ψ be any test function compactly supported in $(0, L)$, and let γ be a positive constant. Set

$$g_\gamma(x) = x^\gamma \rho(x), \qquad \varphi(x) = \int_0^x \rho(\tau)d\tau,$$

and

$$\tilde{\varphi}(t) = \varphi(x) - \varphi(L).$$

It follows that

$$(x^{\gamma+1}\tilde{\varphi}(x))' = (\gamma+1)x^\gamma \tilde{\varphi}(x) + g_{\gamma+1}(x),$$

and hence,

$$\begin{aligned} < g_{\gamma+1}, \psi > &= \int_0^L g_{\gamma+1}(x)\psi(x)dx \\ &= \int_0^L \left((x^{\gamma+1}\tilde{\varphi}(x))' - (\gamma+1)x^\gamma\tilde{\varphi}(x)\right)\psi(x)dx \\ &= -\int_0^L \left(x^{\gamma+1}\tilde{\varphi}(x)\psi'(x) + (\gamma+1)x^\gamma\tilde{\varphi}(x)\psi(x)\right)dx. \end{aligned}$$

Then, we have

$$\begin{aligned} | < g_{\gamma+1}, \psi > | &\leq \left(\|\tilde{\varphi}\|_{0,\gamma+1,L} + (\gamma+1)\|\tilde{\varphi}\|_{0,\gamma,L}\right)\|\psi\|_{H^1(0,L)} \\ &\leq (L+\gamma+1)\|\tilde{\varphi}\|_{0,\gamma,L}\|\psi\|_{H^1(0,L)}. \end{aligned}$$

Therefore,

$$\|g_{\gamma+1}\|_{H^{-1}(0,L)} \leq (L+\gamma+1)\|\tilde{\varphi}\|_{0,\gamma,L}. \tag{5.20}$$

Thus, using Theorem 3.7, we have that for any $\gamma > \max\{\gamma_0, \frac{3}{4}\}$ there exists a constant $C_\gamma > 0$ such that

$$\begin{aligned} &\|\rho\|_{-1,\gamma+1,L} = \|g_{\gamma+1}\|_{H^{-1}(0,L)} \\ &\leq (L+\gamma+1)\,C_\gamma^{-1}\left\{\|\Phi_m[\varphi]\|_{0,\gamma,L_m} + \frac{L_m^{\gamma+\frac{1}{2}}}{\sqrt{2\gamma+1}}|\Phi_m[\varphi](L_m)|\right\}. \end{aligned} \tag{5.21}$$

Using (2.13), we have

$$\Phi_m[\varphi](L_m) = \frac{1}{F(c_0)J_0} I_m[\rho](L_m^2). \tag{5.22}$$

Replacing (5.22) in (5.21) and using (5.17), with $2\gamma - 3/2$ replaced by γ, we obtain

$$\|\rho\|_{-1,\gamma+1,L} \leq \frac{(L+\gamma+1)}{\sqrt{2}|J_0 F(c_0)|} C_\gamma^{-1} \left\{ 1 + \sqrt{2} \frac{L_m}{\sqrt{2\gamma+1}} ||T_{L^2_m}|| \right\} \|I_m[\rho]\|_{1,\frac{\gamma}{2}-\frac{1}{4},L^2_m} .$$

Therefore, setting

$$C_2 = \frac{(L+\gamma+1)}{\sqrt{2}|J_0 F(c_0)|} C_\gamma^{-1} \left\{ 1 + \sqrt{2} \frac{L_m}{\sqrt{2\gamma+1}} ||T_{L^2_m}|| \right\},$$

we obtain (3.9). The proof of Theorem 3.9 is achieved. □

6 Numerical reconstruction results

This section is devoted to the proof of Theorems 3.10 and 3.11.

Proof of Theorem 3.10. Let ρ be a function in $C^0([0, L])$, and let us consider the functions $\{\varphi_j\}_{j\geq 1}$ defined in (3.13)-(3.15).

First, we note that for all $k \geq 1$ we have

$$\varphi_k(x) = 0, \qquad \forall x \notin [\beta^k L, \beta^{k-1} L). \tag{6.1}$$

Then, we can define the sequence $\{\psi_p\}_{p\in\mathbb{N}^*}$ as

$$\psi_p(x) = \sum_{j=1}^{p} \varphi_j(x) \quad \forall x \in \mathbb{R}.$$

Using (6.1) we have that for all $x \in \mathbb{R} \setminus (0, L)$,

$$\psi_p(x) = 0 \quad \forall p \in \mathbb{N}^*,$$

and hence,

$$\lim_{p\to+\infty} \psi_p(x) = 0 \quad \forall x \in \mathbb{R} \setminus (0, L).$$

Besides that, we consider the *ceiling function*

$$\lceil x \rceil = \min\{k \in \mathbb{Z} \mid k \geq x\},$$

i.e., $\lceil x \rceil$ is the smallest integer not less than x.

Next, we define

$$k^*(x) = \left\lceil \frac{\ln(x/L)}{\ln(\beta)} \right\rceil \quad \forall x \in (0, L). \tag{6.2}$$

Then, we have

$$x \in [\beta^{k^*(x)}L, \beta^{k^*(x)-1}L), \ \ \forall x \in (0, L).$$

Therefore, we obtain for $x \in (0, L)$

$$\psi_p(x) = \varphi_{k^*(x)}(x) \quad \forall p \geq k^*(x),$$

and hence,

$$\lim_{p\to+\infty} \psi_p(x) = \varphi_{k^*(x)}(x) \quad \forall x \in (0, L). \tag{6.3}$$

Thus, the series in (3.16) is convergent, i.e. the function $\tilde{\varphi}$ is well defined.

On the other hand, by replacing (3.10) in (2.12) we obtain

$$\beta_j = \beta_0\beta^j, \ j = 1, ..., m.$$

By using (6.1) we have

$$\tilde{\varphi}(x) = 0, \qquad \forall x \in \mathbb{R} \setminus [0, L),$$

and combining with (5.6), we get

$$\tilde{\Phi}_m[\tilde{\varphi}](t/\beta_0) = \sum_{j=1}^{m} a_j\tilde{\varphi}(\beta^j t) \quad \forall t \geq 0. \tag{6.4}$$

By replacing $t = 0$ and $t = \beta_0 L_m$ in (6.4), and using (3.11), (3.13), and (3.16), we obtain

$$\tilde{\Phi}_m[\tilde{\varphi}](0) = g(0), \quad \text{and} \quad \tilde{\Phi}_m[\tilde{\varphi}](L_m) = g(\beta_0 L_m) = 0.$$

Now, if we take $t \in (0, \beta_0 L_m) = (0, L/\beta^m)$, we have

$$\beta^j t \in (0, \beta^{j-m}L), \ \ \text{for } j \in \{1, 2, ..., m\}.$$

We need to consider two cases, $t < L$ and $t \geq L$.

Case $t < L$. In this case we have

$$\beta^j t \in (0, L), \ \ \text{for } j \in \{1, 2, ..., m\},$$

and

$$k^*(\beta^j t) = j + k^*(t).$$

Thus, replacing (6.3) in (6.4), we obtain

$$\tilde{\Phi}_m[\tilde{\varphi}](t/\beta_0) = \sum_{j=1}^{m} a_j\varphi_{j+k^*(t)}(\beta^j t),$$

and hence, using (3.15) with $k+1 = m + k^*(t)$, we obtain

$$\begin{aligned}\tilde{\Phi}_m[\tilde{\varphi}](t/\beta_0) &= \sum_{j=1}^{m-1} a_j \varphi_{j+k^*(t)}(\beta^j t) + a_m \varphi_{m+k^*(t)}(\beta^m t) \\ &= \sum_{j=1}^{m-1} a_j \varphi_{j+k^*(t)}(\beta^j t) + \left(g\,(t) - \sum_{j=1}^{m-1} a_j \varphi_{j+k^*(t)} \left(\beta^j t\right) \right) \\ &= g(t).\end{aligned}$$

Case $t \geq L$. Let us set

$$k_*(x) = \left\lfloor \frac{\ln(x/L)}{\ln(1/\beta)} \right\rfloor \quad \forall x \geq L,$$

where

$$\lfloor x \rfloor = \max\{k \in \mathbb{Z} \;\middle|\; k \leq x\},$$

is the floor function, i.e., it is the largest integer not greater than x. Thus, we have

$$\beta^{k_*(x)+1} x < L \leq \beta^{k_*(x)} x \quad \forall x \geq L, \tag{6.5}$$

and

$$k^*(\beta^{k_*(t)+1} t) = 1. \tag{6.6}$$

Then, we infer from (6.5) that

$$\beta^j t \geq L \qquad \forall j \leq k_*(t),$$

$$\beta^j t < L \quad \forall j \geq k_*(t) + 1.$$

By using (6.4) and (6.3), it follows that

$$\begin{aligned}\tilde{\Phi}_m[\tilde{\varphi}](t/\beta_0) &= \sum_{j=k_*(t)+1}^{m} a_j \varphi_{k^*(\beta^j t)} \left(\beta^j t\right) \\ &= \sum_{j=1}^{m-k_*(t)} a_{j+k_*(t)} \varphi_{k^*(\beta^{j+k_*(t)} t)} \left(\beta^{j+k_*(t)} t\right).\end{aligned}$$

From (6.6), we have

$$k^*(\beta^{j+k_*(t)} t) = j \quad \forall j \geq 1,$$

and hence, from (3.13)-(3.14), with $k+1 = m - k_*(t)$, we obtain

$$
\begin{aligned}
\tilde{\Phi}_m[\tilde{\varphi}](t/\beta_0) &= \sum_{j=1}^{m-k_*(t)} a_{j+k_*(t)}\varphi_j\left(\beta^{j+k_*(t)}t\right)\\
&= \sum_{j=1}^{m-k_*(t)-1} a_{j+k_*(t)}\varphi_j\left(\beta^{j+k_*(t)}t\right) + a_m\varphi_{m-k_*(t)}\left(\beta^m t\right)\\
&= \sum_{j=1}^{m-k_*(t)-1} a_{j+k_*(t)}\varphi_j\left(\beta^{j+k_*(t)}t\right)\\
&+ \left(g(t) - \sum_{j=1}^{m-k_*(t)-1} a_{k_*(t)+j}\varphi_j\left(\beta^{j+k_*(t)}t\right)\right)\\
&= g(t).
\end{aligned}
$$

It remains to prove (3.18). Replacing (3.11) in (3.17) and using (2.13), we get

$$
\begin{aligned}
\tilde{\Phi}_m[\tilde{\varphi}](t/\beta_0) &= \frac{\tilde{I}_m[\rho](t^2/\beta_0^2) - \tilde{I}_m[\rho](L_m^2)}{J_0F(c_0)}\\
&= \tilde{\Phi}_m[\varphi](t/\beta_0) - \frac{\tilde{I}_m[\rho](L_m^2)}{J_0F(c_0)} \quad \forall t \in [0, \beta_0 L_m]\,, \qquad (6.7)
\end{aligned}
$$

where $\varphi(x) = \int_0^x \rho(\tau)d\tau$. Using $\tilde{\Phi}_m[1] = 1$ and Theorem 3.2 we obtain (3.18), i.e.

$$
\tilde{\varphi}(x) = \varphi(x) - \frac{\tilde{I}_m[\rho](L_m^2)}{J_0F(c_0)} \quad \forall x \in [0, L]\,.
$$

This completes the proof of Theorem 3.10. □

Proof of Theorem 3.11. Let ρ be a function in $C^0([0, L])$, and let $\{\varphi_j\}_{j\geq 1}$ be defined in (3.13)-(3.15). Using (3.18), we obtain

$$
\tilde{\varphi}(x) = \varphi(x) - \varphi(L) \quad \forall x \in [0, L], \qquad (6.8)
$$

where $\varphi = \int_0^x \rho(\tau)d\tau$ and $\tilde{\varphi}$ has been defined in (3.16).

Recall that the family of norms $||\cdot||_{[a,b)}$ (for $0 \leq a < b < \infty$) satisfies (3.19)-(3.20). Using (3.16), (6.2)-(6.3) and (6.8), we obtain

$$
\|\varphi(\cdot) - \varphi(L)\|_{[\beta^{k+1}L, \beta^k L)} = \|\varphi_{k+1}\|_{[\beta^{k+1}L, \beta^k L)}\,. \qquad (6.9)
$$

Let us prove that for any $k \geq 0$, we have

$$\|\varphi_{k+1}\|_{[\beta^{k+1}L,\beta^k L)} \leq \frac{C(\beta^m)}{a_m^{k+1}} \|g\|_{[\beta^{k+1}\beta_0 L_m,\beta_0 L_m)} . \tag{6.10}$$

The proof of (6.10) is done by induction on k.

Case $k = 0$. Using (3.13) and (3.19)-(3.20), we have

$$\|\varphi_1\|_{[\beta L,L)} \leq \frac{C(\beta^m)}{a_m} \|g\|_{[\beta\beta_0 L_m,\beta_0 L_m)} ,$$

as desired.

Assume now that for all $j = 1, ..., k$ (with $k \geq 1$), we have

$$\|\varphi_j\|_{[\beta^j L,\beta^{j-1} L)} \leq \frac{C(\beta^m)}{a_m^j} \|g\|_{[\beta^j\beta_0 L_m,\beta_0 L_m)} , \tag{6.11}$$

and let us prove (6.10).

Case $k+1 \leq m$. Using (3.14) and (3.19)-(3.20), we obtain

$$\|\varphi_{k+1}\|_{[\beta^{k+1}L,\beta^k L)} \leq \frac{1}{a_m} \Bigg(C(\beta^m) \|g\|_{[\beta^{k+1}\beta_0 L_m,\beta^k\beta_0 L_m)} + \sum_{j=1}^{k} a_{m-k-1+j} C\left(\frac{\beta^{k+1}}{\beta^j}\right) \|\varphi_j\|_{[\beta^j L,\beta^{j-1} L)} \Bigg) .$$

Using induction hypothesis (6.11), we have

$$\begin{aligned}
\|\varphi_{k+1}\|_{[\beta^{k+1}L,\beta^k L)} &\leq \frac{1}{a_m} \Bigg(C(\beta^m) \|g\|_{[\beta^{k+1}\beta_0 L_m,\beta^k\beta_0 L_m)} \\
&\quad + \sum_{j=1}^{k} a_{m-k-1+j} C\left(\frac{\beta^{k+1}}{\beta^j}\right) \frac{C(\beta^m)}{a_m^j} \|g\|_{[\beta^j\beta_0 L_m,\beta_0 L_m)} \Bigg) \\
&\leq \frac{C(\beta^m)}{a_m^{k+1}} \Bigg(a_m^k \|g\|_{[\beta^{k+1}\beta_0 L_m,\beta^k\beta_0 L_m)} \\
&\quad + \sum_{j=1}^{k} a_{m-k-1+j} C\left(\frac{\beta^{k+1}}{\beta^j}\right) \|g\|_{[\beta^j\beta_0 L_m,\beta_0 L_m)} \Bigg) \\
&\leq \frac{C(\beta^m)}{a_m^{k+1}} \left(a_m^k + \sum_{j=1}^{k} a_{m-k-1+j} C\left(\frac{\beta^{k+1}}{\beta^j}\right) \right) \|g\|_{[\beta^{k+1}\beta_0 L_m,\beta_0 L_m)} .
\end{aligned} \tag{6.12}$$

Note that

$$C(u) \leq 1 \quad \forall u \in (0,1), \tag{6.13}$$

for $C(\cdot)$ is nondecreasing and $C(1) = 1$. Therefore,

$$C\left(\frac{\beta^{k+1}}{\beta^j}\right) \leq 1 \quad \forall j \in \{1, ..., k\}.$$

Thus, we obtain

$$\|\varphi_{k+1}\|_{([\beta^{k+1},\beta^k L)} \leq \frac{C(\beta^m)}{a_m^{k+1}} \|g\|_{[\beta^{k+1}\beta_0 L_m,\beta_0 L_m)}.$$

This proves (6.10) for all $k = \{0, ..., m-1\}$.

Case $k+1 > m$**.** Replacing φ_{k+1} by expression in (3.15) and using (3.19)-(3.20) and the induction hypothesis, we obtain

$$\begin{aligned}
\|\varphi_{k+1}\|_{[\beta^{k+1}L,\beta^k L)} \leq \frac{1}{a_m}\Big(C(\beta^m)\,\|g\|_{[\beta^{k+1}\beta_0 L_m,\beta^k\beta_0 L_m)} \\
+ \sum_{j=1}^{m-1} a_j C\left(\frac{\beta^m}{\beta^j}\right)\frac{C(\beta^m)}{a_m^{j+k-m+1}}\|g\|_{[\beta^{j+k-m+1}\beta_0 L_m,\beta_0 L_m)}\Bigg) \\
= \frac{C(\beta^m)}{a_m^{k+1}}\Big(a_m^k\,\|g\|_{[\beta^{k+1}\beta_0 L_m,\beta^k\beta_0 L_m)} \\
+ \sum_{j=1}^{m-1} a_j C\left(\frac{\beta^m}{\beta^j}\right)\|g\|_{[\beta^{j+k-m+1}\beta_0 L_m,\beta_0 L_m)}\Bigg) \\
\leq \frac{C(\beta^m)}{a_m^{k+1}}\left(a_m^k + \sum_{j=1}^{m-1} a_j C\left(\frac{\beta^m}{\beta^j}\right)\right)\|g\|_{[\beta^{k+1}\beta_0 L_m,\beta_0 L_m)},
\end{aligned} \tag{6.14}$$

and with (6.13) we infer that $C(\beta^m/\beta^j) \leq 1$ for all $j \in \{1, ..., m-1\}$. This completes the proof of (6.10).

On the other hand, using (3.17) and (3.20), we obtain

$$\|g\|_{[\beta^{k+1}\beta_0 L_m,\beta_0 L_m)} \leq C(\beta_0)\left\|\tilde{\Phi}_m[\tilde{\varphi}]\right\|_{[\beta^{k+1}L_m,L_m)}.$$

By replacing (6.8) in (6.10), we obtain

$$\|\varphi_{k+1}\|_{[\beta^{k+1}L,\beta^k L)} \leq C(\beta_0)\frac{C(\beta^m)}{a_m^{k+1}}\left\|\tilde{\Phi}_m[\varphi](\cdot) - \tilde{\Phi}_m[\varphi](L_m)\right\|_{[\beta^{k+1}L_m,L_m)},$$

and by replacing in (6.9), we obtain (3.22). This completes the proof of Theorem 3.11.

□

7 Numerical results

In this section we discuss the numerical implementation of the scheme developed when proving Theorem 3.10.

Firstly, we define $\{\alpha\}_{j=1}^{m}$ as in (3.10), and let

$$F_j = \begin{cases} 0 & j = 0, \\ F\left(\dfrac{\alpha_j + \alpha_{j+1}}{2}\right) & j = 1, ..., m-1, \\ F(c_0) & j = m, \end{cases}$$

where F is Hill's function defined in (1.3). Next, we set

$$a_j = \frac{F_j - F_{j-1}}{F(c_0)}, \qquad j = 1, ..., m. \tag{7.1}$$

Since F is increasing and $0 \le F(x) < 1$ for all $x \ge 0$, we infer that $a_j > 0$ for all $j = 1, ..., m$, and that

$$\sum_{j=1}^{m} a_j = 1.$$

The corresponding approximation F_m of Hill's function is shown graphically in Figure 2.

Recall that a non-regular mesh in the interval $[0, L]$ was introduced when proving Theorem 3.10. Now, let us start defining

$$\mathcal{P}_{q,1} = \left\{ (x_0, x_1, \ldots, x_q) \in \mathbb{R}^{q+1} \colon\ x_j \in [\beta L, L),\ x_0 = \beta L,\ x_{j-1} < x_j,\ \forall j = 1, ..., q \right\},$$

and its representative vector

$$\mathbf{P}_1 = (x_0, x_1, \ldots, x_q) \in \mathbb{R}^{q+1}.$$

Next, introduce the sets

$$\mathcal{P}_{q,j} = \left\{ x \,\middle|\, \beta^{j-1} x \in \mathcal{P}_{q,1} \right\}, \qquad j \ge 1,$$

and denote their corresponding representative vectors by

$$\mathbf{P}_j = \beta^{j-1} \mathbf{P}_1 = \beta \mathbf{P}_{j-1} \in \mathbb{R}^{q+1}.$$

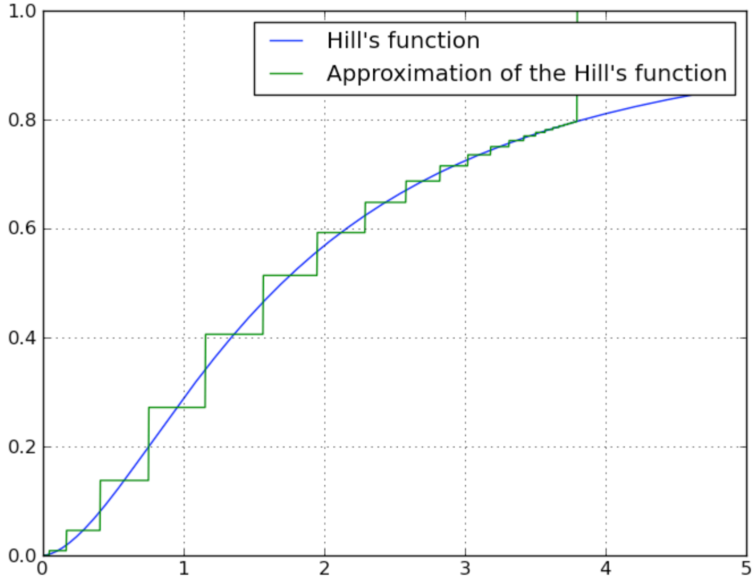

Figure 2. Hill's function and its approximation

Let us fix some $p \geq 1$. Our aim is to recover the function ρ on the mesh

$$\Sigma_{p,q} = \cup_{j=1}^{p} \mathcal{P}_{q,j}, \tag{7.2}$$

where the corresponding representative vector is given by

$$\mathbf{P} = (\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_p) \in \mathbb{R}^{p+q+1}.$$

By using (3.13)-(3.15), we can define the vectors $\mathbf{G}_1, \mathbf{G}_2, ..., \mathbf{G}_m \in \mathbb{R}^{q+1}$ inductively as follows:

$$(\mathbf{G}_1)_s = \frac{1}{a_m} g\left(\frac{(\mathbf{P}_1)_s}{\beta^m}\right), \quad s = 1, ..., q+1, \tag{7.3}$$

and for $k = 1, ..., m-1$:

$$(\mathbf{G}_{k+1})_s = \frac{1}{a_m}\left(g\left(\frac{(\mathbf{P}_{k+1})_s}{\beta^m}\right) - \sum_{j=1}^{k} a_{m-k-1+j}(\mathbf{G}_j)_s\right) \quad s = 1, ..., q+1. \tag{7.4}$$

Finally, we define the vectors $\mathbf{G}_k \in \mathbb{R}^{q+1}$ for $k = m, ..., p-1$, by

$$(\mathbf{G}_{k+1})_s = \frac{1}{a_m}\left(g\left(\frac{(\mathbf{P}_{k+1})_s}{\beta^m}\right) - \sum_{j=1}^{m-1} a_j \left(\mathbf{G}_{j+k-m+1}\right)_s\right) \qquad s = 1, ..., q+1. \tag{7.5}$$

Introduce the vector

$$\mathbf{G} = (\mathbf{G}_1, \mathbf{G}_2, \cdots, \mathbf{G}_p) \in \mathbb{R}^{p+q+1},$$

which represents a discretization of the function $\tilde{\varphi}$ given by Theorem 3.10 on the mesh defined by $\Sigma_{p,q}$,; that is, $((\mathbf{P})_s, (\mathbf{G})_s)_{s=1}^{p+q+1}$ is a discretization of the curve $(x, \tilde{\varphi}(x))$, $x \in (0, L)$.

Therefore, using (3.18) and applying a forward difference scheme, we obtain an approximation of the curve $(x, \rho(x))$, $x \in (0, L)$, through the vectors $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^{p+q}$ given by

$$(\mathbf{X})_s = (\mathbf{P})_s, \quad (\mathbf{Y})_s = \max\left\{\frac{(\mathbf{G})_{s+1} - (\mathbf{G})_s}{(\mathbf{P})_{s+1} - (\mathbf{P})_s}, 0\right\} \quad s = 1, ..., p+q. \tag{7.6}$$

It should be noted that the maximum function was considered in (7.6) because of the positivity restriction on the density function.

7.1 Examples

Let us consider

$$\rho(x) = \frac{8a^8x^7}{(x^8+a^8)^2}, \qquad \varphi(x) = \int_0^x \rho(\tau)d\tau = \frac{x^8}{x^8+a^8}, \tag{7.7}$$

with $a = 1.5$. Figures 3 and 4 show functions $\rho(x)$ and $\varphi(x)$ defined in (7.7) and their approximations obtained by the previous procedure.

Remark 7.1. If we consider any discretization of (2.4) on a given mesh, one has to solve a system like

$$A\vec{y} = \vec{g}.$$

Obviously, the system depends strongly on the choice of the mesh.

We notice that the matrix A may not be invertible. While it is difficult to give a general criterion for the invertibility of A in terms of the mesh, Theorem 3.10 guarantees that the matrix A is indeed invertible when the non-regular mesh described in (7.2) is used for the discretization of system (2.4).

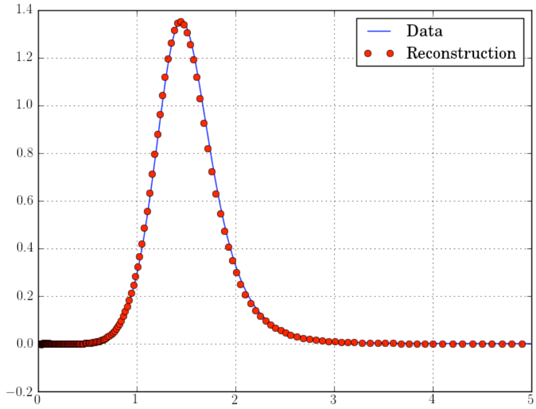

Figure 3. The target function ρ and its approximation.

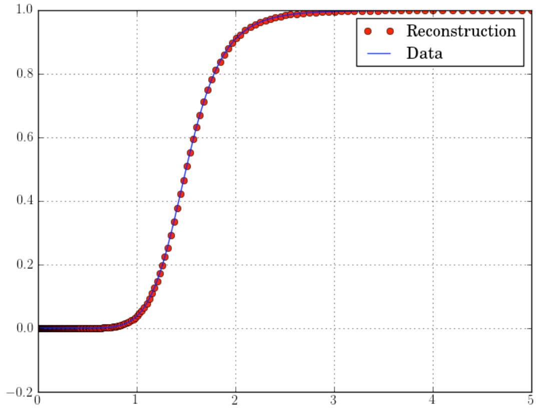

Figure 4. The target function φ and its approximation.

Let us now consider the example studied in [6]. To this end, we define

$$I(t) = \begin{cases} 0, & t \in (0, t_{Delay}), \\ I_{Max}\left[1 + \left(\dfrac{K_I}{t - t_{Delay}}\right)^{n_I}\right]^{-1}, & t > t_{Delay}, \end{cases} \tag{7.8}$$

with $t_{Delay} = 30[ms]$, $n_I \simeq 2.2$, $I_{Max} = 150[pA]$ and $K_I \simeq 100[ms]$. The current given in (7.8) is a sigmoidal function with short delay (Figure 5**B**), which is similar to the profiles encountered in some practical situations (see e.g., [1], [3] or [11]).

A: approximation of ρ. **B**: current $I(t)$ as defined by (7.8)

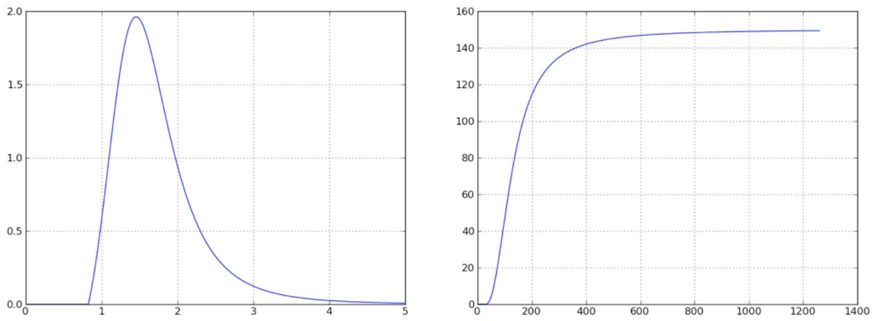

Figure 5. Approximation of the function $\rho(x)$ with current $I(t)$ as defined by (7.8)

The numerical solution corresponding to these data is shown in Figure 5**A**. It should be noted that the numerical solution given here is perfectly consistent with those obtained in [3].

8 Polynomial approximation of Hill's function

In this section we consider the same inverse problem with another approximation of the kernel in (2.2), for which we keep the function c and replace Hill's function F by a polynomial approximation around c_0.

More precisely, let P_m be the standard Taylor polynomial expansion of degree m of (1.3) around c_0; that is, $P_m \in \mathbb{R}[X]$, $\deg(P_m) \leq m$ and

$$F(x) = P_m(x - c_0) + O(|x - c_0|^{m+1}). \tag{8.1}$$

A new approximation for the kernel is defined by

$$PK_m(t, x) = P_m(c(t, x) - c_0), \tag{8.2}$$

where $c(t, x)$ is the solution of (2.3), given by

$$c(t, x) = c_0 - c_0 \left(\frac{2}{L}\right)^{\frac{1}{2}} \left(\sum_{k=0}^{+\infty} \frac{e^{-\mu_k^2 Dt}}{\mu_k} \psi_k(x)\right), \tag{8.3}$$

with

$$\mu_k = \frac{2k+1}{2L}\pi, \tag{8.4}$$

and

$$\psi_k(x) = \left(\frac{2}{L}\right)^{1/2} \sin(\mu_k x). \tag{8.5}$$

Besides, we define the total current associated with this polynomial approximation as follows:

$$PI_m[\rho](t) = J_0 \int_0^L \rho(x) PK_m(t,x)dx \quad \forall t > 0. \tag{8.6}$$

Next, we present our main result regarding the operator $PI_M : L^2(0,L) \to L^\infty(0,+\infty)$; it asserts that the identifiability for the operator PI_m holds when $m \leq 8$.

Theorem 8.1. *Let $m \leq 8$ be a given integer. Then*

$$Ker\ PI_m = \{0\},$$

where

$$Ker\ PI_m = \left\{ f \in L^2(0,L) \middle| \ PI_m[f](t) = 0 \ \forall t > 0 \right\}.$$

8.1 Proof of Theorem 8.1.

We can without loss of generality assume that $J_0 = 1$. Let us start noting that $\{\psi_k\}_{k \geq 0}$ is an orthonormal basis in $L^2(0,L)$. Thus, for any $f \in L^2(0,L)$, we can write

$$f(x) = \sum_{k \geq 0} < f, \psi_k > \psi_k(x) \qquad \text{in } L^2(0,L), \tag{8.7}$$

where

$$< f, \psi_k >= \int_0^L f(x)\psi_k(x)dx.$$

We write

$$P_m(z) = \alpha_0 + \alpha_1 z + \cdots + \alpha_m z^m,$$

where $\alpha_j \in \mathbb{R}$, for all $j = 0, 1, ..., m$, and introduce the set

$$\Lambda_m = \left\{ \sum_{j=1}^{k} \mu_{n_j}^2 \ \middle| n_j \geq 0, \ 1 \leq k \leq m \right\}. \tag{8.8}$$

Let $\varepsilon > 0$ be a given positive constant. For any $\rho \in L^2(0, L)$ and for all $s \geq 0$, we have

$$PI_m[\rho](\varepsilon + s) = \alpha_0 \int_0^L \rho(x)dx$$

$$+ \sum_{j=1}^{m} \alpha_j (-c_0 \sqrt{\frac{2}{L}})^j \int_0^L \left(\sum_{k \geq 0} \frac{e^{-\mu_k^2 D(\varepsilon + s)}}{\mu_k} \psi_k(x) \right)^j \rho(x)dx$$

$$= \alpha_0 \int_0^L \rho(x)dx$$

$$+ \sum_{j=1}^{m} \alpha_j (-c_0 \sqrt{\frac{2}{L}})^j \sum_{k_1,\dots,k_j \geq 0} e^{-\sum_{p=1}^{j} \mu_{k_p}^2 D(s+\varepsilon)} \int_0^L (\Pi_{p=1}^j \frac{\psi_{k_p}(x)}{\mu_{k_p}}) \rho(x)dx.$$

Note that the convergence of the last series is fully justified, as for any $j \in \{1, ..., m\}$ and any $s \geq 0$, we have

$$\sum_{k_1,\dots,k_j \geq 0} e^{-\sum_{p=1}^{j} \mu_{k_p}^2 D(s+\varepsilon)} \left| \int_0^L (\Pi_{p=1}^j \frac{\psi_{k_p}(x)}{\mu_{k_p}}) \rho(x)dx \right|$$

$$\leq ||\rho||_{L^1(0,L)} \left(\sum_{k \geq 0} \sqrt{\frac{2}{L}} \frac{e^{-\mu_k^2 D\varepsilon}}{\mu_k} \right)^j < \infty, \quad (8.9)$$

Therefore, there is a family $\{a_\lambda(\varepsilon, \rho)\}_{\lambda \in \Lambda_m}$ such that

$$\sum_{\lambda \in \Lambda_m} |a_\lambda(\varepsilon, \rho)| < \infty, \quad (8.10)$$

and

$$PI_m[\rho](\varepsilon + s) \quad = \quad \alpha_0 \int_0^L \rho(x)dx + \sum_{\lambda \in \Lambda_m} a_\lambda(\varepsilon, \rho) e^{-\lambda D s} \quad \forall s \geq 0. \quad (8.11)$$

Lemma 8.2. *Let $\rho \in L^2(0, L)$ be a given function, such that*

$$PI_m[\rho](t) = 0 \quad \forall t > 0. \quad (8.12)$$

Then

$$\int_0^L \rho(x)dx = 0, \tag{8.13}$$

and

$$a_\lambda(\varepsilon, \rho) = 0 \quad \forall \lambda \in \Lambda_m. \tag{8.14}$$

Proof. First, the series in (8.11) is uniformly convergent for $s \geq 0$, by (8.10).

Define $\{\lambda_k\}_{k\geq 1}$ as

$$\lambda_1 = \min\Big\{\lambda \in \Lambda_m\Big\}, \qquad \lambda_{k+1} = \min\Big\{\lambda \in \Lambda_m \setminus \{\lambda_1, ..., \lambda_k\}\Big\}, \tag{8.15}$$

and note that this defines an increasing sequence $0 < \lambda_1 < \lambda_2 < \cdots$

Using definition 8.15, we can rewrite (8.11) as

$$PI_m[\rho](\varepsilon + s) = \alpha_0 \int_0^L \rho(x)dx + \sum_{k\geq 1} a_{\lambda_k}(\varepsilon, \rho)e^{-\lambda_k D\varepsilon}e^{-\lambda_k Ds} \quad \forall s \geq 0. \tag{8.16}$$

Set

$$S_j(s) = \sum_{k\geq j} a_{\lambda_k}(\varepsilon, \rho)e^{-\lambda_k D\varepsilon}e^{-\lambda_k Ds} \quad \forall s \geq 0.$$

Then we have that

$$|S_j(s)| \leq e^{-\lambda_j Ds}\left(\sum_{k\geq j} |a_{\lambda_k}(\varepsilon, \rho)|\, e^{-\lambda_k D\varepsilon}\right). \tag{8.17}$$

Plugging (8.16) into (8.12), it follows that

$$\alpha_0 \int_0^L \rho(x)dx + S_1(s) = 0 \quad \forall s > 0. \tag{8.18}$$

Passing to the limit as $s \to +\infty$ in (8.18), and noting that $\alpha_0 = F(c_0) \neq 0$ and that $S_1(s) \to 0$ by (8.17), we obtain (8.13).

The proof of (8.14) is done by induction on $j \geq 1$.

Case j=1. Plugging (8.13) in (8.18) and multiplying by $e^{\lambda_1 Ds}$, we obtain

$$a_{\lambda_1}(\varepsilon, \rho)e^{-\lambda_1 D\varepsilon} + e^{\lambda_1 Ds}S_2(s) = 0 \quad \forall s > 0. \tag{8.19}$$

But, using (8.17), we also have that

$$|e^{\lambda_1 Ds} S_2(s)| \le Ce^{-(\lambda_2-\lambda_1)Ds}.$$

Thus, noting that $\lambda_1 < \lambda_2$ and passing to the limit as $s \to \infty$ in (8.19), we obtain

$$a_{\lambda_1}(\varepsilon, \rho) = 0.$$

Case j=n+1. Assume that

$$a_{\lambda_j}(\varepsilon, \rho) = 0 \quad \forall j = \{1, ..., n\}. \tag{8.20}$$

Plugging (8.20) and (8.13) in (8.12), we infer that

$$a_{\lambda_{n+1}}(\varepsilon, \rho)e^{-\lambda_{n+1}D\varepsilon} + e^{\lambda_{n+1}Ds} S_{n+2}(s) = 0, \quad \forall s > 0. \tag{8.21}$$

On the other hand, using (8.17), we have that

$$|e^{\lambda_{n+1}Ds} S_{n+2}(s)| \le Ce^{-(\lambda_{n+2}-\lambda_{n+1})Ds}.$$

Thus, noting that $\lambda_{n+1} < \lambda_{n+2}$ and passing to the limit as $s \to \infty$ in (8.21), we infer that

$$a_{\lambda_{n+1}}(\varepsilon, \rho) = 0.$$

This yields (8.14). This complete the proof of Lemma 8.2. □

Lemma 8.3. *Let* $\{\mu_k\}_{k\ge 0}$ *be the sequence defined in* (8.4). *Assume that*

$$\mu_{n_1}^2 + \cdots + \mu_{n_k}^2 = \mu_n^2. \tag{8.22}$$

for some $k \ge 1$ *and* $n, n_1, ..., n_k \ge 0$. *Then*

$$k = 1 \textit{ mod } 8. \tag{8.23}$$

Proof. We have

$$\mu_n^2 = \frac{\pi^2}{4L^2}(4\varphi(n) + 1),$$

where $\varphi(n) = n^2 + n$. Thus, substituting this expression of $\mu_{n_i}^2$ in (8.22) yields

$$k + 4\sum_{i=1}^{k} \varphi(n_i) = 1 + 4\varphi(n).$$

Noticing that $\varphi(n)$ is an even number for all $n \in \mathbb{N}$, we obtain (8.23). □

Proof of the Theorem 8.1. Let $\rho \in L^2(0,L)$ be a given function such that (8.12) holds. From Lemma 8.2 we infer that (8.14) holds. If $m \le 8$, using Lemma 8.3 we have that for all $n \ge 0$, all $k \in \{2,...,m\}$, and all $n_1,...,n_k \ge 0$,

$$\mu_{n_1}^2 + \cdots + \mu_{n_k}^2 \neq \mu_n^2.$$

Then, with $\lambda = \mu_n^2 \in \Lambda_m$, we obtain

$$a_{\mu_n^2}(\varepsilon,\rho) = e^{-\mu_n^2 D\varepsilon}\alpha_1(-c_0\sqrt{\frac{2}{L}})\int_0^L \frac{\psi_n(x)}{\mu_n}\rho(x)dx.$$

Since $\alpha_1 = F'(c_0) \neq 0$ (F being increasing), we infer that

$$< \psi_n, \rho >= 0, \quad \forall n \ge 0,$$

and hence, with (8.7), that

$$\rho = 0.$$

This completes the proof of Theorem 8.1. □

Corollary 8.4. *Let $m \le 8$ be a given integer. If $\rho \in L^2(0,L)$ is such that*

$$PI_m[\rho](t) = 0 \quad \forall t \in (0,\delta),$$

for some $\delta > 0$, then

$$\rho \equiv 0.$$

Proof. It is sufficient to notice that the map $t \to PI_m[\rho](t)$ is analytic on $(0,+\infty)$, and to apply Theorem 8.1. □

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