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FINITE-TIME STABILIZATION OF A NETWORK OF STRINGS

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Abstract. We investigate the finite-time stabilization of a tree-shaped network of strings. Transparent boundary conditions are applied at all the external nodes. At any internal node, in addition to the usual continuity conditions, a modified Kirchhoff law incorporating a damping term αu_t with a coefficient α that may depend on the node is considered. We show that for a convenient choice of the sequence of coefficients α, any solution of the wave equation on the network becomes constant after a finite time. The condition on the coefficients proves to be sharp at least for a star-shaped tree. Similar results are derived when we replace the transparent boundary condition by the Dirichlet (resp. Neumann) boundary condition at one external node. Our results lead to the finite-time stabilization even though the systems may not be dissipative.

1. **Introduction.** Solutions of certain ODE $\dot{x} = f(x)$ may reach the equilibrium state in finite time. This phenomenon, when combined with the stability, was termed *finite-time* stability in [4, 11].

A *finite-time stabilizer* is a feedback control for which the closed-loop system is finite-time stable around some equilibrium state. In some sense, it satisfies an exact controllability objective with a control in feedback form. On the other hand, a finite-time stabilizer may be seen as an exponential stabilizer yielding an arbitrarily large decay rate for the solutions to the closed-loop system. Indeed, any solution of the closed-loop system can be estimated as

$$||x(t)|| \le h(||x_0||)\mathbf{1}_{[0,T]}(t) \le h(||x_0||)e^{-\lambda(t-T)}$$

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where $h(\delta) \to 0$ as $\delta \to 0$, and $\lambda > 0$ is arbitrarily large. This explains why some efforts were made in the last decade to construct finite-time stabilizers for controllable systems, including the linear ones. See [15] for some recent developments and up-to-date references, and [3] for some connections with Lyapunov theory.

To the best knowledge of the authors, the analysis of the finite-time stabilization of PDE is not developed yet. However, since [14], it is well-known that solutions of the wave equation on certain bounded domains may disappear when using *transparent* boundary conditions. For instance, the solution of the 1-D wave equation

$$u_{tt} - c^2 u_{xx} = 0, \qquad \text{in } (0,L) \times (0,T), \tag{1}$$
$$c u_x(L,t) = -u_t(L,t), \qquad \text{in } (0,T), \tag{2}$$
$$c u_x(0,t) = u_t(0,t), \qquad \text{in } (0,T), \tag{3}$$
$$(u(0), u_t(0)) = (u^0, u^1), \qquad \text{in } (0,L), \tag{4}$$

is finite-time stable in the space $\{(u,v) \in H^1(0,L) \times L^2(0,L);\ c\big(u(0) + u(L)\big) + \int_0^L v(x)dx = 0\}$, with $T = L/c$ as extinction time (see e.g. [12, Theorem 0.5] for the details.) The condition (2) is "transparent" in the sense that a wave $u(x,t) = f(x - ct)$ traveling to the right satisfies (2) and leaves the domain at $x = L$ without generating any reflected wave. Note that the solution issued from any state $(u^0, u^1) \in H^1(0,L) \times L^2(0,L)$ is not necessarily vanishing, but constant, for $t \geq L/c$. Note also that if we replace (3) by the boundary condition $u(0,t) = 0$ (or $u_x(0,t) = 0$), then a finite-time extinction still occurs (despite the fact that waves bounce at $x = 0$) with an extinction time $T = 2L/c$. We refer to [5] for the analysis of the finite-time extinction property for a nonhomogeneous string with a viscous damping at one extremity, to [8] for the finite-time stabilization of a string with a moving boundary, to [16] (resp. [17]) for the finite-time stabilization of a system of conservation laws on an interval (resp. on a tree-shaped network).

The finite-time stability of (1)-(4) is easily established when writing (1) as a system of two transport equations

$$d_t + c d_x = 0,$$
$$s_t - c s_x = 0.$$

where $d := u_t - c u_x$ and $s := u_t + c u_x$ stand for the Riemann invariants for the wave equation written as a first order hyperbolic system. The boundary conditions (2) and (3) yield $d(0,t) = s(L,t) = 0$ (and hence $d(.,t) = s(.,t) = 0$ for $t \geq L/c$), while the boundary conditions (2) and $u(0,t) = 0$ yield $s(L,t) = 0$ and $d(0,t) = -s(0,t)$ (and hence $s(.,t) = 0$ for $t \geq L/c$ and $d(.,t) = 0$ for $t \geq 2L/c$).

The stabilization of networks of strings has been considered in e.g. [1, 2, 7, 9, 10, 19, 22]. In [10], the authors considered a star of vibrating strings, and derived the finite-time stability (resp. the exponential stability) when transparent boundary conditions are applied at all external nodes (resp. at all external nodes but one, which is changing as times proceeds). For a more general network, we guess that the finite time stability cannot hold without the introduction of additional feedback controls at the internal nodes. Indeed, it is proved here that for a bone-shaped tree, if the feedback controls are applied only at the external nodes, then the finite-time stability fails.

The aim of this paper is to investigate the *finite-time* stabilization of a *tree-shaped* network of strings. At each internal node n connecting k edges, we assume that the usual continuity condition hold:

$$u_i(n,t) = u_j(n,t), \qquad \text{for all indices } i,j \text{ of edges having } n \text{ as one end,} \tag{5}$$

while the usual Kirchhoff law is modified by incorporating a *damping term* inside:

$$\sum_i c_i u_{i,x}(n,t) - c_{i_0} u_{i_0,x}(n,t) = -\alpha(n) u_t(n,t). \tag{6}$$

In (6), the sum is over the indices i of the edges having n as initial point, i_0 denotes the index of the edge having n as final point (the tree being oriented), $\alpha(n) \in \mathbb{R}$ is a coefficient depending on the node n, and $u(n,t) := u_i(n,t)$ (for any i). The case $\alpha = 0$ corresponds to the usual (conservative) Kirchhoff law.

Note that we can assume without loss of generality that the length of each edge is one, by scaling the variable x and the coefficient c_i along each edge.

Even if the finite-time stabilization of 2×2 hyperbolic systems on tree-shaped networks was already considered in [17] (and applied to the regulation of water flows in networks of canals, with $k-1$ controls at any node connecting k canals), the novelty (and difficulty) here comes from the fact that only *one* control is applied at each internal node. The present work can be seen as a first step in the understanding of the finite-time stabilization of systems of conservation laws with few controls. In practice, on account of possible budget limitations, it is important that the number of interior controls be chosen as small as possible.

A natural guess is that the finite-time stability cannot hold if one can find in the tree a pair of adjacent nodes that are free of any control, because of the (partial but standing) bounces of waves at these nodes. This conjecture will be demonstrated here for a star-shaped tree and a bone-shaped tree.

Actually, we shall prove that the finite-time stabilization can be achieved for a very particular choice of the coefficient α at each internal node. One of the main results proved in this paper is the following

Theorem 1.1. *Consider any tree-shaped network of strings, with transparent boundary conditions at all the external nodes but one (the root), where a homogeneous Dirichlet boundary condition is applied, and with the continuity conditions and the modified Kirchhoff law at the internal nodes. If at each internal node n connecting k edges we have*

$$\alpha(n) = k - 2, \tag{7}$$

then each solution of the wave equation on the network becomes null after some finite time. Conversely, the condition (7) is also necessary for the finite-time stability of the system on a star-shaped tree.

Similar results will be obtained when replacing at the root the homogeneous Dirichlet boundary condition by the homogeneous Neumann boundary condition or by the transparent boundary condition. The necessity of the condition $\alpha = k-2$ for the finite-time stability of a star-shaped tree is obtained by doing some explicit computation of the discrete spectrum of the underlying operator. The same approach gives for a bone-shaped tree a necessary and sufficient condition for the finite-time stability, which differs slightly from those stated in Theorem 1.1.

The condition $\alpha(n) = k-2$ can be interpreted as follows: the wave emanating from n along any edge is the sum of the waves arriving at n along the $k-1$ others edges (see below (34)). The node n is *transparent* in the sense that there is transmission without attenuation nor reflection of all the waves that reach it.

The well-posedness of the system is also completely characterized in terms of the coefficients α (see below Theorem 2.1). We note that the well-posedness of the system may hold even if the system is not dissipative, and that our finite-time stability results hold in a class of systems that are not necessarily dissipative.

Finally, we point out that the recent papers [18, 21] contain some results similar to ours. (Note that the finite-time stability is termed *super-stability* in [18, 21].) The paper [21] is concerned with star-shaped trees, while the paper [18] considers very general networks. When comparing the results in [18] with ours, we observe the following:

1. The wave equation in [18] takes the form
$$\rho_i u_{i,tt} - T_i u_{i,xx} = 0, \quad \text{in } (0,1)\times(0,T) \tag{8}$$
where $T_i, \rho_i > 0$ are given coefficients, while the modified Kirchhoff law at an internal node of a tree is written
$$\sum_i T_i u_{i,x}(n,t) - T_{i_0} u_{i_0,x}(n,t) = -\alpha(n) u_t(n,t). \tag{9}$$
Note that $(T_i, \rho_i) := (c_i, c_i^{-1})$ is a possible choice (among others) to yield (12) (see below) on the edge i.
2. The well-posedness of the complete system is derived in [18, Proposition 2.1] under the *sufficient* condition $\alpha(n) \le 0$ for all internal nodes n, which ensures that the system is *dissipative*, while a *sharp* condition is given in our Theorem 2.1.
3. When $(T_i, \rho_i) = (c_i, c_i^{-1})$ for all i, the condition (3.7) in [18, Theorem 3.2] which gives the values of $\alpha(n)$ at all the internal nodes ensuring the finite-time stability is the same as our condition (92) in Theorem 4.1. However, it is also assumed in [18, Theorem 3.2] that
$$\alpha(n) \le 0 \qquad \text{for all internal nodes } \ n. \tag{10}$$
This imposes some restrictions on the coefficients T_i, ρ_i or on the shape of the tree. For instance, for uniform data ($T_i = \rho_i = 1$ for all i), then the condition (10) (with α_n as in (92)) is satisfied only if $k_n = 2$ for all internal nodes n, i.e. only if the tree is a *line*, while our Theorem 4.1 is valid for *any tree*. On the other hand, by choosing other pairs of positive coefficients (T_i, ρ_i) with $T_i/\rho_i = c_i^2$, Theorem 3.2 in [18] gives modified Kirchhoff laws (9) different from (6) that may lead to the finite-time stability.
4. Our approach is based upon the analysis of the Riemann invariants and the propagation of waves, while [18] is based upon spectral theory.

The paper is outlined as follows. In Section 2, we provide a sharp condition on the coefficients $\alpha(n)$ for the system to be well-posed. It is obtained by expressing the conditions (5)-(6) at the internal nodes in terms of the Riemann invariants. In Section 3, we consider the particular case of a tree with two internal nodes to obtain sharp results for the finite-time stability. In Section 4, we derive the finite-time stability results when the coefficients α are chosen as in (7). Finally, we prove in Section 5 the necessity of that condition for tree-shaped networks.

2. **Well-posedness.**

We introduce some notations inspired by [6]. Let $\mathcal{T}$ be a tree, whose *vertices* (or *nodes*) are numbered by the index $n \in \mathcal{N} = \{0, ..., N\}$, and whose *edges* are numbered by the index $i \in \mathcal{I} = \{1, ..., N\}$. We choose a simple vertex (i.e. an external node), called the *root* of $\mathcal{T}$ and denoted by $\mathcal{R}$, and which corresponds to the index $n = 0$. The edge containing $\mathcal{R}$ has $i = 1$ as index, and its other endpoint has for index $n = 1$. We choose an orientation of the edges in the tree such that $\mathcal{R}$ is the “first” encountered vertex. The *depth* d of the tree is the number of generations ($d = 1$ for a tree reduced to a single edge, $d = 2$ for a star-shaped tree, etc.) Once the orientation of the tree is chosen, each point of the i-th edge (of length 1) is identified with a real number $x \in [0, 1]$. The points $x = 0$ and $x = 1$ are termed the *initial point* and the *final point* of the i-th edge, respectively. Renumbering the edges if needed, we can assume that the edge of index i has as final point the vertex with the (same) index $n = i$ for all $i \in \mathcal{I}$. (See Figure 1.) The set of indices of

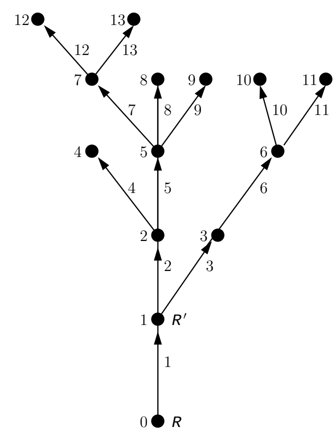

FIGURE 1. A tree with 14 nodes, a depth equal to 5, with simple nodes $\mathcal{N}_S = \{0, 4, 8, 9, 10, 11, 12, 13\}$ and multiple nodes $\mathcal{N}_M = \{1, 2, 3, 5, 6, 7\}$.

simple and multiple nodes are denoted by $\mathcal{N}_S$ and $\mathcal{N}_M$, respectively.

For $n \in \mathcal{N}_M$ we denote by $\mathcal{I}_n$ the set of indices of those edges having the vertex of index n as initial point. As we consider a network of strings whose constants c_i may vary from one edge to another one, the case $\#(\mathcal{I}_n) = 1$ (one child) is possible. The number of edges having the vertex of index n as one as their extremities, also termed the *degree* of the vertex n, is

$$k_n := \#(\mathcal{I}_n) + 1 \geq 2. \tag{11}$$

We consider the following system

$$u_{i,tt} - c_i^2 u_{i,xx} = 0, \qquad t > 0,\ 0 < x < 1,\ i \in \mathcal{I} \tag{12}$$
$$(u_i(.,0), u_{i,t}(.,0)) = (u_i^0, u_i^1), \qquad i \in \mathcal{I} \tag{13}$$

with the following boundary conditions

$$c_n u_{n,x}(1,t) = -u_{n,t}(1,t), \qquad t > 0,\ n \in \mathcal{N}_S \setminus \{0\}, \tag{14}$$
$$\sum_{i \in \mathcal{I}_n} c_i u_{i,x}(0,t) - c_n u_{n,x}(1,t) = -\alpha_n u_{n,t}(1,t), \qquad t > 0,\ n \in \mathcal{N}_M, \tag{15}$$
$$u_i(0,t) = u_n(1,t), \qquad t > 0,\ n \in \mathcal{N}_M,\ i \in \mathcal{I}_n, \tag{16}$$

where the sequence $(\alpha_n)_{n \in \mathcal{N}_M}$ is still to be defined. For the boundary condition at the root $\mathcal{R}$, we shall consider one of the following conditions

$$u_1(0,t) = 0, \qquad t > 0 \qquad \text{(Dirichlet boundary condition)}; \tag{17}$$
$$u_{1,x}(0,t) = 0, \qquad t > 0 \qquad \text{(Neumann boundary condition)}; \tag{18}$$
$$c_1 u_{1,x}(0,t) = u_{1,t}(0,t), \qquad t > 0 \quad \text{(Transparent boundary condition)}. \tag{19}$$

Let

$$\mathcal{H} = \big\{(u_i, v_i)_{i\in\mathcal{I}} \in \prod_{i\in\mathcal{I}}[H^1(0,1)\times L^2(0,1)];\ u_i(0) = u_n(1)\ \forall n\in\mathcal{N}_M,\ \forall i\in\mathcal{I}_n\big\}$$

and $\mathcal{H}_0 = \big\{(u_i, v_i)_{i\in\mathcal{I}} \in \mathcal{H};\ u_1(0) = 0\big\}$. The spaces $\mathcal{H}$ and $\mathcal{H}_0$ are Hilbert spaces when endowed with the scalar product

$$\big((u_i, v_i)_{i\in\mathcal{I}}, (\tilde{u}_i, \tilde{v}_i)_{i\in\mathcal{I}}\big)_{\mathcal{H}} := \sum_{i\in\mathcal{I}}\int_0^1 \big(u_i(x)\tilde{u}_i(x) + u_{i,x}(x)\tilde{u}_{i,x}(x) + v_i(x)\tilde{v}_i(x)\big)dx.$$

Replacing $u_{i,t}$ by v_i and dropping the variable t, conditions (14) - (19) may be rewritten respectively as

$$\begin{aligned}
c_n u_{n,x}(1) &= -v_n(1), \qquad n\in\mathcal{N}_S\setminus\{0\}, && (20)\\
\sum_{i\in\mathcal{I}_n} c_i u_{i,x}(0) - c_n u_{n,x}(1) &= -\alpha_n v_n(1), \qquad n\in\mathcal{N}_M, && (21)\\
u_i(0) &= u_n(1), \qquad n\in\mathcal{N}_M,\ i\in\mathcal{I}_n, && (22)\\
u_1(0) &= 0, && (23)\\
u_{1,x}(0) &= 0, && (24)\\
c_1 u_{1,x}(0) &= v_1(0). && (25)
\end{aligned}$$

If $t\in\mathbb{R}^+ \to (u_i, v_i)_{i\in\mathcal{I}} \in \mathcal{D}(A_T)$ is continuous (for the definition of $\mathcal{D}(A_T)$, see below), using $v_i = u_{i,t}$, (22) and (23) we obtain

$$\begin{aligned}
v_i(0) &= v_n(1), \qquad n\in\mathcal{N}_M,\ i\in\mathcal{I}_n, && (26)\\
v_1(0) &= 0. && (27)
\end{aligned}$$

Introduce the operator A_D, A_N and A_T defined as

$$\begin{aligned}
A_D((u_i, v_i)_{i\in\mathcal{I}}) &= (v_i, c_i^2 u_{i,xx})_{i\in\mathcal{I}},\\
A_N((u_i, v_i)_{i\in\mathcal{I}}) &= (v_i, c_i^2 u_{i,xx})_{i\in\mathcal{I}},\\
A_T((u_i, v_i)_{i\in\mathcal{I}}) &= (v_i, c_i^2 u_{i,xx})_{i\in\mathcal{I}},
\end{aligned}$$

with respective domains

$$\begin{aligned}
\mathcal{D}(A_D) &= \{(u_i, v_i)_{i\in\mathcal{I}} \in \prod_{i\in\mathcal{I}}[H^2(0,1)\times H^1(0,1)];\ (20)-(22),\ (23)\\
&\qquad \text{and } (26)-(27) \text{ hold}\} \subset \mathcal{H}_0,\\
\mathcal{D}(A_N) &= \{(u_i, v_i)_{i\in\mathcal{I}} \in \prod_{i\in\mathcal{I}}[H^2(0,1)\times H^1(0,1)];\ (20)-(22),\ (24)\\
&\qquad \text{and } (26) \text{ hold}\} \subset \mathcal{H},\\
\mathcal{D}(A_T) &= \{(u_i, v_i)_{i\in\mathcal{I}} \in \prod_{i\in\mathcal{I}}[H^2(0,1)\times H^1(0,1)];\ (20)-(22),\ (25)\\
&\qquad \text{and } (26) \text{ hold}\} \subset \mathcal{H}.
\end{aligned}$$

The main result in this section is concerned with the well-posedness of system (12)-(16) and (17) (or (18), or (19)).

Theorem 2.1. *Let $\mathcal{T}$ be a tree and let $(\alpha_n)_{n\in\mathcal{N}_M}$ be a given family of real numbers. Then A_T generates a strongly continuous semigroup of operators on $\mathcal{H}$ if, and only if,*

$$\alpha_n \neq k_n \qquad \forall n\in\mathcal{N}_M, \tag{28}$$

where k_n is the degree of the vertex n defined in (11). The same conclusion holds for A_N on $\mathcal{H}$ (resp. for A_D on $\mathcal{H}_0$).

Remark 1. (1) As a consequence of Theorem 2.1, system (12)-(16) and (19) is *not well-posed* when $\alpha_n = k_n$ for some $n \in \mathcal{N}_M$. The initial data has to fulfill some compatibility conditions for the solution to exist. Consider the simplest case of a tree with two edges ($N = 2$) and $c_1 = c_2 = 1$. From (35) (see below), we have

$$v_1^0(1-t) - u_{1,x}^0(1-t) + v_2^0(t) + u_{2,x}^0(t) = 0 \quad \text{for a.e. } t \in (0,1). \tag{29}$$

This is a *necessary* condition on the initial data for the condition (15)-(16) to be satisfied. In other words, for initial data not satisfying the condition (29), there is *no solution* to (12)-(16) and (19).

(2) Furthermore, even for initial data satisfying (29), the corresponding solutions are *not unique* for positive times. Indeed, taking 0 as initial data, and picking any function $h \in C^1(\mathbb{R})$ with $h(t) = 0$ for $t \le 0$, then the trajectory candidate

$$\begin{aligned} u_1(x,t) &= h(t+x-1), \quad 0 \le x \le 1, t \ge 0, \\ u_2(x,t) &= h(t-x), \quad 0 \le x \le 1, t \ge 0 \end{aligned}$$

satisfies (12)-(16) and (19) with $\alpha_1 = k_1 = 2$. Thus, there is no uniqueness of trajectories.

(3) On the other hand, system (12)-(16) and (19) is *well-posed* when $\alpha_n \ne k_n$ for all $n \in \mathcal{N}_M$. Note that the system is not necessarily dissipative, i.e. the energy is not necessarily nonincreasing (even if the system is finite-time stable). Consider for instance a star-shaped tree with 3 edges ($N = 3$), and $c_1 = c_2 = c_3 = 1$ for simplicity. Pick $\alpha_1 := k_1 - 2 = 1$. (According to Theorem 4.1, the system is finite-time stable to constant functions.) Then a direct computation gives for any solution $(u_i, v_i)_{1 \le i \le 3}$ of (12)-(16) and (19)

$$\frac{d}{dt}\frac{1}{2}\sum_{i=1}^{3}\int_0^1 (u_{i,t}^2 + u_{i,x}^2)dx = u_{1,t}^2(1,t) - (u_{1,t}^2(0,t) + u_{2,t}^2(1,t) + u_{3,t}^2(1,t)).$$

This expression is *positive* for $t > 0$ small enough if the initial data $(u_i^0, v_i^0)_{1\le i\le 3} \in \mathcal{D}(A_T)$ is chosen so that

$$|v_1^0(1)|^2 > |v_1^0(0)|^2 + |v_2^0(1)|^2 + |v_3^0(1)|^2.$$

In that case, the energy may increase before reaching the value 0 in finite time.

Proof. We sketch the proof only for A_T. We need a preliminary result about the Riemann invariants around an internal node. Consider any internal node connecting edges whose indices range over $\{1, ..., k\}$ (to simplify the notations). Consider any solution of (12) satisfying

$$u_1(1,t) = u_2(0,t) = \cdots = u_k(0,t) \tag{30}$$

$$c_2 u_{2,x}(0,t) + \cdots + c_k u_{k,x}(0,t) - c_1 u_{1,x}(1,t) = -\alpha u_{1,t}(1,t) \tag{31}$$

Introduce the Riemann invariants

$$d_i(x,t) := u_{i,t}(x,t) - c_i u_{i,x}(x,t), \tag{32}$$

$$s_i(x,t) := u_{i,t}(x,t) + c_i u_{i,x}(x,t) \tag{33}$$

for all $i \in \mathcal{I}$. Then the following result holds.

Lemma 2.2. *1. If $\alpha \neq k$, then $s_1(1,t), d_2(0,t), ..., d_k(0,t)$ can be expressed in a unique way as functions of $d_1(1,t), s_2(0,t), ..., s_k(0,t)$. In particular, if $\alpha = k-2$, we obtain*

$$s_1(1,t) = \sum_{i=2}^{k} s_i(0,t). \tag{34}$$

2. If $\alpha = k$, then the existence of a solution to (12) and (30)-(31) implies

$$d_1(1,t) + \sum_{i=2}^{k} s_i(0,t) = 0. \tag{35}$$

As a consequence, the initial data of a smooth solution should satisfy the compatibility condition

$$v_1^0(1-c_1t) - c_1u_{1,x}^0(1-c_1t) + \sum_{i=2}^{k}[v_i^0(c_it) + c_iu_{i,x}^0(c_it)] = 0 \qquad \text{for } 0 \leq t \leq \min_{1\leq i\leq k} c_i^{-1}. \tag{36}$$

Proof of Lemma 2.2. Using Riemann invariants, we see that (12) and (30)-(31) are transformed into

$$d_{i,t} + c_id_{i,x} = 0, \qquad i = 1, ..., k, \tag{37}$$
$$s_{i,t} - c_is_{i,x} = 0, \qquad i = 1, ..., k, \tag{38}$$
$$s_1(1,t) + d_1(1,t) = s_2(0,t) + d_2(0,t) = \cdots = s_k(0,t) + d_k(0,t), \tag{39}$$
$$\sum_{i=2}^{k}[s_i(0,t) - d_i(0,t)] - (s_1(1,t) - d_1(1,t)) = -\alpha(s_1(1,t) + d_1(1,t)). \tag{40}$$

To simplify the notations, we write s_1 for $s_1(1,t)$, s_2 for $s_2(0,t)$, etc. Then (39)-(40) can be written

$$s_1 + d_1 = d_i + s_i, \qquad i = 2, ..., k, \tag{41}$$
$$(1-\alpha)s_1 + d_2 + \cdots + d_k = (1+\alpha)d_1 + s_2 + \cdots + s_k \tag{42}$$

We readily infer from (41) that

$$s_1 - d_2 = -d_1 + s_2, \tag{43}$$
$$d_2 - d_3 = -s_2 + s_3, \tag{44}$$
$$\vdots$$
$$d_{k-1} - d_k = -s_{k-1} + s_k. \tag{45}$$

Adding the $k-1$ equations in (41) results in

$$(k-1)s_1 - \sum_{i=2}^{k} d_i = (1-k)d_1 + \sum_{i=2}^{k} s_i$$

Subtracting this last equation from (42), we obtain

$$2\sum_{i=2}^{k} d_i = (k+\alpha)d_1 + (k+\alpha-2)s_1 = 2d_1 + (k+\alpha-2)(d_1+s_1).$$

Combined to the relation $d_1 + s_1 = d_k + s_k$, this yields

$$\sum_{i=2}^{k} d_i = d_1 + (\frac{k+\alpha}{2} - 1)(d_k + s_k).$$

Using this relation in (42) together with the relation $s_1 = d_k + s_k - d_1$, we obtain

$$(k-\alpha)d_k = 2d_1 + 2\sum_{i=2}^{k-1} s_i + (\alpha - k + 2)s_k. \tag{46}$$

Thus, if $\alpha \neq k$, we infer from (43)-(46) that $s_1(1,t), d_2(0,t), ..., d_k(0,t)$ can be expressed in a unique way as functions of $d_1(1,t), s_2(0,t), ..., s_k(0,t)$. In particular, if $\alpha = k-2$, then (46) becomes

$$d_k = d_1 + \sum_{i=2}^{k-1} s_i. \tag{47}$$

Adding (43),(44),...,(45) and (47) yields (34). Finally, if $\alpha = k$, then (46) gives (35). Computing d_i and s_i using (52)-(53) (see below), and replacing s_i^0 and d_i^0 by their expressions in terms of u_i^0 and v_i^0, we obtain (36). □

Let us proceed to the proof of Theorem 2.1. If (28) is not satisfied, picking some initial data $(u_i^0, v_i^0)_{i\in\mathcal{I}} \in \mathcal{D}(A_T)$ that does not satisfy the condition

$$v_n^0(1-c_nt) - c_n u_{n,x}^0(1-c_nt) + \sum_{i\in\mathcal{I}_n} [v_i^0(c_it) + c_i u_{i,x}^0(c_it)] = 0,$$
$$0 \le t \le \min_{i\in\{n\}\cup\mathcal{I}_n} c_i^{-1} \tag{48}$$

around a node $n \in \mathcal{N}_M$ for which $\alpha_n = k_n$ (the existence of such an initial data is obvious, since the conditions (20)-(27) involve only the values of u_i and v_i or their derivatives at the nodes), we infer from Lemma 2.2 that system (12)-(16) and (19) does not admit any solution $(u_i, v_i)_{i\in\mathcal{I}} \in C(\mathbb{R}^+; \mathcal{D}(A_T))$. This shows that A_T is not the generator of a continuous semigroup on $\mathcal{H}$.

Conversely, assume that (28) is satisfied. We aim to construct by a fixed-point procedure a solution to (12)-(16) and (19). Pick any $U^0 = (u_i^0, v_i^0)_{i\in\mathcal{I}} \in \mathcal{H}$ and any $T > \sup_{1\le i\le N} c_i^{-1}$. Set

$$d_i^0 := v_i^0 - c_i u_{i,x}^0, \qquad s_i^0 := v_i^0 + c_i u_{i,x}^0, \qquad i = 1, ..., N.$$

Let $K := \sum_{n\in\mathcal{N}_M} k_n$. Pick any number $\rho \in (0,1)$, and introduce the Hilbert space $\mathcal{E} := L^2_{\rho^t dt}(0,T)^K$ endowed with the norm

$$||(x_1, x_2, ..., x_K)||_{\mathcal{E}} := \left(\sum_{i=1}^{K} \int_0^T |x_i(t)|^2 \rho^t dt\right)^{\frac{1}{2}}.$$

For $n \in \mathcal{N}_M$, let $m_n := \inf_{j\in\mathcal{I}_n} j$, and let $p_1 := 1$ and

$$p_n := 1 + \sum_{m\in\mathcal{N}_M,\, m<n} k_m, \qquad \text{for } n \in \mathcal{N}_M.$$

For instance, with the tree drawn in Figure 1, we have

$$\begin{aligned} \mathcal{N}_M &= \{1,2,3,5,6,7\}, \\ (k_1,k_2,k_3,k_5,k_6,k_7) &= (3,3,2,4,3,3), \end{aligned}$$

$$\begin{array}{rcl} (m_1, m_2, m_3, m_5, m_6, m_7) & = & (2, 4, 6, 7, 10, 12), \\ \text{and} \quad (p_1, p_2, p_3, p_5, p_6, p_7) & = & (1, 4, 7, 9, 13, 16). \end{array}$$

Let

$$X(t) := (x_1(t), ..., x_K(t)) = (..., d_n(1,t), s_{m_n}(0,t), ..., s_{m_n+k_n-2}(0,t), ...)$$

where n ranges over $\mathcal{N}_M$. Note that $x_1(t) = d_1(1,t)$ and $x_K(t) = s_N(0,t)$. For the tree drawn in Figure 1, we have

$$\begin{array}{rcl} X(t) & = & (x_1(t), ..., x_{18}(t)) \\ & = & (d_1(1,t), s_2(0,t), s_3(0,t), d_2(1,t), s_4(0,t), s_5(0,t), d_3(1,t), s_6(0,t), \\ & & d_5(1,t), s_7(0,t), s_8(0,t), s_9(0,t), d_6(1,t), s_{10}(0,t), s_{11}(0,t), d_7(1,t), \\ & & s_{12}(0,t), s_{13}(0,t)). \end{array}$$

Finally, let $\mathcal{S}$ denote the set of the indices of the components of $X(t)$ corresponding to edges whose final points are simple nodes different from the root; that is

$$\mathcal{S} := \{n \in \{2, ..., K\},\ \exists q_n \in \mathcal{N}_S,\ x_n(t) = s_{q_n}(0,t)\}.$$

For the tree in Figure 1, we have $\mathcal{S} = \{5, 11, 12, 14, 15, 17, 18\}$ with

$$(q_5, q_{11}, q_{12}, q_{14}, q_{15}, q_{17}, q_{18}) = (4, 8, 9, 10, 11, 12, 13).$$

Let

$$\mathcal{E}_0 := \{(x_1, ..., x_K) \in \mathcal{E};\ x_1(t) = 0\ \forall t \geq c_1^{-1} \text{ and } \ x_n(t) = 0\ \forall n \in \mathcal{S},\ \forall t \geq c_{q_n}^{-1}\}.$$

We define a map $P : X = (x_1, ..., x_K) \in \mathcal{E}_0 \to \tilde{X} = (\tilde{x}_1, ..., \tilde{x}_K) \in \mathcal{E}_0$ as follows. Pick any $X = (x_1, ..., x_K) \in \mathcal{E}_0$ and any $n \in \mathcal{N}_M$. By Lemma 2.2, there exists a matrix $A_n \in \mathbb{R}^{k_n \times k_n}$ such that the Riemann invariants associated with the solution of (12)-(16) and (19) satisfy

$$\begin{pmatrix} s_n(1,t) \\ d_{m_n}(0,t) \\ \vdots \\ d_{m_n+k_n-2}(0,t) \end{pmatrix} = A_n \begin{pmatrix} d_n(1,t) \\ s_{m_n}(0,t) \\ \vdots \\ s_{m_n+k_n-2}(0,t) \end{pmatrix}.$$

Then, we set for all $t \geq 0$

$$\begin{pmatrix} s_n(1,t) \\ d_{m_n}(0,t) \\ \vdots \\ d_{m_n+k_n-2}(0,t) \end{pmatrix} := A_n \begin{pmatrix} x_{p_n}(t) \\ x_{p_n+1}(t) \\ \vdots \\ x_{p_n+k_n-1}(t) \end{pmatrix}. \tag{49}$$

(Recall that $x_{p_n}(t)$ stands for $d_n(1,t)$.) Next we set

$$d_1(0,t) = 0, \quad t \geq 0, \tag{50}$$

$$s_m(1,t) = 0, \quad m \in \mathcal{N}_S \setminus \{0\}, \quad t \geq 0. \tag{51}$$

Note that $d_i(0,t)$ and $s_i(1,t)$ have been defined for all $i \in \{1, ..., N\}$ and all $t \geq 0$. Next, solving (37)-(38), we set for all $i \in \{1, ..., N\}$ and all $(x,t) \in [0,1] \times \mathbb{R}^+$

$$d_i(x,t) := \begin{cases} d_i^0(x - c_i t) & \text{if } 0 \leq x - c_i t \leq 1, \\ d_i(0, t - c_i^{-1} x) & \text{if } t > x/c_i, \end{cases} \tag{52}$$

and

$$s_i(x,t) := \begin{cases} s_i^0(x + c_i t) & \text{if } 0 \leq x + c_i t \leq 1, \\ s_i(1, t - c_i^{-1}(1-x)) & \text{if } t > (1-x)/c_i. \end{cases} \tag{53}$$

Finally, we set for $n \in \mathcal{N}_M$ and $t \geq 0$

$$\begin{pmatrix} \tilde{x}_{p_n}(t) \\ \tilde{x}_{p_n+1}(t) \\ \vdots \\ \tilde{x}_{p_n+k_n-1}(t) \end{pmatrix} := \begin{pmatrix} d_n(1,t) \\ s_{m_n}(0,t) \\ \vdots \\ s_{m_n+k_n-2}(0,t) \end{pmatrix}. \tag{54}$$

and $P(X) := \tilde{X} = (\tilde{x}_1, ..., \tilde{x}_K)$.

From (50)-(53), we have that

$$\begin{aligned} d_1(1,t) &= 0, \quad \text{for } t \geq c_1^{-1}, & (55)\\ s_m(0,t) &= 0, \quad \text{for } m \in \mathcal{N}_S \setminus \{0\},\ t \geq c_m^{-1}. & (56) \end{aligned}$$

It follows that P is a map from $\mathcal{E}_0$ into itself. Let us check that it is a contraction for ρ small enough. Let $X^1 = (x_1^1, ..., x_K^1)$ and $X^2 = (x_1^2, ..., x_K^2)$ be given in $\mathcal{E}_0$. In what follows, c denotes a constant that may vary from line to line. Then, using (49)-(54), we have that

$$\begin{aligned} ||P(X_1) - P(X_2)||_{\mathcal{E}}^2 &\leq c \sum_{j=1}^{K} \sum_{i=1}^{N} \int_{c_i^{-1}}^{T} |x_j^1(t - c_i^{-1}) - x_j^2(t - c_i^{-1})|^2 \rho^t dt & (57)\\ &\leq c(\max_{i \in \mathcal{I}} \rho^{c_i^{-1}}) ||X^1 - X^2||_{\mathcal{E}}^2. & (58) \end{aligned}$$

This proves that P is a contraction in $\mathcal{E}_0$ for $\rho > 0$ small enough. It follows from the contraction principle that P has a (unique) fixed-point in $\mathcal{E}_0$. It is then easy to check that the Riemann invariants d_i, s_i, $1 \leq i \leq N$, defined along (52)-(53), solve (37)-(38) in the distributional sense and satisfy (39)-(40) almost everywhere. Using again (52)-(53), one has that for any $i \in \mathcal{I}$

$$s_i(x, 0) = s_i^0(x), \quad d_i(x, 0) = d_i^0(x), \quad \text{for a.e. } x \in [0, 1].$$

We can therefore define for all $i \in \mathcal{I}$ and all $T > \sup_{1 \leq i \leq N} c_i^{-1}$ a function $u_i \in H^1((0,1) \times (0,T))$ by

$$u_{i,t} = \frac{1}{2}(s_i + d_i) =: v_i, \quad u_{i,x} = \frac{1}{2c_i}(s_i - d_i),$$

the constant of integration being chosen so that

$$u_i(x,t) = u_i^0(x) + \int_0^t v_i(x,s)ds \quad \text{for a.e. } (x,t) \in (0,1) \times (0,T).$$

Then $(u_i, v_i) \in C(\mathbb{R}^+, H^1(0,1) \times L^2(0,1))$, and (22) follows from (39). We infer that $(u_i, v_i)_{i \in \mathcal{I}}$ is a (weak) solution of (12)-(16) and (19) which is continuous in time with values in $\mathcal{H}$. Set $S(t)U^0 := (u_i(t), v_i(t))_{i \in \mathcal{I}}$. Then it can be seen that $\big(S(t)\big)_{t \geq 0}$ is a strongly continuous semigroup in $\mathcal{H}$ whose generator is A_T. The proof of Theorem 2.1 is complete. □

3. **The example of a tree with two internal nodes.** In this section, we consider a particular case which can be treated in a direct way, and which shows the importance of condition (7) in Theorem 1.1.

To prove on a given example that the finite-time stability (to 0 or to constant functions) does not occur, a way consists in finding an eigenvalue of the underlying operator. Indeed, if we can find an eigenvalue, then the corresponding exponential solution does not steer 0 in finite time. If, in addition, the eigenvalue is different from 0, then the corresponding exponential solution does not take a constant value

$$(u_i(0), v_i(0)) = (u_1(1), v_1(1)) \quad \text{for } 2 \le i \le k_1,$$
$$(u_i(0), v_i(0)) = (u_2(1), v_2(1)) \quad \text{for } k_1 + 1 \le i \le N\}.$$

where $' = d/dx$, $'' = d^2/dx^2$, etc.

Setting $U = (u_i, v_i)_{i\in\mathcal{I}}$, we see that (12)-(16) and (19) may be written as

$$U_t = A_\mathcal{T} U, \tag{61}$$
$$U(0) = U_0 = (u_i^0, u_i^1)_{i\in\mathcal{I}}. \tag{62}$$

If $A_\mathcal{T} U_0 = \lambda U_0$ with $U_0 \neq 0$ and $\lambda \neq 0$, then the solution U of (61)-(62) reads $U(t) = e^{\lambda t} U_0$ (exponential solution), and then the finite-time stabilization to constant functions cannot hold.

Proposition 1. *Let $\mathcal{T}$ denote a tree with N edges and two internal nodes ($\mathcal{N}_M = \{1, 2\}$), and assume that*

$$\alpha_1 \neq k_1 \text{ and } \alpha_2 \neq k_2. \tag{63}$$

Then the operator $A_\mathcal{T}$ has at least one eigenvalue if, and only if,

$$\alpha_1 \neq k_1 - 2 \text{ and } \alpha_2 \neq k_2 - 2. \tag{64}$$

Furthermore, if (64) holds, then the discrete spectrum of $A_\mathcal{T}$ is $\sigma_d(A_\mathcal{T}) = \{\lambda_k;\ k \in \mathbb{Z}\}$ where

$$\lambda_k = \frac{c_2}{2} \log_{-\frac{\pi}{2}} \Big(\frac{(2 + \alpha_1 - k_1)(2 + \alpha_2 - k_2)}{(\alpha_1 - k_1)(\alpha_2 - k_2)} \Big) + i c_2 k \pi. \tag{65}$$

In particular, the finite-time stability to constant functions does not hold for (12)-(16) and (19). Finally, if (60) is satisfied, then the finite-time stability to constant functions holds.

In (65), $\log_{-\frac{\pi}{2}}$ denotes the usual determination of the logarithm in $\mathbb{C} \setminus i\mathbb{R}^-$. Thus

$$\log_{-\frac{\pi}{2}}(z) = \begin{cases} \log|z| & \text{if } z \in (0, +\infty), \\ \log|z| + i\pi & \text{if } z \in (-\infty, 0). \end{cases}$$

Proof. First, $A_\mathcal{T}$ generates a strongly continuous semigroup of operators in $\mathcal{H}$ by (63) and Theorem 2.1. Let $\lambda \in \mathbb{C}$ and $U = (u_i, v_i)_{i\in\mathcal{I}} \in D(A_\mathcal{T})$. Then the equation $A_\mathcal{T} U = \lambda U$ is equivalent to the following system

$$(v_i, c_i^2 u_i'') = \lambda(u_i, v_i) \tag{66}$$
$$c_1 u_1'(0) = v_1(0) \tag{67}$$
$$c_i u_i'(1) = -v_i(1), \qquad 3 \le i \le N \tag{68}$$
$$\sum_{2\le i\le k_1} c_i u_i'(0) - c_1 u_1'(1) = -\alpha_1 v_1(1) \tag{69}$$
$$\sum_{k_1+1\le i\le N} c_i u_i'(0) - c_2 u_2'(1) = -\alpha_2 v_2(1) \tag{70}$$
$$u_i(0) = u_1(1), \qquad 2 \le i \le k_1, \tag{71}$$
$$u_i(0) = u_2(1), \qquad k_1 + 1 \le i \le N. \tag{72}$$

Note that the conditions $v_i(0) = v_1(1)$ for $2 \le i \le k_1$ and $v_i(0) = v_2(1)$ for $k_1 + 1 \le i \le N$ are satisfied whenever (66) and (71)-(72) hold. (66) is easily solved as

$$u_i(x) = a_i e^{\lambda x/c_i} + b_i e^{-\lambda x/c_i},\ v_i = \lambda u_i,\ i \in \mathcal{I}, \tag{73}$$

where $a_i, b_i \in \mathbb{C}$ are constants to be determined. Substituting the above expression of $u_i(x)$ in (67)-(72) yields the system

$$\lambda b_1 = 0, \tag{74}$$
$$\lambda a_i = 0, \qquad 3 \le i \le N, \tag{75}$$
$$\lambda \sum_{2\le i\le k_1} (a_i - b_i) - \lambda(a_1 e^{\lambda/c_1} - b_1 e^{-\lambda/c_1}) = -\alpha_1 \lambda (a_1 e^{\lambda/c_1} + b_1 e^{-\lambda/c_1}), \tag{76}$$
$$\lambda \sum_{k_1+1\le i\le N} (a_i - b_i) - \lambda(a_2 e^{\lambda/c_2} - b_2 e^{-\lambda/c_2}) = -\alpha_2 \lambda (a_2 e^{\lambda/c_2} + b_2 e^{-\lambda/c_2}), \tag{77}$$
$$a_i + b_i = a_1 e^{\lambda/c_1} + b_1 e^{-\lambda/c_1}, \qquad 2 \le i \le k_1, \tag{78}$$
$$a_i + b_i = a_2 e^{\lambda/c_2} + b_2 e^{-\lambda/c_2}, \qquad k_1 + 1 \le i \le N. \tag{79}$$

If $\lambda = 0$, we infer from (78)-(79) and (73) that $u_i(x) = a_1 + b_1$ for all $i \in \mathcal{I}$, i.e. $U = const$, which is excluded. Assume from now on that $\lambda \ne 0$. Then (74)-(79) is equivalent to the system

$$b_1 = 0, \tag{80}$$
$$a_i = 0, \qquad 3 \le i \le N, \tag{81}$$
$$b_2 = a_1 e^{\lambda/c_1} - a_2, \tag{82}$$
$$b_i = a_1 e^{\lambda/c_1}, \qquad 3 \le i \le k_1, \tag{83}$$
$$b_i = a_2 e^{\lambda/c_2} + b_2 e^{-\lambda/c_2}, \qquad k_1 + 1 \le i \le N, \tag{84}$$
$$2a_2 + (\alpha_1 - k_1) e^{\lambda/c_1} a_1 = 0, \tag{85}$$
$$\begin{aligned}&[(-N + k_1 - 1 + \alpha_2) e^{\lambda/c_2} + (N - k_1 - 1 - \alpha_2) e^{-\lambda/c_2}] a_2 \\ &\quad + (-N + k_1 + 1 + \alpha_2) e^{-\lambda/c_2} e^{\lambda/c_1} a_1 = 0.\end{aligned} \tag{86}$$

The existence of a nontrivial solution ($(a_1, a_2) \ne (0,0)$) holds if, and only if, the determinant of the system (85)-(86) in $e^{\lambda/c_1} a_1$ and a_2 vanishes, i.e.

$$(2 + \alpha_1 - k_1)(-N + k_1 + 1 + \alpha_2) e^{-\lambda/c_2} - (\alpha_1 - k_1)(-N + k_1 - 1 + \alpha_2) e^{\lambda/c_2} = 0.$$

Since $-N + k_1 = 1 - k_2$, this can be expressed as

$$(2 + \alpha_1 - k_1)(2 + \alpha_2 - k_2) e^{-\lambda/c_2} - (\alpha_1 - k_1)(\alpha_2 - k_2) e^{\lambda/c_2} = 0.$$

Using (63), the last equation is equivalent to

$$e^{\frac{2\lambda}{c_2}} = \frac{(2 + \alpha_1 - k_1)(2 + \alpha_2 - k_2)}{(\alpha_1 - k_1)(\alpha_2 - k_2)}. \tag{87}$$

(87) has a solution $\lambda \in \mathbb{C}$ if and only if $(2 + \alpha_1 - k_1)(2 + \alpha_2 - k_2) \ne 0$, and in that case the solutions of (87) read

$$\lambda_k = \frac{c_2}{2} \log_{-\frac{\pi}{2}} \Big(\frac{(2 + \alpha_1 - k_1)(2 + \alpha_2 - k_2)}{(\alpha_1 - k_1)(\alpha_2 - k_2)} \Big) + i c_2 k \pi, \quad k \in \mathbb{Z}.$$

Assume finally that (60) holds, e.g. $\alpha_1 = k_1 - 2$ and $\alpha_2 \in \mathbb{R} \setminus \{k_2\}$. Since transparent boundary conditions are applied at all the external nodes, we have

$$\begin{aligned} s_i(1,t) &= 0, \qquad i = 3, ..., N, \ t \ge 0, \\ d_1(0,t) &= 0, \qquad t \ge 0. \end{aligned}$$

This implies

$$\begin{aligned} s_i(x,t) &= 0, \qquad i=3,...,N,\ x\in[0,1],\ t\geq c_i^{-1}, &(88)\\ d_1(x,t) &= 0, \qquad x\in[0,1],\ t\geq c_1^{-1}. &(89)\end{aligned}$$

It follows from (34) and (88) that

$$s_2(0,t)=s_1(1,t), \qquad t\geq \max_{3\leq i\leq k_1} c_i^{-1}.$$

Combined with the continuity condition $u_1(1,t)=u_2(0,t)$, this yields

$$d_2(0,t)=d_1(1,t)=0, \qquad t\geq \max_{i\in\{1\}\cup[3,k_1]} c_i^{-1}.$$

We infer then from Lemma 2.2 (see (43)-(45) and (46)) that $s_2(1,t)=d_{k_1+1}(0,t)=\cdots=d_N(0,t)=0$ for t large enough. This in turn implies $s_1(1,t)=d_3(0,t)=\cdots=d_{k_1}(0,t)=0$ for t large enough. We conclude that for some constants C and T, $u_i(x,t)=C$ for all $i\in\mathcal{I}$, all $x\in[0,1]$ and all $t\geq T$. □

4. **Finite-time extinction.** Pick any tree of depth $d\geq 1$, and define the sequence $(t_i)_{i\in\mathcal{I}}$ as follows

$$\begin{aligned} t_i &= c_i^{-1} \qquad \text{if } i\in\mathcal{N}_S\setminus\{0\}, &(90)\\ t_i &= c_i^{-1}+\max_{j\in\mathcal{I}_i} t_j \qquad \text{if } i\in\mathcal{N}_M. &(91)\end{aligned}$$

Set $T(\mathcal{R})=t_1$. Then it is easily seen that $T(\mathcal{R})$ is the maximum of the quantities

$$c_{i_1}^{-1}+c_{i_2}^{-1}+\cdots+c_{i_p}^{-1},$$

where $p\geq 1$, $i_1=1$, $i_{q+1}\in I_{i_q}$ for $1\leq q\leq p-1$, and the final point of the edge of index i_q is an external node (different from $\mathcal{R}$). Define $T(\mathcal{T})$ as the largest of the $T(\mathcal{R})$'s when the root $\mathcal{R}$ ranges over $\mathcal{N}_S$; that is, we take as root of the tree any external node, change the numbering of the edges and nodes, and define the corresponding sequences $(\mathcal{I}_i)_{i\in\mathcal{I}}$ and $(t_i)_{i\in\mathcal{I}}$. Obviously, $T(\mathcal{R})\leq T(\mathcal{T})\leq 2T(\mathcal{R})$.

Example. Consider again the tree drawn in Figure 1, and assume for simplicity that $c_i=1$ for all $i\in[1,13]$. Then $T(\mathcal{R})=5$ and $T(\mathcal{T})=7$. Indeed, if we take the node of index $n=12$ as (new) root, we obtain $T(\mathcal{R}_{n=12})=7$. Similarly, we see that $T(\mathcal{R}_{n=13})=7$, $T(\mathcal{R}_{n=8})=T(\mathcal{R}_{n=9})=6$, $T(\mathcal{R}_{n=4})=5$, and $T(\mathcal{R}_{n=10})=T(\mathcal{R}_{n=11})=7$.

Theorem 4.1. *Let $\mathcal{T}$ be a tree of root $\mathcal{R}$, and let $T(\mathcal{R})$ and $T(\mathcal{T})$ be as above. Assume that the sequence $(\alpha_n)_{n\in\mathcal{N}_M}$ satisfies the condition*

$$\alpha_n=k_n-2 \qquad n\in\mathcal{N}_M. \tag{92}$$

Pick any initial data $U_0=\{(u_i^0,u_i^1\}_{i\in\mathcal{I}}\in\mathcal{H}$.

(i) If $U_0\in\mathcal{H}_0$, then the solution $(u_i)_{i\in\mathcal{I}}$ of (12)-(16) and (17) satisfies

$$u_i(.,t)\equiv 0, \qquad \forall t\geq 2T(\mathcal{R}),\ \forall i\in\mathcal{I}; \tag{93}$$

(ii) The solution $(u_i)_{i\in\mathcal{I}}$ of (12)-(16) and (18) satisfies for some number $C\in\mathbb{R}$

$$u_i(.,t)\equiv C, \qquad \forall t\geq 2T(\mathcal{R}),\ \forall i\in\mathcal{I}. \tag{94}$$

(iii) The solution $(u_i)_{i\in\mathcal{I}}$ of (12)-(16) and (19) satisfies for some number $C\in\mathbb{R}$

$$u_i(.,t)\equiv C, \qquad \forall t\geq T(\mathcal{T}),\ \forall i\in\mathcal{I}. \tag{95}$$

Remark 2. It is likely that the extinction time T_e (i.e. the least time after which solutions remain constant) is given by $2T(\mathcal{R})$ in the cases (i) and (ii), and $T(\mathcal{T})$ in case (iii), so that the above results are sharp. Actually, for one string, it is well known that $T_e = 2/c_1$ for the solutions of (12)-(16) and (17) (or for the solutions of (12)-(16) and (18)), while $T_e = 1/c_1$ for the solutions of (12)-(16) and (19) (see e.g. [12, 14, 17] and the references therein).

Proof. We use again the Riemann invariants d_i, s_i defined in (32)-(33) that satisfy the transport equations (37)-(38). We need the following

Lemma 4.2. *Let $\mathcal{T}$ be a tree, and let the sequence $(t_i)_{i\in\mathcal{I}}$ be as in (90)-(91). Assume that the sequence $(\alpha_n)_{n\in\mathcal{N}_M}$ satisfies (92). Then for any $U_0 \in \mathcal{H}$ and any solution $(u_i)_{i\in\mathcal{I}}$ of (12)-(16), with corresponding Riemann invariants d_i, s_i, we have for all $i \in \mathcal{I}$*

$$s_i(x,t) = 0 \quad \forall x \in [0,1],\ \forall t \geq t_i. \tag{96}$$

Proof of Lemma 4.2. We argue by induction on the depth d of the tree. If $d = 1$, then there is only one edge ($\mathcal{I} = \{1\}$) and s_1 solves

$$\begin{aligned} s_{1,t} - c_1 s_{1,x} &= 0, \quad t>0,\ 0<x<1, && (97)\\ s_1(1,t) &= 0, \quad t>0, && (98)\\ s_1(.,0) &= s_1^0 := v_1^0 + c_1 u_{1,x}^0. && (99) \end{aligned}$$

Then it is easily seen that

$$s_1(x,t) = \begin{cases} s_1^0(x+c_1t) & \text{if } x+c_1t \leq 1,\\ 0 & \text{if } x + c_1 t \geq 1. \end{cases} \tag{100}$$

Thus

$$s_1(x,t) = 0 \qquad \forall x \in [0,1],\ \forall t \geq c_1^{-1}$$

and (96) is established for $d = 1$.

Assume now Lemma 4.2 established for any tree of depth at most $d-1$, where $d \geq 2$. Pick a tree $\mathcal{T}$ of depth d, and a sequence $(\alpha_n)_{n\in\mathcal{N}_M}$ satisfying $(\mathcal{P})$. Denote by $\mathcal{R}'$ the node of index $n = 1$, and by $\mathcal{T}_i$, for $i = 2, ..., k_1$, the subtree of $\mathcal{T}$ of root $\mathcal{R}'$ and of first edge the edge of $\mathcal{T}$ of index i. Since $\mathcal{T}_i$ is of depth at most $d-1$, we infer from the induction hypothesis that for $i > 1$

$$s_i(x,t) = 0 \qquad \forall x \in [0,1],\ \forall t \geq t_i. \tag{101}$$

It remains to prove (96) for $i = 1$. Since the condition (92) is satisfied for $n = 1$, we infer from (34) that

$$s_1(1,t) = \sum_{i=2}^{k_1} s_i(0,t), \qquad \forall t \geq 0.$$

It follows then from (101) that

$$s_1(1,t) = 0 \qquad \forall t \geq \max_{i\in\mathcal{I}_1} t_i.$$

Finally, using (97), we infer that

$$s_1(x,t) = 0 \qquad \forall x \in [0,1],\ \forall t \geq c_1^{-1} + \max_{i\in\mathcal{I}_1} t_i = t_1.$$

The proof of Lemma 4.2 is complete. □

Let us go back to the proof of Theorem 4.1.

(i) Assume first that $U_0 \in \mathcal{H}_0$, and let $(u_i)_{i\in\mathcal{I}}$ denote the solution of (12)-(16) and (17). From Lemma 4.2, we have that for all $i \in \mathcal{I}$

$$s_i(x,t) = 0 \qquad \forall x \in [0,1],\ \forall t \geq T(\mathcal{R}). \tag{102}$$

From (17), we infer that $d_1(0,t) + s_1(0,t) = 0$ for all $t \geq 0$, and hence

$$d_1(0,t) = 0, \qquad \forall t \geq T(\mathcal{R}).$$

Using (37), we infer that

$$d_1(x,t) = 0, \qquad \forall x \in [0,1],\ \forall t \geq c_1^{-1} + T(\mathcal{R}).$$

Combined with (43)-(45) (with $k = k_1$) and (102), this yields

$$d_2(0,t) = \cdots = d_{k_1}(0,t) = 0, \qquad \forall t \geq c_1^{-1} + \max_{i\in\mathcal{I}_1} c_i^{-1} + T(\mathcal{R}).$$

Using the second definition of $T(\mathcal{R})$ and proceeding inductively, we arrive to

$$d_i(x,t) = 0 \qquad \forall i \in \mathcal{I},\ \forall x \in [0,1],\ \forall t \geq 2T(\mathcal{R}). \tag{103}$$

Gathering together (102) and (103), we infer the existence of some constant $C \in \mathbb{R}$ such that

$$u_i(x,t) = C, \qquad \forall i \in \mathcal{I},\ \forall x \in [0,1],\ \forall t \geq 2T(\mathcal{R}).$$

Using (17), we see that $C = 0$. This proves that solutions of (12)-(16) and (17) are null for $t \geq T(\mathcal{R})$. Combined with the strong continuity of the semigroup $(e^{tA_D})_{t\geq 0}$ in $\mathcal{H}_0$, this yields the finite-time stability.

(ii) Assume now that $u_0 \in \mathcal{H}$ and let $(u_i)_{i\in\mathcal{I}}$ denote the solution of (12)-(16) and (18). From (17), we infer that $d_1(0,t) - s_1(0,t) = 0$ for all $t \geq 0$. The same proof as in (i) then yields

$$s_i(x,t) = d_i(x,t) = 0, \qquad \forall i \in \mathcal{I},\ \forall x \in [0,1],\ \forall t \geq 2T(\mathcal{R}).$$

Thus there exists a constant $C \in \mathbb{R}$ such that

$$u_i(x,t) = C, \qquad \forall i \in \mathcal{I},\ \forall x \in [0,1],\ \forall t \geq 2T(\mathcal{R}).$$

(iii) Pick a solution $(u_i)_{i\in\mathcal{I}}$ of (12)-(16) and (19). Then it follows from Lemma 4.2 that for all $i \in \mathcal{I}$

$$s_i(x,t) = 0 \qquad \forall x \in [0,1],\ \forall t \geq T(\mathcal{R}). \tag{104}$$

For any given $i \in \mathcal{I}$, we pick a sequence $i_1 < i_2 < \cdots < i_p$ such that $i_1 = 1$, $i = i_q$ for some $q \in [1,p]$, and the final point of the edge of index i_p is an external point, that we call $\tilde{\mathcal{R}}$. If we exchange $\mathcal{R}$ and $\tilde{\mathcal{R}}$, we notice that d_i is linked to the $\tilde{s}_j$'s (associated with the new root $\tilde{\mathcal{R}}$) by:

$$d_i(x,t) = \tilde{s}_{i_p - i + 1}(1-x,t).$$

We infer that

$$d_i(x,t) = 0 \qquad \forall x \in [0,1],\ \forall t \geq T(\mathcal{T}). \tag{105}$$

Therefore, there exists a constant $C \in \mathbb{R}$ such that

$$u_i(x,t) = C, \qquad \forall i \in \mathcal{I},\ \forall x \in [0,1],\ \forall t \geq T(\mathcal{T}).$$

The proof of Theorem 4.1 is complete. $\square$

5. **Sharpness of the condition** (92)**.** The condition (92), which is sufficient to yield the finite-time stability, is expected to be also necessary in some situations. (However, we already noticed that it was not necessary for a tree with two internal nodes.)

Here we consider a star-shaped tree, with the homogeneous Dirichlet boundary condition at one external node and the transparent boundary conditions at the other external nodes. We consider any possible value of the coefficient α at the internal node, and exhibit an eigenvalue of the underlying operator when (28) holds and (92) fails.

Assume that $\mathcal{T}$ is a star-shaped tree with N edges ($d = 2$, $k_1 = N$), and consider the boundary conditions (14)-(16) and (17). (See Figure 3.)

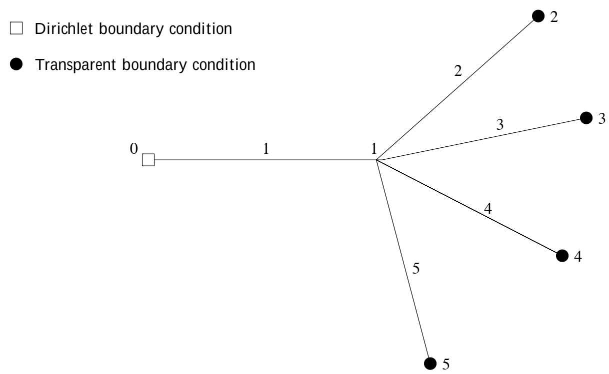

FIGURE 3. A star-shaped tree.

We assume that $\alpha_1 \neq N$, so that the system (12)-(16) and (17) is well-posed in $\mathcal{H}_0$ according to Theorem 2.1. According to Theorem 4.1, there is a finite-time stabilization when $\alpha_1 = N - 2$. We shall show that this condition is sharp, i.e. that a finite-time stabilization cannot hold if $\alpha_1 \notin \{N-2, N\}$.

Let $\alpha_1 \in \mathbb{R}$ be given. The operator A_D reads

$$A_D\big((u_i, v_i)_{i\in\mathcal{I}}\big) = (v_i, c_i^2 u_i'')_{i\in\mathcal{I}}$$

with

$$\begin{aligned} D(A_D) = \{&(u_i, v_i)_{i\in\mathcal{I}} \in \mathcal{H}_0;\ (v_i, c_i^2 u_i'')_{i\in\mathcal{I}} \in \mathcal{H}_0, \\ &c_i u_i'(1) = -v_i(1) \text{ for } 2 \le i \le N, \quad \sum_{2\le i\le N} c_i u_i'(0) - c_1 u_1'(1) = -\alpha_1 v_1(1), \\ &\text{and } \ (u_i(0), v_i(0)) = (u_1(1), v_1(1)) \text{ for } 2 \le i \le N\}, \end{aligned}$$

where $' = d/dx$, $'' = d^2/dx^2$, etc. Setting $U := (u_i, v_i)_{i\in\mathcal{I}}$, we see that (12)-(16) and (17) may be written as

$$U_t = A_D U \tag{106}$$

$$U(0) = U_0 = (u_i^0, u_i^1)_{i\in\mathcal{I}} \tag{107}$$

If $A_D U_0 = \lambda U_0$ with $U_0 \neq 0$, then the solution U of (106)-(107) reads $U(t) = e^{\lambda t} U_0$ (exponential solution), and hence $||U(t)||_{\mathcal{H}} = e^{(\operatorname{Re}\lambda)t} ||U_0||_{\mathcal{H}} > 0$ for all $t \ge 0$. Thus if the operator A_D has *at least* one eigenvalue, then the finite-time stabilization cannot hold.

Proposition 2. *Let $\mathcal{T}$ denote a star-shaped tree with N edges, and assume that $\alpha_1 \neq N$. Then the operator A_D has at least one eigenvalue if, and only if,*

$$\alpha_1 \neq N - 2. \tag{108}$$

Furthermore, if (108) holds, then the discrete spectrum of A_D is $\sigma_d(A_D) = \{\lambda_k;\ k \in \mathbb{Z}\}$ where

$$\lambda_k = \frac{c_1}{2} \log_{-\frac{\pi}{2}}\left(\frac{N-2-\alpha_1}{N-\alpha_1}\right) + ic_1 k\pi \tag{109}$$

In particular, if (108) holds, then the finite-time stabilization of (12)-(16) and (17) in $\mathcal{H}_0$ fails.

Remark 3. (1) It follows from the proof of Proposition 2 that the operator A_D has no eigenvalue when $\alpha_1 = N$. However, as the well-posedness of the system fails by Theorem 2.1, that value of α_1 is not considered for the analysis of the finite-time stability of the system. Actually, there may exist solutions that do not reach a constant value in finite time (pick $h(t) = t$ in Remark 1 (2)).
(2) If we replace the Dirichlet boundary condition $u_1(0,t) = 0$ by the transparent boundary condition $u_{1,t}(0,t) = c_1 u_{1,x}(0,t)$ and take *any* value $\alpha_1 \neq N$, then since $d_1(0,t) = s_2(1,t) = \cdots = s_N(1,t) = 0$ for all $t \geq 0$, we infer from (43)-(45) and (46) that $s_1(1,t) = d_2(0,t) = \cdots = d_N(0,t) = 0$ for all $t \geq \max_{1\leq i\leq N} c_i^{-1}$, so that for some constant $C \in \mathbb{R}$

$$u_i(x,t) = C, \qquad \forall i \in [1,N],\ \forall x \in [0,1],\ \forall t \geq 2 \max_{1\leq i\leq N} c_i^{-1}.$$

Proof. Let $\lambda \in \mathbb{C}$ and $U = (u_i, v_i)_{i\in\mathcal{I}} \in D(A_D)$. Then the equation $A_D U = \lambda U$ is equivalent to the following system

$$(v_i, c_i^2 u_i'') = \lambda(u_i, v_i), \qquad 1 \leq i \leq N, \tag{110}$$

$$u_1(0) = 0, \tag{111}$$

$$c_i u_i'(1) = -v_i(1), \qquad 2 \leq i \leq N, \tag{112}$$

$$\sum_{2\leq i\leq N} c_i u_i'(0) - c_1 u_1'(1) = -\alpha_1 v_1(1), \tag{113}$$

$$u_i(0) = u_1(1), \qquad 2 \leq i \leq N. \tag{114}$$

Note that the conditions $v_1(0) = 0$ and $v_i(0) = v_1(1)$ for $2 \leq i \leq N$ are satisfied whenever (110)-(111) and (114) hold. (110) is easily solved as

$$u_i(x) = a_i e^{\lambda x/c_i} + b_i e^{-\lambda x/c_i},\ v$$

the system

$$b_1 = -a_1, \tag{120}$$

$$a_i = 0, \qquad 2 \le i \le N, \tag{121}$$

$$-(N-1)a_1(e^{\lambda/c_1} - e^{-\lambda/c_1}) - a_1(e^{\lambda/c_1} + e^{-\lambda/c_1}) = -\alpha_1 a_1(e^{\lambda/c_1} - e^{-\lambda/c_1}), \tag{122}$$

$$b_i = a_1(e^{\lambda/c_1} - e^{-\lambda/c_1}), \qquad 2 \le i \le N. \tag{123}$$

The existence of a nontrivial solution ($a_1 \neq 0$) holds if, and only if, the coefficient above a_1 in (122) vanishes, i.e.

$$(-N+\alpha_1)e^{\lambda/c_1} + (N-2-\alpha_1)e^{-\lambda/c_1} = 0. \tag{124}$$

For $\alpha_1 \neq N$, (124) is equivalent to

$$e^{\frac{2\lambda}{c_1}} = \frac{N-2-\alpha_1}{N-\alpha_1}\cdot$$

(5) has a solution $\lambda \in \mathbb{C}$ if and only if $\alpha_1 \neq N-2$, and in that case the solutions of (5) read

$$\lambda_k = \frac{c_1}{2}\log_{-\frac{\pi}{2}}\Big(\frac{N-2-\alpha_1}{N-\alpha_1}\Big) + ic_1 k\pi, \quad k \in \mathbb{Z}. \tag{125}$$

□

Remark 4. For $k \in \mathbb{Z}$ and λ_k as in (125), we introduce the sequence of eigenfunctions $U_k = ((u_{i,k}, v_{i,k}))_{1 \le i \le N, k \in \mathbb{Z}}$ where

$$\begin{aligned} u_{1,k}(x) &= e^{\lambda_k x/c_1} - e^{-\lambda_k x/c_1}, \quad v_{1,k}(x) = \lambda_k u_{1,k}(x), \\ u_{i,k}(x) &= (e^{\lambda_k/c_1} - e^{-\lambda_k/c_1})e^{-\lambda_k x/c_i}, \quad v_{i,k}(x) = \lambda_k u_{i,k}(x), \quad \text{for } 2 \le i \le N. \end{aligned}$$

Then the family $(a_k U_k)_{k\in\mathbb{Z}}$ may fail to be a Riesz basis in $\mathcal{H}_0$ for any choice of the sequence of numbers $(a_k)_{k\in\mathbb{Z}}$. Consider e.g. $N=2$ and $c_2 = c_1/2$. Then, for $N-2 < \alpha_1 < N$,

$$u_{2,k}(x) = (e^{\lambda_k/c_1} - e^{-\lambda_k/c_1})e^{-\log|\frac{N-2-\alpha_1}{N-\alpha_1}|x - i\pi x}e^{-i2k\pi x}.$$

Let $U = (u_i, v_i)_{i=1,2} \in \mathcal{H}_0$ be given. If $(a_k U_k)_{k\in\mathbb{Z}}$ is a Riesz basis in $\mathcal{H}_0$, then U can be expended in terms of the U_k's in $\mathcal{H}_0$ as

$$(u_i, v_i) = \sum_{k\in\mathbb{Z}} d_k a_k (u_{i,k}, v_{i,k}), \quad i = 1, 2$$

for some sequence $(d_k)_{k\in\mathbb{Z}} \in L^2(\mathbb{Z})$. Writing

$$e^{\log|\frac{N-2-\alpha_1}{N-\alpha_1}|x + i\pi x}u_2(x) = \sum_{k\in\mathbb{Z}} c_k e^{-i2k\pi x}$$

we have, by harmonicity, that

$$c_k = d_k a_k (e^{\lambda_k/c_1} - e^{-\lambda_k/c_1}), \quad k \in \mathbb{Z},$$

and hence

$$u_1(x) = \sum_{k\in\mathbb{Z}} \frac{c_k}{e^{\lambda_k/c_1} - e^{-\lambda_k/c_1}}(e^{\lambda_k x/c_1} - e^{-\lambda_k x/c_1})$$

in $L^2(0,1)$. Therefore, u_1 is uniquely determined by the c_k's, and hence by u_2, which is a property much stronger than the conditions $u_1(0) = 0$ and $u_1(1) = u_2(0)$ present in the definition of $\mathcal{H}_0$. This shows that the family $(a_k U_k)_{k\in\mathbb{Z}}$ is not total in $\mathcal{H}_0$.

It is natural to conjecture a decay of all the trajectories like

$$||U(t)||_{\mathcal{H}_0} \leq C(\alpha_1) e^{\frac{c_1}{2} \log\left|\frac{N-2-\alpha_1}{N-\alpha_1}\right| t} ||U(0)||_{\mathcal{H}_0}, \qquad t \geq 0, \tag{126}$$

for $N-2 < \alpha_1 < N$. (Note that $\lim_{\alpha_1 \searrow N-2} \log\left|\frac{N-2-\alpha_1}{N-\alpha_1}\right| = -\infty$.) Without a Riesz basis of eigenvectors in the full space $\mathcal{H}_0$, the validity of (126) seems hard to check.

6. **Conclusion.** In this paper, we addressed the issue of the finite-time stabilization of a network of strings. We showed in particular that, when using the homogeneous Dirichlet boundary condition at the root and transparent boundary conditions at the other external nodes, and incorporating in Kirchhoff law the damping term $\alpha(n)u_t$ with $\alpha(n) = k(n) - 2$ at each internal node n, we obtained the finite-time stability of the system. The condition on the coefficients $\alpha(n)$ proved also to be sharp for a star-shaped tree.

Several questions remain open:

1. Can we obtain a sharp result for a general network? Same question when (12) is replaced by (8)?
2. Can we obtain finite-time stability results for networks containing circuits?

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