Linear relaxation to planar Travelling Waves in Inertial Confinement Fusion

L. Monsaingeon

Abstract
We study linear stability of planar travelling waves for a scalar reaction-diffusion equation with non-linear anisotropic diffusion. The mathematical model is derived from the full thermo-hydrodynamical model describing the process of Inertial Confinement Fusion. We show that solutions of the Cauchy problem become planar exponentially fast with rate $s(\varepsilon', k) > 0$, where $\varepsilon' = \frac{T_{\text{min}}}{T_{\text{max}}} \ll 1$ is a small temperature ratio and $k \gg 1$ the transversal wrinkling wavenumber of perturbations. We rigorously recover in some particular limit $(\varepsilon', k) \to (0, +\infty)$ a dispersion relation $s(\varepsilon', k) \sim \gamma_0 k^\alpha$ previously computed heuristically and numerically in some physical models of Inertial Confinement Fusion.

1 Introduction

The aim of Inertial Confinement Fusion - ICF in short - is to concentrate extreme temperature and pressure conditions in plasma to a single point in order to trigger a nuclear fusion reaction. The corresponding models couple thermal reaction-diffusion equations with hydrodynamical effects. For such extreme temperature regimes ($T \sim 10^7 K$) the heat conductivity coefficient $\lambda$ in the usual diffusion term $\nabla \cdot (\lambda \nabla T)$ cannot be considered as constant, but must be taken of the form

$$\lambda = \lambda(T) = T^{m-1}$$

for some fixed conductivity exponent $m > 1$ (the value $m = 7/2$ is often considered in ICF, corresponding to the Spitzer electronic heat conductivity). Scaling at the hot side we always take in the sequel temperature values $\varepsilon' = T_{\text{min}} < T < T_{\text{max}} = 1$, and consider the limit of small temperature ratio $\varepsilon' \ll 1$ (which is indeed true in ICF, typically $\frac{T_{\text{min}}}{T_{\text{max}}} \sim 10^{-3}$). An important consequence is here the strong variation of the diffusion length scale: since temperature varies of several orders of magnitude so does the thermal conductivity $\lambda = T^{m-1}$, and the effective diffusion length scales are thus very different.
in the cold and hot regions. The physically relevant wavenumbers are therefore \( 1 \ll k \ll 1/(\varepsilon')^{m-1} \), corresponding to wavelengths \( \frac{1}{k} \) between the shortest and longest diffusive length scales. The second important feature of ICF is the propagation of spherical temperature waves, which concentrate a laser driven input of energy to the center of a spherical fuel target. These waves have a very particular spatial structure consisting in three separate zones: in the most inner one (the target) an initial amount of unburned fuel is at rest \( T \approx \varepsilon' \), in the outer one the fuel has been transformed into hot plasma by the laser \( T \approx 1 \), and in the intermediate region temperature evolves between \( \varepsilon' \) and 1 according to a specific power law.

When the temperature ratio \( \varepsilon' \) is small a very thin boundary layer, called ablation front, separates in the wave the cold region from the intermediate one. In the theoretical situation the geometry is purely radial and the ablation front should be a sphere shrinking in time towards its center. However, this ideal scenario is unstable due to the so-called Rayleigh-Taylor and Darrieus-Landau effects, which are purely hydrodynamical and concentrate at the ablation front: should the latter slightly deviate from its ideal spherical geometry, wrinkling may amplify, centripetal waves may stop propagating and the desired fusion reaction may not be triggered. A so-called transversal mass ablation effect fortunately tends to stabilize the system. This phenomenon can be roughly explained as follows: whereas the diffusion in the radial direction is responsible for the particular structure of the wave, the orthoradial diffusion across the wrinkled front simply tends to reduce its unstable transversal oscillations.

A self-consistent linear analysis of the full thermo-hydrodynamical model [11], performed in the relevant frequency regime for large conductivity exponents \( m \rightarrow +\infty \), showed that the transversal mass ablation should lead to a stabilizing contribution

\[
s(\varepsilon', k) \sim \nu k^{1-\frac{1}{m-1}}, \quad \nu = \text{cst} > 0
\]

in the full dispersion relation \( (s > 0 \) corresponding to an exponential damping rate). It was suggested in [8] that this stabilization mechanism can actually be investigated looking at a much simpler model, namely the purely diffusive relaxation of thermal waves, free from hydrodynamical effects. A first rigorous study of an approximated non-Galilean model [6] for finite conductivity exponents led to a very similar dispersion relation in the regime \( 1 \ll k^2 \ll 1/(\varepsilon')^{m-1} \), except for a logarithmic correction. In this paper we derive a purely thermal model suitably approximating the full one, and rigorously recover the anticipated self-consistent dispersion relation \( s \sim \nu_m k^{1-\frac{1}{m-1}} \) for finite conductivity exponents \( m > 3 \) encompassing the physical case \( m = 7/2 \) (in some frequency regime discussed later and relating \( k \gg 1 \) to \( \varepsilon' \ll 1 \)).
2 Model and contents

The spherical geometry being difficult to investigate we perform our study for the following planar configuration: \( x \in \mathbb{R} \) will denote the infinite longitudinal direction of propagation (mimicking the radial depth of penetration in the target for the real setting) and \( y \in \mathbb{R} \) the transversal one. This planar approximation is legitimate in ICF since the relevant wavelengths are much smaller than the radius of the target. Neglecting the transversal velocity \( \mathbf{V} = (V,0) \) we consider now the following 2 dimensional thermo-hydrodynamical model for \((t,x,y) \in \mathbb{R}^3\)

\[
\begin{align*}
\rho \partial_t T - \nabla \cdot (\lambda \nabla T) + \rho V \partial_x T &= f(T), \quad (2.1a) \\
\partial_t \rho + \partial_x (\rho V) &= 0, \quad (2.1b) \\
\rho T &= 1, \quad (2.1c) \\
T(t, -\infty, y) &= \varepsilon', \quad T(t, +\infty, y) = 1. \quad (2.1d)
\end{align*}
\]

Here \( T \) is the reduced temperature of the plasma scaled at the hot side \( (T_{\text{min}} = \varepsilon' \ll 1 = T_{\text{max}}) \), \( \rho \) its density, \( V \in \mathbb{R} \) its longitudinal velocity, \( \lambda = \lambda(T) = T^{-m-1} \) the reduced non-linear heat conductivity, and \( f(T) \) a non-linear reaction term of ignition type modelling input of energy by the laser in the outer region. Equations (2.1a)-(2.1b) correspond to conservation of energy and mass, and (2.1c) is a quasi-isobaric approximation \( p \approx \text{cst} \) which is common in ICF. Since we neglected the transversal velocity conservation of momentum is not required here to close the system, which is however Galilean invariant. Boundary conditions (2.1d) simply state that the medium is at rest at \( x = -\infty \) and totally combusted at \( x = +\infty \). Temperature waves propagate now in the negative \( x \) direction (the hot medium invades the cold one), and the transversal wrinkling of the front will be later taken into account considering a periodicity condition in the \( y \) direction.

Considering (2.1b) as a Schwarz condition for crossed derivatives we may define \( X(t,x,y) \) such that \( \partial_t X = -\rho V \) and \( \partial_x X = \rho \): this new Lagrangian coordinate is mass-weighted with respect to the Eulerian one as

\[
X(t,x,y) := \int_0^x \rho(t,z,y)dz - \int_0^t \rho V(\tau,0,y)d\tau. \quad (2.2)
\]

One easily checks that for physical \( \rho \geq 1/T \geq 1 \) this defines a proper change of coordinates \((t,x,y) \rightarrow (t,X,y)\) such that \( X(t, \pm\infty, y) = \pm\infty \), and inverting the Jacobian yields the usual characteristic \( \frac{dx}{dt} = V(t,x,y) \) for Lagrangian particles. The ideal spherical geometry corresponds here to planar longitudinal waves \( \partial_y = 0 \): from now on we use the

Lagrangian approximation: \( \partial_y X = 0. \quad (2.3) \)

This is of course a drastic approximation since coupling of the conservation of mass and energy occurs precisely through this \( \partial_y X \) term, but however more
subtle than neglecting every transversal variation $\partial_y = 0$ directly in the equations: transversal diffusion is still effective and will indeed induce the stabilization mechanism explained in the Introduction. Throughout the rest of the paper we will work in these Lagrangian coordinates and write $x$ instead of $X$ with a clear abuse of notations. Written in these new coordinates with the Lagrangian approximation (2.3), conservation of energy (2.1a) is uncoupled from the hydrodynamics

$$\rho \partial_t T - \rho \partial_x (\lambda \rho \partial_x T) - \partial_y (\lambda \partial_y T) = f(T). \quad (2.4)$$

Since $\rho = 1/T$ and $\lambda = T^{m-1}$ equation (2.4) is a scalar reaction-diffusion one, in which the difference between longitudinal and transversal diffusions clearly appears.

As previously mentioned, temperature evolves in some portion of the wave profile according to a specific power law. More precisely, $\nu = T^{m-1}$ behaves linearly with respect to the Eulerian longitudinal coordinate in this intermediate region (this is very similar to the Porous Media Equation, see e.g. [5]). However, the Lagrangian coordinates are mass-weighted and stretched with respect to the Eulerian ones: the suitable variable turns to be

$$\mu := T^{m-2}$$

which we will show to behave linearly in $x$ in the intermediate region (existence and qualitative properties of the wave solution will be investigated in Section 3). Since we investigate wave propagation it will also be more convenient to work in the Lagrangian wave frame $x + ct$, in which the wave is simply a planar stationary solution: in terms of the unknown $\mu = T^{m-2}$ equation (2.4) and boundary conditions (2.1d) read in the translated frame

$$\partial_t \mu + c \partial_x \mu - \left( \mu \partial_{xx} \mu + \frac{1}{m-2} \partial_x \mu \right)^2 - \left( \mu^{m-2} \partial_{yy} \mu + \frac{2}{m-2} \mu^{m-2} (\partial_y \mu)^2 \right) = G(\mu) \quad (2.5a)$$

$$\mu(t, -\infty, y) = \varepsilon, \quad \mu(t, +\infty, y) = 1, \quad \varepsilon := (\varepsilon')^{m-2} \ll 1. \quad (2.5b)$$

The right-hand side $G(\mu) := (m-2)\mu f(\mu^{\frac{1}{m-2}})$ corresponds to the initial reaction term $f(T)$ and is again of ignition type: $G$ is smooth and there exists a threshold $\theta \in ]0, 1[$ such that

$$\mu \in [0, \theta] : \quad G \equiv 0,$n
$$\mu \in [\theta, 1[ : \quad G > 0,$n
$$\mu = 1 : \quad G = 0 \quad \text{and} \quad G'(1) < 0.$$

In order to investigate linear stability with respect to wrinkled perturbations $U(t, x, y)$ we linearize at the planar travelling wave $\mu_0(t, x, y) = p(x)$
with an additional $2\pi/k$ periodicity condition in the $y$ direction:

$$\partial_t U - \left( p\partial_{xx} U + p^{\frac{m}{m-2}} \partial_{yy} U \right) + \left( c - \frac{2p'}{m-2} \right) \partial_x U - p'' U = \frac{dG}{d\mu}(p) U, \quad (2.6a)$$

$$U(t, \pm\infty, y) = 0, \quad (2.6b)$$

$$U(t, x, y + 2\pi/k) = U(t, x, y). \quad (2.6c)$$

Note again the difference between the longitudinal diffusion $p\partial_{xx}$ and the transversal one $p^{\frac{m}{m-2}} \partial_{yy}$ in (2.6a). Expanding now in transversal Fourier modes $U(t, x, y) = u(x)e^{-st+iky}$ leads to

$$-pu'' + \left( c - \frac{2p'}{m-2} \right) u' + \left( k^2 p^{\frac{m}{m-2}} - p'' - \frac{dG}{d\mu}(p) \right) u = su, \quad (2.7a)$$

$$u(\pm\infty) = 0, \quad (2.7b)$$

where $'=d/dx$. Boundary conditions for perturbations (2.7b) simply come from the original boundary conditions $\mu(t, -\infty, y) = \varepsilon$ and $\mu(t, +\infty, y) = 1$.

Let us stress at this point that the wave speed $c = c_\varepsilon > 0$ and profile $p = p_\varepsilon(x) > 0$ are uniquely determined by the small parameter $\varepsilon > 0$ (this should be no surprise since the reaction term is of ignition type, see Section 3). We will see that the planar wave satisfies $p(-\infty) = \varepsilon$ exponentially fast with rate $r = c_\varepsilon/\varepsilon$: when $\varepsilon \to 0$ we have $r \to +\infty$ and the wave degenerates into a free boundary solution, $p_\varepsilon \to p_0$ and $p_0(x) \equiv 0$ for $x$ sufficiently negative. This is illustrated in Figure 1 and is again very similar to the Porous Media Equation scenario. Physically relevant perturbations should therefore not disturb the reference wave ahead of the front, and as a consequence we only investigate perturbations with maximal decay $u(-\infty) = 0$.

For fixed temperature ratio $\varepsilon > 0$ and wavenumber $k > 0$ only some particular values of $s = s(\varepsilon, k)$ allow the connection $u(-\infty) = u(+\infty) = 0$, and since we set $s$ to be the damping rate we also want to compute the smallest such $s$ (corresponding to optimal stability). We will see that the wave satisfies $p(x) \geq \varepsilon > 0$: problem (2.7) is hence uniformly elliptic, and finding the minimal $s$ is clearly a principal eigenvalue problem. As usual in elliptic theory, signed solutions play an important role and the eigenfunction associated with minimal eigenvalue $s$ will be indeed positive.

Let us recall that the relevant frequency regime in ICF is $1 \ll k \ll 1/(\varepsilon')^{m-1} = 1/\varepsilon^{\frac{m-1}{m-2}}$; in the whole paper we will always assume that

$$1 \ll k \ll \frac{1}{\varepsilon^{\frac{1}{m-2}}}, \quad (2.8)$$

which is a slightly more restrictive regime if $m > 2$. For some large $k_0 > 0$ and small $\eta > 0$ let us define the frequency (open) set

$$U := \left\{ \varepsilon, k > 0 : k > k_0 \text{ and } k < \frac{\varepsilon^{\eta}}{\varepsilon^{\frac{1}{m-2}}} \right\}. \quad (2.9)$$
Taking \((\varepsilon, k) \to (0, +\infty)\) in \(U\) clearly implies regime (2.8). The parameter \(\eta > 0\) is needed for technical reasons, but may be chosen as small as desired so that there is no real loss of information when restricting (2.8) to \((\varepsilon, k) \in U\).

Our main result is the following

**Theorem 1.** Fix a reaction term \(G\) of ignition type and a conductivity exponent \(m > 3\): for any \(\eta > 0\) there exists \(k_0 > 0\) such that, for any \((\varepsilon, k) \in U\) defined in (2.9), there exists a principal eigenvalue \(s = s(\varepsilon, k) > 0\) such that problem (2.7) has a positive solution \(u(x)\) with maximal decay when \(x \to -\infty\).

Moreover \(s(\varepsilon, k) > 0\) is the smallest eigenvalue, there exists no other principal eigenvalue, and the associated eigenspace is one-dimensional. Finally there exists \(\gamma_0 = \gamma_0(m, G) > 0\) such that

\[
s(\varepsilon, k) \sim \gamma_0 k^{1 - \frac{1}{m-1}}
\]

when \((\varepsilon, k) \to (0, +\infty)\) in \(U\).

This means that the wave is linearly stable with respect to wrinkled perturbations and that any solution of the non-linear Cauchy problem will become planar exponentially fast with rate at least \(s(\varepsilon, k) > 0\). Note that our result does not tell anything regarding perturbations in the longitudinal direction: invariance of the wave solutions under \(x\)-translations results as usual in a non-trivial element \(dp/dx\) in the kernel of the linearized operator, i.e. in a zero eigenvalue. For fixed \(\varepsilon > 0\) non-linear techniques can be used to establish stability of the wave with respect to almost any such perturbation (see e.g.\([9, 10]\)), but one has then to face the degeneracy into free boundary when \(\varepsilon \to 0\). The main interest of our result is that it is uniform in \(\varepsilon \to 0\), see (2.10).

The paper is organized as follows. In Section 3 we establish existence of the planar travelling wave and investigate its linear behaviour and the degeneracy into free boundary when \(\varepsilon \to 0^+\). Section 4 is devoted to the construction of solutions with maximal decay at \(-\infty\) in the cold region \(p = O(\varepsilon)\) for \(s = \gamma k^{1 - \frac{1}{m-1}}\) and fixed \(\gamma\). This is done expanding the solution \(u = u_0 + \varepsilon u_1 + \varepsilon u_2\) in the suitable scale \(\xi = \frac{x}{\varepsilon}\), and we also compute \((u, du/dx)\) at the exit of the boundary layer. Section 5 is a formal limit \((\varepsilon, k) \to (0, +\infty)\) performed in the linear region \(dp/dx \approx cst > 0\) at scale \(\zeta = k^{1 - \frac{1}{m-1}} x\). Due to the the degeneracy into free boundary \(p_\varepsilon \to p_0\) the resulting limiting problem is singular at \(\zeta = 0\), but will nonetheless yield the asymptotic coefficient \(\gamma_0\) in (2.10) as the principal eigenvalue of some singular Sturm-Liouville problem on the half line \(\zeta > 0\).

In Section 6 we rigorously justify this limit by proving that the real physical setting \(\varepsilon > 0, k < +\infty\) automatically matches the formal asymptotic problem when \((\varepsilon, k) \to (0, +\infty)\) in \(U\). In other words, suitably choosing \(\gamma = \gamma(\varepsilon, k)\) allows to connect the maximal decay solution \(u(-\infty) = 0\) of Section 4 with the stable branch \(u(+\infty) = 0\) of Section 3 and of course \(\gamma(\varepsilon, k) \to \gamma_0\). This last section also contains the proof of Theorem 1.
3 Planar travelling wave and boundary layer

In this section we study the aforementioned travelling wave at which the problem is linearized. Introducing \( \mu(t, x, y) = p(x) \) in (2.5a) leads to

\[
-pp'' + \left( c - \frac{p'}{m-2} \right) p' = G(p), \quad p(-\infty) = \varepsilon, \quad p(+\infty) = 1,
\]

and in order to eliminate invariance under \( x \)-translations we impose the additional pinning condition

\[
p(0) = \theta.
\]

Proposition 3.1. Let \( \varepsilon \in ]0, \theta[: \)

1. There exists a unique speed \( c = c_\varepsilon > 0 \) such that problem (3.1)- (3.2) admits a solution \( p = p_\varepsilon(x) \). This solution is unique and satisfies \( 0 < p'_\varepsilon < (m-2)c_\varepsilon \).

2. When \( \varepsilon \to 0^+ \) we have that \( c_\varepsilon \to c_0 > 0 \) and \( p_\varepsilon(.) \to p_0(.) \) uniformly on \( \mathbb{R} \), where \( p_0 \) solves

\[
-x \in ]-\infty, -\frac{\theta}{(m-2)c_0} \] : \[ p_0(x) = 0 \]
\[
x \in ]-\frac{\theta}{(m-2)c_0}, 0] \] : \[ p_0(x) = \theta + (m - 2)c_0 x \]
\[
x \in [0, +\infty] \] : \[ \begin{cases} -p_0 p''_0 + \left( c_0 - \frac{p'_0}{m-2} \right) p'_0 = G(p_0) \\ p_0(0) = \theta, \quad p_0(+\infty) = 1 \end{cases} \]

This kind of results is well-known - see e.g. [2, 4] - and relies on simple ODE techniques, the proof will only be sketched here for the sake of completeness.

Convergence \( p_\varepsilon \to p_0 \) to the free boundary solution is portrayed in Figure 1.

Proof. Equation (3.1) being invariant under translations we may use the Sliding Method from [3] to see that any solution must be increasing in \( x \), and therefore that \( \varepsilon < p < 1 \). Setting \( p' = U(p) > 0 \) for \( p \in ]\varepsilon, 1[ \), (3.1) leads to

\[
-p \frac{dU}{dp} U(p) + \left( c - \frac{U(p)}{m-2} \right) U(p) = G(p)
\]

with \( U(\varepsilon) = p'(-\infty) = 0 \) and \( U(1) = p'(+\infty) = 0 \). For \( p \leq \theta \) the reaction term can be omitted and this first order Cauchy problem can be explicitly integrated over \( p \in ]\varepsilon, \theta[ \) as

\[
p \leq \theta \quad \Rightarrow \quad U(p) = (m - 2)c \left[ 1 - \left( \frac{\varepsilon}{p} \right)^{\frac{1}{m-2}} \right],
\]

and the pinning condition \( p(0) = \theta \) yields

\[
p'(0) = U(\theta) = (m - 2)c \left[ 1 - \left( \frac{\varepsilon}{\theta} \right)^{\frac{1}{m-2}} \right].
\]
For fixed $c > 0$ the Cauchy problem

$$x \geq 0 : \begin{cases} -pp'' + \left( c - \frac{p'}{m-2} \right) p' = G(p) \\ p(0) = \theta \\ p'(0) = \alpha \end{cases}$$

is independent of $\varepsilon$, and shooting with respect to $\alpha$ there exists a unique $\alpha = \alpha_0(c) > 0$ such that the corresponding solution satisfies $p(+\infty) = 1$. The solution on $x < 0$ therefore matches the one on $x > 0$ if and only if

$$(m - 2)c \left( 1 - \left( \frac{\varepsilon}{\theta} \right)^{\frac{1}{m-2}} \right) = p'(0) = \alpha_0(c).$$

One then shows that this fixed point equation in $c$ has a unique solution $c = c_\varepsilon$ ($\alpha_0$ is decreasing in $c$) associated with a unique profile $p = p_\varepsilon$, and convergence of $c_\varepsilon \to c_0$ and $p_\varepsilon \to p_0$ on $]-\infty, 0[$ when $\varepsilon \searrow 0$ follows by monotonicity.

We establish now precise asymptotes on $p_\varepsilon \to p_0$ when $\varepsilon \to 0$. Scaling $x = \varepsilon \xi$ and $q(\xi) = \frac{p_\varepsilon(\varepsilon \xi)}{\varepsilon}$ (3.3) reads

$$\begin{cases} \frac{dq}{d\xi} = \frac{dp}{dx} = (m - 2)c \left( 1 - \frac{1}{q^{m-2}} \right) \\ q(-\infty) = 1 \end{cases}.$$ 

This shows that $p_\varepsilon$ presents a boundary layer of thickness $\varepsilon$ in which $p = O(\varepsilon)$, and $p'$ varies from $p'(-\infty) = 0$ when $p_\varepsilon = \varepsilon$ (thus $q = 1$) to $(m - 2)c[1 + o(1)] > 0$ when $p_\varepsilon \gg \varepsilon$ (thus $q \gg 1$) as pictured in Figure 1.

![Figure 1: monotonic convergence $p_\varepsilon \to p_0$ and $\varepsilon$-boundary layer.](image)

From now on we choose the origin at the slope discontinuity by setting

$$x_\theta := \frac{\theta}{(m - 2)c_0}, \quad p_0(x_\theta) = \theta, \quad (3.4)$$

and we are out of the boundary layer as soon as $x \gg \varepsilon$. To the right $p_\varepsilon$ is linear, $p'_\varepsilon(x) \sim_{\varepsilon \to 0} (m - 2)c_0$ and $p_\varepsilon(x) \sim_{\varepsilon \to 0} (m - 2)c_0x$. We will refer in the
following to the set where $\varepsilon \ll p < \theta$ indistinctly as the “hot zone” or “linear zone”, and call “cold zone” the set where $p = O(\varepsilon)$ (Figure 1 should make this terminology self-explanatory). In addition, plugging (3.3) into (3.1) yields an expression of

$$p''_\varepsilon = \frac{1}{p_\varepsilon} \left( c - \frac{U(p_\varepsilon)}{m - 2} \right) U(p_\varepsilon) = \frac{1}{p_\varepsilon} \left( \frac{1}{m - 2} \right) \frac{U(p_\varepsilon)}{p_\varepsilon} \quad (p_\varepsilon \leq \theta). \quad (3.5)$$

for any $\varepsilon > 0$. As a consequence of (3.3)-(3.5) we obtain the following asymptotes in the linear zone when $\varepsilon \to 0^+$:

$$p_\varepsilon(x) \sim (m - 2)c_0x,$$

$$p'_\varepsilon(x) \sim (m - 2)c_0,$$

$$p''_\varepsilon(x) \sim \frac{A}{\varepsilon} \left( \frac{\varepsilon}{x} \right)^{1 + \frac{1}{m - 2}}, \quad (3.6c)$$

with $A := \frac{c_0}{[(m-2)c_0]^{m-2}} > 0$. Asymptotes (3.6) only hold far enough from the boundary layer $x \gg \varepsilon$ and as long as $p_\varepsilon \leq \theta$ (otherwise the reaction term must be accounted for).

Finally, asymptotic analysis of (3.1) at $x = -\infty$ (where $p = \varepsilon$ and $p' = 0$) easily yields for fixed $\varepsilon > 0$ the exponential decay

$$p'_\varepsilon(x) \sim_{x \to -\infty} e^{c_\varepsilon x/\varepsilon}, \quad (3.7)$$

and $|p_\varepsilon(x) - \varepsilon| = O(e^{c_\varepsilon x})$. This will be useful in the next section when building the maximal decay solution in the cold zone.

4 Cold zone and asymptotic expansion

In this section we construct the solution $u$ of (2.7a) with maximal decay in the cold zone in the form of an asymptotic expansion $u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2$, with $u_2 \ll u_1$ in some sense. For the sake of simplicity we omit the subscripts and write here $c = c_\varepsilon$ and $p = p_\varepsilon$. Let us anticipate that perturbations will have a spatial structure resembling the one of the wave, including a boundary layer of thickness $O(\varepsilon)$: we therefore scale again as

$$x = \xi \varepsilon, \quad q(\xi) = \frac{1}{\varepsilon} p(\varepsilon \xi), \quad v(\xi) = u(\varepsilon \xi).$$

The slope of the reference wave solution jumps inside the boundary layer from $dp/dx = dq/d\xi \approx 0$ when $x = -\infty$ to $dp/dx = dq/d\xi \approx (m - 2)c > 0$ when $x \gg \varepsilon$, and this transition is steeper and steeper when $\varepsilon \to 0^+$: we expect a singularity of the second derivative somewhere, which is of course consistent with the slope discontinuity in the asymptotic profile $p_0(x)$. In order to later neglect $p''(x)$ in the linear zone we will have to push the exit of the boundary
layer far enough so that our asymptotic expansion encompasses this singularity. Asymptote (3.6c) shows that in order to do so we need $x \gg \varepsilon^{\frac{1}{m-2}}$: setting

$$x_\varepsilon := \varepsilon^{1-a}, \quad \xi_\varepsilon := x_\varepsilon / \varepsilon, \quad a := \frac{m-2}{m-1} \left(1 + \frac{\eta}{2}\right)$$  \hspace{1cm} (4.1)

for some $\eta > 0$, it is easy to see that $\varepsilon^{\frac{1}{m-2}} \ll x_\varepsilon \ll 1$ holds if $\eta > 0$ is fixed small enough. As a consequence we may safely neglect $p''(x)$ for $x \gg x_\varepsilon$. Here $\eta > 0$ is exactly the one in Theorem 1 and arbitrarily small, and the above definition of $a$ is purely technical. In agreement with Theorem 1 we anticipate that the relevant values of $s$ should be of order $k^{1-\frac{1}{m-1}}$, which is large: through this whole section one should think of $s$ as of $\gamma k^{1-\frac{1}{m-1}}$ for some fixed constant $\gamma$ of order unity and remember that we are interested in $(\varepsilon, k) \to (0, +\infty)$ in the double limit (2.9).

On the cold interval $I = [0, x_\varepsilon]$ we have $p = O(\varepsilon^{1-a})$ for $x \gg x_\varepsilon$, and we may therefore omit the reaction term of ignition type. Recasting (2.7a) in $\xi$ coordinates as

$$Lv = \varepsilon hv,$$

$$L := -q \frac{d^2}{d\xi^2} + \left(c - \frac{2q'}{m-2}\right) \frac{d}{d\xi} - q'',$$

$$h = \left(s - k^2 \varepsilon^{\frac{m}{m-2}} q^{\frac{m}{m-2}}\right),$$  \hspace{1cm} (4.2)

we shall seek below solutions in the form $v = v_0 + \varepsilon v_1 + \varepsilon v_2$ with $v_2 \ll v_1$ in some sense. We will refer to $L$ as the principal operator, which is also the linearized operator for non-wrinkled perturbations $k = 0$.

### 4.1 Maximal decay and principal operator

Let us recall that we investigate maximal decay perturbations $u(-\infty) = 0$: the asymptotic equation associated with (2.7a) for $x = -\infty$ yields two characteristic exponential rates

$$r^\pm = c \pm \sqrt{c^2 + 4\varepsilon^2 (k^2 \varepsilon^{\frac{m}{m-2}} - s)},$$  \hspace{1cm} (4.3)

and maximal decay obviously corresponds to $u(x) = O(e^{r^+ x})$. For $s$ of order $k^{1-\frac{1}{m-1}}$ regime (2.9) implies that $0 < \varepsilon (s - k^2 \varepsilon^{\frac{m}{m-2}}) \ll 1$, and (4.3) consequently yields

$$0 < r^- \ll r^+ \sim \frac{c}{\varepsilon}, \quad r^+ < \frac{c}{\varepsilon}. \hspace{1cm} (4.4)$$

The reference wave solution is as usual determined up to translations and therefore $L[q'] = 0$: as a consequence $v_0(\xi) = q'(\xi)$ is a suitable candidate for the leading order in the asymptotic expansion $v = v_0 + \varepsilon v_1 + \varepsilon v_2$ of $Lv = \varepsilon hv$. 

This is confirmed by (3.7), showing that $q'(\xi) = p'(\varepsilon \xi) \sim e^{c\varepsilon} \ll e^{c\varepsilon + \varepsilon\xi}$ decays fast enough.

In order to work in some fixed functional setting (independent of $\varepsilon$) let us define the following weighted spaces on $I = [-\infty, \xi\varepsilon]$:

$$w(\xi) = \begin{cases} e^{-\frac{c}{2}\varepsilon} & \xi \leq 0 \\ 1 & \xi > 0 \end{cases},$$

$B^0_w = \{ f \in C_b, \ f w \in C_b(I) \}$,

$B^k_w = \{ f \in C^k_b, \ \forall j \leq k : f^{(j)} \in B^0_w \}$,

$B^k_{w,0} = \{ f \in B^k_w, \ f(\xi\varepsilon) = 0 \}$

(recall that $c_\varepsilon \to c_0 > 0$). Here $C^k_b$ denotes the space of functions with continuous bounded first $k$ derivatives, and the $B^k_w$'s are equipped with their usual Banach norm. On one hand (4.4) shows that any maximal decay solution must behave as $e^{\varepsilon r+\xi}$ when $\xi \to -\infty$ with $\varepsilon r+\xi \sim c_0 > c_0/2$, and therefore belongs to $B^0_w$. On the other hand any non maximal decay solution behaves as $v(\xi) \sim e^{\varepsilon r-\xi}$ with $0 < \varepsilon r-\xi \ll c_0/2$: such solutions cannot belong to $B^0_w$, and as a consequence it is legitimate to look for maximal decay solutions only in $B^0_w$.

**Lemma 4.1.** The principal operator $L : B^2_{w,0} \to B^0_w$ is continuously invertible and

$$||L^{-1}|| \leq C\varepsilon^{-a}.$$  \hspace{1cm} (4.5)

**Proof.** Defining $\Phi =: \int_\xi^z \frac{f'}{q}$ it is easy to check that Duhamel’s formula

$$f(\xi) := q'(\xi) \int_\xi^z \left( \int_{-\infty}^z \exp[\Phi(\eta) - \Phi(z)] \frac{g(\eta)}{q(\eta)q'(\eta)} d\eta \right) dz \hspace{1cm} (4.6)$$

yields the unique solution in $B^2_{w,0}$ of $Lf = g$ for any given $g \in B_w$. The estimate for $L^{-1}$ then follows from (3.6)-(3.7) expressed in $\xi = x/\varepsilon$ coordinates and the study of $\Phi$ for $\xi \to -\infty$ and $1 \ll \xi \leq \xi_\varepsilon = \varepsilon^{-a}$ combined with (4.6). \hfill $\Box$

### 4.2 Asymptotic expansion and frequency regime

The expansion $v = v_0 + \varepsilon v_1 + \varepsilon^2 v_2$, with $v_2 \ll v_1$ in some sense, leads to solving $Lv = \varepsilon Hv$ as

$$Lv_0 = 0, \hspace{1cm} (4.7a)$$

$$Lv_1 = Hv_0, \hspace{1cm} (4.7b)$$

$$[L - \varepsilon h]v_2 = \varepsilon Hv_1. \hspace{1cm} (4.7c)$$

In order normalize perturbations $v(\xi\varepsilon) = 1$ we will require in addition that $v_0(\xi\varepsilon) = 1$ and $v_1(\xi\varepsilon) = v_2(\xi\varepsilon) = 0$. 

11
• Resolution of (4.7a): as already discussed the only solution decaying fast enough is \( q' \), and normalizing yields \( v_0 := \frac{q'}{q(\xi_\varepsilon)} \in B^0_w \).

• Resolution of (4.7b): remark that \( h = \left( s - k^2 \varepsilon^\frac{m}{m^2} q^\frac{m}{m^2} \right) \in L^\infty(I) \) implies \( hv_0 \in B^0_w \). Lemma 4.1 then properly defines \( v_1 := L^{-1}[hv_0] \in B^2_{w,0} \).

• Resolution of (4.7c): taking advantage of the linear behaviour of \( q(\xi) \) for large \( \xi \) and regime (2.9) with \( s = \gamma k^{1-\frac{1}{m+1}} \) (for fixed constant \( \gamma \) of order unity), we see that \( h = s - k^2 \varepsilon^\frac{m}{m^2} q^\frac{m}{m^2} \sim s \) uniformly on \( I = [-\infty, \xi_\varepsilon] \). Lemma 4.1 consequently leads to

\[
\|\varepsilon L^{-1}[h']\|_{L(B^2_{w,0})} \leq \varepsilon\|L^{-1}[h]\|_{\infty} \leq C\varepsilon^{1-a}s \ll 1, \tag{4.8}
\]

and \( M := \text{Id} - \varepsilon L^{-1}h \in L(B^2_{w,0}) \) is close to Identity thus invertible. This finally allows to solve \( [L - \varepsilon h]v_2 = \varepsilon hv_1 \) as \( v_2 := M^{-1}(L^{-1}(\varepsilon hv_1)) \in B^2_{w,0} \).

Regime (2.9) also ensures that \( v = v_0 + \varepsilon v_1 + \varepsilon v_2 \) is really an asymptotic expansion in \( B^2_w \), in the sense that \( \|\varepsilon v_2\|_{B^2_w} \ll \|\varepsilon v_1\|_{B^2_w} \ll \|v_0\|_{B^2_w} = \mathcal{O}(1) \).

More precisely, one easily shows using (4.8) and \( M \approx \text{Id} \) that

\[
v_1 = L^{-1}(hv_0) \quad \Rightarrow \quad \|v_1\|_{B^0_{w,2}} \leq C\varepsilon^{1-a}s\|v_0\|_{B^0_w} \ll_{\varepsilon,k} \|v_0\|_{B^2_w},
\]

\[
v_2 \approx L^{-1}(\varepsilon hv_1) \quad \Rightarrow \quad \|v_2\|_{B^0_{w,2}} \leq C\varepsilon^{1-a}s\|v_1\|_{B^0_w} \ll_{\varepsilon,k} \|v_1\|_{B^2_{w,0}}. \tag{4.9}
\]

4.3 Exit boundary conditions

We compute in this section \((v,v')\) at the exit \( \xi = \xi_\varepsilon \) of the cold zone. Since we normalized \( v(\xi_\varepsilon) = 1 \) it is enough to compute \( v' \) and we estimate separately \( v_0'(\xi_\varepsilon) \), \( v_1'(\xi_\varepsilon) \) and \( v_2'(\xi_\varepsilon) \).

• Since we set the exit far enough out of the boundary layer, asymptotes (3.6) hold at \( x_\varepsilon = \varepsilon^{-1-a} \gg \varepsilon \) and therefore \( \varepsilon^2 \frac{d^2 p}{dx^2}(x_\varepsilon) = \mathcal{O}(\varepsilon^{1+\frac{1}{m-1}}) \). In terms of \( \xi = x/\varepsilon \) this corresponds to

\[
v_0'(\xi_\varepsilon) = \mathcal{O} \left( \varepsilon^{a(1+\frac{1}{m-1})} \right). \tag{4.10}
\]

• Let us recall that \( Lv_1 = hv_0 = h\frac{q'}{q(\xi_\varepsilon)} \) and \( h \approx L^\infty(I) \sim s \), so that \( v_1 \) is close in \( B^2_{w,0} \) to the solution of \( Lv = s \frac{q'}{q(\xi_\varepsilon)} \). Using (4.6) it is easy to compute explicitly \( L^{-1}[q'](\xi) = (\xi_\varepsilon - \xi)q'(\xi) \) hence

\[
v_1(\xi) \approx \frac{s(\xi_\varepsilon - \xi)q'(\xi)}{cq'(\xi_\varepsilon)} \quad \text{in} \ B^2_w. \tag{4.11}
\]

Since the \( B^2_{w,0} \) topology controls the first two derivatives we obtain

\[
v_1'(\xi_\varepsilon) \sim \frac{s}{c_\varepsilon q'(\xi_\varepsilon)} \frac{d}{d\xi} \left[ (\xi_\varepsilon - \xi)q'(\xi) \right]_{\xi = \xi_\varepsilon} = -\frac{s}{c_\varepsilon}. \tag{4.12}
\]
Recalling that $$M = (\text{Id} - \varepsilon L^{-1} h) \approx \text{Id}$$ in $$\mathcal{L}(B_{w,0}^2)$$, (4.7c) and (4.11) show that

$$v_2 = (\text{Id} - \varepsilon L^{-1} h)^{-1} L^{-1}[\varepsilon hv_1] \approx \frac{\varepsilon^2 s^2}{c_\varepsilon q'\left(\xi_\varepsilon\right)} L^{-1}[(\xi_\varepsilon - \xi)q']$$

in $$B_{w,0}^2$$. Duhamel Formula (4.6) gives an explicit integral formula for $$L^{-1}[(\xi_\varepsilon - \xi)q']$$: differentiating with respect to $$\xi$$, evaluating at $$\xi = \xi_\varepsilon$$ and taking advantage of the asymptotic behaviour of $$\Phi$$ (for $$\xi \to -\infty$$ and $$1 \ll \xi \leq \xi_\varepsilon$$), one finally estimates

$$v'_2(\xi_\varepsilon) = O\left(\varepsilon^{1-a} s^2\right). \quad (4.13)$$

We claim now that the exit slope $$v'(\xi_\varepsilon)$$ is dominated by $$\varepsilon v'_1(\xi_\varepsilon)$$ in the double limit $$(\varepsilon, k) \to (0, +\infty)$$. For technical reasons we will also need Proposition 4.1. For fixed $$(\varepsilon, k)$$ the quantity $$v'(\xi_\varepsilon)$$ is continuously differentiable with respect to $$\sigma$$ and

$$\frac{\partial}{\partial \sigma}[v'(\xi_\varepsilon)] = -(m - 2)\varepsilon k^{1 - \frac{1}{m-1}} + r(\varepsilon, k, \sigma) \quad (4.15)$$

holds with $$r(\varepsilon, k, \sigma) = o\left(\varepsilon k^{1 - \frac{1}{m-1}}\right)$$ when $$(\varepsilon, k) \to (0, +\infty)$$ and locally uniformly in $$\sigma$$.

Remark that (4.15) is indeed consistent with a formal differentiation of (4.14) with respect to $$\sigma$$. Let us recall that $$s$$ is here of order $$k^{1 - \frac{1}{m-1}} \gg 1$$ as anticipated from Theorem 1: the new parameter

$$\sigma := \frac{s}{(m - 2)c_\varepsilon k^{1 - \frac{1}{m-1}}}$$

should therefore take values of order unity, and the exit conditions are finally summarized in terms of this new parameter $$\sigma$$ by

$$\begin{cases} v(\xi_\varepsilon) = v_0(\xi_\varepsilon) = 1 \\ \frac{dv}{d\xi}(\xi_\varepsilon) \sim \varepsilon \frac{dv_1}{d\xi}(\xi_\varepsilon) \sim \frac{\varepsilon a}{c_\varepsilon} = -(m - 2)\varepsilon k^{1 - \frac{1}{m-1}}. \end{cases} \quad (4.14)$$

For technical reasons we will also need
The main issue in this section is precisely to compute this \( \gamma \) cient to the exit of the boundary layer), and this will relate the asymptotic coeffi-
an asymptotic eigenvalue problem on the half line
coordinates corresponding to a suitable infinite zoom in the linear zone. We obtain
order to keep this paper in a reasonable length.

\[ \text{Proof.} \] Regularity with respect to \( \sigma \) is a classical consequence of the linear de-
pendence on \( \sigma \) of (4.2). Writing \( z = \frac{\partial u}{\partial \sigma} \), estimate (4.15) is just a computation
for the slope \( \frac{\partial v}{\partial x} \), the order one determines here the dependence on \( \sigma \) at the exit, namely \( \frac{\partial v}{\partial x}(\xi_\varepsilon) \sim \varepsilon \frac{\partial v}{\partial x}(\xi_\varepsilon) \). The technical computations are omitted here in
order to keep this paper in a reasonable length. \qed

5 Hot zone and asymptotic problem

In this section we take a formal limit \( \varepsilon = 0, k = +\infty \) in some scaled \( \zeta \) coordi-
nates corresponding to a suitable infinite zoom in the linear zone. We obtain
an asymptotic eigenvalue problem on the half line \( \zeta > 0 \) (\( \zeta = 0 \) corresponding
to the exit of the boundary layer), and this will relate the asymptotic coefficient \( \gamma_0 \) in Theorem 1 to some principal eigenvalue \( \sigma_0 \) of the limiting problem.
The main issue in this section is precisely to compute this \( \sigma_0 \).

We anticipated that \( s \) should be of order \( k^{1-\frac{1}{m-1}} \), which also turns to be
the suitable length-scale to investigate the linear region: scaling (2.7a) as

\[
\zeta = k^{1-\frac{1}{m-1}} x \\
v(\zeta) = \frac{1}{k^{1-\frac{1}{m-1}}} u \left( \frac{\zeta}{k^{1-\frac{1}{m-1}}} \right) \\
\sigma = \frac{s}{k^{1-\frac{1}{m-1}}} (m-2) c k^{1-\frac{1}{m-1}} \\
q(\zeta) = k^{1-\frac{1}{m-1}} p \left( \frac{\zeta}{k^{1-\frac{1}{m-1}}} \right)
\]

(5.1)

for \( k < \infty \) leads to

\[
- q v'' + \left( c - \frac{2q'}{m-2} \right) v' - q'' v = \left( (m-2) c \sigma - q \frac{m}{m-2} + \frac{G'(q/k^{1-\frac{1}{m-1}})}{k^{1-\frac{1}{m-1}}} \right) v
\]

(5.2)

with \( q = q_{\varepsilon,k}(\zeta), p = p_\varepsilon \) and \( c = c_\varepsilon \) depending only on \( \varepsilon \). This scaling is again
Lipschitz \( \frac{\partial q}{\partial x} = \frac{\partial p}{\partial x} \), and the parameter \( \sigma \) takes values of order one.

We recall from Section 3 that in the limit \( \varepsilon \to 0 \) we have \( c_\varepsilon \to c_0 > 0 \),
and the asymptotic wave profile \( p_0(x) \) is exactly linear for \( x \in [0, \frac{\theta}{(m-2)c_0}] \). In \( \zeta \)
coordinates this means \( q(\zeta) \approx q_0(\zeta) = (m-2)c_0 \zeta \) for \( \zeta \in [0, \zeta_k] \), where

\[
\zeta_k := \frac{\theta k^{1-\frac{1}{m-1}}}{(m-2)c_0} \to +\infty.
\]

In the regime (2.9) the exit of the cold zone \( x = x_\varepsilon \) corresponds in \( \zeta \) coordinates to

\[
\zeta_\varepsilon = k^{1-\frac{1}{m-1}} x_\varepsilon = k^{1-\frac{1}{m-1}} \varepsilon^{1-a} \to 0.
\]

Moreover since \( \frac{\partial G}{\partial p}(p_0) \) and \( \frac{\partial^2 p_0}{\partial x^2} \) are of order 1 for \( x \in ]0, +\infty[ \), we may neglect

\[
q''(\zeta) \approx \frac{p_0''(x)}{k^{1-\frac{1}{m-1}}} \to 0, \quad \frac{G'(q(\zeta)/k^{1-\frac{1}{m-1}})}{k^{1-\frac{1}{m-1}}} \approx \frac{G'(p_0(x))}{k^{1-\frac{1}{m-1}}} \to 0
\]
at least formally when \( \varepsilon = 0 \) and \( k = +\infty \). As a matter of fact a singularity of \( \frac{d^2p_0}{dx^2} \) appears inside the boundary layer due to the slope discontinuity of \( p_0 \), but this will be carefully examined in Section 6 (this is precisely the reason why we pushed the exit \( x = x_\varepsilon \) of the cold zone far enough).

Formally taking \( \varepsilon = 0 \), \( k = +\infty \), \( q(\zeta) = q_0(\zeta) = (m - 2)c_0\zeta \), \( \zeta_\varepsilon = 0 \) and \( \zeta_k = +\infty \) in (5.2) we obtain the following asymptotic problem:

\[
\zeta \in ]0, +\infty[; \quad -\zeta v'' + \frac{v'}{m - 2} + bv = \sigma v, \tag{5.3}
\]

with

\[
b(\zeta) = B\zeta^{\frac{m}{m - 2}}, \quad B := \left( (m - 2)c_0 \right)^{\frac{2}{m - 2}} > 0.
\]

Since we were originally interested in perturbations \( u(x) \) vanishing at infinity we also require in (5.3) that \( v(+\infty) = 0 \), which is again a formal but suitable condition.

**Remark 5.1.** For fixed \( \sigma \) and \( \zeta = +\infty \) equation (5.3) reads \(-v'' + B\zeta^{\frac{2}{m - 2}}v = 0\): there exists a unique branch of stable solutions \( v(+\infty) = 0 \) and the other one satisfies \( v(+\infty) = +\infty \), both at least exponentially.

Recalling now that we also look for signed perturbations (as principal eigenfunctions), the parameter \( \sigma \) clearly appears here as a principal eigenvalue to be determined for the Sturm-Liouville problem (5.3). Two main difficulties arise here compared to usual Sturm-Liouville theory: unboundedness \( \zeta \in \mathbb{R}^+ \) and singularity at \( \zeta = 0 \).

Throughout the rest of Section 5 we will consider \( \varepsilon = 0 \), \( k = +\infty \), and the only relevant dependence will be the one on \( \sigma \). In order to keep our notations as light as possible we will write \( p = p_0(x) = (m - 2)c_0x \), \( q = q_0(\zeta) = (m - 2)c_0\zeta \) and \( c = c_0 \). The main result in this section is:

**Theorem 2.** There exists a principal eigenvalue \( \sigma_0 > 0 \) such that problem (5.3) has a positive solution \( v_0 \in C^1([0, +\infty[) \cap C^2([0, +\infty[) \) satisfying boundary conditions \( v_0(0) = 1 \) and \( v(+\infty) = 0 \). This principal eigenfunction satisfies in addition \( v'_0(0) = -(m - 2)\sigma_0 \) and \( v'_0 < 0 \).

Note the \( C^1 \) regularity at \( \zeta = 0 \) in our statement above: this should be seen as a boundary condition, in contrast with \( v_0(0) = 1 \) which is simply a normalization. Moreover, according to scaling (5.1) \( \sigma_0 \) of order unity means \( s \) of order \( k^{1-\frac{1}{m-1}} \): this will yield the asymptotic coefficient \( \gamma_0 = (m - 2)c_0\sigma_0 \) in Theorem 1

### 5.1 The asymptotic principal eigenvalue

In this section we first establish some technical results and then prove Theorem 2. We start by studying the singularity at \( \zeta = 0 \).
Proposition 5.1. For fixed $\sigma \in \mathbb{R}$ there exists a unique solution $v_\sigma \in C^2([0, +\infty[) \cap C^1([0, +\infty[)$ of \([5.3]\) such that $v_\sigma(0) = 1$. It satisfies $v'_\sigma(0) = -(m - 2)\sigma$ and the mapping $\sigma \mapsto v_\sigma(.)$ is $C^1$ from $\mathbb{R}$ into $C^1([0, \zeta_0])$ for any fixed $\zeta_0 > 0$.

Proof. Remarking that the exponent in the zero-th order coefficient $b(\zeta) = B\zeta \frac{2}{m-2}$ is $\frac{m}{m-2} > 1$, solutions of \([5.3]\) “should not see” the coefficient $b(\zeta)$ in the neighbourhood of $\zeta = 0$ and therefore behave as

$$\zeta w'' + \frac{w'}{m - 2} + \sigma w = 0. \quad (5.4)$$

It is easy to obtain a regular solution $w \in C^2([0, +\infty[) \cap C^1([0, +\infty[)$ of \([5.4]\) satisfying $w(0) = 1$ and $w'(0) = -(m - 2)\sigma$. This is to be computed. Moreover, if $b(\zeta)$ is the mapping $B\zeta \frac{2}{m-2}$ such that $w(0) = 1$ and $w'(0) = -(m - 2)\sigma$. The proof consists in two steps: a classical Banach fixed point on the integral formulation for $h$ first yields a solution on $[0, \zeta]$ for fixed small $\zeta > 0$, thus stepping away from the singularity. This solution then extends to the right $[\zeta, +\infty[$, where the ODE is now regular. Uniqueness is obtained as for $w$ since the singularity at $\zeta = 0$ is of the first kind, see again \[7\]. Regularity of $h$ with respect to $\sigma$ is obtained first on $[0, \zeta]$ applying the Implicit Functions Theorem in the integral formulation, and then on $[\zeta, \zeta_0]$ by classical regularity of Cauchy solutions with respect to initial conditions and parameters.

For the sake of clarity we will denote by $v = v_\sigma$ this unique regular solution of \([5.3]\) and will only consider $\sigma \geq 0$.

Proposition 5.2. For $\sigma \geq 0$ only three scenarios are possible for \([5.3]\):

$v(+\infty) = +\infty$, $v(+\infty) = 0$ and $v(+\infty) = -\infty$. Moreover, if $\zeta_\sigma = \left(\frac{\sigma}{B}\right)\frac{m-2}{m}$ is the first time where $b(\zeta) = \sigma$, we have that

1. If there exists $\zeta_1 \geq \zeta_\sigma$ such that $v(\zeta_1) > 0$ and $v'(\zeta_1) \geq 0$, then $v(\zeta) \geq v(\zeta_1) > 0$ on $[\zeta_1, +\infty[$ and $v(+\infty) = +\infty$.

2. If there exists $\zeta_1 \geq \zeta_\sigma$ such that $v(\zeta_1) < 0$ and $v'(\zeta_1) \leq 0$, then $v(\zeta) \leq v(\zeta_1) < 0$ on $[\zeta_1, +\infty[ and v(+\infty) = -\infty$.

Proof. When $\zeta = +\infty$ \([5.3]\) reads $-v'' + B\zeta \frac{2}{m-2} v = 0$ and therefore satisfies the classical Maximum Principle ($B = cst > 0$). As a consequence either $v(+\infty) = \pm \infty$, either $v (+\infty) = cst$, and the only possible finite limit is clearly $v(+\infty) = 0$. Recasting \([5.3]\) as $\left[\zeta \frac{m-2}{m-2} v\right]' = \frac{b-\sigma}{\zeta \frac{m-2}{m-2}} v$ we have by definition $b(\zeta) - \sigma > 0$ for $\zeta > \zeta_\sigma$, and the rest of the statement is easily obtained integrating from $\zeta_1$ to $+\infty$. \(\square\)
Proposition 5.3. For \(\sigma \geq 0\) small enough \(v > 0\) holds on \([0, +\infty[\), and \(v(+\infty) = +\infty\).

Proof. Denoting by \(\varpi\) the solution for \(\sigma = 0, \) (5.3) reads \([\zeta \frac{1}{m-2} \varpi \prime \prime] = B \zeta \frac{3}{m-2} \varpi\). Using boundary conditions \(\varpi(0) = 1\) and \(\varpi'(0) = 0\) it is easy to integrate from \(\zeta = 0\) to \(\zeta > 0\) and see that \(\varpi(+\infty) = +\infty\). This easily extends by continuity to \(\sigma > 0\) small enough remarking that \(\zeta_\sigma := (\frac{\sigma}{B})^{\frac{m-2}{m}} \rightarrow 0^+\) when \(\sigma \rightarrow 0^+\) and using the first case in Proposition 5.2.

Proposition 5.4. For \(\sigma\) large enough there exists a time \(\zeta_- = \zeta_- (\sigma) > 0\) such that \(v(\zeta_-) < 0\).

Proof. Scaling (5.3) as \(t = \sigma \zeta\) and \(y(t) = v \left( \frac{t}{\sigma} \right)\) yields

\[
t \dot{y} + \alpha y + y = \beta t \frac{m}{m-2} y, \quad y(0) = 1, \quad \dot{y}(0) = -\frac{1}{\alpha}
\]

with \(\dot{y} := \frac{dy}{dt}, \alpha = \frac{1}{m-2} \in ]0, 1[\) and \(\beta = \frac{B}{\sigma^2 \frac{m}{m-2}}\). When \(\beta = 0\) we may look for the solution \(y_\alpha(t)\) in the form of a power series, and an easy but technical computation shows that this power series takes negative values \(y_\alpha(t = 2\alpha) < 0\) for any fixed \(\alpha \in ]0, 1[\). This easily extends by continuity to \(\beta > 0\) small, i-e to \(\sigma\) large.

We may now prove our main statement:

Proof. (of Theorem 2). By Propositions 5.3 and 5.4 the quantity

\[
\sigma_0 := \sup_{\sigma \geq 0} \left( \sigma \geq 0, \quad \sigma' \in [0, \sigma] \Rightarrow v_{\sigma'}(.) > 0 \right)
\]

is finite and positive. We claim that \(\sigma_0\) is indeed the desired principal eigenvalue: denoting by \(v_0 := v_{\sigma_0}\) we have by continuity that \(v_0 \geq 0\), and actually \(v_0 > 0\) (unless \(v_0 \equiv 0\), which contradicts \(v_0(0) = 1\)). Stability \(v_0(+\infty) = 0\) is a consequence of Proposition 5.2 (otherwise \(v_0(+\infty) = +\infty\) and there exists a right neighbourhood of \(\sigma_0\) in which \(v_\sigma > 0\), thus contradicting the definition of \(\sigma_0\)). Let \(\zeta_0 := (\frac{\sigma_0}{B})^{\frac{m-2}{m}}\) : by Proposition 5.2 \(v_0 > 0\) must satisfy \(v_0' < 0\) on \([\zeta_0, +\infty[\) (unless it blows at infinity, which is impossible). On \([0, \zeta_0]\) monotonicity is a consequence of \(v_0'(0) = -(m-2)\sigma_0 < 0\) and \(L v_0 = (\sigma_0 - b) v_0 \leq 0\), where \(L = -\zeta \frac{d^2}{d\zeta^2} - \frac{1}{m-2} \frac{d}{d\zeta}\) is elliptic and has no zero-th order coefficient.

5.2 Isolated eigenvalue

We prove here that there are no other eigenvalues close to \(\sigma_0\). More precisely we show that \(\sigma \in [0, \sigma_0[ \Rightarrow v (+\infty) = +\infty\) and that for \(\sigma > \sigma_0\) close enough \(v (+\infty) = -\infty\) holds. The key point is the following monotonicity:

Proposition 5.5. For \(\sigma \in [0, \sigma_0[,\) the mapping \(\sigma \mapsto v_\sigma(.)\) is strictly pointwise decreasing on \([0, +\infty[\).
Proof. Remark that $\zeta = 0$ is irrelevant since we normalized $v_\sigma(0) = 1$ for all $\sigma$. The proof is quite standard: if $\sigma_1 > \sigma_2 \in [0, \sigma_0]$ then $v_1, v_2 > 0$, $\alpha := \frac{v_1}{v_2} > 0$ is well defined and satisfies

$$
\tilde{L}[\alpha] := -\zeta v_2 \alpha'' - \left(2\zeta v'_2 + \frac{v_2}{m-2}\right) \alpha' = (\sigma_1 - \sigma_2)v_1 > 0.
$$

Computing $\alpha(0) = 1$ and $\alpha'(0) = (m-2)(\sigma_2 - \sigma_1) < 0$ shows that $\alpha < 1$ at least in the neighbourhood of $\zeta = 0$. If $\alpha(\zeta^*) \geq 1$ for some $\zeta^* > 0$ then $\alpha$ would attain a non-negative local minimum point $\zeta_m \in ]0, \zeta^*[,$ which would contradict the classical Minimum Principle. Thus $\alpha < 1 \iff v_1 < v_2$ on $]0, +\infty[.$

**Proposition 5.6.** For $\sigma \in ]0, \sigma_0[$ we have that $v_\sigma(+\infty) = +\infty$.

**Proof.** For such $\sigma$ we have by monotonicity $v_\sigma > v_{\sigma_0} > 0$ hence by Proposition 5.2 either $v_\sigma(+\infty) = +\infty$, either $v_\sigma(+\infty) = 0$. Assume by contradiction that $v_\sigma(+\infty) = 0$ for some $\sigma \in ]0, \sigma_0[$: both $v_\sigma$ and $v_{\sigma_0}$ then satisfy the same asymptotic equation when $\zeta \to +\infty$, where they behave similarly. Slowly increasing $\alpha$ starting from zero, we see that $\alpha v_\sigma < v_{\sigma_0}$ holds until a first critical value $\alpha^* \in ]0, 1[$ for which there exists a contact point $\zeta_m$ between $\alpha^*v_\sigma$ and $v_{\sigma_0}$. On the other hand $z := \alpha^*v_\sigma - v_{\sigma_0} \leq 0$ satisfies $Lz = (\sigma - \sigma_0) \alpha^*v_\sigma + \sigma_0 z \leq 0$ with $L = -\zeta \frac{d^2}{d\zeta^2} - \frac{1}{m-2} \frac{d}{d\zeta} + b(\zeta)$, thus contradicting the Minimum Principle.

**Remark 5.2.** In the proof above we only used the fact that $\sigma_0$ is associated with a (stable) positive eigenfunction $v_{\sigma_0}$. As a classical byproduct we retrieve uniqueness of the principal eigenvalue $\sigma_0$: by Proposition 5.6 there are no eigenvalues at all for $\sigma < \sigma_0$, and if there existed an other principal eigenvalue $\sigma_1 > \sigma_0$ associated with a positive eigenfunction $v_{\sigma_1}$ we could repeat exactly the same argument and conclude that $v_\sigma > v_{\sigma_1}$ and $v_\sigma(+\infty) = +\infty$ for all $\sigma < \sigma_1$, which fails for $\sigma = \sigma_0 < \sigma_1$.

**Lemma 5.1.** If $\sigma > \sigma_0$ then $v_\sigma \geq 0$ cannot hold on $\mathbb{R}^+$. 

**Proof.** Assume by contradiction that $v_\sigma \geq 0$ for some $\sigma > \sigma_0$. Once again we must have either $v_\sigma(+\infty) = 0$, either $v_\sigma(+\infty) = +\infty$. Owing to Remark 5.2 above the first case is impossible, and therefore $v_\sigma(+\infty) = +\infty$. Defining $\alpha := \frac{v_\sigma}{v_{\sigma_0}}$ we have $\alpha \geq 0$, $\alpha'(0) = (m-2)(\sigma_0 - \sigma) < 0$ and $\alpha(+\infty) = +\infty$. As a consequence $\alpha$ attains a (non-negative) minimum point at some $\zeta_m > 0$, thus contradicting the classical Minimum Principle and

$$
\tilde{L}[\alpha] := -\zeta v_0 \alpha'' - \left(2\zeta v'_0 + \frac{v_0}{m-2}\right) \alpha' = (\sigma - \sigma_0)v_0 > 0.
$$

**Proposition 5.7.** For $\sigma > \sigma_0$ close enough there holds $v_\sigma(+\infty) = -\infty$. 

18
Proof. By Proposition 5.2 it is enough to show that for such \( \sigma \) there exists a time \( \zeta > \zeta_0 = (\frac{g}{h})^{\frac{m-2}{m}} \) such that \( v_\sigma(\zeta) < 0 \) and \( v_\sigma'(\zeta) < 0 \). For any \( \sigma > \sigma_0 \) we normalized \( v_\sigma(0) = 1 > 0 \), and by Lemma 5.1 \( v_\sigma(.) \) takes negative values at least somewhere in \( \mathbb{R}^+ \): the first time \( \zeta(\sigma) > 0 \) where \( v_\sigma \) vanishes is hence well defined. Since \( v_{\sigma_0} > 0 \) we must have by continuity \( \zeta(\sigma) \rightarrow +\infty \) when \( \sigma \rightarrow \sigma_0^+ \), and in particular \( \zeta_0(\sigma) > \zeta_0 \) since \( \zeta_0 \rightarrow (\frac{g_0}{h})^{\frac{m-2}{m}} < +\infty \). By definition of \( \zeta_0 \) we also have \( v_\sigma > 0 \) on \([0, \zeta_0]\) and \( v_\sigma(\zeta_0) = 0 \) hence \( v_\sigma'(\zeta_0) \leq 0 \), and actually \( v_\sigma'(\zeta_0) < 0 \). Thus the desired behaviour for times \( \zeta \approx \zeta_0(\sigma)^+ \) and \( \sigma > \sigma_0 \) close. \( \square \)

5.3 Analyticity

For \( \sigma > 0 \) equation (5.3) has a left branch of solutions \( v_\sigma(.) \) which are \( C^1 \) at \( \zeta = 0 \) and normalized as \( v_\sigma(0) = 1 \) (see Proposition 5.1). According to Remark 5.1 there also exists for any \( \sigma \) a right branch of stable solutions \( v_\sigma(\sigma, +\infty) = 0 \), which we normalize as \( v_\sigma(\sigma, \zeta_0) = 1 \) for some \( \zeta_0 > 0 \). In this section we show that these two branches are analytical in \( \sigma \), which we will need later for technical reasons.

Proposition 5.8. For any fixed \( \zeta_0 > 0 \) there exists \( R(\zeta_0) > 0 \) such that \( \sigma \mapsto v_\sigma(.) \) is analytical into \( C^1([0, \zeta_0]) \) with radius of convergence at least \( R(\zeta_0) \).

Proof. If no singularity occurred at \( \zeta = 0 \) in (5.3) this would be a classical consequence of the linear dependence of the equation on \( \sigma \). We construct below a solution \( v \in C^1([0, \zeta_0]) \) of (5.3) which is analytical in \( \sigma \) and such that \( v(0) = 1 \). By uniqueness in Proposition 5.1 this solution will agree with \( v_\sigma(.) \).

For \( \sigma \approx \sigma_0 \) let us enlighten \( \sigma - \sigma_0 \) and rewrite (5.3) in the form

\[
L v := -\zeta v'' - \frac{v'}{m-2} + (b - \sigma_0)v = (\sigma - \sigma_0)v. \tag{5.6}
\]

Looking for solutions in power series \( v(\zeta) = \sum_{n \geq 0}(\sigma - \sigma_0)^nv_n(\zeta) \) leads to

\[
\begin{align*}
  n = 0 & : L v_0 = 0 \\
  n \geq 1 & : L v_n = v_{n-1}.
\end{align*}
\]

We naturally set \( v_0 := v_{\sigma_0} \) to be the principal eigenfunction in Theorem 2 and also require \( v_n \) to be \( C^1 \) at \( \zeta = 0 \) and satisfy \( v_n(0) = 0 \) for \( n \geq 1 \) (so that \( v(0) = 1 \) in the end). For any \( g \in C([0, \zeta_0]) \) it is easy to check that Duhamel formula

\[
f(\zeta) = -v_0(\zeta) \int_0^\zeta \frac{1}{v_0^2(s)s^{\frac{m-2}{m}}} \left( \int_0^s v_0(t)t^{\frac{1}{m-2}-1}g(t)dt \right)ds.
\]

yields the unique solution \( Lf = g \) satisfying the above boundary conditions, and that \( ||f||_c \leq C||g||_\infty \) for some \( C(\zeta_0) > 0 \). Equation \( L v_n = v_{n-1} \) is
therefore uniquely solvable as \( v_n = L^{-1}v_{n-1} \) and an induction argument immediately yields \( ||v_n||_{C^1} \leq C^m||v_0||_{\infty} \), thus convergence in \( C \) with radius at least \( R(\zeta_0) := \frac{1}{C^m(\zeta_0)} \).

We have a similar result for the stable right branch:

**Proposition 5.9.** For any fixed \( \zeta_0 > 0 \) there exists \( R(\zeta_0) > 0 \) such that \( \sigma \mapsto v_\sigma(\tau,.) \) is analytical into \( C^b([\zeta_0, +\infty[) \) with radius of convergence at least \( R(\zeta_0) \).

**Proof.** The argument is almost identical, the only difference being that we want now \((v_n)_{n\geq 1}\) to decay at infinity (whereas we had to handle the singularity at \( \zeta = 0 \) for the left branch). The corresponding Duhamel Formula for \( Lf = g \) reads now

\[
f(\zeta) = v_0(\zeta) \int_{\zeta \zeta_0}^{\zeta} \frac{1}{v_0^2(s) s^{\frac{1}{m-2}}} \left( \int_s^{+\infty} v_0(t) t^{\frac{1}{m-2}-1} g(t) dt \right) ds,
\]

and the rest of the proof is straightforward. \( \square \)

### 5.4 Functional setting multiplicities

Let us remind at this point equation (5.3)

\[
\zeta > 0 : \quad -\zeta \frac{d^2 u}{d\zeta^2} - \frac{u'}{m-2} + bu = \sigma u,
\]

which is an eigenvalue problem for \( L := -\zeta \frac{d^2}{d\zeta^2} - \frac{1}{m-2} \frac{d}{d\zeta} + b(\zeta) \). Boundary conditions are (i) \( C^1 \) regularity at \( \zeta = 0 \) and (ii) decay \( v(+\infty) = 0 \). So far we did not define any clear functional setting, and this is the purpose of this section.

Since \( v(+\infty) = 0 \) the largest possible space on which \( L \) could act is clearly a subspace of

\[
E := \left\{ f \in C([0, +\infty[), \quad f(+\infty) = 0 \right\},
\]

and our functional setting should also take into account the singularity \( \zeta = 0 \).

**Proposition 5.10.** For any \( f \in E \) there exists a unique solution \( v \in E \) of \( Lv = f \) having \( C^1 \) regularity at \( \zeta = 0 \). This solution can be computed as

\[
v(\zeta) = L^{-1}[f](\zeta) := \bar{v}(\zeta) \int_{\zeta}^{+\infty} \frac{1}{\bar{v}^2(s) s^{\frac{1}{m-2}}} \left( \int_0^s \bar{v}(t) t^{\frac{1}{m-2}-1} f(t) dt \right) ds,
\]

where \( \bar{v}(\zeta) > 0 \) is the solution in Proposition 5.1 for \( \sigma = 0 \). We have moreover \( v'(0) = -(m-2)f(0) \).
Proof. The $C^1$ regularity at $\zeta = 0$ is easy to retrieve, and one readily checks that $L^{-1}$ maps $E$ to $E$ (or in other words that $v$ decays at infinity if $f$ does).

This very naturally defines the domain

$$D(L) := L^{-1}(E) \subset E,$$

where $L^{-1}$ is given by Proposition 5.10. Any $u \in D(L)$ therefore satisfies both boundary conditions, and let us point out that if $f = L^{-1}g$ with $g \in E$ then $f$ is of course a classical $C^2([0, +\infty[)$ solution of the (singular) ODE.

**Proposition 5.11.** $\sigma_0$ is an eigenvalue of $L : D(L) \subset E \rightarrow E$ with geometric multiplicity $m_g(\sigma_0) = 1$.

This is of course consistent with the one-dimensionality in Theorem 1.

Proof. For $\sigma = \sigma_0$ the principal eigenfunction $v_0 = v_{\sigma_0}$ constructed in Theorem 2 solves the ODE, has the required regularity and decays at infinity, thus belongs to $D(L)$ and is indeed a proper eigenvalue. Regarding it’s geometric multiplicity we recall from Remark 5.1 that for any $\sigma$ there always exists a solution of the ODE which blows (exponentially) at infinity and therefore cannot belong to $E$.

The algebraic multiplicity will also be important in Section 6:

**Proposition 5.12.** $\sigma_0$ has finite algebraic multiplicity $m_a(\sigma_0) = 1$.

Proof. If $v_0$ denotes again our principal eigenfunction and $L_0 := L - \sigma_0$ we need to show that $\text{Ker}(L_0^2) = \text{Ker}(L_0)$ is generated by $v_0$; in other words, that there is no solution $v \in D(L)$ of $Lv = v_0 \iff -\zeta v'' - \frac{\nu}{m-2} + (b - \sigma_0)v = v_0$. Assuming by contradiction that $v$ is such a solution, Variation of Constants $v(\zeta) = \alpha(\zeta)v_0(\zeta)$ and boundary conditions lead to

$$\alpha'(\zeta) = -\frac{1}{\zeta \frac{1}{m-2} v_0^2(\zeta)} \int_0^\zeta \frac{1}{s \frac{1}{m-2} v_0^2(s)} ds.$$ 

Decay $v_0(+\infty) = 0$ (at least exponentially) implies that the integral above is absolutely convergent at infinity, hence that $\alpha'(\zeta) \sim -\frac{C}{v_0^2(\zeta) \frac{1}{m-2} \zeta} \rightarrow -\infty$ with $C = \int_0^{+\infty} s \frac{1}{m-2} v_0^2(s) ds > 0$ (let us also recall that $v_0 > 0$). Using the equation for $v_0$ it is finally easy to see that $\alpha'$ blows so fast at infinity that $v = \alpha v_0$ does too, thus contradicting $v(+\infty) = 0$ and $v \in D(L)$.

**Proposition 5.13.** $\sigma = \sigma_0 \in \mathbb{R}$ is an isolated eigenvalue in $\mathbb{C}$.
Proof. Rewriting \(Lv = \sigma v \iff -\zeta v'' - \frac{1}{m-2}v' + b(\zeta)v = \sigma v\) in the divergence form

\[
-\left(\zeta^{\frac{1}{m-2}}v'\right)' + \frac{b(\zeta)}{\zeta^{1-\frac{1}{m-2}}}v = \frac{\sigma}{\zeta^{1-\frac{1}{m-2}}}v,
\]

we see that the problem is self-adjoint. Any eigenvalue must consequently be real and associated with real eigenfunctions, and it is therefore enough to prove that \(\sigma_0\) is isolated in \(\mathbb{R}\). Let us remind that we are looking for eigenfunctions \(v \in D(L)\), hence \(C^1\) at \(\zeta = 0\). As a result any eigenfunction \(v_\sigma\) must agree with the unique solution in Proposition 5.1, which according to Proposition 5.6 cannot decay at infinity if \(\sigma\) is close to \(\sigma_0\) (and thus \(v_\sigma \notin D(L)\)). \(\square\)

6 Asymptotic matching

In this section we match the physical setting \(\varepsilon > 0, k < +\infty\) with the asymptotic problem formally obtained in Section 5 for \(\varepsilon = 0\) and \(k = +\infty\).

For \(s\) of order \(k^{1-\frac{1}{m-1}}\) we constructed in Section 4 the solution of (2.7a) in \(\xi = x/\varepsilon\) coordinates with maximal decay at \(-\infty\), normalized as \(v(\xi_\varepsilon) = 1\). This was done for \(\varepsilon > 0\) and \(k < +\infty\) (small and large enough) so we may indistinctly switch from \(\xi\) to \(x\) coordinates. This left solution extends to \(x > x_\varepsilon\) and will be denoted below by \(u_l(x) = u_l(\varepsilon, k, s, x)\).

On the right side \(x = +\infty\) we have \(p = 1\), \(p' = p'' = 0\) and the characteristic equation associated with (2.7a) reads \(-r^2 + c_\varepsilon r + (k^2 - s - G'(1)) = 0\), yielding two exponential rates \(r^\pm\). In the regime (2.9) with \(s\) of order \(k^{1-\frac{1}{m-1}} \ll k^2\) (and \(c_\varepsilon \rightarrow c_0 > 0\)) these are

\[
r^\pm \sim \varepsilon k,
\]

and there always exists a unique branch of stable solutions \(u(+\infty) = 0\) which we will denote by \(u_r(x) = u_r(\varepsilon, k, s, x)\).

Instead of parameters \(k \rightarrow +\infty\), \(s = \mathcal{O}\left(k^{1-\frac{1}{m-1}}\right) \rightarrow +\infty\) and \(x\) coordinates we will rather use

\[
\delta = \frac{1}{k^{1-\frac{1}{m-1}}} \rightarrow 0, \quad \sigma = s \frac{\delta}{(m-2)c_\varepsilon} = \mathcal{O}(1), \quad \zeta = x/\delta,
\]

and (5.2) reads

\[
\begin{cases}
Lv := -qv'' + \left(c - 2\frac{q'}{m-2}\right)v' + \left((q - \frac{m}{m-2} - (m-2)c\sigma - q'' - \delta G''(\delta q))\right)v = 0, \\
c = c_\varepsilon > 0, \quad q = q_{\varepsilon, \delta}(\zeta) = \frac{1}{\delta}p_\varepsilon(\delta \zeta).
\end{cases}
\]

Expressed in terms of these new parameters \((\varepsilon, \delta)\) regime (2.9) becomes

\[
\omega := \left\{\varepsilon, \delta > 0 : 0 < \varepsilon^{-\eta \frac{m-2}{m-1} + \frac{1}{m-1}} < \delta < \delta_0\right\},
\]

(6.1)
where $\eta > 0$ is again exactly the one in Theorem 1. In the sequel we shall omit the subscripts $(\epsilon, \delta)$ for convenience, but one should keep in mind that we are concerned here with $\epsilon, \delta > 0$ in the double limit $(\epsilon, \delta) \stackrel{\omega}{\to} (0, 0)$.

Throughout this entire section we will write $v_l(\epsilon, \delta, \sigma, \zeta)$ for the left branch coming from the cold zone and $v_r(\epsilon, \delta, \sigma, \zeta)$ for the stable right branch normalized as $v_r(\zeta_0) = 1$ (for some $\zeta_0 > 0$ to be chosen later). We will also denote by $v_{00}(\sigma, \zeta)$ the regular solution of the formal asymptotic problem (5.3) (defined in Proposition 5.1), and $v_{r0}(\sigma, \zeta)$ the stable right branch of the asymptotic problem normalized as $v_{r0}(\sigma, \zeta_0) = 1$ (see Remark 5.1). We will prove that

\[
\begin{align*}
\frac{\partial v_l(\epsilon, \delta, \sigma, \cdot)}{\partial \sigma}(\epsilon, \delta, \sigma, \cdot) &\to v_{00}(\sigma, \cdot) \quad \text{and} \quad \frac{\partial v_r(\epsilon, \delta, \sigma, \cdot)}{\partial \sigma}(\epsilon, \delta, \sigma, \cdot) \to \frac{\partial v_{r0}(\sigma, \cdot)}{\partial \sigma}(\cdot)
\end{align*}
\]

when $(\epsilon, \delta) \to (0, 0)$ in $\omega$. This should be no surprise since the asymptotic problem was precisely obtained in the (formal) limit $(\epsilon, k) = (0, +\infty)$ or equivalently $(\epsilon, \delta) = (0, 0)$.

Our first step is to establish convergence of the scaled wave profile $q_{\epsilon, \delta} \to q_0$ in the linear zone $\zeta \in [\zeta_\epsilon, \zeta_\delta]$, where $\zeta_\epsilon = x_\epsilon/\delta$ corresponds to the exit $x = x_\epsilon = \epsilon^{1-\alpha}$ of the cold zone and $\zeta_\delta = x_\delta/\delta$ to the exit of the linear zone $x = x_\delta$ defined in (3.4). One can check that

\[
\zeta_\epsilon \to 0 \quad \text{and} \quad \zeta_\delta \to +\infty
\]

when $(\epsilon, \delta) \to (0, 0)$, and the linear zone stretches to $]0, +\infty[$ in this scaling.

**Proposition 6.1.** In the double limit (6.2) we have $C^2$ convergence $\| q_{\epsilon, \delta}(\zeta) - (m-2)c_0\zeta ||_{C^2([\zeta_\epsilon, \zeta_\delta])} \to 0$.

The proof is technical but straightforward when taking advantage of Proposition 3.1 and asymptotes (3.6).

## 6.1 Regularity of the branches

We show here that the branches $v_{l, r}(\epsilon, \delta, \sigma, \cdot)$ can be extended to $(\epsilon, \delta) = (0, 0)$ by the branches $v_{l, r0}(\sigma, \cdot)$ of asymptotic problem (5.3), which was formally obtained for $\epsilon = 0, k = +\infty$ in Section 4.

**Proposition 6.2.** Fix $\zeta_0 > 0$ and $\sigma_*$: then $v_l(\epsilon, \delta, \sigma, \cdot) \to v_{l0}(\sigma, \cdot)$ in $C^1([\zeta_\epsilon, \zeta_0])$ when $(\epsilon, \delta, \sigma) \to (0, 0, \sigma_*)$ in the double limit (6.2).

Note how we handle simultaneously the double limit $(\epsilon, \delta) \to (0, 0)$ and continuity with respect to $\sigma$.

**Proof.** For $\epsilon, \delta > 0$ the left branch $v_l = v_l(\epsilon, \delta, \sigma, \zeta)$ solves (6.1):

\[
Lv_l = 0, \quad L = -q \frac{d^2}{d\zeta^2} + \left( c - \frac{2q'}{m-2} \right) \frac{d}{d\zeta} + \left( \frac{q_m}{m-2}c_\sigma - q'' - \delta G'(\delta q) \right).
\]
while the asymptotic left branch $v_{l0} = v_{l0}(\sigma_*, \zeta)$ solves \(5.3\):

$$L_0v_{l0} = 0, \quad L_0 = -(m-2)c_0\zeta \frac{d^2}{d\zeta^2} - (m-2)c_0 \frac{1}{m-2} \frac{d}{d\zeta} + (m-2)c_0(b(\zeta) - \sigma_*) .$$

Proposition \[6.1\] guarantees that the coefficients of $L$ are very close to those of $L_0$ when \((\varepsilon, \delta, \sigma) \to (0, 0, \sigma_*)\). Boundary conditions \[4.14\] moreover read (in $\zeta$ coordinates) $v_0(\zeta) = 1$ and $v'_0(\zeta) \sim -(m-2)\sigma$, with $\zeta = 0$. According to Proposition \[5.1\] the asymptotic left branch satisfies $v_{l0}(0) = 1, v'_{l0}(0) = -(m-2)\sigma$, so that initial values are also very close. These two branches solve two very similar Cauchy problems, and should therefore stay very close on the interval $[\zeta_0, \zeta_0]$. The difficulty is here that the asymptotic Cauchy problem degenerates at $\zeta = 0$ when $\varepsilon, \delta \to 0$, thus preventing from applying classical regularity of Cauchy solutions. In order to keep this paper in a reasonable length we only sketch the argument below.

The proof relies on a stability argument for $z := v_l - v_{l0}$, which satisfies $z(\zeta) = o(1), z'(\zeta) = o(1)$ and $Lz = (L_0 - L)v_{l0} = o(1).$ Using the very structure of the equation one shows that $z$ stays small in $C^1$ norm for times $\zeta \in [\zeta_0, \zeta^*]$, where $\zeta^* > 0$ stays bounded away from zero when $(\varepsilon, \delta, \sigma) \to (0, 0, \sigma_*)$. This allows first to step away from the singularity $\zeta = 0$ and then to safely apply classical continuity arguments for Cauchy solutions with respect to parameters.

We have a similar result for the $\sigma$-derivative:

**Proposition 6.3.** Let $z_l = \frac{\partial v_l}{\partial \sigma}$ and $z_{l0} = \frac{\partial v_{l0}}{\partial \sigma}$. For fixed $\zeta_0 > 0$ and $\sigma_*$ we have $z_l(\varepsilon, \delta, \sigma, .) \to z_{l0}(\sigma_*, .)$ in $C^1([\zeta_0, \zeta^*_0])$ when $(\varepsilon, \delta, \sigma) \to (0, 0, \sigma_*)$ in the double limit \[6.2\].

**Proof.** The proof is very similar to the previous one: differentiating $Lv_l = 0$ and $L_0v_{l0} = 0$ with respect to $\sigma$ we see that $z_l$ and $z_{l0}$ solve again two very close equations. Initial values are also very close since $\zeta \to 0, z_l(\zeta) = 0, z'_l(\zeta) = -(m-2) + o(1)$ by Proposition \[4.15\], and $v_{l0}(0) = 1, v'_{l0}(0) = -(m-2)\sigma$ implies $z_{l0}(0) = 0, z'_{l0}(0) = -(m-2)$. \[ Q.E.D. \]

The delicate issue for the right branches is to take into account the decay $v_r(+\infty) = 0$, which turns to be a slightly more difficult problem than the previous singularity at $\zeta = 0$ for the left branches. We first establish two decay estimates on $v_r, v'_r$ at infinity uniformly in $\varepsilon, \delta, \sigma$:

**Lemma 6.1.** There exist $\zeta_0 > 0$ and $C > 0$ locally independent of $\sigma$ such that, if $\varepsilon, \delta$ are small enough in the double limit \[6.2\], then $0 \leq v_r(\varepsilon, \delta, \sigma, \zeta) \leq Ce^{-\zeta}$ holds on $[\zeta_0, +\infty[.$

**Proof.** Let us recall that $v_r$ solves the elliptic equation $Lv_r = 0$, with

$$L := -q \frac{d^2}{d\zeta^2} + \left( c - \frac{2q'}{m-2} \right) \frac{d}{d\zeta} + \left( q^m - (m-2)c\sigma - q'' - \delta G'(\delta q) \right).$$
The key point is that the zero-th order coefficient
\[ a_0 := \frac{q}{m} - (m - 2)c\sigma - q'' - \delta G' (\delta q) \left( O(1) - \frac{q''}{q} - \delta G' (\delta q) \right) \] (6.3)
is positive on \([\zeta_0, +\infty[\) if \(\zeta_0\) is chosen large (remember that \(q(\zeta) \sim (m - 2)c_0\zeta\)).
The classical Minimum Principle \(L v_r = 0\) with \(v_r(\zeta_0) = 1\) and \(v_r(+\infty) = 0\) yields positivity, and remarking that \(v_r\) behaves at infinity as \(e^{-r\zeta}\) for some positive \(r \gg 1\) (see the asymptotic equation associated with \(L v_r = 0\) at infinity) it is easy to check that \(v = C e^{-\zeta}\) is a supersolution for \(v_r\) if \(C > 0\) is chosen large enough but independent of \(\varepsilon\) and \(\delta\).

We will also need a similar estimate for the first derivative:

**Lemma 6.2.** There exist \(\zeta_0 > 0\) and \(C > 0\) locally independent of \(\sigma\) such that, if \((\varepsilon, \delta)\) are small enough in the double limit (6.2), then \(|v'_r(\zeta)| \leq \frac{C}{\delta} e^{-\zeta}\) holds on \([\zeta_0, +\infty[\).

**Proof.** Fix any \(\zeta_0 > 0\) large enough as in Lemma 6.1: dividing (6.1) by \(q > 0\) we recast \(L v_r = 0\) as
\[ v''_r + av'_r = bv_r \quad \text{and} \quad \begin{cases} a := \frac{1}{q} \left( \frac{2q'}{m} - c \right), \\ b := \frac{1}{q} \left( q - (m - 2)c\sigma - q'' - \delta G' (\delta q) \right), \end{cases} \]
and obtain a corresponding integral formulation
\[ v'_r(\zeta) = - \int_{\zeta}^{+\infty} \exp[A(s) - A(\zeta)] b(s) v_r(s) ds \]
with \(A(s) := \int^s a\). Taking advantage of the explicit form of \(a(\zeta)\) combined with \(q(\zeta) \geq q(\zeta_0) \approx (m - 2)c_0\zeta_0\) and \(q(s) \leq q(\varepsilon) = 1\) it is easy to estimate
\[ \exp[A(s) - A(\zeta)] = \left( \frac{q(s)}{q(\zeta)} \right)^{\frac{2m}{m - 2}} \exp \left( - \int_{\zeta}^{s} \frac{c}{q} \right) \leq \frac{C}{\delta} \leq 1, \]
and one then easily shows that \(|b(s)| \leq 2q^{\frac{m}{m - 2}} \leq 2 \left( \frac{1}{\delta} \right)^{\frac{m}{m - 2}}\). Our statement finally follows from Lemma 6.1 above.

We have now for the right branch

**Proposition 6.4.** For any \(\sigma_*\) there exits \(\zeta_0 > 0\) large enough such that \(v_r(\varepsilon, \delta, \sigma, \cdot) \rightarrow v_{r_{0}}(\sigma_*, \cdot) \) in \(C^1([\zeta_0, +\infty[\)) when \((\varepsilon, \delta, \sigma) \rightarrow (0, 0, \sigma_*)\) in the double limit (6.2), \(\zeta_0\) can be chosen locally independent of \(\sigma_*\).
Proof. From $Lv_r = 0$ and $L_0v_{r_0} = 0$ we see that $w := v_r - v_{r_0}$ satisfies as before $Lw = (L_0 - L)v_{r_0} := f$ on $[\zeta_0, +\infty[$. Contrarily to the left branch the coefficients of $L$ are not close to those of $L_0$ on the whole interval $[\zeta_0, +\infty[$, but only on $[\zeta_0, \zeta_\delta]$ (see Proposition 6.1). From (5.3) it is however easy to see that $v_{r_0}, v_{r_0}', v_{r_0}''$ decay at least exponentially at infinity, and locally uniformly in $\sigma_*$. According to Lemma 6.2 we have that $v_r, v_r'$ also decay exponentially, and since $\zeta_\delta = \frac{\sigma_0}{\delta} \rightarrow +\infty$ it is clearly enough to prove our statement on the interval $I = [\zeta_0, \zeta_\delta]$.

Remark that (i) the zero-th order coefficient of $L$ can be made larger than 1, see (6.3), (ii) decay of $v_{r_0}$ implies that $Lw = f$ is small in $L^\infty$ norm on $I$ and (iii) $w$ vanishes on the left boundary and is exponentially small on the right one, see Lemma 6.1. Using $\overline{w} := |w|(|\zeta_\delta| + ||f||_\infty = cst$ as a supersolution $L\overline{w} \geq f$ we obtain $|w| \leq \overline{w}$ on $I = [\zeta_0, \zeta_\delta]$ and $\overline{w} = o(1)$ by construction. According to Lemma 6.2 we have that $|v_r'| \leq \frac{C}{\delta} e^{-\zeta_\delta} = o(1)$ on $[\zeta_\delta, +\infty[$, and $v_{r_0}'$ also decays exponentially: thus $|w'| = o(1)$ uniformly on $[\zeta_\delta, +\infty[$. In order to retrieve the same estimate on $[\zeta_0, \zeta_\delta]$ we use the very structure of the equation, integrating backward in time from $\zeta_\delta$ when $w, w'$ are small. \hfill \square

A similar result holds for the $\sigma$ derivative:

**Proposition 6.5.** Let $z_r = \frac{\partial v_r}{\partial \sigma}$ and $z_{r_0} = \frac{\partial v_{r_0}}{\partial \sigma}$: for any fixed $\sigma_*$ there exists $\zeta_0 > 0$ such that $z_r(\varepsilon, \delta, \sigma_\ast) \rightarrow z_{r_0}(\sigma_\ast,.)$ in $C^1([\zeta_0, +\infty[)$ when $(\varepsilon, \delta, \sigma) \rightarrow (0, 0, \sigma_\ast)$ in the double limit (6.2); $\zeta_0$ can be chosen locally independent of $\sigma_*$. \hfill \square

**Proof.** The proof of Proposition 6.4 is easily adapted, the first step being to prove two decay estimates $|z_r(\varepsilon, \delta, \sigma)| \leq C e^{-\zeta}$ and $|z_r'(\varepsilon, \delta, \sigma)| \leq \frac{C}{\delta} e^{-\zeta}$ on $z = \frac{\partial \varepsilon}{\partial \sigma}$ for some fixed exponent $\alpha = \alpha(m) > 0$ (similar to Lemmas 6.1 and 6.2 above).

### 6.2 Evans function and construction of the eigenvalue

In this section we construct the principal eigenfunction $\sigma = \sigma(\varepsilon, \delta)$ connecting maximal decay $v_l(-\infty) = 0$ and $v_r(+\infty) = 0$ in the regime (6.2). Rewriting (6.1) as a first order system

$$\frac{dY}{d\zeta} = A(\varepsilon, \delta, \sigma; \zeta) Y, \quad Y = \begin{pmatrix} v \\ v' \end{pmatrix},$$

we denote by $Y_l(\varepsilon, \delta, \sigma, .)$ and $Y_r(\varepsilon, \delta, \sigma, .)$ the previous left and right branches and define the Evans function

$$E(\varepsilon, \delta, \sigma) := \det \left[ Y_l(\varepsilon, \delta, \sigma, \zeta_0), Y_r(\varepsilon, \delta, \sigma, \zeta_0) \right]$$

(6.4)

for some $\zeta_0 > 0$ to be chosen later. It is clear that the connection holds if and only if $E(\varepsilon, \delta, \sigma) = 0$, thus reducing the problem to finding the zeros $(\varepsilon, \delta, \sigma)$ of $E$. If $\omega$ is defined as in (6.2) let us introduce

$$\hat{\omega} := \omega \cup \{(0, 0)\}, \quad \Omega := \omega \times [\sigma_0 - A, \sigma_0 + A[, \quad \hat{\Omega} := \hat{\omega} \times [\sigma_0 - A, \sigma_0 + A]$$

(6.5)
for some $A > 0$. For $(\varepsilon, \delta, \sigma) \in \Omega$ (hence $\varepsilon, \delta > 0$) there are no singularities in the equations: the two branches $Y_1, Y_r$ are well defined and regular in $(\varepsilon, \delta, \sigma)$, and the Evans function $E(\varepsilon, \delta, \sigma)$ is therefore well defined and $C^1$ on $\Omega$. The set $\hat{\Omega}$ contains critical points of the form $(0, 0, \sigma_*)$, corresponding of course to the asymptotic problem $\varepsilon = 0, k = +\infty$.

**Proposition 6.6.** Fix $\zeta_0 > 0$ large enough: the Evans function extends to a continuous function on $\hat{\Omega}$, again denoted by $E(\varepsilon, \delta, \sigma)$. This extension is continuously differentiable with respect to $\sigma$ in $\hat{\Omega}$.

**Proof.** Since we consider bounded $\sigma \in ]\sigma_0 - A, \sigma_0 + A[$ Propositions 6.2 and 6.4 hold for some $\zeta_0 > 0$ large enough (but independent of $\varepsilon, \delta, \sigma$). The branches $Y_1, Y_r$ thus continuously extend to $Y_{10}, Y_{r0}$ when $(\varepsilon, \delta, \sigma) \rightarrow (0, 0, \sigma_*)$, and this continuously extends $E$ to $\hat{\Omega}$ setting

$$E(0, 0, \sigma) := \det (Y_{10}(\sigma, \zeta_0), Y_{r0}(\sigma, \zeta_0)).$$

By Propositions 5.8 and 5.9 the mapping $\sigma \mapsto E(0, 0, \sigma)$ is analytical hence differentiable in $\sigma$, and Propositions 6.3 and 6.5 finally imply continuity in all three arguments at any asymptotic point $(0, 0, \sigma_*)$.

According to Theorem 2 and by definition of the extension $E(0, 0, \cdot)$ we have of course

$$E(0, 0, \sigma_0) = 0.$$

In order to apply an Implicit Functions Theorem we need to check the usual condition

**Proposition 6.7.** $E$ satisfies $\frac{\partial E}{\partial \sigma}(0, 0, \sigma_0) \neq 0$.

**Proof.** The argument is directly inspired from [1], Lemma 6.2 p.194.

For $\varepsilon = \delta = 0$ the first order system corresponding to (5.3) is

$$\frac{dY}{d\zeta} = A(\sigma, \zeta)Y, \quad Y = \begin{pmatrix} v \\ v' \end{pmatrix}, \quad A(\sigma, \zeta) = \begin{pmatrix} 0 & -1 \\ b-\sigma & -\frac{1}{\zeta} \end{pmatrix},$$

and the Wronskian $W(\sigma, \zeta) = \det (Y_{10}(\sigma, \zeta), Y_{r0}(\sigma, \zeta))$ satisfies

$$\zeta^{\frac{1}{m-2}} W(\sigma, \zeta) = \text{cst} = \zeta_0^{\frac{1}{m-2}} W(\sigma, \zeta_0) = \zeta_0^{\frac{1}{m-2}} E(0, 0, \sigma). \quad (6.6)$$

In the light of Section 5.4 we consider $L = -\zeta \frac{d^2}{d\zeta^2} - \frac{1}{m-2} \frac{d}{d\zeta} + b(\zeta)$ as an unbounded operator $L : D(L) \subset E \rightarrow E$ with (principal) eigenvalue $\sigma_0$. For $\sigma \neq \sigma_0$ and $f \in E$ computing the resolvent $R(\sigma; L)f$ amounts to solving $-\zeta v'' - \frac{v'}{m-2} + (b-\sigma)v = f$ with boundary conditions $v \in D(L)$ corresponding to $C^1$ regularity at $\zeta = 0$ and decay $v(+\infty) = 0$. Variation of Constants and (6.6) yield the unique solution of $(L - \sigma)v = f$ in the form

$$v = \frac{1}{\zeta^{\frac{1}{m-2}} W(\zeta)}(\alpha v_{10} + \beta v_{r0}), \quad (6.7)$$

27
with
\[ \Phi(\zeta) := \frac{1}{\zeta^{m-2}} \int_0^\zeta t^{1-m-2} f(t) dt \] and
\[
\begin{aligned}
\alpha[f](\zeta) &:= \int_\zeta^{+\infty} t^{\frac{1}{m-2}} \Phi(t) v'_0(t) dt, \\
\beta[f](\zeta) &:= \int_0^\zeta t^{\frac{1}{m-2}} \Phi(t) v'_0(t) dt.
\end{aligned}
\]

According to (6.6) and (6.7) the resolvent can therefore be computed as
\[
\forall f \in E, \quad R(\sigma; L)f = v = \frac{1}{E(0,0,\sigma)^{1/m-2}} M(\sigma)f, \quad (6.8)
\]
where \( M(\sigma)f = (\alpha v_0 + \beta v_0) \) and \( \alpha[f], \beta[f] \) are defined above. By Proposition 5.8 and 5.9 the mappings \( \sigma \mapsto v_0(\sigma,.) \) and \( \sigma \mapsto v_0(\sigma,.) \) are analytical, and the family of operators \( M(\sigma) \) is therefore analytical. We constructed \( \sigma_0 \) so that the principal eigenfunction \( v_0 = v_0(\sigma_0,.) = v_0(\sigma_0,.) \in D(L) \) (up to some proportionality factors): it is then the easy to compute \( M(\sigma_0)v_0 \neq 0 \), and \( M(\sigma_0) \) is consequently non-trivial.

By Proposition 5.13 the principal eigenvalue is isolated in \( \mathbb{C} \), and the resolvent \( R(\sigma; L) \) is thus classically meromorphic in some neighbourhood of \( \sigma_0 \) in \( \mathbb{C} \). Since \( E(0,0,\sigma) \) and \( M(\sigma) \) are analytical and \( M(\sigma_0) \neq 0 \) (6.8) shows that the order of the pole \( \sigma = \sigma_0 \) in \( R(\sigma; L) \) equals the order of the zero \( E(0,0,\sigma_0) \): by classical functional calculus arguments the order of the pole in the resolvent is exactly the algebraic multiplicity (see e.g. [12]), which is \( m_a(\sigma_0) = 1 \) by Proposition 5.12.

We may consequently apply the following Implicit Functions Theorem:

**Theorem 3.** There exist \( B \in [0, A[ \), a neighbourhood \( \omega_1 = \{|\varepsilon, \delta| < r_0\} \cap \omega \) of \( \varepsilon = \delta = 0 \) in \( \omega \), and a function \( \sigma(\varepsilon, \delta) \) defined on \( \omega_1 \) such that such that
\[
1. \quad \{ (\varepsilon, \delta, \sigma) \in \omega_1 \times [\sigma_0 - B, \sigma_0 + B] \mid E(\varepsilon, \delta, \sigma) = 0 \} \quad \Leftrightarrow \quad \{ (\varepsilon, \delta) \in \omega_1 \mid \sigma = \sigma(\varepsilon, \delta) \}
\]
\[
2. \quad \sigma \text{ is } C^1 \text{ on } \omega_1 \text{ and } \sigma(\varepsilon, \delta) \to \sigma_0 \text{ when } (\varepsilon, \delta) \to (0,0) \text{ in } \omega_1.
\]

Note that due to the algebraic form (6.2) of \( \omega \) the vicinity condition \( (\varepsilon, \delta) \in \omega_1 \) can be easily relaxed to \( "(\varepsilon, \delta) \in \omega \text{ and } \delta < \delta_0 \text{ small enough}" \), that is to say \( "(\varepsilon, k) \in \mathcal{U} \text{ and } k > k_0 \text{ large enough}" \) as stated in Theorem 1.

**Proof.** This is a non-standard version of the Implicit Functions Theorem since \( E(\varepsilon, \delta, \sigma) \) is not \( C^1 \) in some open set, which \( \Omega \) is not. The argument is based on the usual iterative fixed point strongly relying on Proposition 6.7, but will not be detailed here. \( \square \)
6.3 Proof of Theorem\[1\]

We just defined the function $\sigma(\varepsilon, \delta)$ on $\omega$ so that $E(\varepsilon, \delta, \sigma(\varepsilon, \delta)) = 0$, or in other words that the connection $v(\pm \infty) = 0$ holds in (6.1) with maximal decay to the left if $\sigma = \sigma(\varepsilon, \delta)$. We denote below by $v_{\varepsilon, \delta}(\zeta)$ the corresponding eigenfunction and by $w_{\varepsilon, k}(x)$ its alter-ego in $x$ coordinates, which is a solution of (2.2a) for $\varepsilon > 0$ and $k < \infty$. Our previous scaling $\sigma = \frac{s}{(m-2)c_0k^{1 - \frac{1}{m-1}}}$ and $\delta = \frac{1}{k^{1 - \frac{1}{m-1}}}$ corresponds now to

$$s(\varepsilon, k) := (m - 2)c_0\sigma \left(\varepsilon, \frac{1}{k^{1 - \frac{1}{m-1}}}\right) k^{1 - \frac{1}{m-1}},$$

which is of course the principal eigenvalue in Theorem [1]. This immediately yields the desired asymptotic relation $s \sim \gamma_0 k^{1 - \frac{1}{m-1}}$ with $\gamma_0 := (m - 2)c_0\sigma_0$.

For $\sigma = \sigma(\varepsilon, \delta) \sim \sigma_0 > 0$ the asymptotic equation of (6.1) at $+\infty$ leads to an unstable characteristic exponent $r_+ > 0$ and the stable subspace $v(+\infty) = 0$ has therefore dimension 1. This shows that the eigenspace for $s = s(\varepsilon, k)$ is 1-dimensional as stated in Theorem [1]. We may have equally argued that the space of maximal decay solutions at $-\infty$ is also 1-dimensional for $s = s(\varepsilon, k) \sim \gamma_0 k^{1 - \frac{1}{m-1}}$, see Section 4 and in particular the discussion about $B_w$ spaces page [11]. We prove below the claimed positivity $u_{\varepsilon, k}(x) > 0 \iff v_{\varepsilon, \delta}(\zeta) > 0$ separately on three intervals: in $\zeta$ coordinates on $[\zeta, \zeta_0]$ (for some $\zeta_0$ to be chosen later), again in $\zeta$ coordinates on $[\zeta_0, +\infty[$, and finally in $x$ coordinates on $]-\infty, x_\varepsilon[.$

Since $\sigma(\varepsilon, \delta) \sim \sigma_0$ Proposition 6.2 guarantees that $v_{\varepsilon, \delta}(\zeta) = v_0(\varepsilon, \delta, \sigma_0, \zeta)$ is close to the asymptotic principal eigenfunction $v_0(\sigma_0, \zeta)$ uniformly on $[\zeta, \zeta_0]$ for fixed $\zeta_0 > 0$ when $(\varepsilon, \delta) \to (0, 0)$ in the double limit. By construction we had $v_0(\sigma_0, \zeta) > 0$ on $\mathbb{R}^+$, thus $v_{\varepsilon, \delta}(\zeta) > 0$ on $[\zeta, \zeta_0].$

For $\sigma = \sigma(\varepsilon, \delta)$ equation (6.1) is of the form $L[v_{\varepsilon, \delta}] = 0$, where $L$ is uniformly elliptic on $[\zeta_0, +\infty[$ if $\zeta_0 > 0$. As already discussed the zero-th order coefficient $a_0(\zeta)$ given by (6.3) can moreover be made positive on $[\zeta_0, +\infty[\text{ if } \zeta_0 > 0$ is large enough. We conclude applying the classical Minimum Principle with $v_{\varepsilon, \delta}(\zeta_0) > 0$ and $v_{\varepsilon, \delta}(+\infty) = 0.$

Here the classical Minimum Principle is unfortunately not available at once since the zero-th order coefficient in $Lv = 0$ is not positive, see again (6.3) with $q(-\infty) = \varepsilon/\delta \ll 1$. In order to keep our notations light we omit below the subscripts and one should understand in the following $v(\zeta) = v_{\varepsilon, \delta}(\zeta), u(x) = u_{\varepsilon, k}(x), p = p_\varepsilon(x), c = c_\varepsilon$ and $s = s(\varepsilon, k)$. The asymptotic expansion in the cold zone $v = v_0 + \varepsilon v_1 + \varepsilon v_2$ reads in $x$ coordinates on $]-\infty, x_\varepsilon[.$

$$u = u_0 + \varepsilon u_1 + \varepsilon u_2, \quad u_0 = p'/p'(x_\varepsilon) > 0,$$
and the leading order $u_0$ is positive. According to (4.9) the next orders $\varepsilon u_1, \varepsilon u_2$ are small on $]-\infty, x_{\varepsilon}[$,

\[ s\varepsilon^{1-a} = o(1), \quad \left\{ \begin{array}{l} \|\varepsilon u_1\|_{L^\infty} \leq C\|\varepsilon u_1\|_{B^+_{\varepsilon,0}} \leq C_1 s\varepsilon^{1-a} \\
\|\varepsilon u_2\|_{L^\infty} \leq C\|\varepsilon u_2\|_{B^+_{\varepsilon,0}} \leq C_2 (s\varepsilon^{1-a})^2 \end{array} \right., \quad (6.9) \]

and therefore positivity of $u_0$ should suffice to ensure positivity of the whole expansion at least for times $x \lesssim x_{\varepsilon}$. The asymptotic study of (2.7a) at $-\infty$ however shows that

\[
u \sim e^{r^+x}, \quad 0 < r^+ = \frac{c_\varepsilon + \sqrt{c_\varepsilon^2 + 4\varepsilon (k^2 \varepsilon^{-2} - s)}}{2\varepsilon} < \frac{c_\varepsilon}{\varepsilon},
\]

whereas the planar wave decays as $p'(x) \propto e^{r^-x} \ll e^{r^+x}$. As a consequence the lower order terms $\varepsilon u_1 + \varepsilon u_2$ dominate $u_0$ when $x \to -\infty$ and positivity of $u_0 > 0$ is therefore not enough. We prove below that $\varepsilon u_1 + \varepsilon u_2$ stays negligible on some interval $[x_1, x_{\varepsilon}]$, and use then a suitable comparison principle on $]-\infty, x_{1}$.\n
1. Fix $A > C_1$ with $C_1 > 0$ as in (6.9): monotonicity $u_0' = p''/p'(x_{\varepsilon}) > 0$ and $s\varepsilon^{1-a} \ll 1$ define a unique time $x_1 \in ]-\infty, x_{\varepsilon}[$ such that

\[ u_0(x_1) = A\varepsilon^{1-a}, \quad (6.10) \]

and estimate (6.9) immediately implies

\[ x \in [x_1, x_{\varepsilon}]: \quad u \geq \underbrace{u_0}_{\geq A\varepsilon^{1-a}} - \|\varepsilon u_1\|_{L^\infty} - \|\varepsilon u_2\|_{L^\infty} > 0. \]

2. Let $r_0 := \frac{c_0}{\varepsilon} > 0$ and $\Phi(x) := e^{r_0 x}$: using definition (6.10) of $x_1$ and asymptotes (3.3)-(3.5) we see that $p \sim \varepsilon$, $p' = o(1)$ and $p'' \leq Cs\varepsilon^{-a}$ hold on $]-\infty, x_1]$ and therefore

\[
L[\Phi] = \left[ -pr_0^2 + \left( c - \frac{2p'}{m-2} \right) r_0 + \left( k^2 \varepsilon^{-2} - s - p'' \right) \right] e^{r_0 x} \geq C \left[ -\varepsilon r_0^2 + c_0 r_0 - s \varepsilon^{-a} \right] e^{r_0 x} \geq C \left[ \frac{c_0^2}{4\varepsilon} - s \varepsilon^{-a} \right] e^{r_0 x} > 0 \quad (s\varepsilon^{1-a} \ll 1).\n\]

The function $w := u/\Phi$ then satisfies an elliptic equation $\tilde{L}w := Lu = 0$, where $\tilde{L}$ has therefor positive zero-th order coefficient $\tilde{a}_0 = L[\Phi] > 0$. Since $u \sim e^{r^+x}$ decays faster that $\Phi = e^{r_0 x}$ ($r^+ \approx c_0/\varepsilon > r_0$) we have $w(-\infty) = 0$, and also $w(x_1) = u(x_1)/\Phi(x_1) > 0$: we conclude applying the classical Minimum Principle on $]-\infty, x_{1}]$.\n
30
In order to fully establish Theorem 1 we only have left to prove that

(i) $s(\varepsilon, k)$ is the smallest eigenvalue (not necessarily principal),

(ii) there exists no other principal eigenvalue.

The proof of (i) is very classical and relies on the positivity of the principal eigenfunction $u$ and the fact that, the smaller $s$, the faster a potential eigenfunction must decay on both sides (this can be seen looking at the characteristic equations at $x = \pm \infty$). The argument then goes as follows: if $u$ is a non-trivial eigenfunction associated with $s < s(\varepsilon, k)$ then $w := u/u$ is well-defined, satisfies an elliptic equation $Lw = 0$ with zero-th order coefficient $(s - s)u > 0$, and decays on both sides $u(\pm \infty) = 0$. The classical Maximum/Minimum Principles finally imply $w \equiv 0$, thus contradicting $u \neq 0$.

From (i) there clearly cannot exist a principal eigenvalue $s_1 < s(\varepsilon, k)$, and if there existed a principal eigenvalue $s_1 > s(\varepsilon, k)$ associated with a positive eigenfunction $u_1$ we could apply (i) with $s_1$ instead of $s$ and conclude that $s_1$ is the smallest eigenvalue.

References


