Invariant Gibbs measures for dispersive PDEs

Laurent THOMANN

Université de Lorraine
Institut Élie Cartan, Nancy

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Chapter 2 : Construction of Gibbs measures for PDEs

The Khintchine inequality and the Wiener chaos

Let $(\Omega, \mathcal{F}, \mathbf{p})$ be a probability space, and consider a sequence of independent standard complex Gaussians $g_n \in \mathcal{N}_{\mathbb{C}}(0,1)$ which means that g_n can be written

$$g_n(\omega) = \frac{1}{\sqrt{2}}\big(h_n(\omega) + i\ell_n(\omega)\big),$$

where $\big(h_n(\omega),\, \ell_n(\omega)\big)_{n\geq 1}$ are independent standard real Gaussians $\mathcal{N}_{\mathbb{R}}(0,1)$. Then the following result holds true, known as the Paley-Zygmund inequality or Khintchine inequality

Lemma (Khintchine)

There exists a constant $C > 0$ such that for all $p \geq 2$ and $(c_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$

$$\Big\| \sum_{n=0}^{+\infty} c_n g_n(\omega) \Big\|_{L^p(\Omega)} \leq C\sqrt{p}\Big(\sum_{n=0}^{+\infty} |c_n|^2 \Big)^{1/2}. \tag{1}$$

The Khintchine inequality and the Wiener chaos

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Lemma (**Khintchine**)

There exists a constant $C > 0$ such that for all $p \geq 2$ and $(c_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$

$$\Big\| \sum_{n=0}^{+\infty} c_n g_n(\omega) \Big\|_{L^p(\Omega)} \leq C\sqrt{p}\Big(\sum_{n=0}^{+\infty} |c_n|^2 \Big)^{1/2}. \qquad (2)$$

This result shows that any L^p norm can be controlled by a L^2 norm, which is a genuine smoothing effect. Let us make a parallel with harmonic analysis : in this context Sobolev inequalities can be used, but at the price of loss of derivatives.

Actually, the Khintchine lemma holds for more general centered and localised random variables, like centered Bernoulli r.v. For instance, take $c_n = 1/(n+1)$ and (g_n) a sequence of independent centered Bernoulli r.v. Then the series $\sum_{n=0}^{+\infty} c_n$ diverges, but according to (2) we have $|\sum_{n=0}^{+\infty} c_n g_n(\omega)| < +\infty$, $\mathbf{p}-$a.s., in other words, randomising the signs makes the series a.s. converge.

From the Khintchine lemma we can deduce a large deviations estimate.

Corollary (**Large deviations**)

There exist constants $c, C > 0$ *such that for all* $(c_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$ *and* $\lambda > 0$

$$\mathbf{p}\Big(\omega \in \Omega\ :\ \Big|\sum_{n=0}^{+\infty} c_n g_n(\omega)\Big| > \lambda\Big) \leq C e^{-c\lambda^2/\|c\|_{\ell^2}^2}. \tag{3}$$

Proof : Let $(c_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$, then by (2) we get for all $p \geq 2$

$$\Big\|\sum_{n=0}^{+\infty} c_n g_n(\omega)\Big\|_{L^p(\Omega)} \leq C\sqrt{p}\|c\|_{\ell^2}.$$

Using the Markov inequality, we obtain that for all $\lambda > 0$

$$\mathbf{p}\Big(\omega\ :\ \Big|\sum_{n=0}^{+\infty} c_n g_n(\omega)\Big| > \lambda\Big) \leq \Big(\lambda^{-1}\Big\|\sum_{n=0}^{+\infty} c_n g_n(\omega)\Big\|_{L^p(\Omega)}\Big)^p \leq (C\lambda^{-1}\sqrt{p}\|c\|_{\ell^2})^p.$$

Thus by choosing $p = \delta\lambda^2/\|c\|_{\ell^2}^2$, for δ small enough, we get the bound (3). □

We sometimes need a multilinear version of (2). The result can be proved using hypercontractivity estimate of the Ornstein-Uhlenbeck semi-group and is classical in quantum field theory.

Proposition (**Wiener Chaos**)

Let $c(n_1, \dots, n_k) \in \mathbb{C}$ and $(g_n)_{n\geq 0} \in \mathcal{N}_{\mathbb{C}}(0,1)$ independent standard Gaussians and normalised in L^2. For $k \geq 1$ we define

$$S_k(\omega) = \sum_{n\in\mathbb{N}^k} c(n_1, \dots, n_k)\, g_{n_1}(\omega)\cdots g_{n_k}(\omega).$$

Then for all $p \geq 2$

$$\|S_k\|_{L^p(\Omega)} \leq (p-1)^{\frac{k}{2}}\, \|S_k\|_{L^2(\Omega)}. \qquad (4)$$

This result means that the nonlinear estimates can be reduced to the case $p = 2$. It is useful to control nonlinear terms which are not perfect powers, *e.g.* $\displaystyle\int_{\mathbb{T}} \overline{u}^2 \partial_x(u^2)$: this term appears in the study of DNLS.

The explicit bound (4) in k, p implies the large deviation estimate

$$\mathbf{p}\Big(\omega \in \Omega \,:\, \big|S_k(\omega)\big| > \lambda\Big) \leq C e^{-c\lambda^{2/k}}.$$

Definition of the Gaussian measure in the case of the torus

Let $(\Omega, \mathcal{F}, \mathbf{p})$ be a probability space and $\big(g_n(\omega)\big)_{n\geq 1}$ a sequence of independent complex normalised Gaussians, $g_n \in \mathcal{N}_{\mathbb{C}}(0,1)$. Here we show how we can construct a Gaussian measure on $H^\sigma(\mathbb{T}^d)$.

Consider a Hilbertian basis $(e_n)_{n\geq 1}$ of $L^2(\mathbb{T}^d)$ of eigenfunctions of $(1-\Delta)$. Then

$$(1-\Delta)e_n = \lambda_n^2 e_n, \quad n \geq 1, \quad x \in \mathbb{T}^d,$$

and one has $\lambda_n \sim c n^{1/d}$ when $n \longrightarrow +\infty$.

For $N \geq 1$ we define the random variable

$$\omega \longmapsto \varphi_N(\omega, .) = \sum_{n=1}^{N} \frac{g_n(\omega)}{\lambda_n} e_n(.).$$

Then we have the following result

Proposition

Assume that $\sigma < 1 - d/2$, *then* $(\varphi_N)_{N\geq 1}$ *is a Cauchy sequence in* $L^2(\Omega;\, H^\sigma(\mathbb{T}^d))$. *This enables us to define its limit*

$$\omega \longmapsto \gamma(\omega, .) = \sum_{n\geq 1} \frac{g_n(\omega)}{\lambda_n} e_n(.) \in L^2(\Omega;\, H^\sigma(\mathbb{T}^d)).$$

Notice that the law of γ does not depend on the choice of the Hilbertian basis $(e_n)_{n\geq 1}$.

Proof : We only show that $\gamma \in L^2(\Omega;\, H^\sigma(\mathbb{T}^d))$. For $\sigma \in \mathbb{R}$, we compute

$$\|\gamma(\omega, .)\|^2_{H^\sigma(\mathbb{T}^d)} = \sum_{n\geq 1} \frac{|g_n(\omega)|^2}{\lambda_n^{2-2\sigma}},$$

thus

$$\|\gamma\|^2_{L^2(\Omega;\, H^\sigma(\mathbb{T}^d))} = \sum_{n\geq 1} \frac{1}{\lambda_n^{2-2\sigma}}, \tag{5}$$

and we can conclude that the series converges iff $\sigma < 1 - d/2$, using the asymptotic formula $\lambda_n \sim cn^{1/d}$ when $n \longrightarrow +\infty$. □

Exercise

Let $\sigma \geq 1 - d/2$. Show that for almost all $\omega \in \Omega$, $\|\gamma(\omega, .)\|_{H^\sigma(\mathbb{T}^d)} = +\infty$.
Hint : with an explicit computation, show that

$$\int_\Omega e^{-\|\gamma(\omega,.)\|^2_{H^\sigma(\mathbb{T}^d)}} d\mathbf{p}(\omega) = \mathbf{0}.$$

This result can also be deduced from (5) *using general convergence results on random series in Banach spaces.*

Denote by

$$X^\sigma(\mathbb{T}^d) = \bigcap_{\tau<\sigma} H^\tau(\mathbb{T}^d).$$

We then define the Gaussian probability measure μ on $X^{1-d/2}(\mathbb{T}^d)$ by

$$\mu = \mathbf{p} \circ \gamma^{-1}. \tag{6}$$

In other words, μ is the image of the measure $\mathbf{p}$ under the map

$$\begin{array}{ccc} \Omega & \longrightarrow & X^{1-d/2}(\mathbb{T}^d) \\ \omega & \longmapsto & \gamma(\omega, \cdot) = \displaystyle\sum_{n\geq 1} \frac{g_n(\omega)}{\lambda_n} e_n, \end{array}$$

which means that for all measurable $F : X^{\sigma_c}(\mathbb{T}^d) \longrightarrow \mathbb{R}$

$$\int_{X^{1-d/2}(\mathbb{T}^d)} F(u) \mathrm{d}\mu(u) = \int_\Omega F\big(\gamma(\omega, \cdot)\big) \mathrm{d}\mathbf{p}(\omega).$$

Formally, one has

$$" d\mu = \frac{1}{Z} e^{-H_0(c)} dc d\overline{c} ", \quad H_0(c) = \sum_{n=1}^{+\infty} \lambda_n^2 |c_n|^2, \tag{7}$$

in other words, μ is the Gibbs measure of the linear Schrödinger equation

$$i\partial_t u + (\Delta - 1)u = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{T}^d. \tag{8}$$

Then notation (7) can be understood thanks to the next result.

Proposition

Let $D \geq 1$ and denote by $(e_n)_{1 \leq n \leq D}$ the canonical basis of $\mathbb{R}^D$. Define the measure $\mu = \mathbf{p} \circ \gamma^{-1}$ by

$$\gamma = \sum_{n=1}^{D} \frac{g_n}{\lambda_n} e_n.$$

Then μ is the Gaussian measure

$$d\mu = \frac{1}{Z} e^{-H_0(c)} dL(c), \quad H_0(c) = \sum_{n=1}^{D} \lambda_n^2 |c_n|^2,$$

where dL is the Lebesgue measure in $\mathbb{C}^D$.

Proof : We compute $\mu(A)$ for a cuboid $A = \prod_{n=1}^{D}[\alpha_n, \beta_n] \subset \mathbb{C}^D$, with $\alpha_n = a_n + ic_n$ and $\beta_n = b_n + id_n$. We write $g_n = (h_n + i\ell_n)/\sqrt{2}$ with $h_n, \ell_n \in \mathcal{N}_{\mathbb{R}}(0,1)$. Then we have

$$\begin{aligned}
\mu(A) &= p(\gamma \in A) = p\big(\omega : \gamma(\omega) \in A)\big) \\
&= \int_\Omega \mathbf{1}_{\{\gamma(\omega)} \in A\} d\mathbf{p}(\omega) \\
&= \prod_{n=1}^{D} \int_{\{\frac{g_n(\omega)}{\lambda_n} \in [\alpha_n, \beta_n]\}} d\mathbf{p}(\omega) \\
&= \prod_{n=1}^{D} \int_{\{\frac{h_n(\omega)}{\sqrt{2}\lambda_n} \in [a_n, b_n]\}} d\mathbf{p}(\omega) \int_{\{\frac{\ell_n(\omega)}{\sqrt{2}\lambda_n} \in [c_n, d_n]\}} d\mathbf{p}(\omega) \\
&= \prod_{n=1}^{D} \Big(\frac{\lambda_n^2}{\pi} \int_{a_n}^{b_n} \int_{c_n}^{d_n} \mathrm{e}^{-\lambda_n^2(x_n^2 + y_n^2)} dx_n dy_n \Big) \\
&= \frac{(\prod_{n=1}^{D} \lambda_n)^2}{\pi^D} \int_A \mathrm{e}^{-\sum_{n=1}^{D} \lambda_n^2 |c_n|^2} dL(c),
\end{aligned}$$

which was the claim. □

Let's come back to the measure μ defined in (6). We summarize its main properties :

- $\mu(X^{1-d/2}(\mathbb{T}^d)) = 1$ (μ is a probability measure) ;
- $\mu(H^{1-d/2}(\mathbb{T}^d)) = 0$ (the support of μ is composed of rough functions, see the previous exercise). Actually, this shows that the function $x \longmapsto \sum_{n\geq 1} \frac{g_n(\omega)}{\lambda_n} e_n(x)$ has almost surely the same Sobolev regularity than the function $x \longmapsto \sum_{n\geq 1} \frac{1}{\lambda_n} e_n(x)$. However there is a regularisation at the L^p scale, as will be seen in Chapter 3 ;

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- Let $\sigma < 1 - d/2$. Then for any open set, $B \subset H^\sigma(\mathbb{T}^d)$, $B \neq \emptyset$, we have $\mu(B) > 0$;
- The previous construction can easily be adapted to the case of a compact manifold $\mathcal{M}$, where $(e_n)_{n\geq 1}$ is a Hilbertian basis of $L^2(\mathcal{M})$ of eigenfunctions of the Laplacian : $(1 - \Delta)e_n = \lambda_n^2 e_n$, $n \geq 1$. The asymptotic of the $\lambda_n \sim cn^{1/d}$ is given by the Weyl formula.

We stress that the support of μ is rough when $d \geq 2$. The regularity of the support of a Gibbs measure is given by the linear part of the equation (even in the nonlinear case). In general, if there is few dispersion or if the dimension increases, then the support of the measure becomes rough.

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We conclude this paragraph with an elementary result.

Proposition

The measure μ defined in (6) is invariant by the flow of the linear Schrödinger equation (8).

Proof : Denote by Φ the flow of (8), then

$$\Phi\big(t, \gamma(\omega, x)\big) = \sum_{n \geq 1} \frac{e^{-it\lambda_n^2} g_n(\omega)}{\lambda_n} e_n(x),$$

and we observe that this r.v. has the same law as γ because of the rotation invariance of the complex Gaussians. □

Singular measures and perturbations

Consider the family of measures

$$d\mu_\beta = \frac{1}{Z_\beta} e^{-\beta H_0(c)} dc d\overline{c},$$

with $\beta > 0$. What happens when β varies ? To answer this question we will need the Kakutani theorem.

Theorem

Consider the infinite tensor products of probability measures on $\mathbb{R}^{\mathbb{N}}$

$$\mu_i = \bigotimes_{n\in\mathbb{N}} \mu_{n,i}, \qquad i = 1, 2.$$

Then the measures μ_1 *and* μ_2 *on* $\mathbb{R}^{\mathbb{N}}$ *endowed with its cylindrical Borel* σ*-algebra are absolutely continuous with respect each other,* $\mu_1 \ll \mu_2$*, and* $\mu_2 \ll \mu_1$*, if and only if the following holds :*

(i) *The measures* $\mu_{n,1}$ *and* $\mu_{n,2}$ *are for each* n *absolutely continuous with respect to each other : there exists two functions* $g_n \in L^1(\mathbb{R}, d\mu_{n,2})$*,* $h_n \in L^1(\mathbb{R}, d\mu_{n,1})$ *such that*

$$d\mu_{n,1} = g_n d\mu_{n,2}, \qquad d\mu_{n,2} = h_n d\mu_{n,1}.$$

(ii) *The functions* g_n *are such that the infinite product*

$$\prod_{n\in\mathbb{N}} \int_{\mathbb{R}} g_n^{1/2} d\mu_{n,2}$$

is convergent (i.e. positive).

Theorem

Furthermore, if any of the condition above is not satisfied (i.e. if the two measures μ_1 *and* μ_2 *are not absolutly continuous with respect to each other), then the two measures are mutually singular* ($\mu_1 \perp \mu_2$) *: there exists a set* $A \subset \mathbb{R}^{\mathbb{N}}$ *such that*

$$\mu_1(A) = 1, \qquad \mu_2(A) = 0.$$

An application of the previous results yields

Corollary

Let $(e_n)_{n\geq 1}$ be a Hilbertian basis of $L^2(\mathbb{T}^d)$. Then

(*i*) *Consider $\alpha_n, \beta_n > 0$ and the measures $\mu = \mathbf{p}\circ\gamma^{-1}$ and $\nu = \mathbf{p}\circ\psi^{-1}$ with*

$$\gamma = \sum_{n=1}^{+\infty} \frac{g_n}{\alpha_n} e_n, \qquad \psi = \sum_{n=1}^{+\infty} \frac{g_n}{\beta_n} e_n.$$

Then the mesures μ and ν are absolutely continuous with respect to each other if and only if

$$\sum_{n=1}^{+\infty} (\frac{\alpha_n}{\beta_n} - 1)^2 < +\infty.$$

(*ii*) *Consider $\lambda_n > 0$ and the measures $\mu_\beta = \mathbf{p}\circ\gamma_\beta^{-1}$ with*

$$\gamma_\beta = \sum_{n=1}^{+\infty} \frac{g_n}{\beta\lambda_n} e_n.$$

Assume that $\beta \neq \beta'$, then the measures μ_β and $\mu_{\beta'}$ are singular.

Exercise

Let $\beta, \beta' > 0$ *with* $\beta \neq \beta'$. *Construct an explicit set* A *such that* $\mu_\beta(A) = 1$ *and* $\mu_{\beta'}(A) = 0$.

An another natural question is the behaviour of μ under transformations.

In the case of translations, the answer is given by the Cameron-Martin theorem, and we state it only in the particular case of the measure (6). Recall that μ is a probability measure on $X^{1-d/2}(\mathbb{T}^d)$.

Theorem (Cameron-Martin)

Given $h \in X^{1-d/2}(\mathbb{T}^d)$, define the shifted measure μ^h by $\mu^h = \mu(. - h)$. Then, the measure μ^h is mutually absolutely continuous with respect to μ if and only if $h \in H^1(\mathbb{T}^d)$.

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We say that a measure μ is quasi-invariant under a transformation T if μ and $T\#\mu$ are mutually absolutely continuous, or equivalently that their zero measure sets are preserved. This is a natural extension of the (rigid) concept of invariant measure, and this notion is particularly relevant in infinite dimension.

For more analysis of Gaussian measures on Hilbert or Banach spaces, we refer to [Janson] and [Kuo].

Regularity results for random series in L^p spaces

We state here known convergence results on the convergence of random series in Banach spaces (in L^p actually). The following result is a combination of results of Hoffman-Jorgensen and Maurey-Pisier. For an introduction on this topic, we refer to the books of Marcus-Pisier, J.-P. Kahane and to the book of Li and Queffélec. See also Imekraz-Robert-Thomann and references therein.

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Theorem

Let $p \in [2, +\infty)$ and $(F_n)_{n\geq 0} \in L^p(\mathbb{R}^d)$. Assume that $(g_n)_{n\geq 0} \in \mathcal{N}_{\mathbb{C}}(0,1)$ is i.i.d. and that $(\varepsilon_n)_{n\geq 0} \in \{-1, 1\}$ is an i.i.d. Rademacher sequence. The following statements are equivalent :

- (i) *the series $\sum \varepsilon_n F_n$ converges almost surely in $L^p(\mathbb{R}^d)$,*
- (ii) *the series $\sum g_n F_n$ converges almost surely in $L^p(\mathbb{R}^d)$,*
- (iii) *the function $\displaystyle\sum_{n\geq 0} |F_n|^2$ belongs to $L^{\frac{p}{2}}(\mathbb{R}^d)$.*

There are also continuity results for random series (Paley-Zygmund, Salem-Zygmund, ...).

Application : We define the so-called special Hermite function by

$$\varphi_n(z) = \frac{1}{\sqrt{\pi n!}} z^n e^{-|z|^2/2}, \qquad n \geq 0,$$

and the Gaussian random variable

$$\eta(\omega, z) = \sum_{n=0}^{+\infty} g_n(\omega)\varphi_n(z) = \frac{1}{\sqrt{\pi}} \Big(\sum_{n=0}^{+\infty} \frac{z^n g_n(\omega)}{\sqrt{n!}} \Big) e^{-|z|^2/2}.$$

Proposition

Let $2 \leq p < +\infty$. Then $\eta(\omega, .) \notin L^p(\mathbb{C})$ for almost all $\omega \in \Omega$.

Proof : We simply observe that $\sum_{n=0}^{+\infty} |\varphi_n(z)|^2 \equiv 1$ and that a random series either converges a.s. or diverges a.s. $\square$

Nonlinearities

We now turn to the nonlinear Schrödinger equation on $\mathbb{T}^d$,

$$i\partial_t u + (\Delta - 1)u = |u|^{p-1}u, \quad (t,x) \in \mathbb{R} \times \mathbb{T}^d.$$

The Hamiltonian of this equation is

$$H = \int_{\mathbb{T}^d} (|u|^2 + |\nabla u|^2) + \frac{2}{p+1} \int_{\mathbb{T}^d} |u|^{p+1}.$$

We denote by μ the Gaussian measure which corresponds to the linear problem (8). We are able to construct a Gibbs measure to this problem in the following cases :

- In dimension $d = 1$: μ is supported in $X^{1/2}(\mathbb{T})$. A Sobolev imbedding argument yields $\int_{\mathbb{T}} |u|^{p+1} < +\infty$, $\mu-$a.s. and one can define a Gibbs measure by
$$d\rho(u) = \exp(-\frac{2}{p+1}\|u\|^{p+1}_{L^{p+1}(\mathbb{T})})d\mu(u).$$
- In dimension $d = 2$: μ is supported in $X^0(\mathbb{T}^2)$. In this case $\int_{\mathbb{T}^2} |u|^{p+1} = +\infty$, $\mu-$a.s. because $\int_{\mathbb{T}^2} |u|^2 = +\infty$, $\mu-$a.s. Therefore, the construction is more difficult and has been done for $p = 3$ by Bourgain with a Wick renormalisation of the non-linearity. This can be extended to any $p \in 2\mathbb{N} + 1$ (see Oh-Thomann).
- In dimension $d \geq 3$: the situation is unclear to me.

The construction of Gibbs measures of focusing equations is harder in general. Actually, if we set $\mathrm{d}\rho(u) = G(u)\mathrm{d}\mu(u)$ we have to check that the density is integrable with respect to μ, *i.e.* $G \in L^p(\mathrm{d}\mu)$. This induces some restrictions on the degree of the non-linearity and needs renormalisation arguments. There are also non existence results by Brydges-Slade.

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For the mathematical construction of Gibbs measures or more generally Wiener measures for dispersive PDEs, we refer to P. Zhidkov , Lebowitz-Rose-Speer, B. Bidégaray, J. Bourgain, and more recently to N. Tzvetkov, Burq-Tzvetkov, Thomann-Tzvetkov, Burq-Thomann-Tzvetkov, T. Oh, Tzvetkov-Visciglia, Bourgain-Bulut and Oh-Thomann. M. Sy constructs a measure supported on smooth functions for BO.