Invariant Gibbs measures for dispersive PDEs

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Introduction

Consider a dynamical system. We try to understand the long time behaviour of a family of trajectories, rather than the behaviour of each trajectory one by one.

An important tool is the existence of an invariant measure of probability. Here we are concerned with the dynamical system given by the flow of a Hamiltonian dispersive partial differential equation.

In this context, we can sometimes define a Gibbs measure. It turns out that such a measure is very efficient to obtain qualitative information about the long time behaviour of the solution of the PDE, and even to prove almost sure global existence results.

The plan of these lectures is the following :

1. In Chapter 1 we introduce the notion of invariant measure and review some classical results of ergodic theory. We illustrate some of the issues with examples in finite dimension.
2. In Chapter 2 we present the main ideas used in the construction of Gibbs measures for PDEs.
3. In Chapter 3 we show how one can use Gibbs measures to construct strong global solutions of Schrödinger equations.
4. In Chapter 4 we present some compactness methods which allow to construct weak global solutions of Schrödinger equations at very low regularity.

Here are some references, from which the material of Chapter 1 has been taken :

- For an introduction to the notion of recurrence, we refer to the article of F. Béguin in Images des Mathématiques (science popularization, in French).
- Lectures on dynamical systems by Y. Benoist and F. Paulin (in French).
- The books of Bunimovich *et al.* and of Cornfeld *et al.*.
- The book of Y. Coudène (in French).
- Lectures on invariant measures by C. Liverani.
- The book of V. V. Nemytskii and V. V. Stepanov.
- Lectures on ergodic theory on the blog of T. Tao.

Here are some other lectures and surveys concerning the study and the use of Gibbs measures for dispersive PDEs :

- Lectures on invariant measures and PDEs by A. Nahmod.
- Lectures on Gibbs measures and PDEs by T. Oh.
- My habilitation thesis, Chapter 1 and Chapter 2 (in French).
- Lectures on the wave equation with random data by N. Tzvetkov.
- The book of P. Zhidkov.

Invariant Gibbs measures for dispersive PDEs

Chapter 1 : Invariant measures and ODEs

Definition

Definition

(*i*) *Consider a space X, a measure μ on X and a measurable map $T : X \longrightarrow X$. The measure μ is called invariant with respect to T if for any $\mu-$measurable set A*

$$\mu\big(T^{-1}(A)\big) = \mu(A).$$

One also says that T preserves the measure μ.

(*ii*) *Consider a space X and one parameter group $(\Phi(t,.))_{t\in\mathbb{R}}$ with $\Phi(t,.) : X \longrightarrow X$. A measure μ defined in the space X is called invariant with respect to $(\Phi(t,.))_{t\in\mathbb{R}}$ if for any $\mu-$measurable set A*

$$\mu\big(\Phi(t,A)\big) = \mu(A), \quad t \in \mathbb{R}.$$

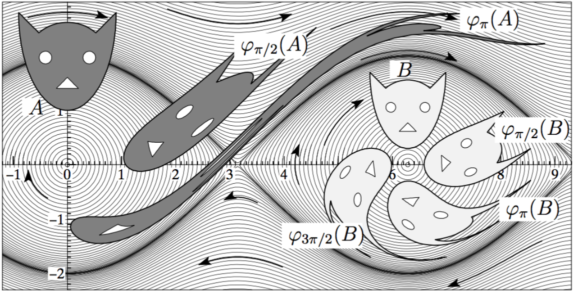

Figure: Conservation of the area for the pendulum (picture taken on the web).

Remark

- *In general, for a measurable map* $T : X \longmapsto X$*, we can define the measure* $T_{\#}\mu := \mu \circ T^{-1}$*, called the pushforward measure, and it satisfies*

$$\int_X F(x) d(T_{\#}\mu)(x) = \int_X F\big(T(x)\big) d\mu(x),$$

for all measurable $F : X \longrightarrow \mathbb{R}$*. In general* $\mu \circ T$ *is not a measure.*

- *One can always transform a continuous dynamical system* $\big(\Phi(t,.)\big)_{t\in\mathbb{R}}$ *into a discrete one by setting* $T := \Phi(\tau,.)$ *for some* $\tau \in \mathbb{R}$*.*

Exercise

1. *For a measurable set A, define $\mu_0 = \delta_A$ and set $\mu_t = \Phi(t)\#\mu_0$. Check that $\mu_t = \delta_{\Phi(t,A)}$.*
2. *If μ is invariant with respect to T, check that in general $\mu(A) \leq \mu(T(A))$.*
3. *Assume that $X = \mathbb{R}$ and $T \equiv 0$. Find the invariant probability measures.*
4. *Assume that $X = \mathbb{R}$. Find some maps T which leave the Lebesgue measure invariant.*

A first natural question is the existence of an invariant measure of probability, in general. An answer is given by the Krylov-Bogoliuboff theorem, in the case of a compact space X.

Theorem (Krylov and Bogoliuboff)

Let X be a compact metric space and consider $T : X \longrightarrow X$ a continuous map (resp. $\Phi(t,.) : X \longrightarrow X$). Then there exists an invariant probability measure.

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Let X be a compact metric space and consider $T : X \longrightarrow X$ a continuous map (resp. $\Phi(t,.) : X \longrightarrow X$). Then there exists an invariant probability measure.

Proof : Consider a discrete dynamical system T.
Consider any probability measure ν on X and define the following sequence of measures $(\mu_n)_{n\in\mathbb{N}^*}$ defined by

$$\mu_n = \frac{1}{n}\sum_{k=0}^{n-1} \nu \circ T^{-k} = \frac{1}{n}\sum_{k=0}^{n-1} T^k_{\#}\nu.$$

We have $\mu_n(X) = 1$, therefore by the Riesz theorem the sequence $(\mu_n)_{n\geq 1}$ admits a subsequence which converges weakly to a probability measure μ. It is then easy to check (exercise) that μ is invariant under the transform T. $\square$

The Liouville theorem

The problem with this result is the lack of information of the obtained invariant measure. For instance, if x_0 is an equilibrium point (fixed point) of the system, the Dirac measure δ_{x_0} is invariant, and we may be interested in less trivial examples. Moreover, this result only holds in compact spaces X.

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We may also be interested in studying families of measures, for instance having a density with respect to some reference measure, such as the Lebesgue measure. We now present a useful result which helps us to go into this direction in the context of ODEs : the Liouville theorem.

Let $\Omega \subset \mathbb{R}^d$ be an open set and $F : \Omega \longrightarrow \mathbb{R}^d$ a C^∞ function. Consider the ordinary differential equation

$$\begin{cases} \dot{x}(t) = \dfrac{dx}{dt}(t) = F(x(t)), \\ x(0) = x_0. \end{cases}$$

We assume that for all $x_0 \in \mathbb{R}$ the system has a unique solution $\Phi(t, x_0)$, such that $\Phi(0, x_0) = x_0$ and which is defined for all $t \in \mathbb{R}$. The family $(\Phi(t, .)_{t\in\mathbb{R}}$ is a one parameter group of diffeomorphisms such that $\Phi(0, .) = id$, $\Phi(t, \Phi(s, .)) = \Phi(t + s, .)$ for all $s, t \in \mathbb{R}$.

Theorem (**Liouville**)

Denote by dx the Lebesgue measure on Ω *and let* $g : \Omega \longrightarrow [0, +\infty)$ *a* $\mathcal{C}^\infty$ *function. The flow* $\Phi(t, .)$ *preserves the measure gdx if and only if*

$$\sum_{k=1}^{d} \frac{\partial}{\partial x_k}\big(gF_k\big) = 0. \tag{1}$$

Proof : Denote by $\mathcal{C}_c^\infty(\Omega, \mathbb{R})$ the set of $\mathcal{C}^\infty$ functions with compact support in Ω. By density of $\mathcal{C}_c^\infty(\Omega, \mathbb{R})$ in $L^1(\Omega)$, it is enough to show that for all $t \in \mathbb{R}$ and $f \in \mathcal{C}_c^\infty(\Omega, \mathbb{R})$ we have

$$\int_\Omega f(x)g(x)dx = \int_\Omega f(\Phi(t, x))g(x)dx.$$

Since $\Phi(0,x)=x$, it is enough to show that

$$\frac{d}{dt}\Big(\int_\Omega f(\Phi(t,x))g(x)dx\Big)=0.$$

Since $\Phi(t,.)$ is a group of diffeomorphisms, it is enough to show that

$$I=\frac{d}{dt}\Big(\int_\Omega f(\Phi(t,x))g(x)dx\Big)_{t=0}=0,$$

(possibly replace $f(\Phi(t_0,.))$ with $\widetilde{f}\in\mathcal{C}_c^\infty(\Omega,\mathbb{R})$ at $t=t_0$ in the case $t_0\neq 0$).

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Now we compute

$$I=\int_\Omega\sum_{k=1}^d F_k(x)\frac{\partial f}{\partial x_k}(x)g(x)dx=-\int_\Omega\Big(\sum_{k=1}^d\frac{\partial}{\partial x_k}(gF_k)\Big)(x)f(x)dx,$$

hence the result. □

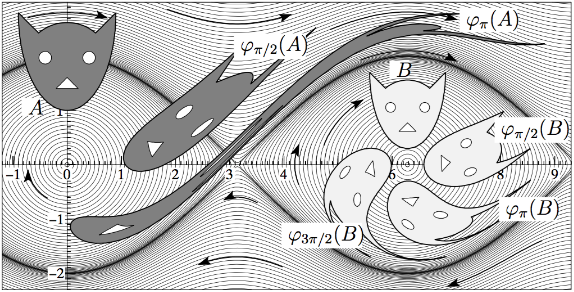
An important class of examples of such systems is given by the Hamiltonian equations. Let $\Omega \subset \mathbb{R}^{2d}$ and $H : \Omega \longrightarrow \mathbb{R}$, $(x, y) \longmapsto H(x, y)$ a $\mathcal{C}^\infty$ function, then the equations corresponding to the Hamiltonian H are

$$\begin{cases} \dot{x}_k(t) = \dfrac{\partial H}{\partial y_k}\big(x(t), y(t)\big), \quad 1 \leq k \leq d \\ \dot{y}_k(t) = -\dfrac{\partial H}{\partial x_k}\big(x(t), y(t)\big), \quad 1 \leq k \leq d. \end{cases}$$

Such a system satisfies condition (1) with $g \equiv 1$, because

$$\frac{\partial}{\partial x_k}\Big(\frac{\partial H}{\partial y_k}(x,y)\Big) - \frac{\partial}{\partial y_k}\Big(\frac{\partial H}{\partial x_k}(x,y)\Big) = 0.$$

In conclusion, the Lebesgue measure is preserved by the flow.



Example

In the phase space $\mathbb{R}^2$, consider the Hamiltonian $H(x, y) = (x^2 + y^2)/2$ and the corresponding system (harmonic oscillator)

$$\begin{cases} \dot{x}(t) = \dfrac{\partial H}{\partial y}\big(x(t), y(t)\big) = y(t), \\ \dot{y}(t) = -\dfrac{\partial H}{\partial x}\big(x(t), y(t)\big) = -x(t). \end{cases} \tag{2}$$

Example

Then by the Liouville theorem, the Lebesgue measure $dxdy$ is invariant by the flow as well as $f(H)dxdy$ for any nonnegative measurable function f. Examples of such measures are :

- The Gaussian measure $d\mu = \frac{1}{2\pi} e^{-(x^2+y^2)/2} dxdy$;
- The measure $d\mu = \chi(x^2 + y^2)dxdy$ where $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$ with $0 \leq \chi \leq 1$.

Other measures are preserved, namely :

- The Dirac measure δ_0, since 0 is an equilibrium.
- The uniform measure on any circle $x^2 + y^2 = R$.

Let us notice that, thanks to the ergodic decomposition measure, an invariant measure can be decomposed into a sum of invariant ergodic measures.

The Poincaré theorem for a discrete dynamical system

Theorem (Poincaré)

Let $(X, \mathcal{B}, \mu)$ be a probability space and let $T : X \longrightarrow X$ be a map which preserves the probability measure μ.

(i) *Let $A \in \mathcal{B}$ be such that $\mu(A) > 0$, then there exists $k \geq 1$ such that $\mu(A \cap T^k(A)) > 0$.*

(ii) *Let $B \in \mathcal{B}$ be such that $\mu(B) > 0$, then for μ-almost all $x \in B$, the orbit $\big(T^n(x)\big)_{n \in \mathbb{N}}$ enters infinitely many times in B.*

(iii) *Assume that T is invertible. Let $C \in \mathcal{B}$ be such that $\mu(C) > 0$, then*

$$\limsup_{n \to \infty} \mu(C \cap T^n(C)) \geq \mu(C)^2.$$

Proof : (i) By the invariance of μ, $\sum_{n\in\mathbb{N}}\mu(T^{-n}(A))=\sum_{n\in\mathbb{N}}\mu(A)=+\infty$, which implies that there exists $m<n$ such that $\mu(T^{-n}(A)\cap T^{-m}(A))>0$. Then,

$$T^{-n}(A)\cap T^{-m}(A)\subset T^{-n}(A)\cap T^{-n}(T^{n-m}(A))=T^{-n}(A\cap T^{n-m}(A)),$$

and by invariance of the measure,
$0<\mu\big(T^{-n}(A)\cap T^{-m}(A)\big)\leq\mu\big(A\cap T^{n-m}(A)\big)$, hence the result.

Proof : (i) By the invariance of μ, $\sum_{n\in\mathbb{N}} \mu(T^{-n}(A)) = \sum_{n\in\mathbb{N}} \mu(A) = +\infty$, which implies that there exists $m < n$ such that $\mu(T^{-n}(A) \cap T^{-m}(A)) > 0$. Then,

$$T^{-n}(A) \cap T^{-m}(A) \subset T^{-n}(A) \cap T^{-n}(T^{n-m}(A)) = T^{-n}(A \cap T^{n-m}(A)),$$

and by invariance of the measure,
$0 < \mu\big(T^{-n}(A) \cap T^{-m}(A)\big) \leq \mu\big(A \cap T^{n-m}(A)\big)$, hence the result.

(ii) If not, there exists $n_0 \geq 1$ such that the set

$$A = \{x \in B \ : \ \forall n \geq n_0, \ T^n(x) \notin B\}$$

satisfies $\mu(A) > 0$. Since one can replace T by T^{n_0}, one can assume $n_0 = 1$. Then by (i), there exists $k \geq 1$ such that $\mu(A \cap T^k(A)) > 0$, which is a contradiction.

(iii) (Taken from [Tao].) Let $N \geq 1$. Then

$$\int_X \sum_{n=1}^{N} \mathbf{1}_{T^n(C)} d\mu = N\mu(C),$$

and by Cauchy-Schwarz

$$\int_X \Big(\sum_{n=1}^{N} \mathbf{1}_{T^n(C)}\Big)^2 d\mu \geq N^2 \mu(C)^2. \tag{3}$$

We expand the left hand side of (3), thus

$$\int_X \Big(\sum_{n=1}^{N} \mathbf{1}_{T^n(C)}\Big)^2 d\mu = \sum_{n=1}^{N} \sum_{m=1}^{N} \mu\big(T^n(C) \cap T^m(C)\big).$$

Since T is invertible, for $m \geq n$, $\mu\big(T^n(C) \cap T^m(C)\big) = \mu\big(C \cap T^{m-n}(C)\big)$, hence for all $1 \leq k \leq N$

$$
\begin{aligned}
\int_X \Big(\sum_{n=1}^{N} 1_{T^n(C)}\Big)^2 d\mu &= N\mu(C) + 2\sum_{1\le n<m\le N} \mu(C\cap T^{m-n}(C)) \\
&= N\mu(C) + 2\sum_{j=1}^{N-1}(N-j)\mu(C\cap T^j(C)) \\
&\le N\mu(C) + 2\sum_{j=1}^{k-1}(N-j)\mu(C\cap T^j C) + (N-k)^2 \sup_{n\ge k}\mu\big(C\cap T^n(C)\big) \\
&\le N\mu(C) + 2N(k-1)\mu(C) + (N-k)^2 \sup_{n\ge k}\mu\big(C\cap T^n(C)\big).
\end{aligned}
$$

Now, by the previous line and (3) and the choice $N=k^2$,

$$
k^4\mu(C)^2 \le k^2\mu(C) + 2k^3\mu(C) + (k^2-k)^2 \sup_{n\ge k}\big(C\cap T^n(C)\big).
$$

Finally, we divide by k^4 and take the limit $k\longrightarrow+\infty$, which concludes the proof. $\square$

Exercise

Check what this means for the harmonic oscillator in $\mathbb{R}^2$, $T = \Phi(1, .)$ *and* $A, B, C = B\big((1,1), 1/2\big)$. *More generally consider the system with Hamiltonian*

$$H(x, y) = (x^2 + y^2)^2/2,$$

for which the speed of rotation of a point depends on the distance to the origin.

Corollary

Let X be a separable space (which means that it contains a countable, dense subset). Let μ a probability measure, and $T : X \longrightarrow X$ a map which preserves μ. Then μ-almost all point of X is recurrent for T, i.e. for μ-almost all point of X there exists a sequence $(n_k)_{k\geq 0}$ going to infinity such that

$$T^{n_k}(x) \longrightarrow x, \quad \textit{when} \;\; k \longrightarrow +\infty.$$

Proof : Let $\mathcal{C}$ be a countable dense of open subsets of X. By the Poincaré theorem, for all $B \in \mathcal{C}$ there exists a μ-negligible set N_B such that every $x \in B \backslash N_B$, the orbit $\big(T^n(x)\big)_{n\in\mathbb{N}}$ enters infinitely many times in B. The set $N = \bigcup_{B\in\mathcal{C}} N_B$ is also μ-negligible and by construction every point in N^c is recurrent. □

Example

Consider the differential equation

$$\dot{x}(t) = F(x(t)), \quad \text{in } \Omega \subset \mathbb{R}^d,$$

and its flow $\Phi(t, .)$. Assume that there exists an invariant probability measure ρ. Then for ρ-almost all $x \in \Omega$ there exists a sequence of times $(t_n)_{n \geq 0}$ going to infinity such that

$$\|\Phi(t_n, x) - x\| \longrightarrow 0.$$

Example

In the phase space $\mathbb{R}^2$, consider the Hamiltonian $H(x,y) = xy$ and the corresponding system

$$\begin{cases} \dot{x}(t) = \dfrac{\partial H}{\partial y}\big(x(t),y(t)\big) = x(t), \\ \dot{y}(t) = -\dfrac{\partial H}{\partial x}\big(x(t),y(t)\big) = -y(t). \end{cases}$$

Again, by the Liouville theorem, the Lebesgue measure $dxdy$ is invariant by the flow (as well as $f(H)dxdy$ for any nonnegative measurable function f), but there is no absolutely continuous measure probability measure which is invariant as well. By contradiction, assume that such a measure μ exists. Consider the square $S_0 = \{1 \leq x \leq 2,\ \ 1 \leq y \leq 2\}$ and S_t its image by the flow. Then for $t \geq 1$, $S_0 \cap S_t = \emptyset$, but this is in contradiction with the Poincaré theorem.

However, there are invariant probability measures, as for example δ_0.

The Poincaré theorem for a continuous dynamical system

What can be said about a continuous dynamical system ? Before we give an answer, we need the following definition :

Definition

Let $A \subset B \subset \mathbb{R}$. We say that A is relatively dense in B if there exists a positive number ℓ such that for all $a \in B$ we have $[a, a+\ell] \cap A \neq \emptyset$.

Then we have the following result

Theorem (Khintchine)

Let $0 < \lambda < 1$. Consider $\Phi(t, .) : X \longrightarrow X$ which preserves a probability measure μ. Then for a relatively dense set of $t \in \mathbb{R}$

$$\mu(A \cap \Phi(t, A)) \geq \lambda \mu(A)^2. \qquad (4)$$

Example

Consider the harmonic oscillator (2) in $\mathbb{R}^2$, set $\mu = dx$ and let $A \subset \mathbb{R}^2$ a set of positive measure (for instance $A = B((1,1), 1/2)$). Denote by $I \subset \mathbb{R}$ the largest interval containing 0 for which (4) holds true. Then, by $2\pi-$periodicity of the flowmap, (4) holds for $J = \bigcup_{k\in\mathbb{Z}} \big(I + 2\pi k\big)$ which is a relatively dense set in $\mathbb{R}$.

Exercise

Check that the Khintchine theorem may not hold if $\Phi(t,.)$ is replaced with $\Phi(g(t),.)$ for some functions g.

The Birkhoff-Khintchine theorem

Consider a measurable set $E \subset X$ and denote by $\mathbf{1}_E$ the indicator function. Then the set of instants of time of the interval $(0, T)$ for which the points $\Phi(t,x) \in E$ is given by $\int_0^T \mathbf{1}_E(\Phi(t,x))dt$. Then the following results implies that the mean

$$\frac{1}{T}\int_0^T \mathbf{1}_E(\Phi(t,x))dt$$

has a limit when $T \longrightarrow +\infty$.

Theorem (Birkhoff-Khintchine)

Consider $\Phi(t) : X \longrightarrow X$ which preserves a probability measure μ. Then for any function F in $L^1(X, d\mu)$

$$\frac{1}{T}\int_0^T F(\Phi(t,x))dt \longrightarrow G(x), \quad \text{when} \quad T \longrightarrow +\infty,$$

for μ-almost all $x \in X$ and in $L^1(X, d\mu)$.

Under the assumptions of the Birkhoff-Khintchine theorem, let us prove that

$$\int_X F(x)d\mu = \int_X G(x)d\mu \qquad (5)$$

Let $T > 0$, then by the invariance of μ

$$\begin{aligned} \int_X \Big(\frac{1}{T}\int_0^T F(\Phi(t,x))dt\Big)d\mu &= \frac{1}{T}\int_0^T \Big(\int_X F(\Phi(t,x))d\mu\Big)dt \\ &= \frac{1}{T}\int_0^T \Big(\int_X F(x)d\mu\Big)dt \\ &= \int_X F(x)d\mu. \end{aligned}$$

As a consequence, letting $T \longrightarrow +\infty$, with the Birkhoff-Khintchine theorem we obtain (5).

As the next results shows, this procedure allows to construct constant of motions.

Proposition

The function G in the Birkhoff-Khintchine theorem is defined μ-almost everywhere and is invariant by the flow, i.e. it is constant along every trajectory on which it is defined :

$$G(\Phi(t,x)) = G(x).$$

Proof : Let $t_0 \in \mathbb{R}$ and $T > 0$, then by the flow property

$$G(\Phi(t_0,x)) - G(x) = \lim_{T \to +\infty} \Big(\frac{1}{T} \int_0^T F(\Phi(t+t_0,x))dt - \frac{1}{T} \int_0^T F(\Phi(t,x))dt \Big).$$

Then by a change of variable

$$\begin{aligned}
\frac{1}{T}\int_0^T F(\Phi(t+t_0,x))dt - \frac{1}{T}\int_0^T F(\Phi(t,x))dt \\
= \frac{1}{T}\int_{t_0}^{T+t_0} F(\Phi(t,x))dt - \frac{1}{T}\int_0^T F(\Phi(t,x))dt \\
= \frac{T+t_0}{T}\left[\frac{1}{T+t_0}\int_0^{T+t_0} F(\Phi(t,x))dt\right] - \frac{1}{T}\int_0^{t_0} F(\Phi(t,x))dt \\
- \frac{1}{T}\int_0^T F(\Phi(t,x))dt.
\end{aligned}$$

Finally by taking the limit $T \longrightarrow +\infty$ we get $G(\Phi(t_0,x)) - G(x) = 0$. □

Example

Consider the harmonic oscillator

$$\begin{cases} \dot{x}(t) = y(t), \\ \dot{y}(t) = -x(t). \end{cases}$$

The flowmap reads

$$\begin{aligned} \Phi\big(t,(x;y)\big) &= \begin{pmatrix} x\cos t + y\sin t \\ -x\sin t + y\cos t \end{pmatrix} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &:= R(t)(x;y). \end{aligned}$$

The Gaussian probability measure on $\mathbb{R}^2$ given by

$$d\mu = \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)} dxdy$$

is invariant by the flow.

Example

Let F be such that $\int_{\mathbb{R}^2} |F(x;y)| e^{-\frac{1}{2}(x^2+y^2)} dxdy < \infty$. Then by the Birkhoff-Khintchine theorem, there exists G such that for almost all $(x,y) \in \mathbb{R}^2$

$$\frac{1}{T}\int_0^T F\big(R(t)(x;y)\big)\,dt \longrightarrow G(x,y), \quad \text{when} \quad T \longrightarrow +\infty.$$

Actually, by writing $T = 2\pi k + r$ with $0 \leq r < 2\pi$ and using that $R(t+2\pi) = R(t)$, we see that

$$G(x;y) = \frac{1}{2\pi}\int_0^{2\pi} F\big(R(t)(x;y)\big)\,dt,$$

and G satisfies

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} G(x;y) e^{-\frac{1}{2}(x^2+y^2)} dxdy = \frac{1}{2\pi}\int_{\mathbb{R}^2} F(x;y) e^{-\frac{1}{2}(x^2+y^2)} dxdy.$$

Gibbs measures in finite dimension

It is now time to give the following definition

Definition

Let $H : \mathbb{R}^{2d} \longrightarrow \mathbb{R}$ *by a smooth function and* $\beta > 0$ *such that* $e^{-\beta H} \in L^1(\mathbb{R}^{2d}, dx)$*. Then the probability measure on* $\mathbb{R}^{2d}$

$$d\mu_\beta = \frac{1}{Z_\beta} e^{-\beta H} dx, \quad Z_\beta = \int_{\mathbb{R}^{2d}} e^{-\beta H(x)} dx, \tag{6}$$

is called a Gibbs measure with energy H*. In the context of statistical physics this mesure is also called the Maxwell-Boltzmann, or the canonical distribution. The coefficient* Z_β *is called the partition function.*

Such a measure is invariant under the Hamiltonian dynamics defined by H, but it is not the only one. However, a Gibbs measure has a particular status compared to the other invariant measures as we will see.

Variational characterisation of Gibbs measures

This paragraph is inspired from [Oh-Quastel]. Given a probability measure $d\rho = gdx$ that is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^{2d}$, we define its entropy $S(g)$ and average energy $\langle H(g)\rangle$ by

$$S(g) = -\int_{\mathbb{R}^{2d}} g(x)\log\big(g(x)\big)dx \qquad \text{and} \qquad \langle H(g)\rangle = \int_{\mathbb{R}^{2d}} H(x)g(x)dx,$$

where H is the Hamiltonian for the underlying dynamics and we set $M(g) = \displaystyle\int_{\mathbb{R}^{2d}} g(x)dx = 1$. For a given $C \in \mathbb{R}$, we assume that there exists a unique $\beta > 0$ such that $\langle H(\mu_\beta)\rangle = C$ where μ_β is as in (6).

Now we consider the following maximisation problem

$$\max_{\langle H(g)\rangle = C,\ M(g)=1} S(g). \tag{7}$$

Proposition

The Gibbs measure μ_β is the unique maximizer of the problem (7).

In statistical mechanics, the equilibrium configuration of the system is dictated by the maximization of the entropy, according to the second law of thermodynamics.

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Proof : By the Lagrange multiplier method, there exist $\beta, \delta \in \mathbb{R}$ such that

$$dS(g) = \beta d\langle H(g)\rangle + \delta dM(g),$$

i.e.

$$\int_{\mathbb{R}^{2d}} \big(\log g(x) + 1 + \delta + \beta H(x)\big) f(x)dx = 0,$$

for all test functions f. Thus, $g(x) = \mathrm{e}^{-1-\delta-\beta H(x)}$. Moreover, by the mass constraint $M(g) = 1$, we must have $g(x) = Z_\beta^{-1}\mathrm{e}^{-\beta H(x)}$.
Therefore, if there is any extremal point for the entropy functional, it has to be the Gibbs measure μ_β.

Also, by a direct computation, we have $d^2S(g)(h,h) = -\int_{\mathbb{R}^{2d}} \frac{h^2}{g}dx \leq 0$,
hence we get uniqueness of the maximizer. □

Gibbs measures and the Langevin equation

The next paragraph is taken from the book [Pavliotis].

We consider the Langevin equation which describes the motion of a particle that is subject to friction and stochastic forcing

$$\ddot{q} = -\nabla V(q) - \gamma \dot{q} + \sqrt{2\gamma\beta^{-1}}\dot{W}, \quad q \in \mathbb{R}^d. \tag{8}$$

This is Newton's equation of motion with two additional terms, a linear dissipation term $\gamma\dot{q}$ and a stochastic forcing $\sqrt{2\gamma\beta^{-1}}\dot{W}$. The parameters of the equation are the friction coefficient $\gamma > 0$ and the temperature $\beta^{-1} = k_B T$, where k_B denotes Boltzmann's constant and T the absolute temperature.

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Introducing the momentum $p = \dot{q}$, we can write the Langevin equation (12) as a system of first-order stochastic differential equations in phase space $(q, p) \in \mathbb{R}^{2d}$:

$$\begin{cases} dq = pdt, \\ dp = -\nabla V(q)dt - \gamma pdt + \sqrt{2\gamma\beta^{-1}}dW. \end{cases} \tag{13}$$

The position and momentum $\{q, p\}$ define a Markov process with generator

$$\mathcal{L} = p \cdot \nabla_q - \nabla_q \cdot V \nabla_p + \gamma(-p\nabla_p + \beta^{-1}\Delta_p). \tag{14}$$

Now consider the Hamiltonian

$$H(p, q) = \frac{1}{2}|p|^2 + V(q). \tag{15}$$

This quantity, as well as any function of it, is invariant under the deterministic Hamiltonian dynamics. This in turns leads to many probability measures that are invariant by the Hamiltonian flow.

However, the presence of noise and dissipation in (13) results in selecting a unique invariant distribution :

Theorem

Let V be a smooth confining potential. Then the Markov process with generator (14) is ergodic. The unique invariant distribution is

$$d\mu_\beta = \frac{1}{Z_\beta} e^{-\beta H(p,q)} dpdq,$$

where H is the Hamiltonian (15), and the normalization factor Z_β is the partition function

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Observe that this measure is independent of the friction coefficient $\gamma > 0$.

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Observe that this measure is independent of the friction coefficient $\gamma > 0$.

We will not discuss here stochastic ODEs and PDEs, and we refer to the book of Kuksin and Shirikyan to go into this direction.

- Common features but different philosophy.

An example of renormalisation in the construction of a Gibbs measure

Example

Let us consider the Schrödinger equation, with periodic boundary conditions

$$i\partial_t u + (\Delta - 1)u = \varepsilon |u|^2 u, \quad (t,x) \in \mathbb{R} \times \mathbb{T},$$

with $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ and $\varepsilon \in \{0, 1, -1\}$. This equation derives from the Hamiltonian

$$H(u) = \int_{\mathbb{T}} \Big(|u|^2 + |\nabla u|^2 + \frac{\varepsilon}{2} |u|^4 \Big) dx,$$

and can be written

$$\begin{cases} \dot{u} = -i \dfrac{\delta H}{\delta \overline{u}}, \\ \dot{\overline{u}} = i \dfrac{\delta H}{\delta u}. \end{cases}$$

Now, we consider the restriction of this Hamiltonian on the space

$$E = \{ u = c_0 + c_1 e^{ix}, \;\; c_0, c_1 \in \mathbb{C} \}.$$

Example

This induces the ODE

$$i\partial_t u + (\Delta - 1)u = \varepsilon \Pi\big(|u|^2 u\big), \quad (t,x) \in \mathbb{R} \times \mathbb{T}, \quad u \in E, \qquad (16)$$

where Π is the orthogonal projector on E. In coordinates we get

$$\begin{aligned} H_E(u) = H(\Pi u) &= \int_{\mathbb{T}} \Big(|c_0 + c_1 e^{ix}|^2 + |ic_1 e^{ix}|^2 + \frac{\varepsilon}{2}|c_0 + c_1 e^{ix}|^4 \Big) dx \\ &= |c_0|^2 + 2|c_1|^2 + \frac{\varepsilon}{2}(|c_0|^4 + |c_1|^4 + 4|c_0|^2|c_1|^2). \end{aligned}$$

The equation (16) is equivalent to the system

$$\begin{cases} i\dot{c}_0 = c_0\big(1 + \varepsilon(|c_0|^2 + 2|c_1|^2)\big), \\ i\dot{c}_1 = c_1\big(2 + \varepsilon(2|c_0|^2 + |c_1|^2)\big). \end{cases}$$

Example

Observe that

$$M = \int_{\mathbb{T}} |u|^2 dx = |c_0|^2 + |c_1|^2$$

is a constant of motion.
By the Liouville theorem, the Lebesgue measure $dc_0 d\overline{c_0} dc_1 d\overline{c_1}$ is invariant.

• Assume that $\varepsilon = 0$ (linear case). We can define the Gibbs measure

$$d\mu = e^{-(|c_0|^2 + 2|c_1|^2)} dc_0 d\overline{c_0} dc_1 d\overline{c_1}$$

which is finite and also invariant by the flow.
• Assume that $\varepsilon = 1$ (defocusing case). We can define the Gibbs measure

$$d\rho = e^{-H} dc_0 d\overline{c_0} dc_1 d\overline{c_1} = e^{-\frac{1}{2}(|c_0|^4 + |c_1|^4 + 4|c_0|^2|c_1|^2)} d\mu$$

which is finite and also invariant by the flow. Observe that $\rho \ll \mu$. This argument can be adapted for the construction of a Gibbs measure for the complete Schrödinger equation.

Example

• Assume that $\varepsilon = -1$ (focusing case). How to define an analogous version of ρ? It can be given by

$$\begin{aligned} d\rho &= \chi(|c_0|^2 + |c_1|^2) e^{-H} dc_0 d\overline{c_0} dc_1 d\overline{c_1} \\ &= \chi(|c_0|^2 + |c_1|^2) e^{\frac{1}{2}(|c_0|^4 + |c_1|^4 + 4|c_0|^2|c_1|^2)} d\mu, \end{aligned}$$

where $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$ with $0 \leq \chi \leq 1$. Here one uses that M is a constant of motion to prove the invariance of ρ. Observe again that $\rho \ll \mu$. By taking $\chi(t) = e^{-t^K}$ with $K > 2$ the construction still works and the new measure satisfies $\rho(B) > 0$ for any open set $B \subset \mathbb{C}^2$.

A generalisation of this argument has been made by Lebowitz-Rose-Speer for the Schrödinger equation. However, in the infinite dimensional context, the proof is harder since $\|u\|_{L^4}$ can not be controlled by $\|u\|_{L^2}$. We will not give more details here, see *e.g.* Lebowitz-Rose-Speer, Tzvetkov, ... for such constructions.