Invariant Gibbs measures for dispersive PDEs

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Contents

Introduction

Consider a dynamical system. We try to understand the long time behaviour of a family of trajectories, rather than the behaviour of each trajectory one by one. An important tool is the existence of an invariant measure of probability. Here we are concerned with the dynamical system given by the flow of a Hamiltonian dispersive partial differential equation. In this context, we can sometimes define a Gibbs measure. It turns out that such a measure is very efficient to obtain qualitative information about the long time behaviour of the solution of the PDE, and even to prove almost sure global existence results.

The plan of these lectures is the following:

1. In Chapter 1 we introduce the notion of invariant measure and review some classical results of ergodic theory. We illustrate some of the issues with examples in finite dimension.

2. In Chapter 2 we present the main ideas used in the construction of Gibbs measures for PDEs.

3. In Chapter 3 we show how one can use Gibbs measures to construct strong global solutions of Schrödinger equations.

4. In Chapter 4 we present some compactness methods which allow to construct weak global solutions of Schrödinger equations at very low regularity.

Here are some references, from which the material of Chapter 1 has been taken:

- For an introduction to the notion of recurrence, we refer to the article [5] of F. Béguin in Images des Mathématiques (science popularization, in French).

- Lectures on dynamical systems by Y. Benoist and F. Paulin (in French) [8].

- The books of Bunimovich *et al.* [17] and of Cornfeld *et al.* [28].

- The book of Y. Coudène (in French) [29].

- Lectures on invariant measures by C. Liverani [52].

- The book of V. V. Nemytskii and V. V. Stepanov [60].

- Lectures on ergodic theory on the blog of T. Tao [77].

Here are some other lectures and surveys concerning the study and the use of probabilistic arguments and Gibbs measures for dispersive PDEs:

- Lectures on invariant measures and PDEs by A. Nahmod [58].
- Lectures on Gibbs measures and PDEs by T. Oh [62].
- My habilitation thesis, Chapter 1 and Chapter 2 (in French) [78].
- Lectures on the wave equation with random data by N. Tzvetkov [84].
- The book of P. Zhidkov [89].

1 Invariant measures and ODEs

1.1 Definition

Definition 1.1. *(i)* Consider a space X, a measure μ on X and a measurable map $T : X \longrightarrow X$. The measure μ is called invariant with respect to T if for any $\mu-$measurable set A

$$\mu\big(T^{-1}(A)\big) = \mu(A).$$

One also says that T preserves the measure μ.

(ii) Consider a space X and one parameter group $(\Phi(t,.))_{t\in\mathbb{R}}$ with $\Phi(t,.) : X \longrightarrow X$. A measure μ defined in the space X is called invariant with respect to $(\Phi(t,.))_{t\in\mathbb{R}}$ if for any $\mu-$measurable set A

$$\mu\big(\Phi(t,A)\big) = \mu(A), \quad t \in \mathbb{R}.$$

Remark 1.2. • In general, for a measurable map $T : X \longmapsto X$, we can define the measure $T_{\#}\mu := \mu \circ T^{-1}$, called the pushforward measure, and it satisfies

$$\int_X F(x) d(T_{\#}\mu)(x) = \int_X F\big(T(x)\big) d\mu(x),$$

for all measurable $F : X \longrightarrow \mathbb{R}$. In general $\mu \circ T$ is not a measure.

• One can always transform a continuous dynamical system $\big(\Phi(t,.)\big)_{t\in\mathbb{R}}$ into a discrete one by setting $T := \Phi(\tau,.)$ for some $\tau \in \mathbb{R}$.

Exercise 1.3.

1. For a measurable set A, define $\mu_0 = \delta_A$ and set $\mu_t = \Phi(t)\#\mu_0$. Check that $\mu_t = \delta_{\Phi(t,A)}$.
2. If μ is invariant with respect to T, check that in general $\mu(A) \leq \mu(T(A))$.
3. Assume that $X = \mathbb{R}$ and $T \equiv 0$. Find the invariant probability measures.
4. Assume that $X = \mathbb{R}$. Find some maps T which leave the Lebesgue measure invariant.

A first natural question is the existence of an invariant measure of probability, in general. An answer is given by the Krylov-Bogoliuboff theorem, in the case of a compact space X.

Theorem 1.4 (Krylov and Bogoliuboff). Let X be a compact metric space and consider $T : X \longrightarrow X$ a continuous map (resp. $\Phi(t,.) : X \longrightarrow X$). Then there exists an invariant probability measure.

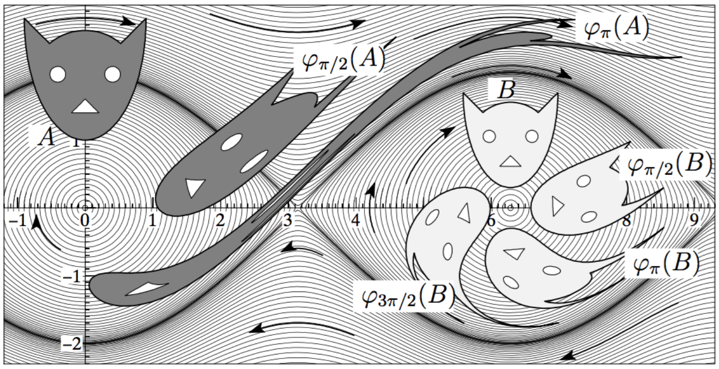

Figure 1.1: Conservation of the area for the pendulum (picture taken on the web).

Proof. We sketch the proof in the case of a discrete dynamical system T. Consider any probability measure ν on X and define the following sequence of measures $(\mu_n)_{n\in\mathbb{N}^*}$ defined by

$$\mu_n = \frac{1}{n}\sum_{k=0}^{n-1} \nu \circ T^{-k} = \frac{1}{n}\sum_{k=0}^{n-1} T^k_{\#}\nu.$$

We have $\mu_n(X) = 1$, therefore by the Riesz theorem (this can also be viewed as a particular case of the Prokhorov theorem, see Theorem 4.4) the sequence $(\mu_n)_{n\geq 1}$ admits a subsequence which converges weakly to a probability measure μ. It is then easy to check (exercise) that μ is invariant under the transform T (see [52] for more details). □

The problem with this result is the lack of information of the obtained invariant measure. For instance, if x_0 is an equilibrium point (fixed point) of the system, the Dirac measure δ_{x_0} is invariant, and we may be interested in less trivial examples. Moreover, this result only holds in compact spaces X.

We may also be interested in studying families of measures, for instance having a density with respect to some reference measure, such as the Lebesgue measure. In the next section we present a useful result which helps us to go into this direction in the context of ODEs: the Liouville theorem.

1.2 The Liouville theorem

Let $\Omega \subset \mathbb{R}^d$ be an open set and $F : \Omega \longrightarrow \mathbb{R}^d$ a C^∞ function. Consider the ordinary differential equation

$$\begin{cases} \dot{x}(t) = \dfrac{dx}{dt}(t) = F(x(t)), \\ x(0) = x_0. \end{cases}$$

We assume that for all $x_0 \in \mathbb{R}$ the system has a unique solution $\Phi(t, x_0)$, such that $\Phi(0, x_0) = x_0$ and which is defined for all $t \in \mathbb{R}$. The family $(\Phi(t, .)_{t\in\mathbb{R}}$ is a one parameter group of diffeomorphisms such that $\Phi(0, .) = id$, $\Phi(t, \Phi(s, .)) = \Phi(t + s, .)$ for all $s, t \in \mathbb{R}$.

Theorem 1.5 (Liouville). Denote by dx the Lebesgue measure on Ω and let $g : \Omega \longrightarrow [0,+\infty)$ a $\mathcal{C}^\infty$ function. The flow $\Phi(t,.)$ preserves the measure gdx if and only if

$$\sum_{k=1}^{d} \frac{\partial}{\partial x_k}\big(gF_k\big) = 0. \tag{1.1}$$

Proof. Denote by $\mathcal{C}_c^\infty(\Omega,\mathbb{R})$ the set of $\mathcal{C}^\infty$ functions with compact support in Ω. By density of $\mathcal{C}_c^\infty(\Omega,\mathbb{R})$ in $L^1(\Omega)$, it is enough to show that for all $t \in \mathbb{R}$ and $f \in \mathcal{C}_c^\infty(\Omega,\mathbb{R})$ we have

$$\int_\Omega f(x)g(x)dx = \int_\Omega f(\Phi(t,x))g(x)dx.$$

Since $\Phi(0,x) = x$, it is enough to show that

$$\frac{d}{dt}\Big(\int_\Omega f(\Phi(t,x))g(x)dx\Big) = 0.$$

Since $\Phi(t,.)$ is a group of diffeomorphisms, it is enough to show that

$$I = \frac{d}{dt}\Big(\int_\Omega f(\Phi(t,x))g(x)dx\Big)_{t=0} = 0,$$

(possibly replace $f(\Phi(t_0,.))$ with $\widetilde{f} \in \mathcal{C}_c^\infty(\Omega,\mathbb{R})$ at $t = t_0$ in the case $t_0 \neq 0$). Now we compute

$$I = \int_\Omega \sum_{k=1}^{d} F_k(x)\frac{\partial f}{\partial x_k}(x)g(x)dx = -\int_\Omega \Big(\sum_{k=1}^{d} \frac{\partial}{\partial x_k}(gF_k)\Big)(x)f(x)dx,$$

hence the result. □

An important class of examples of such systems is given by the Hamiltonian equations. Let $\Omega \subset \mathbb{R}^{2d}$ and $H : \Omega \longrightarrow \mathbb{R}$, $(x,y) \longmapsto H(x,y)$ a $\mathcal{C}^\infty$ function, then the equations corresponding to the Hamiltonian H are

$$\begin{cases} \dot{x}_k(t) = \dfrac{\partial H}{\partial y_k}\big(x(t),y(t)\big), \quad 1 \le k \le d \\ \dot{y}_k(t) = -\dfrac{\partial H}{\partial x_k}\big(x(t),y(t)\big), \quad 1 \le k \le d. \end{cases}$$

Such a system satisfies condition (1.1) with $g \equiv 1$, because

$$\frac{\partial}{\partial x_k}\Big(\frac{\partial H}{\partial y_k}(x,y)\Big) - \frac{\partial}{\partial y_k}\Big(\frac{\partial H}{\partial x_k}(x,y)\Big) = 0.$$

In conclusion, the Lebesgue measure is preserved by the flow (see the illustration in Figure 1.1).

Example 1.6. In the phase space $\mathbb{R}^2$, consider the Hamiltonian $H(x,y) = (x^2+y^2)/2$ and the corresponding system (harmonic oscillator)

$$\begin{cases} \dot{x}(t) = \dfrac{\partial H}{\partial y}\big(x(t),y(t)\big) = y(t), \\ \dot{y}(t) = -\dfrac{\partial H}{\partial x}\big(x(t),y(t)\big) = -x(t). \end{cases} \tag{1.2}$$

Then by the Liouville theorem, the Lebesgue measure $dxdy$ is invariant by the flow as well as $f(H)dxdy$ for any nonnegative measurable function f. Examples of such measures are:

- The Gaussian measure $d\mu = \dfrac{1}{2\pi}\mathrm{e}^{-(x^2+y^2)/2}dxdy$;
- The measure $d\mu = \chi(x^2+y^2)dxdy$ where $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$ with $0 \leq \chi \leq 1$.

Other measures are preserved, namely:

- The Dirac measure δ_0, since 0 is an equilibrium.
- The uniform measure on any circle $x^2+y^2=R$.

Let us notice that, thanks to the ergodic decomposition measure, an invariant measure can be decomposed into a sum of invariant ergodic measures (see [55]).

Exercise 1.7. In the phase space $\mathbb{R}^2$, consider an Hamiltonian of the form $H(x,y) = h(x)$. Which property of the Lebesgue mesure plays here a role in the invariance under the Hamiltonian flow?

1.3 The Poincaré recurrence theorem and applications

1.3.1 The Poincaré theorem for a discrete dynamical system

Theorem 1.8 (Poincaré). Let $(X, \mathcal{B}, \mu)$ be a probability space and let $T : X \longrightarrow X$ be a map which preserves the probability measure μ.

(i) Let $A \in \mathcal{B}$ be such that $\mu(A) > 0$, then there exists $k \geq 1$ such that $\mu(A \cap T^k(A)) > 0$.

(ii) Let $B \in \mathcal{B}$ be such that $\mu(B) > 0$, then for μ-almost all $x \in B$, the orbit $\big(T^n(x)\big)_{n\in\mathbb{N}}$ enters infinitely many times in B.

(iii) Assume that T is invertible. Let $C \in \mathcal{B}$ be such that $\mu(C) > 0$, then

$$\limsup_{n\to\infty} \mu(C \cap T^n(C)) \geq \mu(C)^2.$$

Proof. (i) By the invariance of μ, $\sum_{n\in\mathbb{N}} \mu(T^{-n}(A)) = \sum_{n\in\mathbb{N}} \mu(A) = +\infty$, which implies that there exists $m < n$ such that $\mu(T^{-n}(A) \cap T^{-m}(A)) > 0$. Then,

$$T^{-n}(A) \cap T^{-m}(A) \subset T^{-n}(A) \cap T^{-n}(T^{n-m}(A)) = T^{-n}(A \cap T^{n-m}(A)),$$

and by invariance of the measure, $0 < \mu\big(T^{-n}(A) \cap T^{-m}(A)\big) \leq \mu\big(A \cap T^{n-m}(A)\big)$, hence the result.

(ii) If not, there exists $n_0 \geq 1$ such that the set

$$A = \{x \in B \;\; \forall n \geq n_0,\; T^n(x) \notin B\}$$

satisfies $\mu(A) > 0$. Since one can replace T by T^{n_0}, one can assume $n_0 = 1$. Then by (i), there exists $k \geq 1$ such that $\mu(A \cap T^k(A)) > 0$, which is a contradiction.

(iii) (Taken from [77].) Let $N \geq 1$. Then

$$\int_X \sum_{n=1}^{N} \mathbf{1}_{T^n(C)} d\mu = N\mu(C),$$

and by Cauchy-Schwarz

$$\int_X \big(\sum_{n=1}^{N} \mathbf{1}_{T^n(C)}\big)^2 d\mu \geq N^2 \mu(C)^2. \tag{1.3}$$

We expand the left hand side of (1.3), thus

$$\int_X \big(\sum_{n=1}^{N} \mathbf{1}_{T^n(C)}\big)^2 d\mu = \sum_{n=1}^{N}\sum_{m=1}^{N} \mu\big(T^n(C)\cap T^m(C)\big).$$

Since T is invertible, for $m \geq n$, $\mu\big(T^n(C)\cap T^m(C)\big) = \mu\big(C\cap T^{m-n}(C)\big)$, hence for all $1 \leq k \leq N$

$$\begin{aligned}
\int_X \big(\sum_{n=1}^{N} 1_{T^n(C)}\big)^2 d\mu &= N\mu(C) + 2\sum_{1\leq n<m\leq N} \mu\big(C\cap T^{m-n}(C)\big) \\
&= N\mu(C) + 2\sum_{j=1}^{N-1}(N-j)\mu\big(C\cap T^j(C)\big) \\
&\leq N\mu(C) + 2\sum_{j=1}^{k-1}(N-j)\mu(C\cap T^jC) + (N-k)^2 \sup_{n\geq k}\mu\big(C\cap T^n(C)\big) \\
&\leq N\mu(C) + 2N(k-1)\mu(C) + (N-k)^2 \sup_{n\geq k}\mu\big(C\cap T^n(C)\big).
\end{aligned}$$

Now, by the previous line and (1.3) and the choice $N = k^2$,

$$k^4\mu(C)^2 \leq k^2\mu(C) + 2k^3\mu(C) + (k^2-k)^2 \sup_{n\geq k}\big(C\cap T^n(C)\big).$$

Finally, we divide by k^4 and take the limit $k \longrightarrow +\infty$, which concludes the proof. □

Exercise 1.9. Check what this means for the harmonic oscillator in $\mathbb{R}^2$, $T = \Phi(1,.)$ and $A, B, C = B\big((1,1), 1/2\big)$. More generally consider the system with Hamiltonian

$$H(x,y) = (x^2+y^2)^2/2,$$

for which the speed of rotation of a point depends on the distance to the origin.

Corollary 1.10. Let X be a separable space (which means that it contains a countable, dense subset). Let μ a probability measure, and $T : X \longrightarrow X$ a map which preserves μ. Then μ-almost all point of X is recurrent for T, *i.e.* for μ-almost all point of X there exists a sequence $(n_k)_{k\geq 0}$ going to infinity such that

$$T^{n_k}(x) \longrightarrow x, \quad \text{when } k \longrightarrow +\infty.$$

Proof. Let $\mathcal{C}$ be a countable dense of open subsets of X. By the Poincaré theorem, for all $B \in \mathcal{C}$ there exists a μ-negligible set N_B such that every $x \in B\backslash N_B$, the orbit $\big(T^n(x)\big)_{n\in\mathbb{N}}$ enters infinitely many times in B. The set $N = \bigcup_{B\in\mathcal{C}} N_B$ is also μ-negligible and by construction every point in N^c is recurrent. □

Example 1.11. Consider the differential equation

$$\dot{x}(t) = F(x(t)), \quad \text{in } \Omega \subset \mathbb{R}^d,$$

and its flow $\Phi(t,.)$. Assume that there exists an invariant probability measure ρ. Then for ρ-almost all $x \in \Omega$ there exists a sequence of times $(t_n)_{n\geq 0}$ going to infinity such that

$$\|\Phi(t_n, x) - x\| \longrightarrow 0.$$

Example 1.12. In the phase space $\mathbb{R}^2$, consider the Hamiltonian $H(x,y) = xy$ and the corresponding system

$$\begin{cases} \dot{x}(t) = \dfrac{\partial H}{\partial y}\big(x(t), y(t)\big) = x(t), \\ \dot{y}(t) = -\dfrac{\partial H}{\partial x}\big(x(t), y(t)\big) = -y(t). \end{cases}$$

Again, by the Liouville theorem, the Lebesgue measure $dxdy$ is invariant by the flow (as well as $f(H)dxdy$ for any nonnegative measurable function f), but there is no absolutely continuous measure probability measure which is invariant as well. By contradiction, assume that such a measure μ exists. Consider the square $S_0 = \{1 \leq x \leq 2, \ \ 1 \leq y \leq 2\}$ and S_t its image by the flow. Then for $t \geq 1$, $S_0 \cap S_t = \emptyset$, but this is in contradiction with the Poincaré theorem.

However, there are invariant probability measures, as for example δ_0.

1.3.2 The Poincaré theorem for a continuous dynamical system

What can be said about a continuous dynamical system? Before we give an answer, we need the following definition:

Definition 1.13. Let $A \subset B \subset \mathbb{R}$. We say that A is relatively dense in B if there exists a positive number ℓ such that for all $a \in B$ we have $[a, a+\ell] \cap A \neq \emptyset$.

Then we have the following result (for a proof see [60, page 453])

Theorem 1.14 (Khintchine). Let $0 < \lambda < 1$. Consider $\Phi(t,.) : X \longrightarrow X$ which preserves a probability measure μ. Then for a relatively dense set of $t \in \mathbb{R}$

$$\mu(A \cap \Phi(t, A)) \geq \lambda \mu(A)^2. \tag{1.4}$$

Example 1.15. Consider the harmonic oscillator (1.2) in $\mathbb{R}^2$, set $\mu = dx$ and let $A \subset \mathbb{R}^2$ a set of positive measure (for instance $A = B((1,1), 1/2)$). Denote by $I \subset \mathbb{R}$ the largest interval containing 0 for which (1.4) holds true. Then, by $2\pi-$periodicity of the flowmap, (1.4) holds for $J = \bigcup_{k\in\mathbb{Z}} \big(I + 2\pi k\big)$ which is a relatively dense set in $\mathbb{R}$.

Exercise 1.16. Check that the result of Theorem 1.14 may not hold if $\Phi(t,.)$ is replaced with $\Phi(g(t),.)$ for some functions g.

1.4 The Birkhoff-Khintchine theorem

Consider a measurable set $E \subset X$ and denote by $\mathbf{1}_E$ the indicator function. Then the set of instants of time of the interval $(0, T)$ for which the points $\Phi(t, x) \in E$ is given by $\int_0^T \mathbf{1}_E(\Phi(t, x))dt$. Then the following results implies that the mean

$$\frac{1}{T}\int_0^T \mathbf{1}_E(\Phi(t, x))dt$$

has a limit when $T \longrightarrow +\infty$.

Theorem 1.17 (Birkhoff-Khintchine). Consider $\Phi(t) : X \longrightarrow X$ which preserves a probability measure μ. Then for any function F in $L^1(X, d\mu)$

$$\frac{1}{T}\int_0^T F(\Phi(t, x))dt \longrightarrow G(x), \quad \text{when} \quad T \longrightarrow +\infty,$$

for μ-almost all $x \in X$ and in $L^1(X, d\mu)$.

For a proof we refer to [60, page 460] or to [8, page 29] (written for a discrete dynamical system). In the discrete setting, this result is a kind of law of large numbers (see [47, Chapter 16]).

Under the assumptions of Theorem 1.17, let us prove that

$$\int_X F(x)d\mu = \int_X G(x)d\mu \tag{1.5}$$

Let $T > 0$, then by the invariance of μ

$$\begin{aligned}
\int_X \Big(\frac{1}{T}\int_0^T F(\Phi(t, x))dt\Big)d\mu &= \frac{1}{T}\int_0^T \Big(\int_X F(\Phi(t, x))d\mu\Big)dt \\
&= \frac{1}{T}\int_0^T \Big(\int_X F(x)d\mu\Big)dt \\
&= \int_X F(x)d\mu.
\end{aligned}$$

As a consequence, letting $T \longrightarrow +\infty$, with Theorem 1.17 we obtain (1.5).

As the next results shows, this procedure allows to construct constant of motions (see [60, page 469]).

Proposition 1.18. The function G in Theorem 1.17 is defined μ-almost everywhere and is invariant by the flow, *i.e.* it is constant along every trajectory on which it is defined:

$$G(\Phi(t, x)) = G(x).$$

Proof. Let $t_0 \in \mathbb{R}$ and $T > 0$, then by the flow property

$$G(\Phi(t_0, x)) - G(x) = \lim_{T\to+\infty}\Big(\frac{1}{T}\int_0^T F(\Phi(t + t_0, x))dt - \frac{1}{T}\int_0^T F(\Phi(t, x))dt\Big).$$

Then by a change of variable

$$\begin{aligned}
&\frac{1}{T}\int_0^T F(\Phi(t+t_0,x))dt - \frac{1}{T}\int_0^T F(\Phi(t,x))dt \\
&\qquad = \frac{1}{T}\int_{t_0}^{T+t_0} F(\Phi(t,x))dt - \frac{1}{T}\int_0^T F(\Phi(t,x))dt \\
&\qquad = \frac{T+t_0}{T}\frac{1}{T+t_0}\int_0^{T+t_0} F(\Phi(t,x))dt - \frac{1}{T}\int_0^{t_0} F(\Phi(t,x))dt - \frac{1}{T}\int_0^T F(\Phi(t,x))dt.
\end{aligned}$$

Finally by taking the limit $T \longrightarrow +\infty$ we get $G(\Phi(t_0,x)) - G(x) = 0$. □

Example 1.19. Consider the harmonic oscillator

$$\begin{cases} \dot{x}(t) = y(t), \\ \dot{y}(t) = -x(t). \end{cases}$$

The flowmap reads

$$\Phi\big(t,(x;y)\big) = \begin{pmatrix} x\cos t + y\sin t \\ -x\sin t + y\cos t \end{pmatrix} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} := R(t)(x;y).$$

The Gaussian probability measure on $\mathbb{R}^2$ given by

$$d\mu = \frac{1}{2\pi}\mathrm{e}^{-\frac{1}{2}(x^2+y^2)}dxdy$$

is invariant by the flow. Let F be such that $\int_{\mathbb{R}^2}|F(x;y)|\mathrm{e}^{-\frac{1}{2}(x^2+y^2)}dxdy < \infty$. Then by the Birkhoff-Khintchine theorem, there exists G such that for almost all $(x,y)\in\mathbb{R}^2$

$$\frac{1}{T}\int_0^T F\big(R(t)(x;y)\big)dt \longrightarrow G(x,y), \quad \text{when} \quad T\longrightarrow +\infty.$$

Actually, by writing $T = 2\pi k + r$ with $0 \le r < 2\pi$ and using that $R(t+2\pi) = R(t)$, we see that

$$G(x;y) = \frac{1}{2\pi}\int_0^{2\pi} F\big(R(t)(x;y)\big)dt,$$

and G satisfies

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} G(x;y)\mathrm{e}^{-\frac{1}{2}(x^2+y^2)}dxdy = \frac{1}{2\pi}\int_{\mathbb{R}^2} F(x;y)\mathrm{e}^{-\frac{1}{2}(x^2+y^2)}dxdy.$$

1.5 Gibbs measures in finite dimension

It is now time to give the following definition

Definition 1.20. Let $H : \mathbb{R}^{2d} \longrightarrow \mathbb{R}$ by a smooth function and $\beta > 0$ such that $\mathrm{e}^{-\beta H} \in L^1(\mathbb{R}^{2d}, dx)$. Then the probability measure on $\mathbb{R}^{2d}$

$$d\mu_\beta = \frac{1}{Z_\beta}\mathrm{e}^{-\beta H}dx, \quad Z_\beta = \int_{\mathbb{R}^{2d}} \mathrm{e}^{-\beta H(x)}dx, \tag{1.6}$$

is called a Gibbs measure with energy H. In the context of statistical physics this mesure is also called the Maxwell-Boltzmann, or the canonical distribution. The coefficient Z_β is called the partition function.

As we have seen before, such a measure is invariant under the Hamiltonian dynamics defined by H, but it is not the only one. However, a Gibbs measure has a particular status compared to the other invariant measures as we will see.

1.5.1 Variational characterisation of Gibbs measures

This paragraph is inspired from [66]. Given a probability measure $d\rho = gdx$ that is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^{2d}$, we define its entropy $S(g)$ and average energy $\langle H(g)\rangle$ by

$$S(g) = -\int_{\mathbb{R}^{2d}} g(x)\log\big(g(x)\big)dx \qquad \text{and} \qquad \langle H(g)\rangle = \int_{\mathbb{R}^{2d}} H(x)g(x)dx,$$

where H is the Hamiltonian for the underlying dynamics and we set $M(g) = \displaystyle\int_{\mathbb{R}^{2d}} g(x)dx = 1$. For a given $C \in \mathbb{R}$, we assume that there exists a unique $\beta > 0$ such that $\langle H(\mu_\beta)\rangle = C$ where μ_β is as in (1.6). Now we consider the following maximisation problem

$$\max_{\langle H(g)\rangle = C,\, M(g)=1} S(g), \tag{1.7}$$

and we have the following result.

Proposition 1.21. The Gibbs measure μ_β is the unique maximizer of the problem (1.7).

In statistical mechanics, the equilibrium configuration of the system is dictated by the maximization of the entropy, according to the second law of thermodynamics.

Proof. By the Lagrange multiplier method, there exist $\beta, \delta \in \mathbb{R}$ such that

$$dS(g) = \beta d\langle H(g)\rangle + \delta dM(g),$$

i.e.

$$\int_{\mathbb{R}^{2d}} \big(\log g(x) + 1 + \delta + \beta H(x)\big) f(x)dx = 0,$$

for all test functions f. Thus, we conclude that $g(x) = \mathrm{e}^{-1-\delta-\beta H(x)}$. Moreover, by the mass constraint $M(g) = 1$, we must have $g(x) = Z_\beta^{-1}\mathrm{e}^{-\beta H(x)}$. Therefore, if there is any extremal point for the entropy functional, it has to be the Gibbs measure μ_β. Also, by a direct computation, we have $d^2S(g)(h,h) = -\displaystyle\int_{\mathbb{R}^{2d}} \frac{h^2}{g}dx \le 0$, hence we get uniqueness of the maximizer. □

1.5.2 Gibbs measures and the Langevin equation

The next paragraph is taken from the book [70, Chapter 6]. For more details, we refer to it.

We consider the Langevin equation which describes the motion of a particle that is subject to friction and stochastic forcing

$$\ddot{q} = -\nabla V(q) - \gamma \dot{q} + \sqrt{2\gamma\beta^{-1}}\dot{W}, \quad q \in \mathbb{R}^d. \tag{1.8}$$

This is Newton's equation of motion with two additional terms, a linear dissipation term $\gamma\dot{q}$ and a stochastic forcing $\sqrt{2\gamma\beta^{-1}}\dot{W}$. The parameters of the equation are the friction coefficient $\gamma > 0$ and the temperature $\beta^{-1} = k_B T$, where k_B denotes Boltzmann's constant and T the absolute temperature. The Langevin equation describes the dynamics of a particle that moves according to Newton's second law and is in contact with a thermal reservoir that is at equilibrium at time $t = 0$ at temperature β^{-1}.

Introducing the momentum $p = \dot{q}$, we can write the Langevin equation (1.8) as a system of first-order stochastic differential equations in phase space $(q, p) \in \mathbb{R}^{2d}$:

$$\begin{cases} dq = pdt, \\ dp = -\nabla V(q)dt - \gamma pdt + \sqrt{2\gamma\beta^{-1}}dW. \end{cases} \tag{1.9}$$

The position and momentum $\{q, p\}$ define a Markov process with generator

$$\mathcal{L} = p \cdot \nabla_q - \nabla_q \cdot V \nabla_p + \gamma(-p\nabla_p + \beta^{-1}\Delta_p). \tag{1.10}$$

Now consider the Hamiltonian

$$H(p, q) = \frac{1}{2}|p|^2 + V(q). \tag{1.11}$$

This quantity, as well as any function of it, is invariant under the deterministic Hamiltonian dynamics. This in turns leads to many probability measures that are invariant by the Hamiltonian flow. However, the presence of noise and dissipation in (1.9) results in selecting a unique invariant distribution:

Theorem 1.22. Let V be a smooth confining potential. Then the Markov process with generator (1.10) is ergodic. The unique invariant distribution is

$$d\mu_\beta = \frac{1}{Z_\beta} \mathrm{e}^{-\beta H(p,q)} dpdq,$$

where H is the Hamiltonian (1.11), and the normalization factor Z_β is the partition function

$$Z_\beta = \int_{\mathbb{R}^{2d}} \mathrm{e}^{-\beta H(p,q)} dpdq.$$

Observe that this measure is independent of the friction coefficient $\gamma > 0$.

We will not discuss here stochastic ODEs and PDEs, and we refer to the book of Kuksin and Shirikyan [48] to go into this direction. In the context of SPDEs, there is a stochastic term in the equation, while we handle here deterministic equations with random initial conditions. There are obviously many common features of both approaches, but the philosophy is different: in the first case, typically randomness is a difficulty to tackle, while in the second case is a tool to go beyond the deterministic theory.

1.5.3 An example of renormalisation in the construction of a Gibbs measure

Example 1.23. Let us consider the Schrödinger equation, with periodic boundary conditions

$$i\partial_t u + (\Delta - 1)u = \varepsilon |u|^2 u, \quad (t,x) \in \mathbb{R} \times \mathbb{T},$$

with $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ and $\varepsilon \in \{0, 1, -1\}$. This equation derives from the Hamiltonian

$$H(u) = \int_{\mathbb{T}} \Big(|u|^2 + |\nabla u|^2 + \frac{\varepsilon}{2}|u|^4 \Big) dx,$$

and can be written

$$\begin{cases} \dot{u} = -i\dfrac{\delta H}{\delta \overline{u}}, \\ \dot{\overline{u}} = i\dfrac{\delta H}{\delta u}. \end{cases}$$

Now, we consider the restriction of this Hamiltonian on the space

$$E = \{u = c_0 + c_1 \mathrm{e}^{ix}, \;\; c_0, c_1 \in \mathbb{C}\}.$$

This induces the ODE

$$i\partial_t u + (\Delta - 1)u = \varepsilon \Pi\big(|u|^2 u\big), \quad (t,x) \in \mathbb{R} \times \mathbb{T}, \quad u \in E, \tag{1.12}$$

where Π is the orthogonal projector on E. In coordinates we get

$$\begin{aligned} H_E(u) = H(\Pi u) &= \int_{\mathbb{T}} \Big(|c_0 + c_1 \mathrm{e}^{ix}|^2 + |ic_1 \mathrm{e}^{ix}|^2 + \frac{\varepsilon}{2}|c_0 + c_1\mathrm{e}^{ix}|^4 \Big) dx \\ &= |c_0|^2 + 2|c_1|^2 + \frac{\varepsilon}{2}(|c_0|^4 + |c_1|^4 + 4|c_0|^2|c_1|^2). \end{aligned}$$

The equation (1.12) is equivalent to the system

$$\begin{cases} i\dot{c}_0 = c_0\big(1 + \varepsilon(|c_0|^2 + 2|c_1|^2)\big), \\ i\dot{c}_1 = c_1\big(2 + \varepsilon(2|c_0|^2 + |c_1|^2)\big). \end{cases}$$

Observe that

$$M = \int_{\mathbb{T}} |u|^2 dx = |c_0|^2 + |c_1|^2$$

is a constant of motion.

By the Liouville theorem, the Lebesgue measure $dc_0 d\overline{c_0} dc_1 d\overline{c_1}$ is invariant.

- Assume that $\varepsilon = 0$ (linear case). We can define the Gibbs measure

$$d\mu = \mathrm{e}^{-(|c_0|^2 + 2|c_1|^2)} dc_0 d\overline{c_0} dc_1 d\overline{c_1}$$

which is finite and also invariant by the flow.

- Assume that $\varepsilon = 1$ (defocusing case). We can define the Gibbs measure

$$d\rho = \mathrm{e}^{-H} dc_0 d\overline{c_0} dc_1 d\overline{c_1} = \mathrm{e}^{-\frac{1}{2}(|c_0|^4 + |c_1|^4 + 4|c_0|^2|c_1|^2)} d\mu$$

which is finite and also invariant by the flow. Observe that $\rho \ll \mu$. This argument can be adapted for the construction of a Gibbs measure for the complete Schrödinger equation.

• Assume that $\varepsilon = -1$ (focusing case). How to define an analogous version of ρ? It can be given by

$$d\rho = \chi(|c_0|^2 + |c_1|^2)\mathrm{e}^{-H} dc_0 d\overline{c_0} dc_1 d\overline{c_1} = \chi(|c_0|^2 + |c_1|^2)\mathrm{e}^{\frac{1}{2}(|c_0|^4+|c_1|^4+4|c_0|^2|c_1|^2)} d\mu,$$

where $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$ with $0 \leq \chi \leq 1$. Here one uses that M is a constant of motion to prove the invariance of ρ. Observe again that $\rho \ll \mu$. By taking $\chi(t) = \mathrm{e}^{-t^K}$ with $K > 2$ the construction still works and the new measure satisfies $\rho(B) > 0$ for any open set $B \subset \mathbb{C}^2$.

A generalisation of this argument has been made by Lebowitz-Rose-Speer [50] for the Schrödinger equation. However, in the infinite dimensional context, the proof is harder since $\|u\|_{L^4}$ can not be controlled by $\|u\|_{L^2}$. We will not give more details here, see *e.g.* [50, 82, 19] for such constructions.

2 Construction of Gibbs measures for PDEs

2.1 The Khintchine inequality and the Wiener chaos

Let $(\Omega, \mathcal{F}, \mathbf{p})$ be a probability space, and consider a sequence of independent standard complex Gaussians $g_n \in \mathcal{N}_{\mathbb{C}}(0,1)$ which means that g_n can be written

$$g_n(\omega) = \frac{1}{\sqrt{2}}\big(h_n(\omega) + i\ell_n(\omega)\big),$$

where $\big(h_n(\omega),\, \ell_n(\omega)\big)_{n\geq 1}$ are independent standard real Gaussians $\mathcal{N}_{\mathbb{R}}(0,1)$. Then the following result holds true (see [22, Lemma 3.1] for a proof), known as the Paley-Zygmund inequality or Khintchine inequality

Lemma 2.1 (Khintchine). There exists a constant $C > 0$ such that for all $p \geq 2$ and $(c_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$

$$\|\sum_{n=0}^{+\infty} c_n g_n(\omega)\|_{L^p(\Omega)} \leq C\sqrt{p}\big(\sum_{n=0}^{+\infty} |c_n|^2\big)^{1/2}. \tag{2.1}$$

This result shows that any L^p norm can be controlled by a L^2 norm, which is a genuine smoothing effect. Let us make a parallel with harmonic analysis: in this context Sobolev inequalities can be used, but at the price of loss of derivatives.

Actually, Lemma 2.1 holds for more general centered and localised random variables, like centered Bernoulli r.v. For instance, take $c_n = 1/(n+1)$ and (g_n) a sequence of independent centered Bernoulli r.v. Then the series $\sum_{n=0}^{+\infty} c_n$ diverges, but according to (2.1) we have $|\sum_{n=0}^{+\infty} c_n g_n(\omega)| < +\infty$, $\mathbf{p}-$a.s., in other words, randomising the signs makes the series a.s. converge.

From Lemma 2.1 we can deduce a large deviations estimate.

Corollary 2.2 (Large deviations). There exist constants $c, C > 0$ such that for all $(c_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$ and $\lambda > 0$

$$\mathbf{p}\Big(\omega \in \Omega \,:\, \Big|\sum_{n=0}^{+\infty} c_n g_n(\omega)\Big| > \lambda\Big) \leq C\mathrm{e}^{-c\lambda^2/\|c\|^2_{\ell^2}}. \tag{2.2}$$

Proof. Let $(c_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$, then by (2.1) we get for all $p \geq 2$ $\big\| \sum_{n=0}^{+\infty} c_n g_n(\omega)\big\|_{L^p(\Omega)} \leq C\sqrt{p}\|c\|_{\ell^2}$.

Using the Markov inequality, we obtain that for all $\lambda > 0$

$$\mathbf{p}\big(\omega \,:\, \big|\sum_{n=0}^{+\infty} c_n g_n(\omega)\big| > \lambda\,\big) \leq (\lambda^{-1}\big\|\sum_{n=0}^{+\infty} c_n g_n(\omega)\big\|_{L^p(\Omega)})^p \leq (C\lambda^{-1}\sqrt{p}\|c\|_{\ell^2})^p\,.$$

Thus by choosing $p = \delta\lambda^2/\|c\|^2_{\ell^2}$, for δ small enough, we get the bound (2.2). □

We sometimes need a multilinear version of (2.1). The result can be proved using hypercontractivity estimate of the Ornstein-Uhlenbeck semi-group and is classical in quantum field theory (see [72, Theorem I.22]).

Proposition 2.3 (Wiener Chaos). Let $c(n_1,\dots,n_k) \in \mathbb{C}$ and $(g_n)_{n\geq 0} \in \mathcal{N}_{\mathbb{C}}(0,1)$ independent standard Gaussians and normalised in L^2. For $k \geq 1$ we define

$$S_k(\omega) = \sum_{n\in\mathbb{N}^k} c(n_1,\dots,n_k)\, g_{n_1}(\omega)\cdots g_{n_k}(\omega).$$

Then for all $p \geq 2$

$$\|S_k\|_{L^p(\Omega)} \leq (p-1)^{\frac{k}{2}}\,\|S_k\|_{L^2(\Omega)}. \tag{2.3}$$

This result means that the nonlinear estimates can be reduced to the case $p = 2$. It is useful to control nonlinear terms which are not perfect powers, *e.g.* $\int_{\mathbb{T}} \overline{u}^2\partial_x(u^2)$: this term appears in the study of DNLS, see [79].

The explicit bound (2.3) in k, p implies the large deviation estimate

$$\mathbf{p}\Big(\omega \in \Omega \,:\, \Big|S_k(\omega)\Big| > \lambda\Big) \leq C\mathrm{e}^{-c\lambda^{2/k}}.$$

2.2 Definition of the Gaussian measure in the case of the torus

Let $(\Omega, \mathcal{F}, \mathbf{p})$ be a probability space and $\big(g_n(\omega)\big)_{n\geq 1}$ a sequence of independent complex normalised Gaussians, $g_n \in \mathcal{N}_{\mathbb{C}}(0,1)$. Here we show how we can construct a Gaussian measure on $H^\sigma(\mathbb{T}^d)$.

Consider a Hilbertian basis $(e_n)_{n\geq 1}$ of $L^2(\mathbb{T}^d)$ of eigenfunctions of $(1-\Delta)$. Then

$$(1-\Delta)e_n = \lambda_n^2 e_n, \quad n \geq 1, \quad x \in \mathbb{T}^d,$$

and one has $\lambda_n \sim cn^{1/d}$ when $n \longrightarrow +\infty$.

For $N \geq 1$ we define the random variable

$$\omega \longmapsto \varphi_N(\omega, .) = \sum_{n=1}^{N} \frac{g_n(\omega)}{\lambda_n} e_n(.),$$

then we have the following result

Proposition 2.4. Assume that $\sigma < 1-d/2$, then $(\varphi_N)_{N\geq 1}$ is a Cauchy sequence in $L^2\big(\Omega;\, H^\sigma(\mathbb{T}^d)\big)$. This enables us to define its limit

$$\omega \longmapsto \gamma(\omega,.) = \sum_{n\geq 1} \frac{g_n(\omega)}{\lambda_n} e_n(.) \in L^2\big(\Omega;\, H^\sigma(\mathbb{T}^d)\big).$$

Notice that the law of γ does not depend on the choice of the Hilbertian basis $(e_n)_{n\geq 1}$.

Proof. We only show that $\gamma \in L^2\big(\Omega;\, H^\sigma(\mathbb{T}^d)\big)$. For $\sigma \in \mathbb{R}$, we compute

$$\|\gamma(\omega,.)\|^2_{H^\sigma(\mathbb{T}^d)} = \sum_{n\geq 1} \frac{|g_n(\omega)|^2}{\lambda_n^{2-2\sigma}},$$

thus

$$\|\gamma\|^2_{L^2(\Omega;\, H^\sigma(\mathbb{T}^d))} = \sum_{n\geq 1} \frac{1}{\lambda_n^{2-2\sigma}}, \tag{2.4}$$

and we can conclude that the series converges iff $\sigma < 1 - d/2$, using the asymptotic formula $\lambda_n \sim cn^{1/d}$ when $n \longrightarrow +\infty$. □

Exercise 2.5. Let $\sigma \geq 1 - d/2$. Show that for almost all $\omega \in \Omega$, $\|\gamma(\omega,.)\|_{H^\sigma(\mathbb{T}^d)} = +\infty$. Hint: with an explicit computation, show that

$$\int_\Omega \mathrm{e}^{-\|\gamma(\omega,.)\|^2_{H^\sigma(\mathbb{T}^d)}} d\mathbf{p}(\omega) = \mathbf{0}.$$

This result can also be deduced from (2.4) using general convergence results on random series in Banach spaces (see [42, Section 5]).

Denote by

$$X^\sigma(\mathbb{T}^d) = \bigcap_{\tau<\sigma} H^\tau(\mathbb{T}^d).$$

We then define the Gaussian probability measure μ on $X^{1-d/2}(\mathbb{T}^d)$ by

$$\mu = \mathbf{p} \circ \gamma^{-1}. \tag{2.5}$$

In other words, μ is the image of the measure $\mathbf{p}$ under the map

$$\begin{array}{rcl} \Omega & \longrightarrow & X^{1-d/2}(\mathbb{T}^d) \\ \omega & \longmapsto & \gamma(\omega,\cdot) = \displaystyle\sum_{n\geq 1} \frac{g_n(\omega)}{\lambda_n} e_n, \end{array}$$

which means that for all measurable $F : X^{\sigma_c}(\mathbb{T}^d) \longrightarrow \mathbb{R}$

$$\int_{X^{1-d/2}(\mathbb{T}^d)} F(u)\mathrm{d}\mu(u) = \int_\Omega F\big(\gamma(\omega,\cdot)\big)\mathrm{d}\mathbf{p}(\omega).$$

Formally, one has

$$\text{"} d\mu = \frac{1}{Z}\mathrm{e}^{-H_0(c)} dc d\overline{c} \text{"}, \quad H_0(c) = \sum_{n=1}^{+\infty} \lambda_n^2 |c_n|^2, \tag{2.6}$$

in other words, μ is the Gibbs measure of the linear Schrödinger equation

$$i\partial_t u + (\Delta - 1)u = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{T}^d. \tag{2.7}$$

Then notation (2.6) can be understood thanks to the next result.

Proposition 2.6. Let $D \geq 1$ and denote by $(e_n)_{1\leq n\leq D}$ the canonical basis of $\mathbb{R}^D$. Define the measure $\mu = \mathbf{p} \circ \gamma^{-1}$ by

$$\gamma = \sum_{n=1}^{D} \frac{g_n}{\lambda_n} e_n.$$

Then μ is the Gaussian measure

$$d\mu = \frac{1}{Z} \mathrm{e}^{-H_0(c)} dL(c), \quad H_0(c) = \sum_{n=1}^{D} \lambda_n^2 |c_n|^2,$$

where dL is the Lebesgue measure in $\mathbb{C}^D$.

Proof. We compute $\mu(A)$ for a cuboid $A = \prod_{n=1}^{D} [\alpha_n, \beta_n] \subset \mathbb{C}^D$, with $\alpha_n = a_n + ic_n$ and $\beta_n = b_n + id_n$. We write $g_n = (h_n + i\ell_n)/\sqrt{2}$ with $h_n, \ell_n \in \mathcal{N}_{\mathbb{R}}(0,1)$. Then we have

$$\begin{aligned}
\mu(A) &= p(\gamma \in A) = p\big(\omega : \gamma(\omega) \in A)\big) \\
&= \int_{\Omega} \mathbf{1}_{\{\gamma(\omega)} \in A\} d\mathbf{p}(\omega) \\
&= \prod_{n=1}^{D} \int_{\{\frac{g_n(\omega)}{\lambda_n} \in [\alpha_n, \beta_n]\}} d\mathbf{p}(\omega) \\
&= \prod_{n=1}^{D} \int_{\{\frac{h_n(\omega)}{\sqrt{2}\lambda_n} \in [a_n, b_n]\}} d\mathbf{p}(\omega) \int_{\{\frac{\ell_n(\omega)}{\sqrt{2}\lambda_n} \in [c_n, d_n]\}} d\mathbf{p}(\omega) \\
&= \prod_{n=1}^{D} \Big(\frac{\lambda_n^2}{\pi} \int_{a_n}^{b_n} \int_{c_n}^{d_n} \mathrm{e}^{-\lambda_n^2 (x_n^2 + y_n^2)} dx_n dy_n \Big) \\
&= \frac{(\prod_{n=1}^{D} \lambda_n)^2}{\pi^D} \int_A \mathrm{e}^{-\sum_{n=1}^{D} \lambda_n^2 |c_n|^2} dL(c),
\end{aligned}$$

which was the claim. □

Let's come back to the measure μ defined in (2.5). We summarize its main properties:

- $\mu(X^{1-d/2}(\mathbb{T}^d)) = 1$ (μ is a probability measure) ;
- $\mu(H^{1-d/2}(\mathbb{T}^d)) = 0$ (The support of μ is composed of rough functions, see Exercise 2.5.) Actually, this shows that the function $x \longmapsto \sum_{n\geq 1} \frac{g_n(\omega)}{\lambda_n} e_n(x)$ has almost surely the same Sobolev regularity than the function $x \longmapsto \sum_{n\geq 1} \frac{1}{\lambda_n} e_n(x)$. However there is a regularisation at the L^p scale, as will be seen in Chapter 3 ;

- Let $\sigma < 1 - d/2$. Then for any open set, $B \subset H^\sigma(\mathbb{T}^d)$, $B \neq \emptyset$, we have $\mu(B) > 0$;
- The previous construction can easily be adapted to the case of a compact manifold $\mathcal{M}$, where $(e_n)_{n\geq 1}$ is a Hilbertian basis of $L^2(\mathcal{M})$ of eigenfunctions of the Laplacian: $(1-\Delta)e_n = \lambda_n^2 e_n$, $n \geq 1$. The asymptotic of the $\lambda_n \sim cn^{1/d}$ is given by the Weyl formula.

We stress that the support of μ is rough when $d \geq 2$. The regularity of the support of a Gibbs measure is given by the linear part of the equation (even in the nonlinear case). In general, if there is few dispersion or if the dimension increases, then the support of the measure becomes rough.

We conclude this paragraph with an elementary result.

Proposition 2.7. The measure μ defined in (2.5) is invariant by the flow of the linear Schrödinger equation (2.7).

Proof. Denote by Φ the flow of (2.7), then

$$\Phi\big(t, \gamma(\omega, x)\big) = \sum_{n\geq 1} \frac{\mathrm{e}^{-it\lambda_n^2} g_n(\omega)}{\lambda_n} e_n(x),$$

and we observe that this r.v. has the same law as γ because of the rotation invariance of the complex Gaussians. □

2.3 Singular measures and perturbations

Consider the family of measures

$$d\mu_\beta = \frac{1}{Z_\beta} \mathrm{e}^{-\beta H_0(c)} dc d\bar{c},$$

with $\beta > 0$. What happens when β varies? To answer this question we will need the Kakutani theorem (see [45]).

Theorem 2.8. Consider the infinite tensor products of probability measures on $\mathbb{R}^{\mathbb{N}}$

$$\mu_i = \bigotimes_{n\in\mathbb{N}} \mu_{n,i}, \qquad i = 1, 2.$$

Then the measures μ_1 and μ_2 on $\mathbb{R}^{\mathbb{N}}$ endowed with its cylindrical Borel σ-algebra are absolutely continuous with respect each other, $\mu_1 \ll \mu_2$, and $\mu_2 \ll \mu_1$, if and only if the following holds:

(i) The measures $\mu_{n,1}$ and $\mu_{n,2}$ are for each n absolutely continuous with respect to each other: there exists two functions $g_n \in L^1(\mathbb{R}, d\mu_{n,2})$, $h_n \in L^1(\mathbb{R}, d\mu_{n,1})$ such that

$$d\mu_{n,1} = g_n d\mu_{n,2}, \qquad d\mu_{n,2} = h_n d\mu_{n,1}.$$

(ii) The functions g_n are such that the infinite product

$$\prod_{n\in\mathbb{N}} \int_{\mathbb{R}} g_n^{1/2} d\mu_{n,2}$$

is convergent ($i.e.$ positive).

Furthermore, if any of the condition above is not satisfied (*i.e.* if the two measures μ_1 and μ_2 are not absolutly continuous with respect to each other), then the two measures are mutually singular ($\mu_1 \perp \mu_2$): there exists a set $A \subset \mathbb{R}^{\mathbb{N}}$ such that

$$\mu_1(A) = 1, \qquad \mu_2(A) = 0.$$

An application of the previous results yields

Corollary 2.9. Let $(e_n)_{n\geq 1}$ be a Hilbertian basis of $L^2(\mathbb{T}^d)$. Then

(*i*) Consider $\alpha_n, \beta_n > 0$ and the measures $\mu = \mathbf{p} \circ \gamma^{-1}$ and $\nu = \mathbf{p} \circ \psi^{-1}$ with

$$\gamma = \sum_{n=1}^{+\infty} \frac{g_n}{\alpha_n} e_n, \qquad \psi = \sum_{n=1}^{+\infty} \frac{g_n}{\beta_n} e_n.$$

Then the mesures μ and ν are absolutely continuous with respect to each other if and only if

$$\sum_{n=1}^{+\infty} (\frac{\alpha_n}{\beta_n} - 1)^2 < +\infty.$$

(*ii*) Consider $\lambda_n > 0$ and the measures $\mu_\beta = \mathbf{p} \circ \gamma_\beta^{-1}$ with

$$\gamma_\beta = \sum_{n=1}^{+\infty} \frac{g_n}{\beta \lambda_n} e_n.$$

Assume that $\beta \neq \beta'$, then the measures μ_β and $\mu_{\beta'}$ are singular.

Exercise 2.10. Let $\beta, \beta' > 0$ with $\beta \neq \beta'$. Construct an explicit set A such that $\mu_\beta(A) = 1$ and $\mu_{\beta'}(A) = 0$.

An another natural question is the behaviour of μ under transformations.

In the case of translations, the answer is given by the Cameron-Martin theorem, and we state it only in the particular case of the measure (2.5). Recall that μ is a probability measure on $X^{1-d/2}(\mathbb{T}^d)$.

Theorem 2.11 (Cameron-Martin). Given $h \in X^{1-d/2}(\mathbb{T}^d)$, define the shifted measure μ^h by $\mu^h = \mu(. - h)$. Then, the measure μ^h is mutually absolutely continuous with respect to μ if and only if $h \in H^1(\mathbb{T}^d)$.

For more details and applications, see [66].

We say that a measure μ is quasi-invariant under a transformation T if μ and $T\#\mu$ are mutually absolutely continuous, or equivalently that their zero measure sets are preserved. This is a natural extension of the (rigid) concept of invariant measure, and this notion is particularly relevant in infinite dimension. For quasi-invariance of Gaussian measures under the flow of dispersive PDEs, we refer to the recent papers [83, 69].

For more analysis of Gaussian measures on Hilbert or Banach spaces, we refer to [43] and to [49].

2.4 Regularity results for random series in L^p spaces

We state here known convergence results on the convergence of random series in Banach spaces (in L^p actually). The following result is a combination of results of Hoffman-Jorgensen [41] and Maurey-Pisier [57]. For an introduction on this topic, we refer to the books of Marcus-Pisier [56], J.-P. Kahane [44] and to the book of Li and Queffélec [51]. See also Imekraz-Robert-Thomann [42] and references therein.

Theorem 2.12. Let $p \in [2, +\infty)$ and $(F_n)_{n \geq 0} \in L^p(\mathbb{R}^d)$. Assume that $(g_n)_{n \geq 0} \in \mathcal{N}_{\mathbb{C}}(0,1)$ is i.i.d. and that $(\varepsilon_n)_{n \geq 0} \in \{-1, 1\}$ is an i.i.d. Rademacher sequence.

The following statements are equivalent:

(i) the series $\sum \varepsilon_n F_n$ converges almost surely in $L^p(\mathbb{R}^d)$,

(ii) the series $\sum g_n F_n$ converges almost surely in $L^p(\mathbb{R}^d)$,

(iii) the function $\sum_{n \geq 0} |F_n|^2$ belongs to $L^{\frac{p}{2}}(\mathbb{R}^d)$.

There are also continuity results for random series (Paley-Zygmund, Salem-Zygmund, ...). See references in [42].

Application: We define the so-called special Hermite function by

$$\varphi_n(z) = \frac{1}{\sqrt{\pi n!}} z^n \mathrm{e}^{-|z|^2/2}, \qquad n \geq 0, \tag{2.8}$$

and the Gaussian random variable

$$\eta(\omega, z) = \sum_{n=0}^{+\infty} g_n(\omega) \varphi_n(z) = \frac{1}{\sqrt{\pi}} \Big(\sum_{n=0}^{+\infty} \frac{z^n g_n(\omega)}{\sqrt{n!}} \Big) \mathrm{e}^{-|z|^2/2}.$$

Proposition 2.13. Let $2 \leq p < +\infty$. Then $\eta(\omega, .) \notin L^p(\mathbb{C})$ for almost all $\omega \in \Omega$.

Proof. We simply observe that $\sum_{n=0}^{+\infty} |\varphi_n(z)|^2 \equiv 1$ and that a random series either converges a.s. or diverges a.s. □

Exercise 2.14. Let $(\varphi_n)_{n \geq 0}$ be defined by (2.8) and consider the random variable

$$\gamma(\omega, z) = \sum_{n=0}^{+\infty} \frac{g_n(\omega)}{\sqrt{2(n+1)}} \varphi_n(z) = \frac{1}{\sqrt{2\pi}} \Big(\sum_{n=0}^{+\infty} \frac{z^n g_n(\omega)}{\sqrt{(n+1)!}} \Big) \mathrm{e}^{-|z|^2/2}.$$

Show that, for $2 \leq p < +\infty$, $\gamma(\omega, .) \in L^p(\mathbb{C})$ for a.a. $\omega \in \Omega$, but that $\gamma(\omega, .) \notin L^2(\mathbb{C})$ for a.a. $\omega \in \Omega$. (In the case $2 \leq p < +\infty$, see also Lemma 3.16 for a proof using the Khintchine inequality (2.1), and in the case $p = 2$, see Exercise 2.5.)

2.5 Nonlinearities

We now turn to the nonlinear Schrödinger equation on $\mathbb{T}^d$,

$$i\partial_t u + (\Delta - 1)u = |u|^{p-1}u, \quad (t,x) \in \mathbb{R} \times \mathbb{T}^d.$$

The Hamiltonian of this equation is

$$H = \int_{\mathbb{T}^d} (|u|^2 + |\nabla u|^2) + \frac{2}{p+1}\int_{\mathbb{T}^d} |u|^{p+1}.$$

We denote by μ the Gaussian measure which corresponds to the linear problem (2.7). We are able to construct a Gibbs measure to this problem in the following cases:

- In dimension $d = 1$: μ is supported in $X^{1/2}(\mathbb{T})$. A Sobolev imbedding argument yields $\displaystyle\int_{\mathbb{T}} |u|^{p+1} < +\infty$, $\mu-$a.s. and one can define a Gibbs measure by

$$\mathrm{d}\rho(u) = \exp\big(-\frac{2}{p+1}\|u\|^{p+1}_{L^{p+1}(\mathbb{T})}\big)\mathrm{d}\mu(u).$$

- In dimension $d = 2$: μ is supported in $X^0(\mathbb{T}^2)$. In this case $\displaystyle\int_{\mathbb{T}^2} |u|^{p+1} = +\infty$, $\mu-$a.s. because $\displaystyle\int_{\mathbb{T}^2} |u|^2 = +\infty$, $\mu-$a.s. Therefore, the construction is more difficult and has been done for $p = 3$ by Bourgain [11] with a Wick renormalisation of the non-linearity. This can be extended to any $p \in 2\mathbb{N} + 1$, see [68] and references therein.

- In dimension $d \geq 3$: the situation is unclear to me.

The construction of Gibbs measures of focusing equations is harder in general. Actually, if we set $\mathrm{d}\rho(u) = G(u)\mathrm{d}\mu(u)$ we have to check that the density is integrable with respect to μ, *i.e.* $G \in L^p(\mathrm{d}\mu)$. This induces some restrictions on the degree of the non-linearity and needs renormalisation arguments. There are also non existence results, see Brydges-Slade [16].

For the mathematical construction of Gibbs measures or more generally Wiener measures for dispersive PDEs, we refer to P. Zhidkov [89], Lebowitz-Rose-Speer [50], B. Bidégaray [9], J. Bourgain [10, 11], and more recently to N. Tzvetkov [80, 81, 82], Burq-Tzvetkov [21, 23], Thomann-Tzvetkov [79], Burq-Thomann-Tzvetkov [19], T. Oh [63, 64], Tzvetkov-Visciglia [85], Bourgain-Bulut [13] and Oh-Thomann [68]. We also mention the recent result of Sy [76] who constructs an invariant measure for the Benjamin-Ono equation, which is supported on smooth functions.

3 Almost sure global wellposedness of the LLL equation below $L^2(\mathbb{C})$

In this chapter, we show how we can use a Gibbs measure to prove almost sure global existence results. We will present the method on the Landau Lowest Level (LLL) equation. The results are taken from Germain-Hani-Thomann [36] and some analysis from Germain-Hani-Thomann [35] and Gérard-Germain-Thomann [34].

3.1 Introduction

For $1 \leq p \leq +\infty$ we define the Bargmann-Fock spaces

$$F^p(\mathbb{C}) = \left\{ u(z) = \mathrm{e}^{-\frac{|z|^2}{2}} f(z) \ , \ f \text{ entire holomorphic} \right\} \cap L^p(\mathbb{C}) \ .$$

In the sequel we consider the Lowest Landau Level equation which reads

$$\begin{cases} i\partial_t u = \Pi(|u|^2 u), & (t, z) \in \mathbb{R} \times \mathbb{C}, \\ u(0, z) = u_0(z), \end{cases} \tag{LLL}$$

where Π is the orthogonal projector on $F^2(\mathbb{C})$.

This equation is used in the description of fast rotating Bose-Einstein condensates, see *e.g.* [2, 3, 61], the book [1] and references therein. The equation (LLL) can be obtained as the restriction of the continuous resonant equation (CR) which was introduced by Faou-Germain-Hani [33] and further studied in [35, 36]. The equation (CR) (and therefore (LLL)) can be derived from the Gross-Pitaevskii equation with partial confinement, see Hani-Thomann [39].

Let $z = x + iy$. Denote by H the harmonic oscillator $H = -\partial_x^2 - \partial_y^2 + x^2 + y^2$. A Hilbertian basis of normalized eigenfunctions of H for $F^2(\mathbb{C})$ is given by the so-called special Hermite functions defined for $n \geq 0$ by

$$\varphi_n(z) = \frac{1}{\sqrt{\pi n!}} z^n \mathrm{e}^{-|z|^2/2},$$

and which satisfy

$$H\varphi_n = 2(n+1)\varphi_n.$$

Therefore, every $u \in F^2(\mathbb{C})$ can be decomposed in a series

$$u = \sum_{n=0}^{+\infty} c_n \varphi_n. \tag{3.1}$$

We are able to explicitly compute the kernel of Π

$$\sum_{n=0}^{+\infty} \varphi_n(z)\overline{\varphi_n(w)} = \frac{1}{\pi}\Big(\sum_{n=0}^{+\infty}\frac{1}{n!}(z\overline{w})^n\Big)\mathrm{e}^{-|z|^2/2-|w|^2/2} = \frac{1}{\pi}\mathrm{e}^{z\overline{w}-|z|^2/2-|w|^2/2}.$$

As a consequence,

$$[\Pi u](z) = \frac{1}{\pi}\mathrm{e}^{-\frac{|z|^2}{2}}\int_{\mathbb{C}} \mathrm{e}^{z\overline{w}-\frac{|w|^2}{2}}u(w)\,dL(w),$$

where dL stands for the Lebesgue measure on $\mathbb{C}$.

We define the trilinear operator $\mathcal{T}$ by

$$\mathcal{T}(u_1,u_2,u_3) = \Pi(u_1u_2\overline{u_3}). \tag{3.2}$$

The equation (LLL) is Hamiltonian: indeed, introducing the functional

$$\begin{aligned}\mathcal{E}(u_1,u_2,u_3,u_4) &\overset{def}{=} \langle \mathcal{T}(u_1,u_2,u_3)\,,\,u_4\rangle_{L^2(\mathbb{C})}\\ &= \int_{\mathbb{C}}(u_1u_2\overline{u_3}\overline{u_4})(z)dL(z)\end{aligned}$$

and setting

$$\mathcal{E}(u) := \mathcal{E}(u,u,u,u) = \int_{\mathbb{C}}|u(z)|^4dL(z) = \|u\|^4_{L^4(\mathbb{C})},$$

then (LLL) derives from the Hamiltonian $\mathcal{E}$ given the symplectic form

$$\omega(f,g) = \mathfrak{Im}\int_{\mathbb{C}} f\overline{g}\,dL,$$

so that (LLL) is equivalent to

$$i\partial_t u = \frac{1}{2}\frac{\partial\mathcal{E}(u)}{\partial\bar{u}}.$$

The family $(\varphi_n)_{n\geq 0}$ is particularly well adapted in the study of the operator $\mathcal{T}$ since on has (see [35, Lemma 7.1])

$$\mathcal{T}(\varphi_{n_1},\varphi_{n_2},\varphi_{n_3}) = \alpha_{n_1,n_2,n_3,n_4}\varphi_{n_4}, \qquad n_4 = n_1+n_2-n_3, \tag{3.3}$$

with

$$\alpha_{n_1,n_2,n_3,n_4} = \mathcal{E}(\varphi_{n_1},\varphi_{n_2},\varphi_{n_3},\varphi_{n_4}) = \frac{\pi}{2}\frac{(n_1+n_2)!}{2^{n_1+n_2}\sqrt{n_1!n_2!n_3!n_4!}}\mathbf{1}_{n_1+n_2=n_3+n_4}.$$

Using (3.3) we can prove that $\mathrm{e}^{itH}\mathcal{T}(u_1,u_2,u_3) = \mathcal{T}(\mathrm{e}^{itH}u_1,\mathrm{e}^{itH}u_2,\mathrm{e}^{itH}u_3)$, and therefore with the change of unknowns $v = \mathrm{e}^{itH}u$ we see that (LLL) is equivalent to the equation

$$i\partial_t v + Hv = \Pi(|v|^2v), \quad (t,z)\in\mathbb{R}\times\mathbb{C}. \tag{3.4}$$

3.2 Some deterministic results

3.2.1 Well-posedness of the LLL equation

Define the harmonic Sobolev spaces for $s \in \mathbb{R}$, by

$$\mathcal{H}^s = \mathcal{H}^s(\mathbb{C}) = \big\{ u \in \mathcal{S}'(\mathbb{C}),\ H^{s/2}u \in L^2(\mathbb{C}) \big\}.$$

This is a weighted Sobolev norm. In the Bargmann-Fock space, it simply corresponds to a weighted L^2-norm. Set $\langle z \rangle = (1+|z|^2)^{1/2}$, then we have (see [34] for a proof).

Lemma 3.1. Let $s \in \mathbb{R}$. There exists $C > 0$ such that for all $u \in F^2(\mathbb{C}) \cap \mathcal{H}^s(\mathbb{C})$

$$\frac{1}{C} \|\langle z \rangle^s u\|_{L^2(\mathbb{C})} \leq \|u\|_{\mathcal{H}^s(\mathbb{C})} \leq C \|\langle z \rangle^s u\|_{L^2(\mathbb{C})}.$$

Exercise 3.2. Prove Lemma 3.1 in the particular case where $s \in 2\mathbb{N}$. *Hint:* use the decomposition (3.1), and the relations $z\varphi_n = \sqrt{n+1}\varphi_{n+1}$ and $H\varphi_n = 2(n+1)\varphi_n$.

Proposition 3.3. The following quantities are conservation laws for (LLL):

$$\mathcal{E}(u) = \int_{\mathbb{C}} |u(z)|^4 dL(z) \qquad \text{(Hamiltonian)}$$

$$M(u) = \int_{\mathbb{C}} |u(z)|^2 dL(z) \qquad \text{(Mass)}$$

$$P(u) = \int_{\mathbb{C}} (|z|^2 - 1)|u(z)|^2\, dL(z) \qquad \text{(Angular momentum)}$$

$$Q(u) = \int_{\mathbb{C}} z|u|^2(z) dL(z) \qquad \text{(Magnetic momentum).}$$

Notice that the $\mathcal{H}^1$ norm is also preserved, since in coordinates we can check that

$$\int_{\mathbb{C}} |H^{1/2}u(z)|^2 dL(z) = 2\int_{\mathbb{C}} |z|^2|u(z)|^2\, dL(z) = 2(P(u) + M(u)).$$

An important tool in the study of the (LLL) equation are the hypercontractivity inequalities of Carlen [26].

Proposition 3.4. Assume that $1 \leq p \leq q \leq \infty$. Then $F^p(\mathbb{C}) \subset F^q(\mathbb{C})$ and

$$\Big(\frac{q}{2\pi}\Big)^{1/q} \|u\|_{L^q(\mathbb{C})} \leq \Big(\frac{p}{2\pi}\Big)^{1/p} \|u\|_{L^p(\mathbb{C})}, \tag{3.5}$$

with optimal constants.

This result can be understood as smoothing estimate in the L^p scale. Compare with the Khintchine Lemma 2.1.

Proof. We prove prove the result for $p = 1$ and $q = +\infty$, see [90, Corollary 2.8]. Write $u(z) = f(z)\mathrm{e}^{-|z|^2/2}$ where f is entire. By the Cauchy formula, for all $r > 0$,

$$|f(0)| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(r\mathrm{e}^{i\theta})| d\theta.$$

Thus by integration in $r>0$

$$|f(0)|\int_0^{+\infty} r\mathrm{e}^{-r^2/2}dr \leq \frac{1}{2\pi}\int_0^{2\pi}\int_0^{+\infty} |f(r\mathrm{e}^{i\theta})|r\mathrm{e}^{-r^2/2}drd\theta,$$

in other words

$$|u(0)| = |f(0)| \leq \frac{1}{2\pi}\int_0^{2\pi}\int_0^{+\infty} |f(r\mathrm{e}^{i\theta})|r\mathrm{e}^{-r^2/2}drd\theta = \\ = \frac{1}{2\pi}\int_{\mathbb{C}} |f(z)|\mathrm{e}^{-|z|^2/2}dL(z) = \frac{1}{2\pi}\|u\|_{L^1(\mathbb{C})}.$$

More generally, for any $z\in\mathbb{C}$ and f we apply the previous inequality to the entire function

$$w \longmapsto f(z-w)\mathrm{e}^{w\overline{z}-|z|^2/2},$$

and deduce the announced bound $\|u\|_{L^\infty(\mathbb{C})} \leq \frac{1}{2\pi}\|u\|_{L^1(\mathbb{C})}$. □

As a consequence, we observe that for all $u\in F^2(\mathbb{C})$

$$\mathcal{E}(u) = \|u\|^4_{L^4(\mathbb{C})} \leq \frac{1}{2\pi}\|u\|^4_{L^2(\mathbb{C})}.$$

We refer to the book [90] for more analysis on Bargmann-Fock spaces.

Exercise 3.5.

1. Show that with a slight modification in the previous proof one can also obtain the case $q=\infty$ and any $p\geq 1$.
2. Prove directly the inequality (3.5) for $(q,p)=(\infty,2)$. *Hint:* use the identity

$$\int_{\mathbb{C}} \mathrm{e}^{-|w|^2+aw+c\overline{w}}\, dL(w) = \pi\mathrm{e}^{ac}.$$

We are now able to show that (LLL) is globally well-posed in $F^p(\mathbb{C})$ with $2\leq p\leq 4$.

Proposition 3.6 (Gérard-Germain-LT [34]). Assume that $2\leq p\leq 4$. The equation (LLL) is globally well-posed for data in $F^p(\mathbb{C})$ and such data lead to solutions in $\mathcal{C}^\infty\big(\mathbb{R},F^p(\mathbb{C})\big)$.
Moreover, there exists $C=C(\|u_0\|_{L^p(\mathbb{C})})>0$ such that

$$\|u(t)-u_0\|_{L^p(\mathbb{C})} \leq C|t|^{4/p-1}, \qquad \|u(t)-u_0\|_{L^2(\mathbb{C})} \leq C|t|, \qquad \forall t\in\mathbb{R}. \tag{3.6}$$

Proof. First we observe that for any $p\geq 1$, the projector Π has a unique bounded extension to L^p, which is given by the kernel $\frac{1}{\pi}\mathrm{e}^{-\frac{|z|^2}{2}-\frac{|w|^2}{2}+\overline{w}z}$. Actually, the operator with kernel $\frac{1}{\pi}\mathrm{e}^{-\frac{|z|^2}{2}-\frac{|w|^2}{2}+\overline{w}z}$ is, by definition, bounded on $L^2(\mathbb{C})$. A simple computation shows that it is also bounded on $L^\infty(\mathbb{C})$. By interpolation, it is then bounded on $L^p(\mathbb{C})$ for any $p\in[2,\infty]$. Since it is self-adjoint, it is also, by duality, bounded on L^p for any $p\in[1,2]$.

Local well-posedness is obtained by a fixed point argument from the following a priori estimate: using successively the boundedness of Π, Hölder's inequality, and (3.5),

$$\big\|\Pi(|u|^2u)\big\|_{L^p} \leq C_1\big\||u|^2u\big\|_{L^p} = C_1\|u\|^3_{L^{3p}} \leq C_2\|u\|^2_{L^4}\|u\|_{L^p}.$$

The previous inequality shows that the lifespan of the solution only depend on the L^4 norm which is preserved, hence we get global well-posedness.

Let us now prove the bound (3.6). We write $u = u_0 + v$, then for $t \geq 0$ we have

$$v(t) = -i \int_0^t \mathcal{T}(u_0 + v)(s) ds.$$

We take the L^2-norm and get with the help of (3.5)

$$\|v(t)\|_{L^2(\mathbb{C})} \leq C_1 t \|u_0 + v\|^3_{L^6(\mathbb{C})} \leq C_2 t (\|u_0\|^3_{L^6(\mathbb{C})} + \|v\|^3_{L^6(\mathbb{C})}) \leq C_3 t (\|u_0\|^3_{L^p(\mathbb{C})} + \|v\|^3_{L^4(\mathbb{C})}).$$

Therefore, by the conservation of the energy, we obtain $\|v(t)\|_{L^2(\mathbb{C})} \leq Ct$ which is the second bound. The first bound follows from interpolation with the energy. □

3.2.2 KAM results for a perturbed equation

The next result can be found in [34]. In the sequel, we consider the (non-local) perturbation of the (LLL) equation

$$i\partial_t u + \nu M u = \varepsilon \Pi(|u|^2 u), \quad (t, z) \in \mathbb{R} \times \mathbb{C}, \tag{3.7}$$

where $\nu, \varepsilon > 0$ are small and where M is the (Hermite) multiplier, defined by $M\varphi_j = \xi_j \varphi_j$ with $-1 \leq \xi_j \leq 1$.

Notice that M and H commute and that we have the following conservation laws :

$$\int_{\mathbb{C}} |u(z)|^2 dL(z), \qquad \int_{\mathbb{C}} \overline{u} H u(z) dL(z), \qquad \nu \int_{\mathbb{C}} \overline{u} M u(z) dL(z) + \varepsilon \int_{\mathbb{C}} |u(z)|^4 dL(z),$$

which are the L^2 and $\mathcal{H}^1$ norms as well the Hamiltonian (there are other conservation laws).

Using the commutation of M and H, as well as the relation

$$\mathrm{e}^{itH} \mathcal{T}(u_1, u_2, u_3) = \mathcal{T}(\mathrm{e}^{itH} u_1, \mathrm{e}^{itH} u_2, \mathrm{e}^{itH} u_3),$$

we see that (3.7) is equivalent to the equation ($v = \mathrm{e}^{itH} u$)

$$i\partial_t v + Hv + \nu M v = \Pi(|v|^2 v), \quad (t, z) \in \mathbb{R} \times \mathbb{C}. \tag{3.8}$$

The abstract KAM result [38, Theorem 2.3] can directly be applied to the equation (3.8) and hence (3.7) (see also [38, Theorem 6.6] for a similar statement for the Schrödinger equation with harmonic potential).

Theorem 3.7. Let $n \geq 1$ be an integer and set $\mathcal{A} = [-1, 1]^{n+1}$. There exist $\varepsilon_0 > 0$, $\nu_0 > 0$, $C_0 > 0$ and, for each $\varepsilon < \varepsilon_0$, a Cantor set $\mathcal{A}_\varepsilon \subset \mathcal{A}$ of asymptotic full measure when $\varepsilon \to 0$, such that for each $\xi \in \mathcal{A}_\varepsilon$ and for each $C_0 \varepsilon \leq \nu < \nu_0$, the solution of

$$i\partial_t u + \nu M u = \varepsilon \Pi(|u|^2 u), \quad (t, z) \in \mathbb{R} \times \mathbb{C}, \tag{3.9}$$

with initial datum

$$u_0(z) = \sum_{j=0}^{n} I_j^{1/2} e^{i\theta_j} \varphi_j(z), \tag{3.10}$$

with $(I_0, \cdots, I_n) \subset (0,1]^{n+1}$ and $\theta \in \mathbb{T}^{n+1}$, is quasi periodic with a quasi period $\omega^\star$ close to $\omega_0 = (2j+2)_{j=0}^n$: $|\omega^\star - \omega_0| < C\nu$.
More precisely, when θ covers $\mathbb{T}^n$, the set of solutions of (3.9) with initial datum (3.10) covers a $(n+1)$-dimensional torus which is invariant by (3.9). Furthermore this torus is linearly stable.

Notice that one already knew that the equation (3.7) is globally well-posed for such initial conditions.

3.2.3 Control of Sobolev norms for a perturbed equation

The next result can be found in [34]. We define the Hermite multiplier M by $M\varphi_j = m_j \varphi_j$, where $(m_j)_{j\in\mathbb{N}}$ is a bounded sequence of real numbers chosen in the following classes: for any $k \geq 1$, we define the class

$$\mathcal{W}_k = \Big\{(m_j)_{j\in\mathbb{N}} : \; m_j = \frac{\tilde{m}_j}{(j+1)^k} \;\text{ with }\; \tilde{m}_j \in [-1/2, 1/2]\Big\}$$

which is endowed with the product Lebesgue (probability) measure. Consider the problem

$$i\partial_t u + Mu = \Pi(|u|^2 u), \quad (t,z) \in \mathbb{R} \times \mathbb{C}. \tag{3.11}$$

The following almost global existence result is proved in [37, Theorem 1.1].

Theorem 3.8. Let $k, r \in \mathbb{N}$. There exists a set $\mathcal{B}_k \subset \mathcal{W}_k$ of measure 1 such that if $(m_j)_{j\in\mathbb{N}} \in \mathcal{B}_k$ there exists $s_0 \in \mathbb{N}$ such that for any $s \geq s_0$, there are $\varepsilon_0 > 0$, $c > 0$, such that for any $\varepsilon \in (0, \varepsilon_0)$, for any $u_0 \in \mathcal{H}^s(\mathbb{C})$ with

$$\|u_0\|_{\mathcal{H}^s(\mathbb{C})} \leq \varepsilon,$$

the equation (3.11) with initial datum u_0 has a unique global solution $u \in \mathcal{C}^\infty\big(\mathbb{R}, \mathcal{H}^s(\mathbb{C})\big)$ and it satisfies

$$\|u(t)\|_{\mathcal{H}^s(\mathbb{C})} \leq 2\varepsilon, \quad |t| \leq c\varepsilon^{-r}.$$

To prove this result, we apply [37, Theorem 1.1] to the equation $i\partial_t v + Hv + Mv = \Pi(|v|^2 v)$, obtained with the change unknown $v = \mathrm{e}^{itH} u$.
By the result of Lemma 3.1, Theorem 3.8 shows that if the initial condition is strongly localised in space, then the corresponding solution remains localised for large times.

3.3 Statement of the probabilistic results

Set

$$X^0_{hol}(\mathbb{C}) := \big(\cap_{\sigma>0} \mathcal{H}^{-\sigma}(\mathbb{C})\big) \cap (\mathcal{O}(\mathbb{C})\mathrm{e}^{-|z|^2/2}).$$

Define $\gamma \in L^2(\Omega; X^0(\mathbb{C}))$ by

$$\gamma(\omega, z) = \sum_{n=0}^{+\infty} \frac{g_n(\omega)}{\sqrt{2(n+1)}} \varphi_n(z),$$

and for $\beta > 0$ we define $\gamma_\beta = \gamma/\sqrt{\beta}$. Consider the Gaussian probability measure

$$\mu_\beta = (\gamma_\beta)_\# \mathbf{p} := \mathbf{p} \circ \gamma_\beta^{-1}.$$

We will check later in Lemma 3.13 that μ_β is a probability measure on $X^0_{hol}(\mathbb{C})$. Let $2 < p \leq +\infty$, by Lemma 3.16 below, for almost all $\omega \in \Omega$,

$$\gamma(\omega, .) \in F^p(\mathbb{C}) \quad \text{but} \quad \gamma(\omega, .) \notin F^2(\mathbb{C}).$$

As a consequence $\mu_\beta(L^2(\mathbb{C})) = 0$.

Notice that since (LLL) conserves the $\mathcal{H}^1(\mathbb{C})$ norm, μ_β is formally invariant by its flow. More generally, we can define a family $(\rho_\beta)_{\beta>0}$ of probability measures on $X^0_{hol}(\mathbb{C})$ which are formally invariant by (LLL) in the following way: define for $\beta > 0$ the measure ρ_β by

$$d\rho_\beta(u) = C_\beta \mathrm{e}^{-\beta \mathcal{E}(u)} d\mu_\beta(u), \tag{3.12}$$

where $C_\beta > 0$ is a normalising constant (in Lemma 3.16, we will show that $\mathcal{E}(u) < +\infty$, μ_β a.s., which enables us to define this probability measure). By the Kakutani theorem (Theorem 2.8 and Corollary 2.9), the measures ρ_β are mutually singular. Actually, the $(\rho_\beta)_{\beta>0}$ are the Gibbs measures of the equation (3.4).

We are now able to state the following global existence result, which also gives some qualitative information on the long time dynamics.

Theorem 3.9 (Germain-Hani-LT [36]). Let $\beta > 0$. There exists a set $\Sigma \subset X^0_{hol}(\mathbb{C})$ of full ρ_β measure so that for every $u_0 \in \Sigma$ the equation (LLL) with initial condition $u(0) = u_0$ has a unique global solution $u(t) = \Phi(t, u_0)$ such that for any $0 < s < 1/2$

$$u(t) - u_0 \in \mathcal{C}\big(\mathbb{R}; \mathcal{H}^s(\mathbb{C})\big).$$

Moreover, for all $\sigma > 0$ and $t \in \mathbb{R}$

$$\|u(t)\|_{L^3(\mathbb{C})} + \|u(t)\|_{\mathcal{H}^{-\sigma}(\mathbb{C})} \leq C\big(\Lambda(u_0, \sigma) + \ln^{1/2}\big(1 + |t|\big)\big), \tag{3.13}$$

where the constant $\Lambda(u_0, \sigma)$ satisfies the bound $\mu_\beta\big(u_0 : \Lambda(u_0, \sigma) > \lambda\big) \leq C\mathrm{e}^{-c\lambda^2}$.

Furthermore, the measure ρ_β is invariant by Φ: for any ρ_β measurable set $A \subset \Sigma$ and for any $t \in \mathbb{R}$,

$$\rho_\beta(A) = \rho_\beta(\Phi(t, A)).$$

Finally, for all $t \in \mathbb{R}$

$$\|u(t)\|_{L^4(\mathbb{C})} = \|u_0\|_{L^4(\mathbb{C})}.$$

The same result (with the *ad hoc* measures μ and ρ) holds for the perturbed equations (3.8) and (3.11).

Remark 3.10. By the Birkhoff-Kintchine Theorem 1.17 we have for all $k \geq 1$

$$\frac{1}{T}\int_0^T \|u(t)\|^k_{\mathcal{H}^{-\sigma}(\mathbb{C})} dt \longrightarrow G_k(u_0), \quad \text{when} \quad T \longrightarrow +\infty, \tag{3.14}$$

and the fonction G_k is a conservation law: for all $t \in \mathbb{R}$, $G_k(u(t)) = G_k(u_0)$. Moreover

$$\int_{\mathcal{H}^{-\sigma}} G_k(u) d\mu(u) = \int_{\mathcal{H}^{-\sigma}} \|u\|^k_{\mathcal{H}^{-\sigma}(\mathbb{C})} d\mu(u).$$

One even has
$$\frac{1}{T}\int_0^T \mathrm{e}^{\frac{1}{2}\|u(t)\|^2_{\mathcal{H}^{-\sigma}(\mathbb{C})}} dt \longrightarrow G_\infty(u_0), \quad \text{when} \quad T \longrightarrow +\infty,$$
By Theorem 3.9, there may be initial conditions such that $\|u(t)\|_{\mathcal{H}^{-\sigma}(\mathbb{C})}$ may grow like $\ln^{1/2}(t)$, but not many since in mean it stays bounded, by (3.14). Compare with the bound (3.6).

Remark 3.11. Formally, the (LLL) equation looks like the Szegö equation introduced and studied by Gérard and Grellier, but their properties are different. For instance, unlike (3.13) there is no nonlinear smoothing for the Szegö equation, as was shown in [65, Proposition 1.6], therefore it is not clear if an analogous result holds for the Szegö equation.

Remark 3.12. Let us compare the types of results given by the KAM method, the Birkhoff normal form method and the probabilistic methods. Typically, the first two methods concern smooth solutions, while the last yield rough solutions. Observe that a common feature on the three methods is randomness, which appear either in the equation (through the potential) or in the initial conditions. The resonances of the equation play a key role in the first two methods but I do not see where they intervene in the construction of a Gibbs measure.

Let us conclude this section with a few reference concerning the use of Gibbs measure in the construction of global strong solutions to PDEs. In a compact setting: Lebowitz-Rose-Speer [50], Bourgain [11, 10], Zhidkov [89], Tzvetkov [81, 80], Burq-Tzvetkov [23], Oh [63, 64], Burq-Thomann-Tzvetkov [20], Deng [31], Nahmod et al [59], Suzzoni [74], Deng-Tzvetkov-Visciglia [86, 87, 32], Bourgain-Bulut [12, 13, 14, 15], Richards [71] and others. There are also other types of a.s. global wellposedness results, without the use of invariant measures, mainly for the wave equation, but we do not comment on them.

For results in non compact settings, see [6, 7, 24, 25, 53, 54, 88] and references therein.

3.4 Sketch of the proof of the global wellposedness result

In the sequel we fix $\beta = 1$ (say) and write $\mu = \mu_\beta$, $\rho = \rho_\beta$. We only prove Theorem 3.9 for $s = 0$.

Lemma 3.13. The measure μ is a probability measure on $X^0_{hol}(\mathbb{C})$.

Proof. It is enough to show that $\gamma \in X^0_{hol}(\mathbb{C})$, $\mathbf{p}$-a.s. First, for all $\sigma > 0$ we have
$$\int_\Omega \|\gamma\|^2_{\mathcal{H}^{-\sigma}(\mathbb{C})} d\mathbf{p}(\omega) = \int_\Omega \sum_{n=0}^{+\infty} \frac{|g_n|^2}{\big(2(n+1)\big)^{\sigma+1}} d\mathbf{p}(\omega) = C\sum_{n=0}^{+\infty} \frac{1}{(n+1)^{\sigma+1}} < +\infty, \tag{3.15}$$
therefore $\gamma \in \bigcap_{\sigma>0} L^2\big(\Omega\,;\,\mathcal{H}^{-\sigma}(\mathbb{C})\big)$. Next, by [27, Lemma 3.4], for all $A \geq 1$ there exists a set $\Omega_A \subset \Omega$ such that $\mathbf{p}(\Omega_A^c) \leq \exp(-A^\delta)$ and for all $\omega \in \Omega_A$, $\varepsilon > 0$, $n \geq 0$
$$|g_n(\omega)| \leq CA(n+1)^\varepsilon.$$
Then for $\omega \in \bigcup_{A\geq 1}\Omega_A$, $\displaystyle\sum_{n=0}^{+\infty} \frac{z^n g_n(\omega)}{\sqrt{(n+1)!}} \in \mathcal{O}(\mathbb{C})$. □

We first define a smooth version of the usual spectral projector. Let $\chi \in \mathcal{C}_0^\infty(-1,1)$, so that $0 \leq \chi \leq 1$, with $\chi = 1$ on $[-\frac{1}{2}, \frac{1}{2}]$. We define the operators $S_N = \chi\big(\frac{H}{N+1}\big)$ as

$$S_N\big(\sum_{n=0}^{\infty} c_n \varphi_n\big) = \sum_{n=0}^{\infty} \chi\big(\frac{n+1}{N+1}\big) c_n \varphi_n.$$

Then for all $1 < p < +\infty$, the operator S_N is bounded in $L^p(\mathbb{C})$ (see [31, Proposition 2.1] for a proof). This result does not hold true if one replaces S_N with a crude frequency truncation.

3.4.1 Local existence

Recall the definition of $\mathcal{T}$ in (3.2). It will be useful to work with an approximation of (LLL). We consider the dynamical system given by the Hamiltonian $\mathcal{E}_N(u) := \mathcal{E}(S_N u)$. This system reads

$$\begin{cases} i\partial_t u_N = \mathcal{T}_N(u_N), \quad (t,z) \in \mathbb{R} \times \mathbb{C}, \\ u_N(0,z) = u_0(z), \end{cases} \tag{3.16}$$

and $\mathcal{T}_N(u_N) := S_N \mathcal{T}(S_N u, S_N u, S_N u)$. Denote by E_k the space on $\mathbb{C}$ spanned by φ_k. Observe that (3.16) is a finite dimensional dynamical system on $\bigoplus_{k=0}^N E_k$ and that the projection of $u_N(t)$ on its complement is constant. For $N \geq 0$ we define the measures ρ_N by

$$d\rho_N(u) = C_N \mathrm{e}^{-\mathcal{E}_N(u)} d\mu(u),$$

where $C_N > 0$ is a normalising constant. We have the following result

Lemma 3.14. The system (3.16) is globally well-posed in $L^2(\mathbb{C})$. Moreover, the measures ρ_N are invariant by its flow denoted by Φ_N.

Proof. The global existence follows from the conservation of $\|u_N\|_{L^2(\mathbb{C})}$. The invariance of the measures is a consequence of the Liouville theorem and the conservation of $\sum_{k=0}^\infty \lambda_k |c_k|^2$ by the flow of (LLL) (see Theorem 1.5). We refer to [19, Lemma 8.1 and Proposition 8.2] for the details. □

We now state a result concerning dispersive bounds of Hermite functions

Lemma 3.15. For all $2 \leq p \leq +\infty$,

$$\|\varphi_n\|_{L^p(\mathbb{C})} \leq C n^{\frac{1}{2p} - \frac{1}{4}}. \tag{3.17}$$

Proof. By Stirling, we easily get that $\|\varphi_n\|_{L^\infty(\mathbb{C})} \leq C n^{-\frac{1}{4}}$, which is (3.17) for $p = +\infty$; the estimate for $2 \leq p \leq \infty$ follows by interpolation. □

Lemma 3.16. *(i)* For all $2 < p < +\infty$

$$\begin{gathered} \exists C > 0, \exists c > 0, \forall \lambda \geq 1, \forall N \geq 1, \\ \mu\big(u \in X^0_{hol}(\mathbb{C}) : \|S_N u\|_{L^p(\mathbb{C})} > \lambda\big) \leq C e^{-c\lambda^2}, \\ \mu\big(u \in X^0_{hol}(\mathbb{C}) : \|u\|_{L^p(\mathbb{C})} > \lambda\big) \leq C e^{-c\lambda^2}. \end{gathered} \tag{3.18}$$

(ii) For all $2<p<+\infty$, there exists $\delta>0$ such that

$$\exists\, C>0, \exists\, c>0,\ \forall \lambda\geq 1,\ \forall N\geq N_0\geq 1,$$
$$\mu\big(\, u\in X^0_{hol}(\mathbb{C})\,:\, \|(S_N-S_{N_0})u\|_{L^p(\mathbb{C})}>\lambda\,\big)\leq Ce^{-cN_0^{\delta}\lambda^2}\,. \quad (3.19)$$

Proof. We have that

$$\mu\big(\, u\in X^0_{hol}(\mathbb{C})\,:\, \|u\|_{L^p(\mathbb{C})}>\lambda\,\big)=\mathbf{p}\Big(\Big\|\sum_{n=0}^{\infty}\frac{g_n(\omega)}{\sqrt{2(n+1)}}\varphi_n(z)\Big\|_{L^p(\mathbb{C})}>\lambda\Big).$$

Let $q\geq p\geq 2$. Recall here the Khintchine inequality: there exists $C>0$ such that for all real $k\geq 2$ and $(a_n)\in\ell^2(\mathbb{N})$

$$\|\sum_{n\geq 0}g_n(\omega)\,a_n\|_{L^k_{\mathbf{p}}}\leq C\sqrt{k}\Big(\sum_{n\geq 0}|a_n|^2\Big)^{\frac12}, \quad (3.20)$$

if the g_n are iid normalized Gaussians. Applying it to (3.20) we get

$$\|\sum_{n=0}^{\infty}\frac{g_n(\omega)}{\sqrt{2(n+1)}}\varphi_n(z)\|_{L^q_\omega}\leq C\sqrt{q}\Big(\sum_{n=0}^{\infty}\frac{|\varphi_n(z)|^2}{2(n+1)}\Big)^{1/2},$$

and using twice the Minkowski inequality for $q\geq p$ gives

$$\begin{aligned}\|\sum_{n=0}^{\infty}\frac{g_n(\omega)}{\sqrt{2(n+1)}}\varphi_n(z)\|_{L^q_\omega L^p_z} &\leq \|\sum_{n=0}^{\infty}\frac{g_n(\omega)}{\sqrt{2(n+1)}}\varphi_n(z)\|_{L^p_z L^q_\omega}\\ &\leq C\sqrt{q}\Big(\sum_{n=0}^{\infty}\frac{\|\varphi_n(z)\|^2_{L^p(\mathbb{C})}}{\langle n\rangle}\Big)^{1/2}. \end{aligned} \quad (3.21)$$

We are now ready to prove (3.18). Since by Lemma 3.15 we have $\|\varphi_n\|_{L^p(\mathbb{C})}\leq Cn^{\frac{1}{2p}-\frac14}$, we get from (3.21)

$$\|\sum_{n=0}^{\infty}\frac{g_n(\omega)}{\sqrt{2(n+1)}}\varphi_n(z)\|_{L^q_\omega L^p_z}\leq C\sqrt{q}\,.$$

The Bienaymé-Tchebichev inequality gives then

$$\mathbf{p}\left(\|\sum_{n=0}^{\infty}\frac{g_n(\omega)}{\sqrt{2(n+1)}}\varphi_n(z)\|_{L^p(\mathbb{C})}>\lambda\right)\leq(\lambda^{-1}\|\sum_{n=0}^{\infty}\frac{g_n(\omega)}{\sqrt{2(n+1)}}\varphi_n(z)\|_{L^q_\omega L^p_z})^q\leq (C\lambda^{-1}\sqrt{q})^q\,.$$

Thus by choosing $q=\delta\lambda^2\geq 4$, for δ small enough, we get the bound

$$\mathbf{p}\big(\|\sum_{n=0}^{\infty}\frac{g_n(\omega)}{\sqrt{2(n+1)}}\varphi_n(z)\|_{L^p(\mathbb{C})}>\lambda\,\big)\leq Ce^{-c\lambda^2}\,,$$

which is (3.18). □

Remark 3.17. From the previous result we deduce that on the support of μ (resp. ρ) we have $u \in L^4(\mathbb{C})$, thus we get a global existence result. However the invariance of the measures is not directly implied.

Lemma 3.18. Let $p \in [1, \infty[$, then when $N \longrightarrow +\infty$.

$$C_N \mathrm{e}^{-\mathcal{E}_N(u)} \longrightarrow C \mathrm{e}^{-\mathcal{E}(u)} \quad \text{in} \quad L^p(\mathrm{d}\mu(u)).$$

In particular, for all measurable sets $A \subset X^0_{hol}(\mathbb{C})$,

$$\rho_N(A) \longrightarrow \rho(A).$$

Proof. Denote by $G_N(u) = \mathrm{e}^{-\mathcal{E}_N(u)}$ and $G(u) = \mathrm{e}^{-\mathcal{E}(u)}$. By (3.19), we deduce that $\mathcal{E}_N(u) \longrightarrow \mathcal{E}(u)$ in measure, w.r.t. μ. In other words, for $\varepsilon > 0$ and $N \geq 1$ we denote by

$$A_{N,\varepsilon} = \big\{ u \in X^0_{hol}(\mathbb{C}) \ : \ |G_N(u) - G(u)| \leq \varepsilon \big\},$$

then $\mu(A^c_{N,\varepsilon}) \longrightarrow 0$, when $N \longrightarrow +\infty$. Since $0 \leq G, G_N \leq 1$,

$$\begin{aligned} \|G - G_N\|_{L^p_\mu} &\leq \|(G - G_N)\mathbf{1}_{A_{N,\varepsilon}}\|_{L^p_\mu} + \|(G - G_N)\mathbf{1}_{A^c_{N,\varepsilon}}\|_{L^p_\mu} \\ &\leq \varepsilon\big(\mu(A_{N,\varepsilon})\big)^{1/p} + 2\big(\mu(A^c_{N,\varepsilon})\big)^{1/p} \leq C\varepsilon, \end{aligned}$$

for N large enough. Finally, we have when $N \longrightarrow +\infty$

$$C_N = \Big(\int \mathrm{e}^{-\mathcal{E}_N(u)} d\mu(u)\Big)^{-1} \longrightarrow \Big(\int \mathrm{e}^{-\mathcal{E}(u)} d\mu(u)\Big)^{-1} = C,$$

and this ends the proof. $\square$

Remark 3.19. Let us make a comment about the construction of a Gibbs measure for the equation

$$i\partial_t u + Hu = u, \quad (t, z) \in \mathbb{R} \times \mathbb{C}. \tag{3.22}$$

The right thing to do here is to rewrite the equation as $i\partial_t u + (H - 1)u = 0$ and to define the corresponding Gaussian measure. One could also try to construct the measure as a perturbation of μ. A natural candidate for it is the measure $d\rho(u) = C\mathrm{e}^{-\|u\|^2_{L^2(\mathbb{C})}} d\mu(u)$, but this does not work since $\|u\|_{L^2(\mathbb{C})} = +\infty$, μ almost surely. The idea is to define the r.v. $G_N(u) = \|u_N\|^2_{L^2(\mathbb{C})} - \alpha_N$ where $\alpha_N = \sum_{n=1}^N \frac{1}{2(n+1)}$. One can show that $(G_N)_{N\geq 1}$ is a Cauchy sequence in $L^2(X^0_{hol}(\mathbb{C}), d\mu)$ which enables to define its limit G and then the measure $d\rho(u) = C\mathrm{e}^{-G(u)} d\mu(u)$. Namely, for $M \geq N \geq 1$

$$\begin{aligned} \int_{X^0_{hol}(\mathbb{C})} \|G_M(u) - G_N(u)\|^2_{H^{-\sigma}(\mathbb{C})} \mathrm{d}\mu(u) &= \int_\Omega \|G_M(\gamma) - G_N(\gamma)\|^2_{H^{-\sigma}(\mathbb{C})} \mathrm{d}\mathbf{p} \\ &= \int_\Omega \sum_{m,n=N+1}^{M} \frac{(|g_n|^2 - 1)(|g_m|^2 - 1)}{4(n+1)(m+1)}. \end{aligned}$$

Now we use that $\int_\Omega |g_n|^2 \mathrm{d}\mathbf{p} = 1$ and $\int_\Omega (|g_n|^2-1)(|g_m|^2-1)\mathrm{d}\mathbf{p} = 0$ for $n \neq m$, thus

$$\int_{X^0_{hol}(\mathbb{C})} \|G_M(u)-G_N(u)\|^2_{H^{-\sigma}(\mathbb{C})}\mathrm{d}\mu(u) = C\sum_{m,n=N+1}^{M}\frac{1}{(n+1)^2} \leq CN^{-1},$$

hence the result. This renormalisation procedure can be generalised and is known as the Wick ordering, see [68] for more details.

We look for a solution to (LLL) of the form $u = u_0 + v$, thus v has to satisfy

$$\begin{cases} i\partial_t v = \mathcal{T}(u_0+v), & (t,z)\in\mathbb{R}\times\mathbb{C}, \\ v(0,z) = 0, \end{cases} \tag{3.23}$$

with $\mathcal{T}(u) = \mathcal{T}(u,u,u)$. Similarly, we introduce

$$\begin{cases} i\partial_t v_N = \mathcal{T}_N(u_0+v_N), & (t,z)\in\mathbb{R}\times\mathbb{C}, \\ v(0,z) = 0. \end{cases} \tag{3.24}$$

Recall that equation (3.24) is globally well posed in $L^2(\mathbb{C})$, and its flowmap is denoted by Φ_N.

Let $\sigma > 0$ and let us define

$$A(R) = \big\{u_0 \in X^0_{hol}(\mathbb{C}) : \ \|u_0\|_{\mathcal{H}^{-\sigma}(\mathbb{C})} + \|u_0\|_{L^6(\mathbb{C})} \leq R^{1/2}\big\}.$$

Then we have the following result

Lemma 3.20. There exist $c, C > 0$ so that for all $N \geq 0$

$$\rho_N\big(A(R)^c\big) \leq C\mathrm{e}^{-cR}, \quad \rho\big(A(R)^c\big) \leq C\mathrm{e}^{-cR}, \quad \mu\big(A(R)^c\big) \leq C\mathrm{e}^{-cR}.$$

Proof. Observe that we have $\rho_N\big(A(R)^c\big), \rho\big(A(R)^c\big) \leq C\mu\big(A(R)^c\big)$. The result is therefore given by (3.18). □

Proposition 3.21. There exists $c > 0$ such that, for any $R > 0$, $c_0 > 0$, setting $\tau(R) = cR^{-2}$, for any $u_0 \in A(R)$ there exists a unique solution $v \in L^\infty([-\tau,\tau];L^2(\mathbb{C}))$ to the equation (3.23) and a unique solution $v_N \in L^\infty([-\tau,\tau];L^2(\mathbb{C}))$ to the equation (3.24) which furthermore satisfy

$$\|v\|_{L^\infty([-\tau,\tau];L^2(\mathbb{C}))} \leq c_0 R^{-1/2}, \quad \|v_N\|_{L^\infty([-\tau,\tau];L^2(\mathbb{C}))} \leq c_0 R^{-1/2}.$$

As a consequence, for all $|t| \leq cR^{-2}$, if $c_0 \ll 1$

$$\Phi(t,u_0) \in A(R+1), \qquad \Phi_N(t,u_0) \in A(R+1). \tag{3.25}$$

Proof. We only consider the equation (3.23), the other case being similar by the boundedness of S_N on $L^p(\mathbb{C})$. We define the space

$$Z(\tau) = \big\{v \in \mathcal{C}\big([-\tau,\tau];L^2(\mathbb{C})\big) \text{ s.t. } v(0) = 0 \ \text{ and } \ \|v\|_{Z(\tau)} \leq c_0R^{-1/2}\big\},$$

with $\|v\|_{Z(\tau)} = \|v\|_{L^\infty_{[-\tau,\tau]}L^2(\mathbb{C})}$, and for $u_0 \in A(R)$ we define the operator

$$K(v) = -i\int_0^t \mathcal{T}(u_0 + v)ds.$$

We will show that K has a unique fixed point $v \in Z(\tau)$.

We have

$$\begin{aligned}\|K(v)\|_{Z(\tau)} &\leq \tau\big\|\mathcal{T}(u_0+v)\big\|_{Z(\tau)} \\ &\leq C\tau\left(\|\mathcal{T}(u_0,u_0,u_0)\|_Z + \|\mathcal{T}(u_0,u_0,v)\|_Z + \|\mathcal{T}(u_0,v,v)\|_Z + \|\mathcal{T}(v,v,v)\|_Z\right).\end{aligned}$$

We estimate each term. The conjugation plays no role, so we forget it. We only detail the first and the last term.

• Estimate of the trilinear term in v: by (3.5)

$$\|\mathcal{T}(v,v,v)\|_{L^2(\mathbb{C})} \leq C\|v\|^3_{L^6(\mathbb{C})} \leq C\|v\|^3_{L^2(\mathbb{C})}.$$

• Estimate of the constant term in v: for u_0 in $A(R)$

$$\|\mathcal{T}(u_0,u_0,u_0)\|_{L^2(\mathbb{C})} \leq C\|u_0\|^3_{L^6(\mathbb{C})} \leq C\|u_0\|^3_{L^3(\mathbb{C})} \leq CR^{3/2},$$

(recall here that the bound $\|u_0\|_{L^2(\mathbb{C})}$ is forbidden since $\|u_0\|_{L^2(\mathbb{C})} = +\infty$ on the support of μ.)

With these estimates at hand, the result follows by the Picard fixed point theorem. □

Remark 3.22. In [36] we prove a more general result for the complete (CR) equation. In this context, the hypercontractivity estimates are replaced by Strichartz estimates.

3.4.2 Approximation and invariance of the measure

Lemma 3.23. Fix $R \geq 0$. Then for all $\varepsilon > 0$, there exists $N_0 \geq 0$ such that for all $u_0 \in A(R)$ and $N \geq N_0$

$$\big\|\Phi(t,u_0) - \Phi_N(t,u_0)\big\|_{L^\infty([-\tau_1,\tau_1];L^2(\mathbb{C}))} \leq \varepsilon,$$

where $\tau_1 = cR^{-2}$ for some $c > 0$.

Proof. We have

$$v - v_N = -i\int_0^t \big[S_N\big(\mathcal{T}(u_0+v) - \mathcal{T}(u_0+v_N)\big) + (1-S_N)\mathcal{T}(u_0+v)\big]ds.$$

Then we get

$$\|v - v_N\|_{Z(\tau)} \leq C\tau R^2\|v - v_N\|_{Z(\tau)} + \int_{-\tau}^{\tau}\|(1-S_N)\mathcal{T}(u_0+v)\|_{L^2(\mathbb{C})}ds,$$

which in turn implies when $C\tau R^2 \leq 1/2$

$$\|v - v_N\|_{Z(\tau)} \leq 2\int_{-\tau}^{\tau}\|(1-S_N)\mathcal{T}(u_0+v)\|_{L^2(\mathbb{C})}ds.$$

Here we need a bit a compactness to conclude. We refer to [36] for the details. □

Let $D_{i,j} = (i + j^{1/2})^{1/2}$, with $i, j \in \mathbb{N}$ and set $T_{i,j} = \sum_{\ell=1}^{j} \tau_1(D_{i,\ell})$. Let

$$\Sigma_{N,i} := \big\{u_0 \ : \forall j \in \mathbb{N}, \ \ \Phi_N(\pm T_{i,j}, u_0) \in A(D_{i,j+1})\big\},$$

and

$$\Sigma_i := \limsup_{N\to+\infty} \Sigma_{N,i}, \qquad \Sigma := \bigcup_{i\in\mathbb{N}} \Sigma_i.$$

Proposition 3.24. The following holds true:

(*i*) The set Σ is of full ρ measure.

(*ii*) For all $u_0 \in \Sigma$, there exists a unique global solution $u = u_0 + v$ to (LLL). This defines a global flow Φ on Σ.

(*iii*) For all measurable set $A \subset \Sigma$, and all $t \in \mathbb{R}$,

$$\rho(A) = \rho\big(\Phi(t, A)\big).$$

The proof of (*ii*) relies on the invariance of the measure ρ_N under the flow Φ_N. A repeated use of the approximation result of Lemma 3.23 will be crucial to prove (*iii*). The details of the proof can be found in Suzzoni [75, Sections 3.3 and 4].

Let us show how one uses the Gibbs measure to define a global flow and to get the quantitative bound in $\ln^{1/2}(t)$ in Theorem 3.9.

Let $c > 0$ be given by Lemma 3.20. For $T \leq \mathrm{e}^{cR/2}$ we define,

$$\Sigma_R = \bigcap_{k=-[T/\tau]}^{[T/\tau]} \Phi_N\big(-k\tau, B_R\big). \tag{3.26}$$

Now we crucially use the invariance of the measure and get

$$\begin{aligned} \rho_N(X^0_{hol}(\mathbb{R})\backslash\Sigma_R) &\leq (2[T/\tau]+1)\rho_N(X^0(\mathbb{R})\backslash B_R) \\ &\leq CR^2\mathrm{e}^{cR/2}\mathrm{e}^{-cR} \leq C\mathrm{e}^{-cR/4}, \end{aligned}$$

which shows that Σ_R is a big subset of $X^0_{hol}(\mathbb{R})$ when $R \longrightarrow +\infty$. Now, by the definition (3.26) of Σ_R and (3.25), we deduce that for all $|t| \leq T$ and $u_0 \in \Sigma_R$

$$\|\Phi_N(t, u_0)\|_{L^3(\mathbb{C})} + \|\Phi_N(t, u_0)\|_{\mathcal{H}^{-\sigma}(\mathbb{C})} \leq (R+1)^{1/2}.$$

In particular, for $|t| = T \sim \mathrm{e}^{cR/2}$

$$\|\Phi_N(t, u_0)\|_{L^3(\mathbb{C})} + \|\Phi_N(t, u_0)\|_{\mathcal{H}^{-\sigma}(\mathbb{C})} \leq C(\ln|t|+1)^{1/2},$$

and this bound is uniform in $N \geq 1$. The term $\ln^{1/2}(t)$ is reminiscent from the large deviation estimates involving Gaussian random variables.

4 Global weak probabilistic solutions of the LLL equation below $\mathcal{H}^{-1}(\mathbb{C})$

Up to now, we have considered strong probabilistic solutions. We show here how we can construct global probabilistic solutions to PDEs thanks to compactness methods in the space of measures. As an application, we will construct a global dynamics on the support of the white noise measure of the LLL equation which lives at the very low regularity $\mathcal{H}^{-1}(\mathbb{C})$.

4.1 White noise measure and global weak solutions for LLL

Our aim is now to construct weak solutions to the Lowest Landau Level equation

$$\begin{cases} i\partial_t u = \Pi(|u|^2 u), & (t,z) \in \mathbb{R} \times \mathbb{C}, \\ u(0,z) = u_0(z), \end{cases} \tag{LLL}$$

on the support of the white noise measure.

Let us define what we mean by white noise measure in our context. Denote by $(e_n)_{n\geq 0}$ a Hilbertian basis of $L^2(0,1)$ and consider independent standard Gaussians $(g_n)_{n\geq 0}$ on a probability space $(\Omega, \mathcal{F}, \mathbf{p})$. Then it is well-known (see *e.g.* [40, Chapter 2]) that the random series

$$B_t = \sum_{n=0}^{+\infty} g_n \int_0^t e_n(s)ds$$

converges in $L^2(\Omega, \mathcal{F}, \mathbf{p})$ and defines a Brownian motion. The white noise measure is then defined by the map

$$\omega \longmapsto W(t,\omega) = \frac{dB_t}{dt}(\omega) = \sum_{n=0}^{+\infty} g_n(\omega)e_n(t). \tag{4.1}$$

Now consider a Hilbert space $\mathcal{K}$ which is a space of functions on a manifold M and consider a Hilbertian basis $(e_n)_{n\geq 0}$ of $\mathcal{K}$. We define the mean-zero Gaussian white noise (measure) on $\mathcal{K}$ as $\mu = \mathbf{p} \circ W^{-1}$, where

$$W(x,\omega) = \sum_{n=0}^{+\infty} g_n(\omega)e_n(x).$$

Notice that this measure is independent of the choice of the Hilbertian basis of $\mathcal{K}$. It is clear that for all $x \in M$, $\mathbb{E}_{\mathbf{p}}[W(x,\cdot)] = 0$. Moreover, for all $x, y \in M$ we have

$$\mathbb{E}_{\mathbf{p}}[W(x,\cdot)\overline{W(y,\cdot)}] = \sum_{n=0}^{+\infty} e_n(x)\overline{e_n(y)} = \delta(x-y),$$

since the sum in the previous line is the kernel of the identity projector on $\mathcal{K}$. For more details on Gaussian measures on Hilbert spaces, we refer to [43].

Recall the definition of the harmonic Sobolev spaces: for $s \in \mathbb{R}$ we define

$$\mathcal{H}^s = \mathcal{H}^s(\mathbb{C}) = \big\{u \in \mathcal{S}'(\mathbb{C}),\ H^{s/2}u \in L^2(\mathbb{C})\big\},$$

and the norm on $F^2(\mathbb{C}) \cap \mathcal{H}^s(\mathbb{C})$ is a weighted L^2-norm by Lemma 3.1.

Consider the Gaussian random variable

$$\eta(\omega, z) = \sum_{n=0}^{+\infty} g_n(\omega)\varphi_n(z) = \frac{1}{\sqrt{\pi}}\Big(\sum_{n=0}^{+\infty} \frac{z^n g_n(\omega)}{\sqrt{n!}}\Big)\mathrm{e}^{-|z|^2/2},$$

and the measure $\mu = \mathbf{p}\circ\eta^{-1}$. As in Lemma 3.13 we can show that the measure μ is a probability measure on

$$X^{-1}_{hol}(\mathbb{C}) := \big(\cap_{\sigma>1}\mathcal{H}^{-\sigma}(\mathbb{C})\big) \cap (\mathcal{O}(\mathbb{C})\mathrm{e}^{-|z|^2/2}).$$

In Proposition 2.13 we have seen that, for any $2 \leq p < +\infty$, $\eta(\omega, .) \notin F^p(\mathbb{C})$ for a.a. $\omega \in \Omega$.

Since $\|u\|_{L^2(\mathbb{C})}$ is preserved by (LLL), μ is formally invariant under (LLL). We are not able to define a flow at this level of regularity, however using compactness arguments combined with probabilistic methods, we will construct weak solutions.

Theorem 4.1 (Germain-Hani-LT [36]). There exists a set $\Sigma \subset X^{-1}_{hol}(\mathbb{C})$ of full μ measure so that for every $u_0 \in \Sigma$ the equation (LLL) with initial condition $u(0) = u_0$ has a solution

$$u \in \bigcap_{\sigma>1} \mathcal{C}\big(\mathbb{R}\,; \mathcal{H}^{-\sigma}(\mathbb{C})\big).$$

The distribution of the random variable $u(t)$ is equal to μ (and thus independent of $t \in \mathbb{R}$):

$$\mathscr{L}_{X^{-1}_{hol}}\big(u(t)\big) = \mathscr{L}_{X^{-1}_{hol}}\big(u(0)\big) = \mu, \quad \forall\, t \in \mathbb{R}.$$

The proof is based on a compactness argument in the space of measures (the Prokhorov theorem) combined with a representation theorem of random variables (the Skorohod theorem). This approach has been first applied to the Navier-Stokes and Euler equations in Albeverio-Cruzeiro [4] and Da Prato-Debussche [30] and extended to dispersive equations by Burq-Thomann-Tzvetkov [20]. See also Germain-Hani-Thomann [36], Oh-Thomann [68] and Oh-Richards-Thomann [67]. For results in a non compact setting, see Suzzoni [73].

Remark 4.2. For the Szegö equation, using that the $H^{1/2}(\mathbb{T})$ norm is preserved by the flow, the method used in the proof of Theorem 4.1 allows to construct a global dynamics in $\bigcap_{\sigma>0}\mathcal{C}\big(\mathbb{R}\,; H^{-\sigma}_{+}(\mathbb{T})\big)$. See [20] for details.

4.2 The Prokhorov and Skorokhod theorems

We state two basic results, concerning the convergence of random variables. To begin with, recall the following definition (see *e.g.* [47, page 114])

Definition 4.3. Let S be a metric space and $(\rho_N)_{N\geq 1}$ a family of probability measures on the Borel $\sigma-$algebra $\mathcal{B}(S)$. The family (ρ_N) on $(S, \mathcal{B}(S))$ is said to be tight if for any $\varepsilon > 0$ one can find a compact set $K_\varepsilon \subset S$ such that $\rho_N(K_\varepsilon) \geq 1-\varepsilon$ for all $N \geq 1$.

Then, we have the following compactness criterion (see *e.g.* [47, page 114] or [46, page 309])

Theorem 4.4 (Prokhorov). Assume that the family $(\rho_N)_{N\geq 1}$ of probability measures on the metric space S is tight. Then it is weakly compact, *i.e.* there is a subsequence $(N_k)_{k\geq 1}$ and a limit measure ρ_∞ such that for every bounded continuous function $F : S \to \mathbb{R}$,

$$\lim_{k\to\infty} \int_S F(x)d\rho_{N_k}(x) = \int_S F(x)d\rho_\infty(x).$$

In fact, the Prokhorov theorem is stronger: In the case where the space S is separable and complete, the converse of the previous statement holds true, but we will not use this here.

Remark 4.5. Let us make a remark on the case $S = \mathbb{R}^d$. The measure given by the theorem allows mass concentration in a point and the tightness condition forbids the escape of mass to infinity.

The Prokhorov theorem is of different nature compared to the compactness theorems giving the deterministic weak solutions: In the latter case there can be a loss of energy. A weak limit of L^2 functions may lose some mass whereas in the Prokhorov theorem a limit measure is a probability measure.

We now state the Skorokhod theorem

Theorem 4.6 (Skorokhod). Assume that S is a separable metric space. Let $(\rho_N)_{N\geq 1}$ and ρ_∞ be probability measures on S. Assume that $\rho_N \longrightarrow \rho_\infty$ weakly. Then there exists a probability space on which there are $S-$valued random variables $(Y_N)_{N\geq 1}$, Y_∞ such that $\mathcal{L}(Y_N) = \rho_N$ for all $N \geq 1$, $\mathcal{L}(Y_\infty) = \rho_\infty$ and $Y_N \longrightarrow Y_\infty$ a.s.

For a proof, see *e.g.* [46, page 79]. We illustrate this result with two elementary but significant examples:

- Assume that $S = \mathbb{R}$. Let $(Y_N)_{1\leq N\leq\infty}$ be standard Gaussians, *i.e.* $\mathcal{L}(Y_N) = \mathcal{L}(Y_\infty) = \mathcal{N}_{\mathbb{R}}(0,1)$. Then the convergence in law obviously holds, but in general we can not expect the almost sure convergence of the Y_N to Y_∞ (define for example $Y_N = (-1)^N Y_\infty$).
- Assume that $S = \mathbb{R}$. Let $(Y_N)_{1\leq N\leq\infty}$ be random variables. For any random variable Y on $\mathbb{R}$ we denote by $F_Y(t) = P(Y \leq t)$ its cumulative distribution function. Here we assume that for all $1 \leq N \leq \infty$, F_{Y_N} is bijective and continuous, and we prove the Skorokhod theorem in this case. Let U be a r.v. so that $\mathcal{L}(U)$ is the uniform distribution on $[0,1]$ and define the r.v. $\widetilde{Y}_N = F^{-1}_{Y_N}(U)$. We now check that the $\widetilde{Y}_N$ satisfy the conclusion of the theorem. To begin with,

$$F_{\widetilde{Y}_N}(t) = P(\widetilde{Y}_N \leq t) = P(U \leq F_{Y_N}(t)) = F_{Y_N}(t),$$

therefore we have for $1 \leq N \leq \infty$, $\mathcal{L}(Y_N) = \mathcal{L}(\widetilde{Y}_N)$. Now if we assume that $Y_N \longrightarrow Y_\infty$ in law, we have for all $t \in \mathbb{R}$, $F_{Y_N}(t) \longrightarrow F_{Y_\infty}(t)$ and in particular $\widetilde{Y}_N \longrightarrow \widetilde{Y}_\infty$ almost surely.

4.3 General strategy of the proof

Let $(\Omega, \mathcal{F}, \mathbf{p})$ be a probability space and $\big(g_n(\omega)\big)_{n\geq 1}$ a sequence of independent complex normalised Gaussians, $g_n \in \mathcal{N}_{\mathbb{C}}(0,1)$. Let $\mathcal{M}$ be a Riemanian compact manifold and let $(e_n)_{n\geq 1}$ be an Hilbertian basis of $L^2(\mathcal{M})$ (with obvious changes, we can allow $n \in \mathbb{Z}$). Consider one of the equations mentioned in the introduction. Denote by

$$X^\sigma = X^\sigma(\mathcal{M}) = \bigcap_{\tau<\sigma} H^\tau(\mathcal{M}).$$

The general strategy for proving a global existence result is the following:

Step 1: The Gaussian measure $\boldsymbol{\mu}$: We define a measure μ on $X^\sigma(\mathcal{M})$ which is invariant by the flow of the linear part of the equation. The index $\sigma_c \in \mathbb{R}$ is determined by the equation and the manifold $\mathcal{M}$. Indeed this measure can be defined as $\mu = \mathbf{p} \circ \gamma^{-1}$, where $\gamma \in L^2\big(\Omega;\, H^\sigma(\mathcal{M})\big)$ for all $\sigma < \sigma_c$ is a Gaussian random variable which takes the form

$$\gamma(\omega, x) = \sum_{n\geq 1} \frac{g_n(\omega)}{\lambda_n} e_n(x).$$

Here the (λ_n) satisfy $\lambda_n \sim cn^\alpha$, $\alpha > 0$ and are given by the linear part and the Hamiltonian structure of the equation. Notice in particular that for all measurable $F : X^{\sigma_c}(\mathcal{M}) \longrightarrow \mathbb{R}$

$$\int_{X^{\sigma_c}(\mathcal{M})} F(u) \mathrm{d}\mu(u) = \int_\Omega F\big(\gamma(\omega,\cdot)\big) \mathrm{d}\mathbf{p}(\omega). \tag{4.2}$$

Step 2: The invariant measure $\boldsymbol{\rho_N}$: By working on the Hamiltonian formulation of the equation, we introduce an approximation of the initial problem which has a global flow Φ_N, and for which we can construct a measure ρ_N on $X^{\sigma_c}(\mathcal{M})$ which has the following properties

(i) The measure ρ_N is a probability measure which is absolutly continuous with respect to μ

$$\mathrm{d}\rho_N(u) = \Psi_N(u) \mathrm{d}\mu(u).$$

(ii) The measure ρ_N is invariant by the flow Φ_N by the Liouville theorem.

(iii) There exists $\Psi \not\equiv 0$ such that for all $p \geq 1$, $\Psi(u) \in L^p(\mathrm{d}\mu)$ and

$$\Psi_N(u) \longrightarrow \Psi(u), \quad \text{in} \quad L^p(\mathrm{d}\mu).$$

(In particular $\|\Psi_N(u)\|_{L^p_\mu} \leq C$ uniformly in $N \geq 1$.) This enables to define a probability measure on $X^{\sigma_c}(\mathcal{M})$ by

$$\mathrm{d}\rho(u) = \Psi(u) \mathrm{d}\mu(u),$$

which is formally invariant by the equation.

Step 3: The measure ν_N: We abuse notation and write

$$\mathcal{C}\big([-T,T];X^{\sigma_c}(\mathcal{M})\big)=\bigcap_{\sigma<\sigma_c}\mathcal{C}\big([-T,T];H^{\sigma}(\mathcal{M})\big).$$

We denote by $\nu_N=\rho_N\circ\Phi_N^{-1}$ the measure on $\mathcal{C}\big([-T,T];X^{\sigma_c}(\mathcal{M})\big)$, defined as the image measure of ρ_N by the map

$$\begin{array}{ccc} X^{\sigma_c}(\mathcal{M}) & \longrightarrow & \mathcal{C}\big([-T,T];X^{\sigma_c}(\mathcal{M})\big) \\ v & \longmapsto & \Phi_N(t,v). \end{array}$$

In particular, for any measurable $F:\mathcal{C}\big([-T,T];X^{\sigma_c}(\mathcal{M})\big)\longrightarrow\mathbb{R}$

$$\int_{\mathcal{C}\left([-T,T];X^{\sigma_c}\right)}F(u)\mathrm{d}\nu_N(u)=\int_{X^{\sigma_c}}F\big(\Phi_N(t,v)\big)\mathrm{d}\rho_N(v). \tag{4.3}$$

Assume that the corresponding sequence of measures (ν_N) is tight in $\mathcal{C}\big([-T,T];H^{\sigma}(\mathcal{M})\big)$ for all $\sigma<\sigma_c$ (this has to be shown for the considered equation). Therefore, for all $\sigma<\sigma_c$, by the Prokhorov theorem, there exists a measure $\nu_\sigma=\nu$ on $\mathcal{C}\big([-T,T];H^{\sigma}(\mathcal{M})\big)$ so that the weak convergence holds (up to a sub-sequence): For all $\sigma<\sigma_c$ and all bounded continuous $F:\mathcal{C}\big([-T,T];H^{\sigma}(\mathcal{M})\big)\longrightarrow\mathbb{R}$

$$\lim_{N\to\infty}\int_{\mathcal{C}\left([-T,T];H^{\sigma}\right)}F(u)\mathrm{d}\nu_N(u)=\int_{\mathcal{C}\left([-T,T];H^{\sigma}\right)}F(u)\mathrm{d}\nu(u).$$

At this point, observe that if $\sigma_1<\sigma_2$, then $\nu_{\sigma_1}\equiv\nu_{\sigma_2}$ on $\mathcal{C}\big([-T,T];H^{\sigma_1}(\mathcal{M})\big)$. Moreover, by the standard diagonal argument, we can ensure that ν is a measure on $\mathcal{C}\big([-T,T];X^{\sigma_c}(\mathcal{M})\big)$.

Finally, with the Skorokhod theorem, we can construct a sequence of random variables which converges to a solution of the initial problem.

We now state a result which will be useful in the sequel. Assume that ρ_N satisfies the properties mentioned in Step 2.

Proposition 4.7. Let $\sigma<\sigma_c$. Let $p\geq 2$ and $r>p$. Then for all $N\geq 1$

$$\big\|\|u\|_{L^p_TH^\sigma_x}\big\|_{L^p_{\nu_N}}\leq CT^{1/p}\big\|\|v\|_{H^\sigma_x}\big\|_{L^r_\mu}.$$

Let $q\geq 1$, $p\geq 2$ and $r>p$. Then for all $N\geq 1$

$$\big\|\|u\|_{L^p_TL^q_x}\big\|_{L^p_{\nu_N}}\leq CT^{1/p}\big\|\|v\|_{L^q_x}\big\|_{L^r_\mu}.$$

In case $\Psi_N\leq C$, one can take $r=p$ in the previous inequalities.

Proof. We apply (4.3) with the function $u\longmapsto F(u)=\|u\|^p_{L^p_TH^\sigma_x}$. Here and after, we make the abuse of notation

$$\big\|\|u\|_{L^p_TH^\sigma_x}\big\|_{L^p_{\nu_N}}=\|u\|_{L^p_{\nu_N}L^p_TH^\sigma_x}.$$

Then

$$\begin{aligned}
\|u\|^p_{L^p_{\nu_N}L^p_T H^\sigma_x} &= \int_{\mathcal{C}\left([-T,T];X^{\sigma_c}\right)} \|u\|^p_{L^p_T H^\sigma_x} \mathrm{d}\nu_N(u) \\
&= \int_{X^{\sigma_c}} \|\Phi_N(t,v)\|^p_{L^p_T H^\sigma_x} \mathrm{d}\rho_N(v) \\
&= \int_{X^{\sigma_c}} \Big[\int_{-T}^{T} \|\Phi_N(t,v)\|^p_{H^\sigma_x} \mathrm{d}t\Big] \mathrm{d}\rho_N(v) \\
&= \int_{-T}^{T} \Big[\int_{X^{\sigma_c}} \|\Phi_N(t,v)\|^p_{H^\sigma_x} \mathrm{d}\rho_N(v)\Big] \mathrm{d}t, \qquad (4.4)
\end{aligned}$$

where in the last line we used Fubini. Now we use the invariance of ρ_N under Φ_N, and we deduce that for all $t \in [-T,T]$

$$\int_{X^{\sigma_c}} \|\Phi_N(t,v)\|^p_{H^\sigma_x} \mathrm{d}\rho_N(v) = \int_{X^{\sigma_c}} \|v\|^p_{H^\sigma_x} \mathrm{d}\rho_N(v).$$

Therefore, from (4.4) and Hölder we obtain with $1/r_1 + 1/r_2 = 1$

$$\begin{aligned}
\|u\|^p_{L^p_{\nu_N}L^p_T H^\sigma_x} &= 2T \int_{X^{\sigma_c}} \|v\|^p_{H^\sigma_x} \mathrm{d}\rho_N(v) \\
&= 2T \int_{X^{\sigma_c}} \|v\|^p_{H^\sigma_x} \Psi_N(v) \mathrm{d}\mu(v) \\
&\leq 2T \|v\|^p_{L^{pr_1}_\mu H^\sigma_x} \|\Psi_N(v)\|_{L^{r_2}_\mu}.
\end{aligned}$$

Now, let $r > p$, take $r_1 = r/p$ and we can conclude since $\Psi_N(v) \in L^{r_2}(\mathrm{d}\mu)$.
For the proof of the second estimate, we proceed similarly. We take $F(u) = \|u\|^p_{L^p_T L^q_x}$ in (4.3), and use the same arguments as previously. □

4.4 The probabilistic argument of convergence

4.4.1 Definition of $\mathcal{T}(u,u,u)$ on the support of μ

Denote by E_k the space on $\mathbb{C}$ spanned by φ_k. For $N \geq 0$, denote by Π_N the orthogonal projector on the space $\bigoplus_{k=0}^N E_k$ (in this section, we do not need the smooth cut-offs S_N). In the sequel, we denote by $\mathcal{T}(u) = \mathcal{T}(u,u,u)$ and $\mathcal{T}_N(u) = \Pi_N \mathcal{T}(\Pi_N u, \Pi_N u, \Pi_N u)$

Proposition 4.8. For all $p \geq 2$ and $\sigma > 1$, the sequence $\big(\mathcal{T}_N(u)\big)_{N\geq 1}$ is a Cauchy sequence in $L^p\big(X^{-1}_{hol}, \mathcal{B}, d\mu; \mathcal{H}^{-\sigma}(\mathbb{C})\big)$. Namely, for all $p \geq 2$, there exist $\delta > 0$ and $C > 0$ so that for all $1 \leq M < N$,

$$\int_{X^{-1}_{hol}} \|\mathcal{T}_N(u) - \mathcal{T}_M(u)\|^p_{\mathcal{H}^{-\sigma}(\mathbb{C})} \mathrm{d}\mu(u) \leq CM^{-\delta}.$$

We denote by $\mathcal{T}(u) = \mathcal{T}(u,u,u)$ the limit of this sequence and we have for all $p \geq 2$

$$\|\mathcal{T}(u)\|_{L^p_\mu \mathcal{H}^{-\sigma}(\mathbb{C})} \leq C_p. \qquad (4.5)$$

Proof. By the Proposition 2.3 on the Wiener chaos, we only have to prove the statement for $p = 2$.

Firstly, by definition of the measure μ

$$\int_{X^{-1}_{hol}} \|\mathcal{T}_N(u) - \mathcal{T}_M(u)\|^2_{\mathcal{H}^{-\sigma}(\mathbb{C})} \mathrm{d}\mu(u) = \int_{\Omega} \|\mathcal{T}_N\big(\eta(\omega)\big) - \mathcal{T}_M\big(\eta(\omega)\big)\|^2_{\mathcal{H}^{-\sigma}(\mathbb{C})} \mathrm{d}\mathbf{p}(\omega).$$

Therefore, it is enough to prove that $\big(\mathcal{T}_N(\eta)\big)_{N\geq 1}$ is a Cauchy sequence in $L^2\big(\Omega; \mathcal{H}^{-\sigma}(\mathbb{C})\big)$.

Let $1 \leq M < N$ and fix $\sigma > 1$. Then an explicit computation gives

$$\|\mathcal{T}_N(\eta) - \mathcal{T}_M(\eta)\|^2_{\mathcal{H}^{-\sigma}(\mathbb{C})} = \\ = \frac{\pi^2}{64 \cdot 2^\sigma} \sum_{p=0}^{N} \frac{1}{(p+1)^\sigma} \sum_{(n,m)\in A^{(p)}_{M,N} \times A^{(p)}_{M,N}} \frac{(n_1+n_2)!\,(m_1+m_2)!\, g_{n_1} g_{n_2} \overline{g_{n_3} g_{m_1} g_{m_2}} g_{m_3}}{2^{n_1+n_2} 2^{m_1+m_2} p!\, \sqrt{n_1!\, n_2!\, n_3!}\sqrt{m_1!\, m_2!\, m_3!}}$$

where $A^{(p)}_{M,N}$ is the set defined by

$$A^{(p)}_{M,N} = \Big\{ n \in \mathbb{N}^3 \text{ s.t. } 0 \leq n_j \leq N,\ \ n_1 + n_2 - n_3 = p \in \{0 \dots N\}, \\ \big(n_1 > M \text{ or } n_2 > M \text{ or } n_3 > M \text{ or } p > M\big)\Big\}.$$

Now we take the integral over Ω. Here, the key fact is to use that the $(g_n)_{n\geq 0}$ are independent and centred Gaussians: we deduce that each term in the r.h.s. vanishes, unless

Case 1: $(n_1, n_2, n_3) = (m_1, m_2, m_3)$ or $(n_1, n_2, n_3) = (m_2, m_1, m_3)$

or

Case 2: $(n_1, n_2, m_1) = (n_3, m_2, m_3)$ or $(n_1, n_2, m_2) = (n_3, m_1, m_3)$ or $(n_1, n_2, m_3) = (m_1, n_3, m_2)$ or $(n_1, n_2, m_3) = (m_2, n_3, m_1)$.

With a careful inspection of each contribution, we are able to bound the different sums. □

4.4.2 Study of the measure ν_N

Let $N \geq 1$. We then consider the following approximation of (LLL)

$$\begin{cases} i\partial_t u = \mathcal{T}_N(u), \quad (t, z) \in \mathbb{R} \times \mathbb{C}, \\ u(0, z) = u_0(z) \in X^{-1}_{hol}. \end{cases} \tag{4.6}$$

The equation (4.6) is an ODE in the frequencies less than N, whereas for the large frequencies, the solution is constant in time: $(1 - \Pi_N)u(t) = (1 - \Pi_N)u_0$ and for all $t \in \mathbb{R}$.

The main motivation to introduce this system is the following proposition

Proposition 4.9. The equation (4.6) has a global flow Φ_N. Moreover, the measure μ is invariant under Φ_N : For any Borel set $A \subset X^{-1}_{hol}$ and for all $t \in \mathbb{R}$, $\mu\big(\Phi_N(t, A)\big) = \mu\big(A\big)$.

In particular if $\mathscr{L}_{X^{-1}}(v) = \mu$ then for all $t \in \mathbb{R}$, $\mathscr{L}_{X^{-1}}(\Phi_N(t, v)) = \mu$.

Proof. The proof is a direct application of the Liouville Theorem 1.5. □

We denote by ν_N the measure on $\mathcal{C}\big([-T,T];X^{-1}_{hol}\big)$, defined as the image measure of μ by the map

$$\begin{array}{ccc} X^{-1}_{hol} & \longrightarrow & \mathcal{C}\big([-T,T];X^{-1}_{hol}\big) \\ v & \longmapsto & \Phi_N(t,v). \end{array}$$

Lemma 4.10. Let $\sigma>1$ and $p\geq 2$. Then there exists $C>0$ so that for all $N\geq 1$

$$\big\|\|u\|_{W^{1,p}_T\mathcal{H}^{-\sigma}_z}\big\|_{L^p_{\nu_N}}\leq C.$$

Proof. Firstly, we have that for $\sigma>1$, $p\geq 2$ and $N\geq 1$

$$\big\|\|u\|_{L^p_T\mathcal{H}^{-\sigma}_z}\big\|_{L^p_{\nu_N}}\leq C.$$

Indeed, by the definition of ν_N and the invariance of μ by Φ_N we have

$$\|u\|_{L^p_{\nu_N}L^p_T\mathcal{H}^{-\sigma}_z}=(2T)^{1/p}\|v\|_{L^p_\mu\mathcal{H}^{-\sigma}_z}=(2T)^{1/p}\|\eta\|_{L^p_{\mathbf{p}}\mathcal{H}^{-\sigma}_z}.$$

Then, by the Khintchine inequality (3.20) and (3.15), for all $p\geq 2$

$$\|\eta\|_{L^p_{\mathbf{p}}\mathcal{H}^{-\sigma}_z}\leq C\sqrt{p}\|\eta\|_{L^2_{\mathbf{p}}\mathcal{H}^{-\sigma}_z}\leq C.$$

Next, we show that $\big\|\|\partial_t u\|_{L^p_T\mathcal{H}^{-\sigma}_z}\big\|_{L^p_{\nu_N}}\leq C$. By definition of ν_N

$$\begin{aligned}\|\partial_t u\|^p_{L^p_{\nu_N}L^p_T\mathcal{H}^{-\sigma}_z} &= \int_{\mathcal{C}\big([-T,T];X^{-1}_{hol}\big)}\|\partial_t u\|^p_{L^p_T\mathcal{H}^{-\sigma}_z}\mathrm{d}\nu_N(u)\\ &= \int_{X^{-1}_{hol}}\|\partial_t\Phi_N(t,v)\|^p_{L^p_T\mathcal{H}^{-\sigma}_z}\mathrm{d}\mu(v).\end{aligned}$$

Now, since $\Phi_N(t,v)$ satisfies (4.6) and by the invariance of μ, we have

$$\begin{aligned}\|\partial_t u\|^p_{L^p_{\nu_N}L^p_T\mathcal{H}^{-\sigma}_z} &= \int_{X^{-1}_{hol}}\|\mathcal{T}_N(\Phi_N(t,v))\|^p_{L^p_T\mathcal{H}^{-\sigma}_z}\mathrm{d}\mu(v)\\ &= 2T\int_{X^{-1}_{hol}}\|\mathcal{T}_N(v)\|^p_{\mathcal{H}^{-\sigma}_z}\mathrm{d}\mu(v),\end{aligned}$$

and conclude with (4.5) and Proposition 4.8. □

4.4.3 The convergence argument

The importance of Lemma 4.10 above comes from the fact that it allows to establish the following tightness result for the measures ν_N.

Proposition 4.11. Let $T>0$ and $\sigma>1$. Then the family of measures

$$(\nu_N)_{N\geq 1}\quad\text{with}\quad \nu_N=\mathscr{L}_{\mathcal{C}_T\mathcal{H}^{-\sigma}}\big(u_N(t);t\in[-T,T]\big)$$

is tight in $\mathcal{C}\big([-T,T];\mathcal{H}^{-\sigma}(\mathbb{C})\big)$.

Proof. Let $\sigma > 1$. Fix $\sigma > s' > s'' > 1$ and $\alpha > 0$.

We define the space $\mathcal{C}_T^\alpha \mathcal{H}^{-s'} = \mathcal{C}^\alpha\big([-T,T];\mathcal{H}^{-s'}(\mathbb{C})\big)$ by the norm

$$\|u\|_{\mathcal{C}_T^\alpha \mathcal{H}^{-s'}} = \sup_{t_1,t_2\in[-T,T],\, t_1\neq t_2} \frac{\|u(t_1)-u(t_2)\|_{\mathcal{H}_z^{-s'}}}{|t_1-t_2|^\alpha} + \|u\|_{L_T^\infty \mathcal{H}_z^{-s'}},$$

and it is classical that the embedding $\mathcal{C}_T^\alpha \mathcal{H}^{-s'} \subset \mathcal{C}\big([-T,T];\mathcal{H}^{-\sigma}(\mathbb{C})\big)$ is compact.

We now claim that there exists $0<\alpha\ll 1$ so that for all $p\geq 1$ we have the bound

$$\|u\|_{L^p_{\nu_N}\mathcal{C}_T^\alpha \mathcal{H}^{-s'}} \leq C. \tag{4.7}$$

With an interpolation argument we obtain that for some $p \gg 1$

$$\|u\|_{\mathcal{C}_T^\alpha \mathcal{H}^{-s'}} \leq C\|u\|^{1-\theta}_{L_T^p\mathcal{H}^{-s''}}\|u\|^{\theta}_{W_T^{1,p}\mathcal{H}^{-\sigma}} \leq C\|u\|_{L_T^p\mathcal{H}^{-s''}} + C\|u\|_{W_T^{1,p}\mathcal{H}^{-\sigma}},$$

for some small $\alpha>0$. By Lemma 4.10 we then deduce (4.7). Next, let $\delta>0$ and define the subset of $\mathcal{C}_T\mathcal{H}^{-\sigma}$

$$K_\delta = \big\{u\in \mathcal{C}_T\mathcal{H}^{-\sigma} \;\; s.t. \;\; \|u\|_{\mathcal{C}_T^\alpha\mathcal{H}^{-s'}} \leq \delta^{-1}\big\},$$

endowed with the natural topology of $\mathcal{C}_T\mathcal{H}^{-\sigma}$. Thanks to the previous considerations, the set K_δ is compact. Finally, by Markov and (4.7) we get that

$$\nu_N(K_\delta^c) \leq \delta\, \|u\|_{L^1_{\nu_N}\mathcal{C}_T^\alpha\mathcal{H}^{-s'}} \leq \delta C,$$

which shows the tightness of (ν_N). □

The result of Proposition 4.11 enables us to use the Prokhorov theorem: For each $T>0$ there exists a sub-sequence ν_{N_k} and a measure ν on the space $\mathcal{C}\big([-T,T];X^{-1}_{hol}\big)$ so that for all $\tau>1$ and all bounded continuous function $F:\mathcal{C}\big([-T,T];\mathcal{H}^{-\tau}(\mathbb{C})\big)\longrightarrow\mathbb{R}$

$$\int_{\mathcal{C}\big([-T,T];\mathcal{H}^{-\tau}(\mathbb{C})\big)} F(u)\mathrm{d}\nu_{N_k}(u) \longrightarrow \int_{\mathcal{C}\big([-T,T];\mathcal{H}^{-\tau}(\mathbb{C})\big)} F(u)\mathrm{d}\nu(u).$$

By the Skohorod theorem, there exists a probability space $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbf{p}})$, a sequence of random variables $(\widetilde{u}_{N_k})$ and a random variable $\widetilde{u}$ with values in $\mathcal{C}\big([-T,T];X^{-1}_{hol}\big)$ so that

$$\mathscr{L}\big(\widetilde{u}_{N_k};t\in[-T,T]\big) = \mathscr{L}\big(u_{N_k};t\in[-T,T]\big) = \nu_{N_k}, \quad \mathscr{L}\big(\widetilde{u};t\in[-T,T]\big)=\nu, \tag{4.8}$$

and for all $\tau>1$

$$\widetilde{u}_{N_k}\longrightarrow \widetilde{u}, \qquad \widetilde{\mathbf{p}}-\text{a.s. in } \mathcal{C}\big([-T,T];\mathcal{H}^{-\tau}(\mathbb{C})\big). \tag{4.9}$$

We now claim that $\mathscr{L}_{X^{-1}}(u_{N_k}(t)) = \mathscr{L}_{X^{-1}}(\widetilde{u}_{N_k}(t)) = \mu$, for all $t\in[-T,T]$ and $k\geq 1$. Indeed, for all $t\in[-T,T]$, the evaluation map

$$\begin{array}{rccc} R_t : & \mathcal{C}\big([-T,T];X^{-1}_{hol}\big) & \longrightarrow & X^{-1}_{hol} \\ & u & \longmapsto & u(t,.), \end{array}$$

is well defined and continuous.

Thus, for all $t \in [-T,T]$, $u_{N_k}(t)$ and $\widetilde{u}_{N_k}(t)$ have same distribution $(R_t)_{\#}\nu_{N_k}$. By Proposition 4.9, we obtain that this distribution is μ.

Thus from (4.9) we deduce that

$$\mathscr{L}_{X^{-1}}(\widetilde{u}(t)) = \mu, \quad \forall\, t \in [-T,T]. \tag{4.10}$$

Let $k \geq 1$ and $t \in \mathbb{R}$ and consider the r.v. X_k given by

$$X_k = u_{N_k}(t) - R_0(u_{N_k}(t)) + i \int_0^t \mathcal{T}_{N_k}(u_{N_k}) ds.$$

Define $\widetilde{X}_k$ similarly to X_k with u_{N_k} replaced with $\widetilde{u}_{N_k}$. Then by (4.8),

$$\mathscr{L}_{\mathcal{C}_T X^{-1}}(\widetilde{X}_{N_k}) = \mathscr{L}_{\mathcal{C}_T X^{-1}}(X_{N_k}) = \delta_0.$$

In other words, $\widetilde{X}_k = 0$ $\widetilde{\mathbf{p}}-$a.s. and $\widetilde{u}_{N_k}$ satisfies the following equation $\widetilde{\mathbf{p}}-$a.s.

$$\widetilde{u}_{N_k}(t) = R_0(\widetilde{u}_{N_k}(t)) - i \int_0^t \mathcal{T}_{N_k}(\widetilde{u}_{N_k}) ds. \tag{4.11}$$

We now show that we can pass to the limit $k \longrightarrow +\infty$ in (4.11) in order to show that $\widetilde{u}$ is $\widetilde{\mathbf{p}}-$a.s. a solution to (LLL) written in integral form as:

$$\widetilde{u}(t) = R_0(\widetilde{u}(t)) - i \int_0^t \mathcal{T}(\widetilde{u}) ds. \tag{4.12}$$

Firstly, from (4.9) we deduce the convergence of the linear terms in equation (4.11) to those in (4.12). The following lemma gives the convergence of the nonlinear term.

Lemma 4.12. Up to a sub-sequence, the following convergence holds true

$$\mathcal{T}_{N_k}(\widetilde{u}_{N_k}) \longrightarrow \mathcal{T}(\widetilde{u}), \qquad \widetilde{\mathbf{p}} - \text{a.s. in } L^2\big([-T,T]; \mathcal{H}^{-\sigma}(\mathbb{C})\big).$$

Proof. In order to simplify the notations, in this proof we drop the tildes and write $N_k = k$. Let $M \geq 1$ and write

$$\mathcal{T}_k(u_k) - \mathcal{T}(u) = \big(\mathcal{T}_k(u_k) - \mathcal{T}(u_k)\big) + \big(\mathcal{T}(u_k) - \mathcal{T}_M(u_k)\big) + \big(\mathcal{T}_M(u_k) - \mathcal{T}_M(u)\big) + \big(\mathcal{T}_M(u) - \mathcal{T}(u)\big).$$

To begin with, by continuity of the product in finite dimension, when $k \longrightarrow +\infty$

$$\mathcal{T}_M(u_k) \longrightarrow \mathcal{T}_M(u), \qquad \widetilde{\mathbf{p}} - \text{a.s. in } L^2\big([-T,T]; \mathcal{H}^{-\sigma}(\mathbb{C})\big).$$

We now deal with the other terms. It is sufficient to show the convergence in the space $X := L^2\big(\Omega \times [-T,T]; \mathcal{H}^{-\sigma}(\mathbb{C})\big)$, since the almost sure convergence follows after exaction of a sub-sequence.

By definition and the invariance of μ we obtain

$$\begin{aligned}
\| \mathcal{T}_M(u_k) - \mathcal{T}(u_k) \|_X^2 &= \int_{\mathcal{C}([-T,T];X^{-1})} \| \mathcal{T}_M(v) - \mathcal{T}(v) \|^2_{L^2_T \mathcal{H}_z^{-\sigma}} \mathrm{d}\nu_k(v) \\
&= \int_{X^{-1}(\mathbb{C})} \Big\| \mathcal{T}_M\big(\Phi_k(t,g)\big) - \mathcal{T}\big(\Phi_k(t,g)\big) \Big\|^2_{L^2_T \mathcal{H}_z^{-\sigma}} \mathrm{d}\mu(g) \\
&= \int_{X^{-1}(\mathbb{C})} \| \mathcal{T}_M(g) - \mathcal{T}(g)) \|^2_{L^2_T \mathcal{H}_z^{-\sigma}} \mathrm{d}\mu(g) \\
&= 2T \int_{X^{-1}(\mathbb{C})} \| \mathcal{T}_M(g) - \mathcal{T}(g) \|^2_{\mathcal{H}_z^{-\sigma}} \mathrm{d}\mu(g),
\end{aligned}$$

which tends to 0 uniformly in $k \geq 1$ when $M \longrightarrow +\infty$, according to Proposition 4.8.

The term $\big\| \mathcal{T}_M(u) - \mathcal{T}(u) \big\|_X$ is treated similarly. Finally, with the same argument we show

$$\big\| \mathcal{T}_k(u_k) - \mathcal{T}(u_k) \big\|_X \leq C \big\| \mathcal{T}_k(g) - \mathcal{T}(g) \big\|_{L^2_\mu \mathcal{H}^{-\sigma}_z},$$

which tends to 0 when $k \longrightarrow +\infty$. This completes the proof. □

4.4.4 Conclusion of the proof of Theorem 4.1

Define $\widetilde{u}_0 = \widetilde{u}(0) := R_0(\tilde{u})$. Then by (4.10), $\mathscr{L}_{X^{-1}}(\,\widetilde{u}_0\,) = \mu$ and by the previous arguments, there exists $\widetilde{\Omega}' \subset \widetilde{\Omega}$ such that $\widetilde{\mathbf{p}}(\widetilde{\Omega}') = 1$ and for each $\omega' \in \widetilde{\Omega}'$, the random variable $\widetilde{u}$ satisfies the equation

$$\widetilde{u} = \widetilde{u}_0 - i \int_0^t \mathcal{T}(\widetilde{u}) dt, \quad (t,z) \in \mathbb{R} \times \mathbb{C}. \tag{4.13}$$

Set $\Sigma = \widetilde{u}_0(\Omega')$, then $\mu(\Sigma) = \widetilde{\mathbf{p}}(\widetilde{\Omega}') = 1$. It remains to check that we can construct a global dynamics. Take a sequence $T_N \to +\infty$, and perform the previous argument for $T = T_N$. For all $N \geq 1$, let Σ_N be the corresponding set of initial conditions and set $\Sigma = \cap_{N \in \mathbb{N}} \Sigma_N$. Then $\mu(\Sigma) = 1$ and for all $\widetilde{u}_0 \in \Sigma$, there exists

$$\widetilde{u} \in \mathcal{C}\big(\mathbb{R}\,; X^{-1}_{hol}\big),$$

which solves (4.13). This completes the proof of Theorem 4.1.

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