
LONG TIME DYNAMICS FOR THE ONE DIMENSIONAL NON LINEAR SCHRÖDINGER EQUATION

by

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Abstract. — In this article, we first present the construction of Gibbs measures associated to nonlinear Schrödinger equations with harmonic potential. Then we show that the corresponding Cauchy problem is globally well-posed for rough initial conditions in a statistical set (the support of the measures). Finally, we prove that the Gibbs measures are indeed invariant by the flow of the equation. As a byproduct of our analysis, we give a global well-posedness and scattering result for the L^2 critical and super-critical NLS (without harmonic potential).

1. Introduction

The purpose of this work is twofold. First we construct Gibbs measures and prove their invariance by the flow of the nonlinear (focusing and defocusing) Schrödinger equations (defined in a strong sense) in the presence of a harmonic potential. In the construction of these measures, most of the difficulties appear for the focusing case (for which case our results are only true for the cubic non linearity while in the defocusing case we have no restriction on the size of the non linearity). The non linear harmonic oscillator appears as a model in the context of Bose-Einstein condensates and our result gives some insights concerning the long time dynamics of these models. The second purpose of this work is to prove global well-posedness for the L^2 critical and super-critical nonlinear Schrödinger equation (NLS) on \mathbb{R} , with or without harmonic potential, for data of low regularity. Furthermore, we also obtain scattering when there is no harmonic potential. Such kind of result seems to be out of reach of the present critical regularity deterministic methods.

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1.1. The NLS with harmonic potential. — Our analysis here on the NLS with harmonic potential enters into the line of research initiated by Lebowitz-Rose-Speer in [23] and aiming to construct Gibbs measures for Hamiltonian PDE's. This program offers analytic challenges both in the measure construction and in the construction of a well-defined flow on the support of the measure. Usually the support of the measure contains low regularity functions and this fact may be seen as one of the motivations for studying low regularity well-posedness of Hamiltonian PDE's. The approach of [23] has been implemented successfully in several contexts, see *e.g.* Bourgain [4, 5], Zhidkov [43], Tzvetkov [39, 38, 37], Burq-Tzvetkov [7], Oh [26, 27], and the references therein.

A very natural context where one may try to construct Gibbs measures is the Nonlinear Schrödinger equation (NLS) with harmonic potential. Indeed, in this case the spectrum of the linear problem is discrete and the construction of [23] applies at least at the formal level. As we already mentioned this context is natural since the NLS with harmonic potential appears as a model in the Bose-Einstein condensates. As we shall see, it turns out that the construction of [23] provides a Gibbs measure supported by functions for which the corresponding Cauchy problem was not known to be well-posed. In addition the density of the measure can not be evaluated by applying only deterministic arguments such as the Sobolev inequality. All these facts present serious obstructions to make rigorous the Gibbs measure construction.

On the other hand, recent works as [7, 8, 37] showed that by applying more involved probabilistic techniques in combination with the existing deterministic technology for studying these problems one may approach the above difficulties successfully. In particular, in [8, 9] an approach to handle regularities for which the corresponding Cauchy problem is ill-posed is developed. Our goal here is to show that the NLS with harmonic oscillator fits well in this approach. In fact, the eigenfunctions of the linear operator enjoy good estimates which is compensated by the bad separation properties of the spectrum. Such a situation is particularly well adapted for the approach of [8, 9].

We are able to construct Gibbs measures and the corresponding flow for the cubic focusing and arbitrary defocusing NLS in the presence of harmonic potential. The analysis turns out to contain several significant new points with respect to previous works on the subject. Indeed, it seems that it is the first case where the construction and analysis of Gibbs measure with a strong flow can be performed on a non compact phase space. Furthermore, taking into account the low regularity of the initial data, to develop a nice local Cauchy theory at this level of regularity, one has to obtain somehow a gain in terms of derivatives. In the context of wave equations, this gain can be obtained (see [8, 9]) rather easily by first proving a gain at the probabilistic level in terms of L^p regularity, and then balancing this gain on the non linearity and using that the non homogeneous wave propagator itself gains one derivative with respect to the source term regularity. For Schrödinger equations, the situation is much less well behaved. Indeed, no such gain of regularity occurs for the non homogeneous Schrödinger propagator, and the starting point of our analysis was precisely that a gain of derivatives occurs at the probabilistic level in terms of L^p regularity. However, this gain which would allow to perform the analysis for low power nonlinearities ($k \leq 7$) falls well short of what is needed to obtain the full range result ($k < +\infty$) in Theorem 2.4. As a consequence, our analysis requires a full bi-linear analysis at the probabilistic level (see (1.2)). Finally, let us mention some previous works on the non linear harmonic Schrödinger equation (1.1). The (deterministic) Cauchy problem for (1.1) was studied by Oh [28]. We refer to Zhang [42] for blow up and global existence results and Fukuizumi [18] for

the stability of standing waves associated to (1.1). See also Carles [13] for the case of time-dependent potentials and the book of T. Cazenave [14, Chapter 9] for more references.

Let us now describe in more details our results and consider thus the one dimensional non linear Schrödinger equation with harmonic potential

$$(1.1) \quad \begin{cases} i\partial_t u + \partial_x^2 u - x^2 u = \kappa_0 |u|^{k-1} u, & (t, x) \in \mathbb{R} \times \mathbb{R}, \\ u(0, x) = f(x), \end{cases}$$

where $k \geq 3$ is an odd integer and where either $\kappa_0 = 1$ (defocusing case) or $\kappa_0 = -1$ (focusing case). Though we are not aware of physics models involving other non linearities than cubic or quintic, (1.1) appears to be an interesting models on the mathematical point of view. We now state our result concerning (1.1). For more detailed results, see Theorem 2.3 and Theorem 2.4 below.

Theorem 1.1. — *Consider the L^2 Wiener measure on $\mathcal{D}'(\mathbb{R})$, μ , constructed on the harmonic oscillator eigenbasis, i.e. μ is the distribution of the random variable*

$$\sum_{n=0}^{\infty} \sqrt{\frac{2}{2n+1}} g_n(\omega) h_n(x),$$

where $(h_n)_{n=0}^{\infty}$ are the Hermite functions (see (2.1)) and $(g_n)_{n=0}^{\infty}$ is a system of standard independent complex Gaussian random variables. Then in the defocusing case, for any order of nonlinearity, and in the focusing case for the cubic non linearity, the Cauchy problem (1.1) is globally well posed for μ -almost every initial data. Furthermore, in both cases, there exists a Gibbs measure, absolutely continuous with respect to μ , which is invariant by this flow.

The equation (1.1) is a Hamiltonian PDE with a Hamiltonian $J(u)$ (see (2.2) below). As usual the Gibbs measure is a suitable renormalisation of the formal object $\exp(-J(u))du$. Let us recall that the distribution function of a standard (0 mean and 1 variance) Gaussian complex random variable is

$$\frac{1}{\pi} e^{-|z|^2} dL,$$

where dL is the Lebesgue measure on \mathbb{C} .

Notice that the results above are not in the "small data" class of results. Indeed, it follows from our analysis that the measure μ is such that for every $p > 2$ and every $R > 0$ $\mu(u : \|u\|_{L^p} > R) > 0$, i.e. our statistical set contains "many" initial data which are arbitrary large in $L^p(\mathbb{R})$, $p > 2$. Moreover, we use no smallness argument in any place of the proof.

We conjecture that our results hold when x^2 is replaced with a potential $V \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}_+)$, so that $V(x) \sim x^2$ for $|x| \gg 1$ and $|\partial_x^j V(x)| \leq C_j \langle x \rangle^{2-|j|}$ (in particular in such a situation there exists $C > 0$ so that $\lambda_n^2 \sim Cn$).

Let us define the Sobolev spaces \mathcal{H}^s , associated to the harmonic oscillator $-\partial_x^2 + x^2$ via the norm $\|u\|_{\mathcal{H}^s} = \|(-\partial_x^2 + x^2)^{s/2} u\|_{L^2}$. As can easily be seen, for any $s \geq 0$, the Sobolev space of regularity s , $\mathcal{H}^s(\mathbb{R})$ has zero μ measure but for every $s < 0$ the space \mathcal{H}^s is of full μ measure. As a consequence, the initial data in our result is not covered by the present well-posedness theory for (1.1). What is even worse: according to Christ, Colliander, Tao [10] and Burq, Gérard, Tzvetkov [6, Appendix] (notice that these results do not apply stricto sensu to the harmonic oscillator, but the proof can easily be modified), we know that as soon as $k \geq 7$ the system (1.1) is supercritical and there exists no continuous flow on the Sobolev spaces H^s , for $s \in (0, \frac{1}{2} - \frac{2}{k-1})$. As a consequence, even in the local

in time analysis we need to appeal to a bi-linear probabilistic argument. The bi-linear nature of our probabilistic analysis can be seen through the following statement

$$(1.2) \quad \forall \theta < 1/2, \forall t \in \mathbb{R}, \quad \|(e^{-itH}u)^2\|_{\mathcal{H}^\theta} < +\infty, \quad \mu \text{ almost surely.}$$

In our actual proof we do not make use of (1.2) but it was the starting point of our analysis for large k 's. We give the proof of (1.2) in the appendix of this article.

1.2. Global well-posedness and scattering for the “usual” L^2 critical and super-critical NLS on \mathbb{R} . — It turns out that the result described in the previous section has an interesting byproduct. Thanks to the lens transform which has been introduced in [25, 31] (see also [11, 33]), we are able to prove a scattering result for the L^2 critical and super-critical equation

$$(1.3) \quad \begin{cases} i\partial_t u + \partial_x^2 u = |u|^{k-1}u, & k \geq 5, \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \\ u(0, x) = f(x) \end{cases}$$

for $f(x)$ of “super-critical” regularity.

Theorem 1.2. — *The equation (1.3) has for μ -almost every initial data a unique global solution satisfying for any $0 < s < 1/2$,*

$$u(t, \cdot) - e^{it\Delta} f \in C(\mathbb{R}; \mathcal{H}^s(\mathbb{R}))$$

(the uniqueness holds in a space continuously embedded in $C(\mathbb{R}; \mathcal{H}^s(\mathbb{R}))$). Moreover, the solution scatters in the following sense. There exists μ a.s. states $g_\pm \in \mathcal{H}^s(\mathbb{R})$ so that

$$\|u(t, \cdot) - e^{it\Delta}(f + g_\pm)\|_{\mathcal{H}^s(\mathbb{R})} \longrightarrow 0, \quad \text{when } t \longrightarrow \pm\infty.$$

The result of Theorem 1.2 is a large data result and apart from the very recent result by Dodson [16] (global existence and scattering in L^2 for the critical quintic NLS), as far as we know there is no large data scattering results for the problem (1.3) for data which are localized (tending to zero at infinity) but missing H^1 , *i.e.* it seems that the result of Theorem 1.2 is out of reach of the present deterministic results to get scattering. We refer to [24] for deterministic scattering results for (1.3) in Sobolev spaces, H^s , $s \geq 1$. We also refer to [21] for an approach for obtaining scattering results for L^2 critical problems.

The result of Theorem 1.2 is based on a transformation which reduces (1.3) to a problem which fits in the scope of applicability of our previous analysis. However (except in the scale invariant case $k = 5$), the reduced problem is not autonomous which makes the arguments more delicate. In particular there is no conserved energy for the reduced problem. However, we will be able to substitute this lack of conservation law by a monotonicity property which in turn will lead to the fact that, roughly speaking, the measure of a set can not decrease along the flow which is the key of the globalization argument. As a consequence, we are able to carry out the global existence strategy whilst no invariant measure is available (see also Colliander-Oh [15] for results in this direction).

1.3. Plan of the paper. — In the following section, we present in details the construction of the Gibbs measure and we give a detailed measure invariance statement. In Section 3, we give the proof of the approximation property of the Gibbs measure by “finite dimensional” measures. In the next section, we establish a functional calculus of $-\partial_x^2 + x^2$, fundamental for the future analysis. The following two sections are devoted to establishing two families of linear dispersive estimates, namely

the Strichartz and the local smoothing estimates. Here we develop the very classical deterministic estimates but also the more recent stochastic variants of them. The interplay between these two families of estimates is at the heart of our approach. In Section 7, we use the estimates of the two previous sections together with the functional calculus to develop a local Cauchy theory. In Section 8, we present the global arguments, leading to almost sure global well-posedness on the support of the measure. Notice that the path followed here has been much clarified with respect to our previous papers, and consequently is much more versatile. In Section 9, we prove the measure invariance. Section 10 is devoted to the proof of Theorem 1.2. Finally, in an appendix, we gathered some typical properties of the measure μ . These properties are not necessary for the understanding of the proofs of our main results, but they are important in view of understanding the scope of these results.

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2. Hamiltonian formulation and construction of the Gibbs measure

Set $H = -\partial_x^2 + x^2$. Let us recall some elementary facts concerning H (see *e.g.* [29]). The operator H has a self-adjoint extension on $L^2(\mathbb{R})$ (still denoted by H) and has eigenfunctions $(h_n)_{n \geq 0}$ which form a Hilbertian basis of $L^2(\mathbb{R})$ and satisfy $Hh_n = \lambda_n^2 h_n$ with $\lambda_n = \sqrt{2n+1}$. Indeed, h_n are given by the formula

$$(2.1) \quad h_n(x) = (-1)^n c_n e^{x^2/2} \frac{d^n}{dx^n} (e^{-x^2}), \quad \text{with} \quad \frac{1}{c_n} = (n!)^{\frac{1}{2}} 2^{\frac{n}{2}} \pi^{\frac{1}{4}}.$$

The equation (1.1) has the following Hamiltonian

$$(2.2) \quad J(u) = \frac{1}{2} \int_{-\infty}^{\infty} |H^{1/2} u(x)|^2 dx + \frac{\kappa_0}{k+1} \int_{-\infty}^{\infty} |u(x)|^{k+1} dx.$$

Write $u = \sum_{n=0}^{\infty} c_n h_n$. Then in the coordinates $c = (c_n)$ the Hamiltonian reads

$$J(c) = \frac{1}{2} \sum_{n=0}^{\infty} \lambda_n^2 |c_n|^2 + \frac{\kappa_0}{k+1} \int_{-\infty}^{\infty} \left| \sum_{n=0}^{\infty} c_n h_n(x) \right|^{k+1} dx.$$

Let us define the complex vector space E_N by $E_N = \text{span}(h_0, h_1, \dots, h_N)$. Then we introduce the spectral projector Π_N on E_N by

$$\Pi_N \left(\sum_{n=0}^{\infty} c_n h_n \right) = \sum_{n=0}^N c_n h_n.$$

Let $\chi \in \mathcal{C}_0^\infty(-1, 1)$, so that $\chi = 1$ on $[-\frac{1}{2}, \frac{1}{2}]$. Let S_N be the operators

$$(2.3) \quad S_N \left(\sum_{n=0}^{\infty} c_n h_n \right) = \sum_{n=0}^{\infty} \chi \left(\frac{2n+1}{2N+1} \right) c_n h_n = \chi \left(\frac{H}{2N+1} \right) \left(\sum_{n=0}^{\infty} c_n h_n \right).$$

It is clear that $\|S_N\|_{L^2 \rightarrow L^2} = \|\Pi_N\|_{L^2 \rightarrow L^2} = 1$ and we have

$$(2.4) \quad S_N \Pi_N = \Pi_N S_N = S_N, \quad \text{and} \quad S_N^* = S_N.$$

The interest of introducing the smooth cut-off S_N is its better mapping properties on L^p , $p \neq 2$, compared to Π_N (see Proposition 4.1).

Let us now turn to the definition of the Gibbs measure. Write $c_n = a_n + ib_n$. For $N \geq 1$, consider the probability measures on $\mathbb{R}^{2(N+1)}$ defined by

$$d\tilde{\mu}_N = \prod_{n=0}^N \frac{\lambda_n^2}{2\pi} e^{-\frac{\lambda_n^2}{2}(a_n^2 + b_n^2)} da_n db_n,$$

The measure $\tilde{\mu}_N$ defines a measure on E_N via the map

$$(2.5) \quad (a_n, b_n)_{n=0}^N \mapsto \sum_{n=0}^N (a_n + ib_n) h_n,$$

which will still be denoted by $\tilde{\mu}_N$. Notice that $\tilde{\mu}_N$ may be seen as the distribution of the E_N valued random variable

$$(2.6) \quad \omega \mapsto \sum_{n=0}^N \frac{\sqrt{2}}{\lambda_n} g_n(\omega) h_n(x) \equiv \varphi_N(\omega, x),$$

where $(g_n)_{n=0}^N$ is a system of independent, centered, L^2 normalized complex Gaussians on a probability space $(\Omega, \mathcal{F}, \mathbf{p})$.

In order to study convergence properties of φ_N as $N \rightarrow \infty$, we define Sobolev spaces associated to H .

Definition 2.1. — For $1 \leq p \leq +\infty$ and $s \in \mathbb{R}$, we define the space $\mathcal{W}^{s,p}(\mathbb{R})$ via the norm $\|u\|_{\mathcal{W}^{s,p}(\mathbb{R})} = \|H^{s/2}u\|_{L^p(\mathbb{R})}$. In the case $p = 2$ we write $\mathcal{W}^{s,2}(\mathbb{R}) = \mathcal{H}^s(\mathbb{R})$ and if $u = \sum_{n=0}^{\infty} c_n h_n$ we have $\|u\|_{\mathcal{H}^s}^2 = \sum_{n=0}^{\infty} \lambda_n^{2s} |c_n|^2$.

For future references, we state the following key property of the spaces $\mathcal{W}^{s,p}$, which is actually a consequence of the fact that H^{-s} is a pseudo differential operator in a suitable class (which ensures its L^p boundedness)

Proposition 2.2 ([17]). — *For any $1 < p < \infty$, $s \geq 0$, there exists $C > 0$ such that*

$$(2.7) \quad \frac{1}{C} \|u\|_{\mathcal{W}^{s,p}(\mathbb{R})} \leq \| \langle D_x \rangle^s u \|_{L^p(\mathbb{R})} + \| \langle x \rangle^s u \|_{L^p(\mathbb{R})} \leq C \|u\|_{\mathcal{W}^{s,p}(\mathbb{R})}.$$

Let $\sigma > 0$. Then (φ_N) is a Cauchy sequence in $L^2(\Omega; \mathcal{H}^{-\sigma}(\mathbb{R}))$ which defines

$$(2.8) \quad \varphi(\omega, x) = \sum_{n=0}^{\infty} \frac{\sqrt{2}}{\lambda_n} g_n(\omega) h_n(x),$$

as the limit of (φ_N) . Now the map $\omega \mapsto \varphi(\omega, x)$ defines a (Gaussian) measure on $\mathcal{H}^{-\sigma}(\mathbb{R})$ which we shall denote by μ . Notice also that the measure μ can be decomposed into

$$\mu = \mu^N \otimes \tilde{\mu}_N$$

where μ^N is the the distribution of the random variable on E_N^\perp

$$\sum_{n=N+1}^{\infty} \sqrt{\frac{2}{\lambda_n}} g_n(\omega) h_n(x).$$

• **The defocusing case** ($\kappa_0 = 1$) **and** $k \geq 3$. In this case we can define the Gibbs measure ρ by

$$(2.9) \quad d\rho(u) = \exp\left(-\frac{1}{k+1} \|u\|_{L^{k+1}(\mathbb{R})}^{k+1}\right) d\mu(u).$$

We also define its finite dimensional approximations

$$(2.10) \quad \begin{aligned} d\tilde{\rho}_N(u) &= \exp\left(-\frac{1}{k+1} \|S_N u\|_{L^{k+1}(\mathbb{R})}^{k+1}\right) d\tilde{\mu}_N(u), \\ d\rho_N(u) &= \exp\left(-\frac{1}{k+1} \|S_N u\|_{L^{k+1}(\mathbb{R})}^{k+1}\right) d\mu(u) = d\mu^N \otimes d\tilde{\rho}_N. \end{aligned}$$

• **The focusing case** ($\kappa_0 = -1$) **and** $k = 3$. Let $\zeta : \mathbb{R} \rightarrow \mathbb{R}$, $\zeta \geq 0$ be a continuous function with compact support (a cut-off). Define

$$(2.11) \quad \alpha_N = \mathbb{E}(\|\Pi_N u\|_{L^2}^2),$$

and the measures $\tilde{\rho}_N, \rho_N$ as

$$(2.12) \quad \begin{aligned} d\tilde{\rho}_N(u) &= \zeta(\|\Pi_N u\|_{L^2(\mathbb{R})}^2 - \alpha_N) e^{\frac{1}{4} \int_{\mathbb{R}} |S_N u(x)|^4 dx} d\tilde{\mu}_N(u), \\ d\rho_N(u) &= d\mu^N \otimes d\tilde{\rho}_N. \end{aligned}$$

We have the following statement defining the Gibbs measure associated to the equations (1.1).

Theorem 2.3. — (i) *Defocusing case* ($\kappa_0 = 1$) *and* $k < +\infty$. *Let the measure* ρ *be defined by* (2.9).

(ii) *Focusing case* ($\kappa_0 = -1$) *and* $k = 3$: *The sequence*

$$(2.13) \quad G_N(u) = \zeta(\|\Pi_N u\|_{L^2(\mathbb{R})}^2 - \alpha_N) e^{\frac{1}{4} \int_{\mathbb{R}} |S_N u(x)|^4 dx},$$

converges in measure, as $N \rightarrow \infty$, *with respect to the measure* μ . *Denote by* $G(u)$ *the limit of* (2.13) *as* $N \rightarrow \infty$. *Then for every* $p \in [1, \infty[$, $G(u) \in L^p(d\mu(u))$ *and we define* $d\rho(u) \equiv G(u) d\mu(u)$.

In both cases, the sequence $d\rho_N$ *converges weakly to* $d\rho$ *and for any borelian set* $A \subset \mathcal{H}^{-\sigma}$, *we have*

$$(2.14) \quad \lim_{N \rightarrow +\infty} \rho_N(A) = \rho(A).$$

The result in the defocusing case is quite a direct application of the argument of [1]. The construction of the measure in the focusing case is much more involved and is inspired by the work [37] of the third author on the Benjamin-Ono equation. The main difficulty in this construction lies in proving that the weight $G(u)$ belongs to $L^1(d\mu)$. A first candidate for the weight $G(u)$ would have been $\exp(\|u\|_{L^4(\mathbb{R})}^4/4)$, but then the large deviation estimates (see Lemma 3.3) are too weak to ensure the integrability of this weight with respect to the measure $d\mu$. A second guess would have been $\zeta(\|u\|_{L^2}) \exp(\|u\|_{L^4(\mathbb{R})}^4/4)$, as then the same large deviation estimates and Gagliardo-Nirenberg inequalities (using the L^2 bound induced by the ζ cut-off) would ensure this integrability. Unfortunately, on the support of the measure μ , the L^2 norm is almost surely infinite and this choice of weight would lead to a trivial (vanishing) invariant measure. The renormalized (square of the) L^2 norm provides us with an acceptable substitute to this latter choice. Let us also observe that if we vary ζ then we get the support of μ (see

Proposition 3.10 below). Notice also that this choice of weight is reminiscent of Bourgain's work [5] where a similar renormalization is performed at the level of the equation itself rather than the level of the Gibbs measure.

It is now a natural question whether the measure ρ constructed in Theorem 2.3 is indeed invariant by a well-defined flow of (1.1). It turns out to be the case as shows the following statement.

Theorem 2.4. — *Assume that $k = 3$ in the focusing case and $3 \leq k < +\infty$ in the defocusing case. Then the Cauchy problem (1.1) is, for μ -almost every initial data, globally well posed in a strong sense and the Gibbs measure ρ constructed in Theorem 2.3 is invariant under this flow, $\Phi(t)$. More precisely,*

- *There exists a set Σ of full ρ measure and $s < \frac{1}{2}$ (for $k = 3$, $s < \frac{1}{3}$ can be taken arbitrarily close to $\frac{1}{3}$ while for $k \geq 5$, s can be taken arbitrarily close to $\frac{1}{2}$) so that for every $f \in \Sigma$ the equation (1.1) with initial condition $u(0) = f$ has a global solution such that $u(t, \cdot) - e^{-itH} f \in \mathcal{C}(\mathbb{R}; \mathcal{H}^s(\mathbb{R}))$. The solution is unique in the following sense : for every $T > 0$ there is a functional space X_T continuously embedded in $\mathcal{C}([-T, T]; \mathcal{H}^s(\mathbb{R}))$ such that the solution is unique in the class*

$$u(t, \cdot) - e^{-itH} f \in X_T.$$

Moreover, for all $\sigma > 0$ and $t \in \mathbb{R}$

$$\|u(t, \cdot)\|_{\mathcal{H}^{-\sigma}(\mathbb{R})} \leq C(\Lambda(f, \sigma) + \ln^{\frac{1}{2}}(1 + |t|)),$$

and the constant $\Lambda(f, \sigma)$ satisfies the bound $\mu(f : \Lambda(f, \sigma) > \lambda) \leq C e^{-c\lambda^2}$.

- *For any ρ measurable set $A \subset \Sigma$, for any $t \in \mathbb{R}$, $\rho(A) = \rho(\Phi(t)(A))$.*

Notice that in this paper, we had to modify the definition of the finite dimensional approximations measures ρ_N with respect to previous results on the subject (see *e.g.* [4]). Indeed, the lack of continuity of the rough projectors Π_N on our resolution spaces forbid the usual approximation results (see *e.g.* [38, Theorem 1.2]). As a consequence, our new measures enjoy better approximation properties (see (2.14)), but the invariance properties we have to prove are stronger (see Corollary 8.4). We believe nevertheless this new approach is more natural.

3. Proof of Theorem 2.3

In this section we prove Theorem 2.3. As we already mentioned, the main issue is the construction of the measure for (1.1) with $k = 3$ in the focusing case. We fix once for all $\sigma > 0$.

3.1. Preliminaries and construction of the density. — First we recall the following Gaussian bound (Khinchin inequality), which is one of the key points in the study of our random series. See *e.g.* [8, Lemma 4.2.] for a proof in a more general setting. Let us notice that in our particular setting, the random variable being a Gaussian variable of variance $\sum_{n \geq 0} |c_n|^2$, this estimate is also an easy consequence of the growth of the r 'th moments of centered Gaussians (uniform with respect to the variance).

Lemma 3.1. — *Let $(g_n(\omega))_{n \geq 0} \in \mathcal{N}_{\mathbb{C}}(0, 1)$ be independent, complex, L^2 -normalized Gaussian random variables. Then there exists $C > 0$ such that for all $r \geq 2$ and $(c_n) \in \ell^2(\mathbb{N})$*

$$\left\| \sum_{n \geq 0} g_n(\omega) c_n \right\|_{L^r(\Omega)} \leq C \sqrt{r} \left(\sum_{n \geq 0} |c_n|^2 \right)^{\frac{1}{2}}.$$

We will need the following particular case of the bounds on the eigenfunctions (h_n) , proved for example by K. Yajima and G. Zhang [40] (see also H. Koch and D. Tataru [22]).

Lemma 3.2 (Dispersive bound for h_n). — *For every $p \geq 4$ there exists $C(p)$ such that for every $n \geq 0$,*

$$\|h_n\|_{L^p(\mathbb{R})} \leq C(p) \lambda_n^{-\frac{1}{6}}.$$

As a consequence, we may show the following statement. Recall that $\sigma > 0$ was fixed at the beginning of the section.

Lemma 3.3. — *Fix $p \in [4, \infty)$ and $s \in [0, 1/6)$. Then*

$$(3.1) \quad \exists C > 0, \exists c > 0, \forall \lambda \geq 1, \forall N \geq 1, \mu(u \in \mathcal{H}^{-\sigma} : \|S_N u\|_{\mathcal{W}^{s,p}(\mathbb{R})} > \lambda) \leq C e^{-c\lambda^2}.$$

Moreover there exists $\beta(s) > 0$ such that

$$(3.2) \quad \exists C > 0, \exists c > 0, \forall \lambda \geq 1, \forall N \geq N_0 \geq 1,$$

$$\mu(u \in \mathcal{H}^{-\sigma} : \|S_N u - S_{N_0} u\|_{\mathcal{W}^{s,p}(\mathbb{R})} > \lambda) \leq C e^{-cN_0^{\beta(s)} \lambda^2}.$$

Proof. — We have that

$$\begin{aligned} \mu(u \in \mathcal{H}^{-\sigma} : \|S_N u\|_{\mathcal{W}^{s,p}(\mathbb{R})} > \lambda) &= \mathbf{p}(\omega : \left\| \sum_{n=0}^{\infty} \chi\left(\frac{2n+1}{2N+1}\right) \frac{\sqrt{2}}{\lambda_n} g_n(\omega) h_n(x) \right\|_{\mathcal{W}^{s,p}(\mathbb{R})} > \lambda) \\ &= \mathbf{p}(\omega : \left\| \sum_{n=0}^{\infty} \chi\left(\frac{2n+1}{2N+1}\right) \frac{\sqrt{2}}{\lambda_n^{1-s}} g_n(\omega) h_n(x) \right\|_{L^p(\mathbb{R})} > \lambda). \end{aligned}$$

Set

$$f(\omega, x) \equiv \sum_{n=0}^{\infty} \chi\left(\frac{2n+1}{2N+1}\right) \frac{\sqrt{2}}{\lambda_n^{1-s}} g_n(\omega) h_n(x).$$

Then for $q \geq p$, using the Minkowski inequality, we get

$$\|f(\omega, x)\|_{L_x^q L_\omega^p} \leq \|f(\omega, x)\|_{L_x^p L_\omega^q}.$$

By Lemma 3.1 we get

$$\begin{aligned} \|f(\omega, x)\|_{L_x^q} &\leq C \sqrt{q} \left(\sum_{n=0}^{\infty} \chi_0^2 \left(\frac{2n+1}{2N+1} \right) \frac{2}{\lambda_n^{2(1-s)}} |h_n(x)|^2 \right)^{1/2} \\ &\leq C \sqrt{q} \left(\sum_{n=0}^{\infty} \frac{2}{\lambda_n^{2(1-s)}} |h_n(x)|^2 \right)^{1/2}. \end{aligned}$$

Since $s < 1/6$, using Lemma 3.2 and the triangle inequality, we get

$$\|f(\omega, x)\|_{L_\omega^q L_x^p} \leq C\sqrt{q}.$$

Using Bienaymé-Tchebichev inequality, we obtain

$$\mathbf{p}(\omega : \|f(\omega, x)\|_{L_x^p} > \lambda) \leq (\lambda^{-1}\|f(\omega, x)\|_{L_\omega^q L_x^p})^q \leq (C\lambda^{-1}\sqrt{q})^q.$$

Thus by choosing $q = \delta\lambda^2$, for δ small enough, we get the bound

$$\mathbf{p}(\omega : \|f(\omega, x)\|_{L_x^p} > \lambda) \leq Ce^{-c\lambda^2}.$$

This in turn yields (3.1). The proof of (3.2) is very similar. Indeed, in this case, we analyze the function

$$f_{N_0}(\omega, x) \equiv \sum_{n=0}^{\infty} \left(\chi\left(\frac{2n+1}{2N+1}\right) - \chi\left(\frac{2n+1}{2N_0+1}\right) \right) \frac{2}{\lambda_n^{2(1-s)}} g_n(\omega) h_n(x),$$

and we use that there is a negative power of N_0 saving in the estimate. Namely, there is $\gamma(s) > 0$ such that

$$\|f_{N_0}(\omega, x)\|_{L_\omega^q L_x^p} \leq C\sqrt{q}N_0^{-\gamma(s)},$$

which implies (3.2). This completes the proof of Lemma 3.3. \square

With the same arguments one can prove the following statement.

Lemma 3.4. — *Let $\sigma > 0$, then*

$$(3.3) \quad \exists C > 0, \exists c > 0, \forall \lambda \geq 1, \mu(u \in \mathcal{H}^{-\sigma} : \|u\|_{\mathcal{H}^{-\sigma}(\mathbb{R})} > \lambda) \leq Ce^{-c\lambda^2}.$$

Lemma 3.5 (Gagliardo-Nirenberg inequality associated to H). — *For any $s \in (0, 1/6)$, there exists $p < \infty$ and $\theta < 2$ such that*

$$\|u\|_{L^4(\mathbb{R})}^4 \leq C\|u\|_{L^2(\mathbb{R})}^{4-\theta} \|u\|_{W^{s,p}(\mathbb{R})}^\theta.$$

Proof. — First we prove that for any $s \in (0, 1/6)$, there exists $p < \infty$ and $\theta < 2$ such that

$$(3.4) \quad \|u\|_{L^4(\mathbb{R})}^4 \leq C\|u\|_{L^2(\mathbb{R})}^{4-\theta} \|u\|_{W^{s,p}(\mathbb{R})}^\theta,$$

where $W^{s,p}$ is the usual Sobolev space.

Fix $s \in (0, 1/6)$ and write

$$(3.5) \quad \|u\|_{L^4(\mathbb{R})}^4 \leq C\|u\|_{L^2(\mathbb{R})}^2 \|u\|_{L^\infty(\mathbb{R})}^2.$$

Using [32, Proposition A.3], we get that there exists $p \gg 1$ and $\kappa > 0$ such that

$$(3.6) \quad \|u\|_{L^\infty(\mathbb{R})} \leq C\|u\|_{L^p(\mathbb{R})}^\kappa \|u\|_{W^{s,p}(\mathbb{R})}^{1-\kappa},$$

(indeed for large p the derivative loss tends to zero, *i.e.* we may assume that it is smaller than s).

Finally the Hölder inequality and (3.6) implies that for any $q > p$ there exists $\alpha > 0$ such that

$$(3.7) \quad \|u\|_{L^p(\mathbb{R})} \leq C\|u\|_{L^2(\mathbb{R})}^\alpha \|u\|_{L^q(\mathbb{R})}^{1-\alpha} \leq C\|u\|_{L^2(\mathbb{R})}^\alpha \|u\|_{W^{s,p}(\mathbb{R})}^{1-\alpha}.$$

A combination of (3.5), (3.6) and (3.7) yields (3.4).

Finally, to complete the proof of the lemma use that thanks to (2.7)

$$\|u\|_{W^{s,p}(\mathbb{R})} \leq C\|u\|_{\mathcal{W}^{s,p}(\mathbb{R})}.$$

This completes the proof of Lemma 3.5. \square

Denote by

$$F_N(u) = \|\Pi_N u\|_{L^2(\mathbb{R})}^2 - \alpha_N.$$

As in [37], we need the following convergence properties of the sequence $(F_N(u))_{N \geq 0}$.

Lemma 3.6. — *The sequence $(F_N(u))$ is a Cauchy sequence in $L^2(\mathcal{H}^{-\sigma}(\mathbb{R}), d\mu)$. As a consequence, if we denote by $F(u)$ its limit, the sequence $(F_N(u))_{N \geq 0}$ converges to $F(u)$ in measure :*

$$\forall \varepsilon > 0, \lim_{N \rightarrow \infty} \mu(u \in \mathcal{H}^{-\sigma} : |F_N(u) - F(u)| > \varepsilon) = 0.$$

Proof. — Let $N > M \geq 0$, then

$$\|F_N(u) - F_M(u)\|_{L^2(\mathcal{H}^{-\sigma}(\mathbb{R}), d\mu)}^2 = \int_{\Omega} |(\|\varphi_N\|_{L^2(\mathbb{R})}^2 - \alpha_N) - (\|\varphi_M\|_{L^2(\mathbb{R})}^2 - \alpha_M)|^2 d\mathbf{p}(\omega),$$

where φ_N is defined in (2.6). By (2.11) we have

$$(3.8) \quad \|\varphi_N\|_{L^2(\mathbb{R})}^2 - \alpha_N = \sum_{n=0}^N \frac{2}{\lambda_n^2} (|g_n(\omega)|^2 - 1),$$

and therefore

$$(3.9) \quad \|F_N(u) - F_M(u)\|_{L^2(\mathcal{H}^{-\sigma}(\mathbb{R}), d\mu)}^2 = \int_{\Omega} \left| \sum_{n=M+1}^N \frac{2}{\lambda_n^2} (|g_n(\omega)|^2 - 1) \right|^2 d\mathbf{p}(\omega).$$

Now, as the random variables $(g_n(\omega))_{n \geq 0}$ are normalized and independent, for all $n_1 \neq n_2$ we have

$$\int_{\Omega} (|g_{n_1}(\omega)|^2 - 1)(|g_{n_2}(\omega)|^2 - 1) d\mathbf{p}(\omega) = 0,$$

therefore from (3.9) we deduce

$$\|F_N(u) - F_M(u)\|_{L^2(\mathcal{H}^{-\sigma}(\mathbb{R}), d\mu)}^2 = c \sum_{n=M+1}^N \frac{1}{\lambda_n^4} \leq \frac{C}{M+1},$$

as $\lambda_n^2 = 2n + 1$. This proves the first assertion of the lemma.

By the Tchebychev inequality, L^2 convergence implies convergence in measure, hence the result. This completes the proof of Lemma 3.6. \square

The following result is a large deviation bound for the sequence $(F_N(u))$.

Lemma 3.7. — *There exist $C, c > 0$ so that for all $N > M \geq 0$ and $\lambda > 0$*

$$\mu(u \in \mathcal{H}^{-\sigma} : |F_N(u) - F_M(u)| > \lambda) \leq C e^{-c(M+1)^{\frac{1}{2}} \lambda}.$$

Proof. — The result can be viewed as a consequence of a smoothing property of a suitable heat flow, but we give here a direct proof. Define the set

$$B_{M,N} = \{u \in \mathcal{H}^{-\sigma} : |F_N(u) - F_M(u)| > \lambda\}.$$

Then by (3.8) for $N > M$,

$$\begin{aligned} \mu(B_{M,N}) &= \mathbf{p}(\omega : |(\|\varphi_N\|_{L^2(\mathbb{R})}^2 - \alpha_N) - (\|\varphi_M\|_{L^2(\mathbb{R})}^2 - \alpha_M)| > \lambda) \\ (3.10) \quad &= \mathbf{p}(\omega : \left| \sum_{n=M+1}^N \frac{2}{\lambda_n^2} (|g_n(\omega)|^2 - 1) \right| > \lambda). \end{aligned}$$

By the Tchebychev inequality, for all $0 \leq t \leq \frac{\lambda_{M+1}^2}{4}$,

$$\begin{aligned} \mathbf{p}(\omega : \sum_{n=M+1}^N \frac{2}{\lambda_n^2} (|g_n(\omega)|^2 - 1) > \lambda) &\leq e^{-\lambda t} \mathbb{E}[\exp(t \sum_{n=M+1}^N \frac{2}{\lambda_n^2} (|g_n(\omega)|^2 - 1))] \\ (3.11) \quad &= e^{-\lambda t} \prod_{n=M+1}^N \int_{\Omega} e^{\frac{2t}{\lambda_n^2} (|g_n(\omega)|^2 - 1)} d\mathbf{p}(\omega) \\ &= e^{-\lambda t} \prod_{n=M+1}^N e^{-\frac{2t}{\lambda_n^2}} (1 - \frac{2t}{\lambda_n^2})^{-1}. \end{aligned}$$

Now observe that for all $0 \leq x \leq \frac{1}{2}$, $(1-x)^{-1} \leq e^{x+x^2}$, hence (3.11) gives

$$\mathbf{p}(\omega : \sum_{n=M+1}^N \frac{2}{\lambda_n^2} (|g_n(\omega)|^2 - 1) > \lambda) \leq e^{-\lambda t} \exp(4t^2 \sum_{n=M+1}^{\infty} \frac{1}{\lambda_n^4}) \leq \exp(-\lambda t + \frac{Ct^2}{M+1}),$$

as $\lambda_n^2 = 2n+1$. Choose $t = c(M+1)^{\frac{1}{2}}$, with $c > 0$ small enough and deduce

$$\mathbf{p}(\omega : \sum_{n=M+1}^N \frac{2}{\lambda_n^2} (|g_n(\omega)|^2 - 1) > \lambda) \leq Ce^{-c(M+1)^{\frac{1}{2}}\lambda}.$$

Using a slight modification of the previous argument, we can show that

$$\mathbf{p}(\omega : \sum_{n=M+1}^N \frac{2}{\lambda_n^2} (|g_n(\omega)|^2 - 1) < -\lambda) \leq Ce^{-c(M+1)^{\frac{1}{2}}\lambda},$$

and the result follows, by (3.10). This completes the proof of Lemma 3.7. \square

We are now able to define the density $G : \mathcal{H}^{-\sigma}(\mathbb{R}) \rightarrow \mathbb{R}$ (with respect to the measure μ) of the measure ρ . By Lemmas 3.6 and 3.3, we have the following convergences in the μ measure : $F_N(u)$ converges to $F(u)$ and $\|S_N u\|_{L^4(\mathbb{R})}$ to $\|u\|_{L^4(\mathbb{R})}$. Then, by composition and multiplication of continuous functions, we obtain

$$(3.12) \quad G_N(u) \rightarrow \zeta(F(u)) e^{\frac{1}{4} \int_{\mathbb{R}} |u(x)|^4 dx} \equiv G(u),$$

in measure, with respect to the measure μ . As a consequence, G is measurable from $(\mathcal{H}^{-\sigma}(\mathbb{R}), \mathcal{B})$ to \mathbb{R} .

3.2. Integrability of $G_N(u)$. — We now have all the ingredients to prove the following proposition, which is the key point in the proof of Theorem 2.3.

Proposition 3.8. — *Let $1 \leq p < \infty$. Then there exists $C > 0$ such that for every $N \geq 1$,*

$$\left\| \zeta \left(\|\Pi_N u\|_{L^2(\mathbb{R})}^2 - \alpha_N \right) e^{\frac{1}{4} \int_{\mathbb{R}} |S_N u(x)|^4 dx} \right\|_{L^p(d\mu(u))} \leq C.$$

Proof. — Our aim is to show that the integral $\int_0^\infty \lambda^{p-1} \mu(A_{\lambda,N}) d\lambda$ is convergent uniformly with respect to N , where

$$A_{\lambda,N} = \left\{ u \in \mathcal{H}^{-\sigma} : \zeta \left(\|\Pi_N u\|_{L^2(\mathbb{R})}^2 - \alpha_N \right) e^{\frac{1}{4} \int_{\mathbb{R}} |S_N u(x)|^4 dx} > \lambda \right\}.$$

Proposition 3.8 is a straightforward consequence of the following lemma.

Lemma 3.9. — *For any $L > 0$, there exists $C > 0$ such that for every N and every $\lambda \geq 1$,*

$$\mu(A_{\lambda,N}) \leq C \lambda^{-L}.$$

We set

$$N_0 \equiv (\log \lambda)^l,$$

where l is fixed such that $l > \max(2, \frac{1}{\beta(0)} + 1)$ with $\beta(0)$ defined by Lemma 3.3.

Let us first suppose that $N_0 \geq N$. Using Lemma 3.5 and that $\|S_N u\|_{L^2} \leq \|\Pi_N u\|_{L^2}$, we get for $u \in A_{\lambda,N}$,

$$\begin{aligned} \int_{\mathbb{R}} |S_N u(x)|^4 dx &\leq C \|S_N u\|_{L^2(\mathbb{R})}^{4-\theta} \|S_N u\|_{\mathcal{W}^{s,p}(\mathbb{R})}^\theta \\ &\leq C (\log \log \lambda)^{2-\theta/2} \|S_N u\|_{\mathcal{W}^{s,p}(\mathbb{R})}^\theta. \end{aligned}$$

Therefore there exists $\delta > 0$ such that

$$\mu(A_{\lambda,N}) \leq C \mu(u \in \mathcal{H}^{-\sigma} : \|S_N u\|_{\mathcal{W}^{s,p}(\mathbb{R})} > (\log \lambda)^{1/2+\delta}),$$

and using Lemma 3.3, we obtain that for every $L > 0$ there exists C_L such that for every N and λ such that $(\log \lambda)^l \geq N$ one has

$$(3.13) \quad \mu(A_{\lambda,N}) \leq C_L \lambda^{-L}.$$

We next consider the case $N > N_0$. Consider the set

$$B_{\lambda,N} = \left\{ u \in \mathcal{H}^{-\sigma} : \left| (\|\Pi_N u\|_{L^2(\mathbb{R})}^2 - \alpha_N) - (\|\Pi_{N_0} u\|_{L^2(\mathbb{R})}^2 - \alpha_{N_0}) \right| > 1 \right\}.$$

By Lemma 3.7, we get

$$\mu(B_{\lambda,N}) \leq C \exp(-c(\log \lambda)^{l/2}) \leq C_L \lambda^{-L}.$$

Hence it remains to evaluate $\mu(A_{\lambda,N} \setminus B_{\lambda,N})$. Let us observe that for $u \in A_{\lambda,N} \setminus B_{\lambda,N}$ one has

$$\begin{aligned} \|\Pi_{N_0} u\|_{L^2}^2 &= (\|\Pi_N u\|_{L^2}^2 - \alpha_N) - [(\|\Pi_N u\|_{L^2}^2 - \alpha_N) - (\|\Pi_{N_0} u\|_{L^2}^2 - \alpha_{N_0})] + \alpha_{N_0} \\ &\leq C + C \log(N_0) \leq C \log \log \lambda. \end{aligned}$$

Therefore $A_{\lambda,N} \setminus B_{\lambda,N} \subset C_{\lambda,N}$ where

$$C_{\lambda,N} \equiv \left\{ u \in \mathcal{H}^{-\sigma} : \|S_N u\|_{L^4} \geq c[\log \lambda]^{1/4}, \|\Pi_{N_0} u\|_{L^2}^2 \leq C \log \log \lambda \right\}.$$

We next observe that thanks to the triangle inequality $C_{\lambda,N} \subset D_{\lambda,N} \cup E_{\lambda,N}$, where

$$D_{\lambda,N} \equiv \left\{ u \in \mathcal{H}^{-\sigma} : \|S_{N_0}u\|_{L^4} \geq \frac{c}{4}[\log \lambda]^{1/4}, \|\Pi_{N_0}u\|_{L^2}^2 \leq C \log \log \lambda \right\},$$

and

$$E_{\lambda,N} \equiv \left\{ u \in \mathcal{H}^{-\sigma} : \|S_N u - S_{N_0}u\|_{L^4} \geq \frac{c}{4}[\log \lambda]^{1/4} \right\}.$$

The measure of $D_{\lambda,N}$ can be estimated exactly as we did in the analysis of the case $N_0 \geq N$. Finally, using Lemma 3.3, thanks to the choice of N_0 , we get

$$\mu(E_{\lambda,N}) \leq C e^{-cN_0^{\beta(0)}(\log \lambda)^{1/2}} \leq C_L \lambda^{-L}.$$

This ends the proof of the lemma, and Proposition 3.8 follows. \square

We are now able to complete the proof of Theorem 2.3

Proof of Theorem 2.3 (ii). — According to (3.12), we can extract a sub-sequence $G_{N_k}(u)$ so that $G_{N_k}(u) \rightarrow G(u)$, μ a.s. Then by Proposition 3.8 and the Fatou lemma, for all $p \in [1, +\infty)$,

$$\int_{\mathcal{H}^{-\sigma}(\mathbb{R})} |G(u)|^p d\mu(u) \leq \liminf_{k \rightarrow \infty} \int_{\mathcal{H}^{-\sigma}(\mathbb{R})} |G_{N_k}(u)|^p d\mu(u) \leq C,$$

thus $G(u) \in L^p(d\mu(u))$.

Now it remains to check that for any borelian set, $A \subset \mathcal{H}^{-\sigma}$, we have

$$(3.14) \quad \lim_{N \rightarrow +\infty} \int_{\mathcal{H}^{-\sigma}(\mathbb{R})} 1_{u \in A} G_N(u) d\mu(u) = \int_{\mathcal{H}^{-\sigma}(\mathbb{R})} 1_{u \in A} G(u) d\mu(u),$$

which will be implied by

$$(3.15) \quad \lim_{N \rightarrow +\infty} \int_{\mathcal{H}^{-\sigma}(\mathbb{R})} |1_{u \in A}(G_N(u) - G(u))| d\mu(u) = 0.$$

For $N \geq 0$ and $\varepsilon > 0$, we introduce the set

$$B_{N,\varepsilon} = \left\{ u \in \mathcal{H}^{-\sigma}(\mathbb{R}) : |G_N(u) - G(u)| \leq \varepsilon \right\},$$

and denote by $B_{N,\varepsilon}^c$ its complementary.

Firstly, as $1_{u \in A}$ is bounded, for all $N \geq 0$, $\varepsilon > 0$

$$\left| \int_{B_{N,\varepsilon}} 1_{u \in A}(G_N(u) - G(u)) d\mu(u) \right| \leq \varepsilon.$$

Secondly, by Cauchy-Schwarz, Proposition 3.8 and as $G(u) \in L^2(d\mu(u))$, we obtain

$$\begin{aligned} \left| \int_{B_{N,\varepsilon}^c} 1_{u \in A}(G_N(u) - G(u)) d\mu(u) \right| &\leq \|G_N - G(u)\|_{L^2(d\mu)} \mu(B_{N,\varepsilon}^c)^{\frac{1}{2}} \\ &\leq C \mu(B_{N,\varepsilon}^c)^{\frac{1}{2}}. \end{aligned}$$

By (3.12), we deduce that for all $\varepsilon > 0$,

$$\mu(B_{N,\varepsilon}^c) \rightarrow 0, \quad N \rightarrow +\infty,$$

which yields (3.15). This ends the proof of Theorem 2.3 (ii). \square

Notice that (3.15) with $A = \mathcal{H}^{-\sigma}$ gives

$$(3.16) \quad \rho(\mathcal{H}^{-\sigma}(\mathbb{R})) = \lim_{N \rightarrow +\infty} \rho_N(\mathcal{H}^{-\sigma}(\mathbb{R})) = \lim_{N \rightarrow \infty} \tilde{\rho}_N(E_N).$$

Proof of Theorem 2.3 (i). — By the argument giving (3.1), $\|u\|_{L^{p+1}(\mathbb{R})}$ is μ almost surely finite. As a consequence, the measure ρ in the defocusing case is nontrivial. The proof of the weak convergence of $d\rho_N$ to $d\rho$ can be deduced from the proof in the focusing case. This completes the proof of Theorem 2.3. \square

In the focusing case, the measure $\rho = \rho_\zeta$ we have constructed depends on $\zeta \in \mathcal{C}_0^\infty(\mathbb{R})$. We now check that it is in general not trivial. Indeed we have the following result

Proposition 3.10. — *The supports of the measures satisfy*

$$\bigcup_{\zeta \in \mathcal{C}_0^\infty(\mathbb{R})} \text{supp } \rho_\zeta = \text{supp } \mu.$$

Proof. — By construction, it is clear that for all $\zeta \in \mathcal{C}_0^\infty(\mathbb{R})$, the support of ρ_ζ is included in the support of μ .

Let $R \gg 1$ and $\zeta \in \mathcal{C}_0^\infty(\mathbb{R})$ so that $0 \leq \zeta \leq 1$ with $\zeta = 1$ on $|x| \leq R$, and consider the associated measure ρ_ζ . Let $\varepsilon > 0$. We will show that if R is large enough

$$(3.17) \quad \mu(u \in \mathcal{H}^{-\sigma} : |F(u)| \leq R) \geq 1 - \varepsilon,$$

which, as the density ρ_ζ does not vanish on the set $\{u \in \mathcal{H}^{-\sigma} : |F(u)| \leq R\}$ will yield the result. Write

$$(3.18) \quad \{u \in \mathcal{H}^{-\sigma} : |F(u)| > R\} \subset \{u \in \mathcal{H}^{-\sigma} : |F_N(u)| > R - 1\} \cup \{u \in \mathcal{H}^{-\sigma} : |F(u) - F_N(u)| > 1\},$$

and

$$\{|F_N(u)| > R - 1\} \subset \{|F_N(u) - F_0(u)| > \frac{R-1}{2}\} \cup \{|F_0(u)| > \frac{R-1}{2}\}.$$

By Lemma 3.7 and by the direct estimate

$$\mu(u \in \mathcal{H}^{-\sigma} : |F_0(u)| > \frac{R-1}{2}) \leq Ce^{-cR},$$

we obtain that (uniformly in N)

$$(3.19) \quad \mu(u \in \mathcal{H}^{-\sigma} : |F_N(u)| > R - 1) \leq Ce^{-cR} \leq \varepsilon/2,$$

if R is large enough. By Lemma 3.6, if N is large enough, we also have

$$(3.20) \quad \mu(u \in \mathcal{H}^{-\sigma} : |F(u) - F_N(u)| > 1) \leq \varepsilon/2.$$

Hence from (3.18)-(3.20) we deduce (3.17). This in turn completes the proof of Proposition 3.10. \square

Let us remark that in the construction of the measure ρ in the focusing case one may replace the assumption of compact support on ζ by a sufficiently rapid decay as for example $\zeta(x) \sim \exp(-|x|^K)$, $|x| \gg 1$ with K large enough (see [23]).

4. Functional calculus of H

We recall the classical Mehler formula for $|t| < \pi/2$, $f \in L^1(\mathbb{R})$, (see e.g. [13] and references therein)

$$(4.1) \quad e^{-itH}(f) = \frac{1}{|\sin(2t)|^{1/2}} \int_{-\infty}^{\infty} e^{i \frac{(x^2/2+y^2/2)\cos(2t)-xy}{\sin(2t)}} f(y) dy.$$

One may check (4.1) by a direct computation. The explicit representation of the kernel of $\exp(-itH)$ given by (4.1) will allow us to develop the functional calculus of H which will be of importance in several places of our proof of Theorem 2.4. The representation of e^{-itH} given by (4.1) is also the key point of the proof of the local in time (deterministic) Strichartz estimates of the next section. The goal of this section is to prove the following statement.

Proposition 4.1. — *Consider for $\varphi \in \mathcal{S}(\mathbb{R})$ the operator $\varphi(h^2H)$. Then, for any $1 \leq p \leq +\infty$ and any $|\alpha| < 1$, there exists $C > 0$ such that for any $0 < h \leq 1$, $\|\langle x \rangle^{-\alpha} \varphi(h^2H) \langle x \rangle^\alpha\|_{\mathcal{L}(L^p(\mathbb{R}))} \leq C$.*

One may prove Proposition 4.1 by using a suitable pseudo-differential calculus. We present here a direct proof based on the Mehler formula. The result of Proposition 4.1 is a consequence of the following lemma.

Lemma 4.2. — *Let $K(x, y, h)$ be the kernel of the operator $\varphi(h^2H)$. Then there exists $C > 0$ such that for any $0 < h \leq 1$, we have*

$$(4.2) \quad |K(x, y, h)| \leq \frac{C}{h(1 + \frac{(|x|-|y|)^2}{h^2})}.$$

Let us now show how Lemma 4.2 implies Proposition 4.1. By duality, it suffices to consider the case $\alpha \geq 0$.

For $\alpha \geq 0$, we have

$$\begin{aligned} \int_{-\infty}^{\infty} |K(x, y, h)| \langle y \rangle^\alpha dy &\leq C \int_{-\infty}^{\infty} \frac{\langle y \rangle^\alpha}{h(1 + \frac{(|x|-|y|)^2}{h^2})} dy \\ &\leq 2C \int_0^{\infty} \frac{\langle y \rangle^\alpha}{h(1 + \frac{(|x|-y)^2}{h^2})} dy \leq 2C \int_0^{\infty} \frac{1 + |x|^\alpha + ||x| - y|^\alpha}{h(1 + \frac{(|x|-y)^2}{h^2})} dy \\ &\leq 2C \langle x \rangle^\alpha + 2C \int_0^{\infty} \frac{||x| - y|^\alpha}{h(1 + \frac{(|x|-y)^2}{h^2})} dy = 2C \langle x \rangle^\alpha + 2C \int_{-\infty}^{\infty} \frac{|hz|^\alpha}{(1 + z^2)} dz \leq C' \langle x \rangle^\alpha. \end{aligned}$$

On the other hand,

$$\int_{-\infty}^{\infty} |K(x, y, h)| \langle x \rangle^{-\alpha} dx \leq \int_{||x|-|y|| < |y|/2} \cdots + \int_{||x|-|y|| > |y|/2} \cdots.$$

The contribution of the first term is bounded by $C \langle y \rangle^{-\alpha}$, whereas, noticing that in the second integral we have $(|x| - |y|)^2 \geq c(x^2 + y^2)$, the contribution of the second term is also easily bounded by $C \langle y \rangle^{-\alpha}$. Finally, Proposition 4.1 follows by the Schur Lemma. Thus in order to complete the proof of Proposition 4.1, it remains to prove Lemma 4.2.

Proof of Lemma 4.2. — We start from the representation

$$\varphi(h^2 H) = (2\pi)^{-1} \int_{\tau \in \mathbb{R}} e^{i\tau h^2 H} \widehat{\varphi}(\tau) d\tau$$

and thus according to Mehler's formula, we have

$$|K(x, y, h)| \leq C \left| \int_{\tau \in \mathbb{R}} \frac{1}{|\sin(2h^2\tau)|^{1/2}} e^{i\psi(\tau, h, x, y)} \widehat{\varphi}(\tau) d\tau \right|,$$

where

$$\psi(\tau, h, x, y) = -\frac{1}{\sin(2h^2\tau)} \left(\frac{x^2 + y^2}{2} \cos(2h^2\tau) - xy \right).$$

Next, we decompose

$$|K(x, y, h)| \leq \sum_{k \in \mathbb{Z}} \left| \int_{-\frac{\pi}{2} + k\pi < h^2\tau < \frac{\pi}{2} + k\pi} \frac{1}{|\sin(2h^2\tau)|^{1/2}} e^{i\psi(\tau, h, x, y)} \widehat{\varphi}(\tau) d\tau \right|,$$

we make the change of variables $t = h^2\tau - k\pi$ and use that $|\widehat{\varphi}| \leq C\langle x \rangle^{-2}$, since $\widehat{\varphi} \in \mathcal{S}$. Thus

$$\begin{aligned} |K(x, y, h)| &\leq \frac{C}{h} + \sum_{k \in \mathbb{Z} \setminus \{0\}} \left| \int_{-\frac{\pi}{2} + k\pi < h^2\tau < \frac{\pi}{2} + k\pi} \frac{1}{|\sin(2h^2\tau)|^{1/2}} e^{i\psi(\tau, h, x, y)} \widehat{\varphi}(\tau) d\tau \right| \\ &\leq \frac{C}{h} + \frac{1}{h^2} \sum_{k \in \mathbb{Z} \setminus \{0\}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{|\sin(2t)|^{1/2}} \left| \widehat{\varphi}\left(\frac{k\pi + t}{h^2}\right) \right| dt \leq \frac{C}{h}. \end{aligned}$$

As a consequence, it is enough to prove

$$(4.3) \quad |K(x, y, h)| \leq \frac{Ch}{(|x| - |y|)^2}.$$

The key point of the analysis will be the following estimates on the phase function.

Lemma 4.3. — *There exists $C > 0$ such that for any $x, y \in \mathbb{R}$ and any $0 < h \leq 1$ we have*

$$\partial_\tau \psi(x, y, \tau, h) \geq h^2 \frac{x^2 + y^2}{2}.$$

Proof. — Indeed,

$$(4.4) \quad \partial_\tau \psi(x, y, \tau, h) = \frac{2h^2}{\sin^2(2h^2\tau)} \left(\frac{x^2 + y^2}{2} - xy \cos(2h^2\tau) \right)$$

and minimizing with respect to x the expression above gives

$$\partial_\tau \psi(x, y, \tau, h) \geq \frac{2h^2}{\sin^2(2h^2\tau)} \left(\frac{y^2}{2} \sin^2(2h^2\tau) \right) = h^2 y^2.$$

Similarly

$$\partial_\tau \psi(x, y, \tau, h) \geq \frac{2h^2}{\sin^2(2h^2\tau)} \left(\frac{x^2}{2} \sin^2(2h^2\tau) \right) = h^2 x^2$$

and the result of Lemma 4.3 follows. \square

Lemma 4.4. — *There exists $C > 0$ such that for any $x, y \in \mathbb{R}$, any τ , and $0 < h \leq 1$, we have*

$$\partial_\tau \psi(x, y, \tau, h) \geq \frac{h^2(|x| - |y|)^2}{\sin^2(2h^2\tau)}.$$

Proof. — Indeed, this estimate is a straightforward consequence of (4.4) and

$$\frac{x^2 + y^2}{2} - xy \cos(2h^2\tau) \geq \frac{x^2 + y^2}{2} - |xy| = \frac{(|x| - |y|)^2}{2}.$$

This completes the proof of Lemma 4.4. \square

Let us now complete the proof of Lemma 4.2. To estimate K we integrate by parts using the operator $T = \frac{1}{\partial_\tau \psi(x, y, \tau, h)} \partial_\tau$. Notice that according to Lemma 4.4, the singularity of $\frac{1}{|\sin(2h^2\tau)|^{1/2}}$ is harmless and we obtain

$$|K(x, y, h)| \leq C \left| \int_{\tau \in \mathbb{R}} e^{i\psi(x, y, \tau, h)} \partial_\tau \left(\frac{1}{|\sin(2h^2\tau)|^{1/2}} \frac{1}{\partial_\tau \psi} \widehat{\varphi}(\tau) \right) d\tau \right|.$$

In the expression above, we have three contributions according whether the derivative falls on either terms. If the derivative falls on the last term, we obtain a contribution which is, according to Lemma 4.4, bounded by

$$\begin{aligned} C_N \int_{\tau \in \mathbb{R}} \frac{\sin^2(2h^2\tau)}{|\sin(2h^2\tau)|^{1/2} h^2 (|x| - |y|)^2} (1 + |\tau|)^{-N} d\tau \\ \leq C_N \int_{\tau \in \mathbb{R}} \frac{h^3 |\tau|^{3/2}}{h^2 (|x| - |y|)^2} (1 + |\tau|)^{-N} d\tau \leq \frac{Ch}{(|x| - |y|)^2}. \end{aligned}$$

If the derivative falls on the first term, we obtain a contribution which is bounded by

$$\begin{aligned} C \int_{\tau \in \mathbb{R}} \frac{h^2 |\widehat{\varphi}(\tau)| d\tau}{|\sin(2h^2\tau)|^{3/2} |\partial_\tau \psi|} \leq Ch \int_{\tau \in \mathbb{R}} \frac{|\sin|^{1/2}(2h^2\tau)}{(|x| - |y|)^2} |\widehat{\varphi}(\tau)| d\tau \\ \leq C \int_{\tau \in \mathbb{R}} \frac{h |\tau|^{1/2}}{(|x| - |y|)^2} |\widehat{\varphi}(\tau)| d\tau \leq \frac{Ch}{(|x| - |y|)^2}. \end{aligned}$$

Finally, the last case is when the derivative falls on the second term. In this case, using the relation

$$\frac{\partial_\tau^2 \psi}{(\partial_\tau \psi)^2} = -\frac{4h^2 \cos(2h^2\tau)}{\sin(2h^2\tau) \partial_\tau \psi} + \frac{4h^4 xy}{\sin(2h^2\tau) (\partial_\tau \psi)^2},$$

Lemma 4.3 and Lemma 4.4, we obtain a contribution which is bounded by

$$\begin{aligned} C \int_{\tau \in \mathbb{R}} \left(\frac{h^2}{|\sin(2h^2\tau)|^{3/2} |\partial_\tau \psi|} + \frac{h^4 |xy|}{|\sin(2h^2\tau)|^{3/2} (\partial_\tau \psi)^2} \right) |\widehat{\varphi}(\tau)| d\tau \\ \leq C \int_{\tau \in \mathbb{R}} \left| \frac{|\sin|^{1/2}(2h^2\tau)}{(|x| - |y|)^2} \widehat{\varphi}(\tau) \right| d\tau \leq \frac{Ch}{(|x| - |y|)^2} \end{aligned}$$

where to estimate $\frac{h^4 xy}{\sin(2h^2\tau) (\partial_\tau \psi)^2}$ we used Lemma 4.3 to estimate one of the $\partial_\tau \psi$ factors and Lemma 4.4 to estimate the other one. This concludes the proof of (4.3) and hence of Lemma 4.2. \square

5. Strichartz estimates

We state the Strichartz inequality (local in time) satisfied by the linear evolution of the Schrödinger equation with harmonic potential.

Lemma 5.1. — *Let us fix $s \in \mathbb{R}$. For every $p \geq 4, q \geq 2$ satisfying $\frac{2}{p} + \frac{1}{q} = \frac{1}{2}$, every $T > 0$, there exists $C > 0$ and such that*

$$(5.1) \quad \|e^{-itH}\|_{\mathcal{H}^s(\mathbb{R}) \rightarrow L^p((0,2\pi); \mathcal{W}^{s,q}(\mathbb{R}))} \leq C.$$

There is also a set of inhomogeneous Strichartz estimates which will not be used here. This result is well-known (see *e.g.* [14]), but let us recall the main line of the proof.

Proof. — Coming back to the definition of the spaces $\mathcal{W}^{s,p}(\mathbb{R})$, we first observe that it suffices to consider the case $s = 0$. We have that $\|e^{-itH}\|_{L^2 \rightarrow L^2} = 1$. Next, as a consequence of (4.1), $\|e^{-itH}\|_{L^1 \rightarrow L^\infty} \leq C/|t|^{1/2}$ for t close to zero, *i.e.* the singularity of $\|e^{-itH}\|_{L^1 \rightarrow L^\infty}$ for $t \sim 0$ is the same as for $\exp(it\partial_x^2)$ and thus (see *e.g.* [19]) e^{-itH} enjoys the same local in time Strichartz estimates as the Schrödinger equation without potential, which is precisely the statement of (5.1). This completes the proof of Lemma 5.1. \square

We need some stochastic improvements of the Strichartz estimates. The following lemma shows that there is a gain of regularity in L^p spaces for the free Schrödinger solution.

Lemma 5.2. — *Let $\varepsilon < \frac{1}{6}$. For any $p, q \geq 4$, there exist $C, c > 0$ such that*

$$\begin{aligned} \forall \lambda \geq 1, \quad \mu(u \in \mathcal{H}^{-\sigma} : \|e^{-itH}u\|_{L^p_{(0,2\pi)} \mathcal{W}^{\varepsilon,q}(\mathbb{R})} > \lambda) &\leq Ce^{-c\lambda^2} \\ \forall \lambda \geq 1, \forall N \geq 1, \quad \tilde{\mu}_N(u \in E_N : \|e^{-itH}u\|_{L^p_{(0,2\pi)} \mathcal{W}^{\varepsilon,q}(\mathbb{R})} > \lambda) &\leq Ce^{-c\lambda^2}. \end{aligned}$$

Proof. — Let us prove the first estimate, the proof of the second being similar. By the definition of μ , we have to show that

$$(5.2) \quad \mathbf{p}(\omega \in \Omega : \|e^{-itH}\varphi\|_{L^p_{(0,2\pi)} \mathcal{W}^{\varepsilon,q}(\mathbb{R})} > \lambda) \leq Ce^{-c\lambda^2}.$$

Now by Lemmas 3.1, 3.2 and Minkowski's inequality, for $r \geq p, q$, we obtain

$$\begin{aligned} \|e^{-itH}\varphi(\omega, \cdot)\|_{L^r(\Omega) L^p_{(0,2\pi)} \mathcal{W}^{\varepsilon,q}(\mathbb{R})} &\leq C \|\langle H \rangle^{\frac{\varepsilon}{2}} e^{-itH}\varphi\|_{L^p_{(0,2\pi)} L^q(\mathbb{R}) L^r(\Omega)} \\ &\leq C\sqrt{r} \left(\sum_{n=0}^{\infty} \lambda_n^{2(\varepsilon-1)} \|h_n\|_{L^q}^2 \right)^{\frac{1}{2}} \\ (5.3) \quad &\leq C\sqrt{r} \left(\sum_{n=0}^{\infty} \lambda_n^{2(\varepsilon-1-\frac{1}{6})} \right)^{\frac{1}{2}}. \end{aligned}$$

Coming back to the definition of λ_n , we get that the sum (5.3) is finite. The estimate (5.2) then follows from the Bienaymé-Tchebychev inequality :

$$\mathbf{p}(\omega \in \Omega : \|e^{-itH}\varphi\|_{L^p_{(0,2\pi)} \mathcal{W}^{\varepsilon,q}(\mathbb{R})} > \lambda) \leq \left(\frac{\|e^{-itH}\varphi\|_{L^r(\Omega) L^p_{(0,2\pi)} \mathcal{W}^{\varepsilon,q}(\mathbb{R})}}{\lambda} \right)^r \leq \left(\frac{C\sqrt{r}}{\lambda} \right)^r,$$

and the choice $r = \varepsilon\lambda^2$ with $\varepsilon > 0$ small enough. This completes the proof of Lemma 5.2. \square

We shall in practice need the following consequence of Lemma 5.2 and (3.3).

Lemma 5.3. — *Let $\sigma > 0$, $0 < \varepsilon < \frac{1}{6}$ and $p, q \geq 4$. Then there exist $C, c > 0$ so that for every $\lambda \geq 1$, every $N \geq 1$,*

$$(5.4) \quad \begin{aligned} \rho(u \in \mathcal{H}^{-\sigma} : \|u\|_{\mathcal{H}^{-\sigma}} > \lambda) &\leq C e^{-c\lambda^2}, \\ \tilde{\rho}_N(u \in E_N : \|u\|_{\mathcal{H}^{-\sigma}} > \lambda) &\leq C e^{-c\lambda^2} \end{aligned}$$

and

$$(5.5) \quad \begin{aligned} \rho(u \in \mathcal{H}^{-\sigma} : \|e^{-itH}u\|_{L^p_{(0,2\pi)}\mathcal{W}^{\varepsilon,q}(\mathbb{R})} > \lambda) &\leq C e^{-c\lambda^2}, \\ \tilde{\rho}_N(u \in E_N : \|e^{-itH}u\|_{L^p_{(0,2\pi)}\mathcal{W}^{\varepsilon,q}(\mathbb{R})} > \lambda) &\leq C e^{-c\lambda^2}. \end{aligned}$$

Proof. — In the defocusing case the proof is a straightforward consequence of the bounds for $\mu, \tilde{\mu}_N$ we have already established. Namely, in this case it is a straightforward consequence of the inequalities

$$\rho(A) \leq \mu(A), \quad \tilde{\rho}_N(A) \leq \tilde{\mu}_N(A).$$

We thus only consider the focusing case which is slightly more delicate. We prove (5.4). By the definition and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \rho(u \in \mathcal{H}^{-\sigma} : \|u\|_{\mathcal{H}^{-\sigma}} > \lambda) &= \int_{\mathcal{H}^{-\sigma}} 1_{\|u\|_{\mathcal{H}^{-\sigma}} > \lambda} G_N(u) d\mu(u) \\ &\leq \|G_N(u)\|_{L^2(d\mu(u))} \mu(u : \|u\|_{\mathcal{H}^{-\sigma}} > \lambda)^{\frac{1}{2}}, \end{aligned}$$

and we obtain

$$\rho_N(u \in \mathcal{H}^{-\sigma} : \|u\|_{\mathcal{H}^{-\sigma}} > \lambda) \leq C e^{-c\lambda^2},$$

and the first claim follows by using (2.14). The proof of the three other claims are similar. This completes the proof of Lemma 5.3. \square

6. Local smoothing effects

The next result is based on the well-known smoothing effect.

Lemma 6.1 (Deterministic smoothing effect). — *Let us fix two positive numbers s and σ such that $s < \sigma < 1/2$. Then there exists $C > 0$ so that*

$$(6.1) \quad \left\| \langle x \rangle^{-\sigma} \sqrt{H^s} e^{-itH} f \right\|_{L^2([0,2\pi] \times \mathbb{R})} \leq C \|f\|_{L^2(\mathbb{R})}.$$

Proof. — Inequality (6.1) is a slight variation of the “usual” local smoothing effect for the harmonic oscillator, namely for $\alpha > 1/2$,

$$(6.2) \quad \left\| \langle x \rangle^{-\alpha} \sqrt{H^{\frac{1}{2}}} e^{-itH} f \right\|_{L^2([0,2\pi] \times \mathbb{R})} \leq C \|f\|_{L^2(\mathbb{R})}.$$

We refer to [40, 41] for a proof of (6.2). Let us fix $\alpha > 1/2$ such that $1 < 2\alpha < \sigma/s$. Take $\theta \in (0, 1)$ such that $\sigma = \theta\alpha$. Then thanks to our choice of α , we have that $s < \frac{\theta}{2}$. Applying (6.2) to h_n gives that

$$\left\| \langle x \rangle^{-\alpha} h_n(x) \right\|_{L^2(\mathbb{R})} \leq C \lambda_n^{-\frac{1}{2}}.$$

Interpolation between the last inequality and the equality $\|h_n\|_{L^2(\mathbb{R})} = 1$ yields that

$$\|\langle x \rangle^{-\sigma} h_n(x)\|_{L^2(\mathbb{R})} \leq C\lambda_n^{-\frac{\theta}{2}}.$$

Since $s < \frac{\theta}{2}$, we obtain that there exists $\delta(s, \sigma) > 0$ such that

$$(6.3) \quad \|\langle x \rangle^{-\sigma} h_n\|_{L^2(\mathbb{R})} \leq C\lambda_n^{-\delta(s, \sigma) - s}.$$

The last estimate in conjunction with [36, Corollary 1.2] implies (6.1) (notice that here we do not need the $\delta(s, \sigma)$ saving in (6.3)). This completes the proof of Lemma 6.1. \square

We also have the following stochastic improvement of the smoothing effect.

Lemma 6.2 (Stochastic smoothing effect). — *Let s, σ be two positive numbers such that $s < \sigma < 1/2$ and $q \geq 2$. Then there exist $C, c > 0$ so that for every $\lambda > 0$, every $N \geq 1$,*

$$(6.4) \quad \begin{aligned} \rho(u \in \mathcal{H}^{-\sigma} : \|\langle x \rangle^{-\sigma} \sqrt{H}^s e^{-itH} u\|_{L^q_{(0, 2\pi)} L^2(\mathbb{R})} > \lambda) &\leq C e^{-c\lambda^2}, \\ \tilde{\rho}_N(u \in E_N : \|\langle x \rangle^{-\sigma} \sqrt{H}^s e^{-itH} u\|_{L^q_{(0, 2\pi)} L^2(\mathbb{R})} > \lambda) &\leq C e^{-c\lambda^2}. \end{aligned}$$

Proof. — Again we only prove the first claim. We compute

$$\langle x \rangle^{-\sigma} \sqrt{H}^s e^{-itH} \varphi(\omega, x) = \sum_{n \geq 0} \frac{\sqrt{2}}{\lambda_n^{1-s}} e^{-it\lambda_n^2} g_n(\omega) \frac{1}{\langle x \rangle^\sigma} h_n(x).$$

Then by Lemma 3.1

$$\|\langle x \rangle^{-\sigma} \sqrt{H}^s e^{-itH} \varphi(\omega, x)\|_{L^r(\Omega)} \leq C\sqrt{r} \left(\sum_{n \geq 0} \frac{1}{\langle \lambda_n \rangle^{2(1-s)}} \left| \frac{h_n(x)}{\langle x \rangle^\sigma} \right|^2 \right)^{\frac{1}{2}}.$$

An application of the Minkowski inequality and (6.3) give

$$\begin{aligned} \|\langle x \rangle^{-\sigma} \sqrt{H}^s e^{-itH} \varphi(\omega, x)\|_{L^r(\Omega; L^q_T L^2(\mathbb{R}))} &\leq C\sqrt{r} \left(\sum_{n \geq 0} \frac{1}{\langle \lambda_n \rangle^{2(1-s)}} \left\| \frac{h_n}{\langle x \rangle^\sigma} \right\|_{L^2(\mathbb{R})}^2 \right)^{\frac{1}{2}} \\ &\leq C\sqrt{r} \left(\sum_{n \geq 0} \frac{1}{\langle \lambda_n \rangle^{2+2\delta(s, \sigma)}} \right)^{\frac{1}{2}} \\ &\leq C\sqrt{r}. \end{aligned}$$

Using the Tchebychev inequality, as we did in the proof of Lemma 5.2 yields

$$(6.5) \quad \mu(u \in \mathcal{H}^{-\sigma} : \|\langle x \rangle^{-\sigma} \sqrt{H}^s e^{-itH} u\|_{L^q_{(0, 2\pi)} L^2(\mathbb{R})} > \lambda) \leq C e^{-c\lambda^2}.$$

Finally we deduce (6.4) from (6.5) as we did in the proof of Lemma 5.3. This completes the proof of Lemma 6.2. \square

7. Local in time results for the nonlinear problem

In this section, we use the linear dispersive estimates established in the previous sections to develop a local Cauchy theory. As the solution we are looking for, will be the sum of the linear solution associated to our initial data and of a smoother term, our functional spaces are naturally *the sum of two spaces*: one which corresponds to the properties of the linear probabilistic solutions, and the other one corresponding to the properties of the deterministic smoother solutions. Fortunately, it turns out that these two spaces have a non trivial intersection which is sufficient to perform the analysis and hence avoid the technicalities involving sum spaces. In the sequel, for conciseness, we shall denote by L_T^q the space $L^q(0, T)$.

7.1. Initial data spaces. — For $\alpha \in \mathbb{R}$, we define the spaces $\mathcal{H}_{\langle x \rangle^\alpha}^s(\mathbb{R})$ equipped with the norm

$$\|u\|_{\mathcal{H}_{\langle x \rangle^\alpha}^s} = \|\langle x \rangle^\alpha \sqrt{H}^s u\|_{L^2}.$$

Recall that e^{-itH} defines the free evolution. We define the spaces for the initial data Y^s

$$Y^s = \{u \in \mathcal{H}^{-\varepsilon/10} : e^{-itH}(u) \in L_{2\pi}^{2(k-1)+\varepsilon} \mathcal{W}_{\frac{s+\varepsilon}{k-1}, r} \cap L_{2\pi}^2 \mathcal{H}_{\langle x \rangle^{-s-\varepsilon/4}}^s\},$$

where s is a positive number satisfying

$$(7.1) \quad \frac{k-3}{2(k-2)} < s < \min\left(\frac{1}{2}, \frac{k-1}{6}\right),$$

$\varepsilon > 0$, is a small number and r is a large number all depending on s to be fixed. The values of ε , r in the definition of the space Y^s (and also the space X_T^s defined in the next section) will be fixed by the analysis of the next two sections. Note that since e^{-itH} is 2π periodic the interval $[0, 2\pi]$ in the definition of Y^s may be replaced by any interval of size 2π . We equip the spaces Y^s with the natural norm

$$\|u\|_{Y^s} = \|u\|_{\mathcal{H}^{-\varepsilon/10}} + \|e^{-itH}u\|_{L_{2\pi}^{2(k-1)+\varepsilon} \mathcal{W}_{\frac{s+\varepsilon}{k-1}, r}} + \|e^{-itH}u\|_{L_{2\pi}^2 \mathcal{H}_{\langle x \rangle^{-s-\varepsilon/4}}^s}.$$

Thanks to Proposition 4.1, we obtain that $\|S_N\|_{Y^s \rightarrow Y^s}$ is bounded, uniformly in N , provided ε is small enough. The main property of the space Y^s we use, is the following Gaussian property.

Lemma 7.1. — *For every s satisfying (7.1) there exists $\varepsilon_0 > 0$ and two positive constants C and c such that for every $N \geq 1$, every $\lambda \geq 1$, every $\varepsilon \in (0, \varepsilon_0)$, every $r \geq 4$, every N*

$$\begin{aligned} \rho(u \in \mathcal{H}^{-\varepsilon/10} : \|u\|_{Y^s} > \lambda) &\leq C e^{-c\lambda^2} \\ \tilde{\rho}_N(u \in E_N : \|u\|_{Y^s} > \lambda) &\leq C e^{-c\lambda^2}. \end{aligned}$$

(recall that the dependence on ε and r of Y^s is implicit).

Proof. — As before, we only prove the first claim. As a consequence of Lemma 6.2, we get that for every $s \in (0, 1/2)$ and every $\varepsilon > 0$,

$$\rho(u \in \mathcal{H}^{-\varepsilon/10} : \|e^{-itH}u\|_{L_{2\pi}^2 \mathcal{H}_{\langle x \rangle^{-s-\varepsilon/4}}^s} > \lambda) \leq C e^{-c\lambda^2}.$$

Next, using Lemmas 3.4 and 5.3, we obtain that for every $s \in (0, (k-1)/6)$ and $\varepsilon > 0$ such that $s + \varepsilon < (k-1)/6$, every $r \geq 4$,

$$\rho(u \in \mathcal{H}^{-\varepsilon/10} : \|u\|_{\mathcal{H}^{-\varepsilon/10}} + \|e^{-itH}u\|_{L_{2\pi}^{2(k-1)+\varepsilon}\mathcal{W}_{\frac{s+\varepsilon}{k-1},r}} > \lambda) \leq Ce^{-c\lambda^2}.$$

This completes the proof of Lemma 7.1. \square

7.2. Solution spaces and linear estimates. — We define the solution spaces of functions on $[-T, T] \times \mathbb{R}$, by

$$X_T^s = \left\{ u \in L_T^\infty \mathcal{H}^{-\varepsilon/10} \cap L_T^{2(k-1)+\varepsilon} \mathcal{W}_{\frac{s+\varepsilon}{k-1},r} \cap L_T^2 \mathcal{H}_{(x)}^{s-s-\varepsilon/4} \right\}$$

where s satisfies (7.1), ε is a small positive number and $r \gg 1$ is a large number, to be chosen in function of s . We equip X_T^s with the norm

$$\|u\|_{X_T^s} = \|u\|_{L_T^\infty \mathcal{H}^{-\varepsilon/10}} + \|u\|_{L_T^{2(k-1)+\varepsilon} \mathcal{W}_{\frac{s+\varepsilon}{k-1},r}} + \|u\|_{L_T^2 \mathcal{H}_{(x)}^{s-s-\varepsilon/4}}.$$

In the next lemma, we state the linear estimates.

Lemma 7.2. — *For any $T \leq 2\pi$,*

$$\|e^{-itH}u\|_{X_T^s} \leq \|u\|_{Y^s},$$

and for any fixed $\tau \in \mathbb{R}$,

$$\|e^{-i\tau H}u\|_{Y^s} = \|u\|_{Y^s}.$$

Moreover, if s satisfies (7.1), there exist $\varepsilon_0 > 0$ and $r_0 \geq 2$ such that for $\varepsilon, \eta \in (0, \varepsilon_0)$, $r > r_0$,

$$(7.2) \quad \left\| \int_0^t e^{-i(t-\tau)H}(F(\tau))d\tau \right\|_{X_T^s} \leq C\|F\|_{L_T^1 \mathcal{H}^{s-\eta}}$$

and for $t \in [-T, T]$

$$(7.3) \quad \left\| \int_0^t e^{-i(t-\tau)H}(F(\tau))d\tau \right\|_{Y^s} \leq C\|F\|_{L_T^1 \mathcal{H}^{s-\eta}}$$

(recall that the dependence on ε and r of X_T^s and Y^s is implicit).

Proof. — The first estimate is a direct consequence of the conservation of the $\mathcal{H}^{-\varepsilon/10}$ -norm by the flow and the definition. The second estimate is a consequence of the time periodicity of the flowthe definition. Let us prove (7.2). We first observe that if s satisfies (7.1) then thanks to the Sobolev inequality and (2.7) we have

$$(7.4) \quad \|u\|_{L_T^{2(k-1)+\varepsilon} \mathcal{W}_{\frac{s+\varepsilon}{k-1},r}} \leq C(\|u\|_{L_T^\infty \mathcal{H}^{s-\eta}} + \|u\|_{L_T^4 \mathcal{W}^{s-\eta, \infty}}),$$

provided the positive numbers ε_0 is small enough and r is large enough. Indeed, thanks to the Sobolev embedding, we have that

$$\|u\|_{L_T^{2(k-1)+\varepsilon} \mathcal{W}_{\frac{s+\varepsilon}{k-1},r}} \leq C\|u\|_{L_T^{2(k-1)+\varepsilon} \mathcal{W}_{\frac{s+\varepsilon}{k-1}+\sigma, \frac{4k-4+2\varepsilon}{2k-6+\varepsilon}}},$$

provided

$$\sigma > \frac{2k-6+\varepsilon}{4k-4+2\varepsilon} - \frac{1}{r}.$$

Observe that the couple $(p, q) = (2(k-1) + \varepsilon, \frac{4k-4+2\varepsilon}{2k-6+\varepsilon})$ satisfies $p \geq 4$ and $\frac{2}{p} + \frac{1}{q} = \frac{1}{2}$. Therefore, (7.4) holds, if we can assure that

$$s - \eta > \frac{2k-6+\varepsilon}{4k-4+2\varepsilon} - \frac{1}{r} + \frac{s+\varepsilon}{k-1}.$$

But the last condition follows from (7.1), provided $0 < \varepsilon, \eta < \varepsilon_0$, if ε_0 is small enough and r large enough. This proves (7.4). Using (7.4), the Strichartz estimates of Lemma 5.1 and the Minkowski inequality, we obtain that

$$\left\| \int_0^t e^{-i(t-\tau)H}(F(\tau))d\tau \right\|_{L_T^{2(k-1)+\varepsilon} \mathcal{W}_{\frac{s+\varepsilon}{k-1}, r}} \leq C \|e^{i\tau H} F(\tau)\|_{L_T^1 \mathcal{H}^{s-\eta}} = C \|F\|_{L_T^1 \mathcal{H}^{s-\eta}}.$$

We next observe that as a consequence of Lemma 6.1, for every s satisfying (7.1) there exists ε_0 such that for $\varepsilon \in (0, \varepsilon_0)$,

$$(7.5) \quad \|e^{-itH}\|_{L^2 \rightarrow L_T^2 \mathcal{H}_{(x)}^{s-s-\varepsilon/4}} \leq C.$$

Using (7.5) and the Minkowski inequality, we obtain that

$$\left\| \int_0^t e^{-i(t-\tau)H}(F(\tau))d\tau \right\|_{L_T^2 \mathcal{H}_{(x)}^{s-s-\varepsilon/4}} \leq C \|F\|_{L_T^1 L^2} \leq C \|F\|_{L_T^1 \mathcal{H}^{s-\eta}}.$$

The proof of (7.2) is completed by the straightforward bound

$$\left\| \int_0^t e^{-i(t-\tau)H}(F(\tau))d\tau \right\|_{L_T^\infty \mathcal{H}^{-\varepsilon/10}} \leq C \|F\|_{L_T^1 \mathcal{H}^{-\varepsilon/10}} \leq C \|F\|_{L_T^1 \mathcal{H}^{s-\eta}}.$$

Let us now prove (7.3). Using (7.4), (7.5) and the Minkowski inequality, we obtain that for $t \in [0, T]$,

$$\left\| \int_0^t e^{-i(t-\tau)H}(F(\tau))d\tau \right\|_{Y^s} \leq C \|e^{-i(t-\tau)H}(\chi(\tau, t)F(\tau))\|_{L_T^1 \mathcal{H}^{s-\eta}} \leq C \|F\|_{L_T^1 \mathcal{H}^{s-\eta}},$$

where $\chi(\tau, t)$ denotes the indicator function of $\tau \in [0, t]$. This completes the proof of Lemma 7.2. \square

7.3. Multilinear estimates. —

Proposition 7.3. — *Assume that s satisfies (7.1) and let $\eta > 0$. There exist $\varepsilon_0 > 0$ and $r_0 \geq 2$ such that the following holds true. For every $\varepsilon \in (0, \varepsilon_0)$, and every $r > r_0$ satisfying $3\varepsilon r > 4(k-1)$, there exists $\kappa > 0$ such that for $T \leq 2\pi$ we have the estimates*

$$(7.6) \quad \|u_1 \cdots u_k\|_{L_T^1 \mathcal{H}^{s-\eta}} \leq CT^\kappa \prod_{j=1}^k \|u_j\|_{X_T^s}$$

and, uniformly in N ,

$$\|S_N((S_N u_1)(S_N u_2) \cdots (S_N u_k))\|_{L_T^1 \mathcal{H}^{s-\eta}} \leq CT^\kappa \prod_{j=1}^k \|S_N u_j\|_{X_T^s} \leq CT^\kappa \prod_{j=1}^k \|u_j\|_{X_T^s}.$$

(we recall again that the dependence on ε and r of X_T^s is implicit).

Proof. — Recall that by (2.3), $S_N = \chi\left(\frac{H}{2N+1}\right)$. Therefore the second inequality is a consequence of the first as thanks to Proposition 4.1 the map S_N is bounded on X_T^s uniformly in N .

Let us now prove the first inequality. Consider a classical Littlewood-Paley decomposition of unity with respect to H ,

$$(7.7) \quad \text{Id} = \sum_N \Delta_N,$$

where the summation is taken over dyadic integers $N = 2^k$, $\Delta_0 = \psi_0(\sqrt{H})$ and for $N \geq 1$, $\Delta_N = \psi(\sqrt{H}/N)$, where ψ_0, ψ are suitable $C_0^\infty(\mathbb{R})$ functions (the support of ψ does not meet zero). Estimate (7.6) is a consequence of the following localized version of it.

Lemma 7.4. — *For any $\delta > 0$, there exists $C > 0$ such that for $N_1 \leq \dots \leq N_k$,*

$$\|\Delta_{N_1}(u_1) \cdots \Delta_{N_k}(u_k)\|_{L_T^1 L^2} \leq CT^\kappa N_k^{-s+\delta} \prod_{j=1}^k \|u_j\|_{X_T^s}.$$

Let us now explain how Lemma 7.4 implies Proposition 7.3. Using the definition of \mathcal{H}^s , after performing (7.7), we can write

$$\|u_1 \cdots u_k\|_{L_T^1 \mathcal{H}^{s-\eta}} \leq C \sum_M \sum_{N_1 \leq \dots \leq N_k} \sum_{\sigma \in S_k} (1+M)^{s-\eta} \|\Delta_M(\Delta_{N_1}(u_{\sigma(1)}) \cdots \Delta_{N_k}(u_{\sigma(k)}))\|_{L_T^1 L^2}.$$

We now observe that Proposition 7.3 is a consequence of Lemma 7.4 (with the choice $\delta = \eta/2$) and the following statement, applied with $\alpha > 0$ small enough.

Lemma 7.5. — *Let $\alpha > 0$. For every $K > 0$, there exists $C > 0$ such that for $M \geq N_k^{1+\alpha}$, $N_1 \leq \dots \leq N_k$,*

$$(7.8) \quad \|\Delta_M(\Delta_{N_1}(u_1) \cdots \Delta_{N_k}(u_k))\|_{L_T^1 L^2} \leq CT(1+M)^{-K} \prod_{j=1}^k \|u_j\|_{X_T^s}.$$

Let us observe that in [6], a similar stronger property (assuming only $M \geq DN_k$, $D \gg 1$) in the context of the analysis on a compact Riemannian manifold is proved, the projectors Δ_N being replaced by the corresponding objects associated to the Laplace-Beltrami operator. In the context of our analysis below the argument is much simpler compared to [6].

Proof of Lemma 7.5. — Since the space X_T^s is embedded in $L_T^\infty \mathcal{H}^{-\varepsilon/10}$ (which is the only L^2 type component of our resolution space), by duality and summing of geometric series the bound (7.8) is a consequence of the eigenfunction bound

$$(7.9) \quad \forall K > 0, \exists C_K : \forall n_1 \leq n_2 \leq \dots \leq n_k \leq n_k^{1+\alpha} \leq m, \quad \left| \int_{\mathbb{R}} h_{n_1} \cdots h_{n_k} h_m \right| \leq C_K (1+m)^{-K}$$

(the argument is trivial in the time variable). By writing $h_m = \frac{1}{(2m+1)^j} H^j h_m$, we make integrations by parts in the left hand-side of (7.9) and obtain

$$\frac{1}{(2m+1)^j} \int_{\mathbb{R}} H^j (h_{n_1} \cdots h_{n_k}) h_m.$$

Starting from the definition of h_n (2.1) we have the relations

$$(7.10) \quad h'_n(x) = \sqrt{\frac{n}{2}} h_{n-1}(x) - \sqrt{\frac{n+1}{2}} h_{n+1}(x)$$

and

$$x h_n(x) = h'_n(x) + \sqrt{2(n+1)} h_{n+1}(x),$$

which implies the bound (for $p \geq 2$),

$$(7.11) \quad \|x^{k_1} \partial_x^{k_2} h_n\|_{L^p(\mathbb{R})} \leq C_{k_1, k_2} (1 + |n|)^{\frac{k_1 + k_2}{2}}.$$

Using (7.11) (applied when extending the powers of H) we obtain that the left hand-side of (7.9) is bounded by $C_j (n_k/m)^j$ which implies the needed bound thanks to our restriction on M . This completes the proof of Lemma 7.5. \square

It remains to prove Lemma 7.4.

Proof of Lemma 7.4. — We take the parameters ε and r in the scope of applicability of Lemma 7.1 and Lemma 7.2. By introducing artificially the weight $\langle x \rangle^{s+\varepsilon/4}$, using the Hölder inequality and Proposition 4.1, we can write

$$(7.12) \quad \|\Delta_{N_1}(u_1) \cdots \Delta_{N_k}(u_k)\|_{L_T^1 L^2} \leq C \|\langle x \rangle^{-s-\varepsilon/4} \Delta_{N_k}(u_k)\|_{L_T^2 L^2} \prod_{j=1}^{k-1} \|\langle x \rangle^{\frac{s+\varepsilon/4}{k-1}} u_j\|_{L_T^{2(k-1)} L^\infty}.$$

We now estimate the right hand-side of (7.12). First, we observe that there exists $\kappa > 0$ such that for $j = 2, \dots, k$, using the Sobolev inequality, and the boundedness on L^r ($1 < r < \infty$) of zero-th order pseudo-differential operators, we can write

$$(7.13) \quad \|\langle x \rangle^{\frac{s+\varepsilon/4}{k-1}} u_j\|_{L_T^{2(k-1)} L^\infty} \leq CT^\kappa \|u_j\|_{L_T^{2(k-1)+\varepsilon} \mathcal{W}_{\frac{s+\varepsilon}{k-1}, r}} \leq CT^\kappa \|u_j\|_{X_T^s},$$

provided $\frac{3\varepsilon}{4(k-1)} r > 1$. Next, using similar arguments as in the proof of Lemma 7.5, we obtain the following statement.

Lemma 7.6. — *For any $\kappa > 0$ and any K , there exists C such that for any M satisfying $M \leq N^{1-\kappa}$, we have*

$$\|\Delta_M \langle x \rangle^{-s-\varepsilon/4} \Delta_N(u)\|_{L_T^2 L^2} \leq C(1 + M + N)^{-K} \|u\|_{X_T^s}.$$

As a consequence of Lemma 7.6, we obtain for arbitrarily small $\kappa > 0$,

$$\begin{aligned} \|\langle x \rangle^{-s-\varepsilon/4} \Delta_{N_k}(u_k)\|_{L_T^2 L^2} &\leq \sum_{M \geq N_k^{1-\kappa}} \|\Delta_M(\langle x \rangle^{-s-\varepsilon/4} \Delta_{N_k}(u_k))\|_{L_T^2 L^2} + CN_k^{-s} \|u_k\|_{X_T^s} \\ &\leq C \sum_{M \geq N_k^{1-\kappa}} \|\Delta_M \frac{\sqrt{H}^s}{M^s}(\langle x \rangle^{-s-\varepsilon/4} \Delta_{N_k}(u_k))\|_{L_T^2 L^2} + CN_k^{-s} \|u_k\|_{X_T^s} \\ &\leq CN_k^{-s+\kappa s} \|\sqrt{H}^s(\langle x \rangle^{-s-\varepsilon/4} \Delta_{N_k}(u_k))\|_{L_T^2 L^2} + CN_k^{-s} \|u_k\|_{X_T^s}. \end{aligned}$$

Using Proposition 4.1, we can write

$$\|\sqrt{H}^s(\langle x \rangle^{-s-\varepsilon/4} \Delta_{N_k}(u_k))\|_{L_T^2 L^2} \leq C \|u_k\|_{L_T^2 \mathcal{H}_{\langle x \rangle^{-s-\varepsilon/4}}^s} + \|[\sqrt{H}^s, \langle x \rangle^{-s-\varepsilon/4}] \Delta_{N_k}(u_k)\|_{L_T^2 L^2}.$$

In order to estimate the commutator contribution, we shall use the Weyl-Hörmander pseudo-differential calculus associated to the metric

$$(7.14) \quad dx^2 + \frac{d\xi^2}{1+x^2+\xi^2}.$$

The symbol classes S^m associated to (7.14) are the spaces of smooth functions on \mathbb{R}^2 satisfying the bounds

$$(7.15) \quad |\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha, \beta} \langle |x| + |\xi| \rangle^{m-\beta}.$$

We refer to [20], Section 18.5, [30] or [3] for a background concerning the analysis we perform now. First, using the functional calculus associated to the operator H , we obtain that for $\sigma \in [0, 1]$ the operator \sqrt{H}^σ is a pseudo-differential operator with symbol in S^σ (in fact even better bounds than (7.15) are enjoyed by the x derivatives of the symbol of \sqrt{H}^σ). We have that $\langle x \rangle^{-s-\varepsilon/4}$ is in S^0 and therefore the commutator $[\sqrt{H}^s, \langle x \rangle^{-s-\varepsilon/4}]$ is a pseudo-differential operator with symbol in S^{s-1} . As a consequence $H^{\varepsilon/8}[\sqrt{H}^s, \langle x \rangle^{-s-\varepsilon/4}]$ is a pseudo-differential operator with symbol in $S^{\varepsilon/8+s-1} \subset S^0$, provided $\varepsilon \ll 1$. Therefore, using the L^2 boundedness of zero order pseudo-differential operators, we obtain that

$$\|[\sqrt{H}^s, \langle x \rangle^{-s-\varepsilon/4}] \Delta_{N_k}(u_k)\|_{L_T^2 \mathcal{H}^{\varepsilon/4}} \leq C \|\Delta_{N_k}(u_k)\|_{L_T^2 L^2}$$

and by duality

$$\|[\sqrt{H}^s, \langle x \rangle^{-s-\varepsilon/4}] \Delta_{N_k}(u_k)\|_{L_T^2 L^2} \leq C \|\Delta_{N_k}(u_k)\|_{L_T^2 \mathcal{H}^{-\varepsilon/4}} \leq CT^{\frac{1}{2}} \|u_k\|_{X_T^s}.$$

Therefore, we obtain the bound

$$(7.16) \quad \|\langle x \rangle^{-s-\varepsilon/4} \Delta_{N_k}(u_k)\|_{L_T^2 L^2} \leq CN_k^{-s+\kappa s} \|u_k\|_{X_T^s}.$$

We now collect (7.12), (7.13) and (7.16) in order to complete the proof of Lemma 7.4. \square

This completes the proof of Proposition 7.3. \square

7.4. Further properties of Y^s with respect to the measure ρ . — From now each time we invoke the space Y^s , we mean that s satisfies (7.1) and ε and r are in the scope of applicability of Lemma 7.1, Lemma 7.2 and Proposition 7.3. Let us next define some auxiliary spaces. Let \tilde{Y}^s be defined by

$$\tilde{Y}^s = \left\{ u \in \mathcal{H}^{-\varepsilon/20} : e^{-itH}(u) \in L_{2\pi}^{2(k-1)+\varepsilon} \mathcal{W}_{\frac{s+\varepsilon}{k-1}, r} \cap L_{2\pi}^2 \mathcal{H}_{\langle x \rangle^{-s-\varepsilon/5}}^s \right\},$$

equipped with the natural norm. The remaining part of this section is devoted to three lemmas needed in the proof of Theorem 2.4. Using the density in L^p , $1 \leq p < \infty$ of the Schwartz class $\mathcal{S}(\mathbb{R})$, as a consequence of Proposition 4.1 (and the fact that the result is straightforward if $f \in \mathcal{S}$), we have the following statement.

Lemma 7.7. — *For every $f \in Y^s$, $\|(1 - S_N)(f)\|_{Y^s} = o(1)_{N \rightarrow \infty}$. A similar statement holds for \tilde{Y}^s .*

One can easily see that the analysis of Lemma 5.2 and Lemma 6.2 implies that φ_N , defined by (2.6) is a Cauchy sequence in $L^2(\Omega; Y^s)$ and thus we may see the measures μ and ρ as finite Borel measures on Y^s . We deduce, thanks to Lemma 7.1

Lemma 7.8. — *There exist $C, c > 0$ such that*

$$\rho(u : \|u\|_{Y^s} > \lambda) + \rho(u : \|u\|_{\tilde{Y}^s} > \lambda) \leq C e^{-c\lambda^2}.$$

We also have the following statement.

Lemma 7.9. — *Assume that $s < s' < s + \frac{\varepsilon}{20}$. Then we have that $\tilde{Y}^s \subset Y^s$ and the embedding is compact. In particular, thanks to Lemma 7.8, for every $\delta > 0$ there exists a compact K of Y^s such that $\rho(Y^s) - \rho(K) < \delta$.*

Notice that as soon as we gain some positive power in H , we gain compactness because powers of H controls both powers of $|D_x|$ and of x . As a consequence, the assumption $s' > s$ ensures that we have compactness in terms of x derivatives and weights in $\langle x \rangle$ for the second norm, whereas it ensures compactness in terms of derivatives in the third norm, while the assumption $s' < s + \frac{\varepsilon}{20} \Rightarrow s' + \frac{\varepsilon}{5} < s + \frac{\varepsilon}{4}$ ensures compactness in terms of weights in $\langle x \rangle$ in this last norm. Finally, since the second and the third term in the definition of Y^s are defined in terms of the free evolution, we may exchange some saving derivatives in H for some compactness in time. We omit the details.

7.5. Local well-posedness results. — Using the results of the previous subsections, we can now get local well-posedness results (uniform with respect to the parameter N) for

$$(7.17) \quad (i\partial_t - H)u = \kappa_0 S_N(|S_N u|^{k-1} S_N u), \quad u(0, x) = u_0(x) \in E_N.$$

Here is a precise statement.

Proposition 7.10. — *There exist $C > 0$, $c \in (0, 1)$, $\gamma > 0$ such that for every $A \geq 1$ and every $N \geq 1$, every $u_0 \in E_N$ satisfying $\|u_0\|_{Y^s} \leq A$ there exists a unique solution of (7.17) on the interval $[-cA^{-\gamma}, cA^{-\gamma}]$ such that $\|u\|_{X_{cA^{-\gamma}}^s} \leq A + A^{-1}$. In addition for $t \in [0, cA^{-\gamma}]$,*

$$(7.18) \quad \|u(t)\|_{Y^s} \leq A + A^{-1}.$$

Moreover, if u and v are two solutions with data u_0 and v_0 respectively, satisfying $\|u_0\|_{Y^s} \leq A$, $\|v_0\|_{Y^s} \leq A$ then $\|u - v\|_{X_{cA^{-\gamma}}^s} \leq C\|u_0 - v_0\|_{Y^s}$ and for $t \in [0, cA^{-\gamma}]$,

$$\|u(t) - v(t)\|_{Y^s} \leq C\|u_0 - v_0\|_{Y^s}.$$

Proof. — We rewrite (7.17) as the integral equation

$$u(t) = e^{-itH}(u_0) + \kappa_0 \int_0^t e^{-i(t-\tau)H} (S_N(|S_N u(\tau)|^{k-1} S_N u(\tau))) d\tau \equiv \Phi_{u_0}(u).$$

Using Lemma 7.2 and Proposition 7.3, we infer the bounds

$$(7.19) \quad \|\Phi_{u_0}(u)\|_{X_T^s} \leq \|u_0\|_{Y^s} + CT^\kappa \|u\|_{X_T^s}^k$$

and (after some algebraic manipulations on $|u|^{k-1}u - |v|^{k-1}v$)

$$(7.20) \quad \|\Phi_{u_0}(u) - \Phi_{u_0}(v)\|_{X_T^s} \leq CT^\kappa \|u - v\|_{X_T^s} (\|u\|_{X_T^s}^{k-1} + \|v\|_{X_T^s}^{k-1}).$$

Therefore if we choose T as $T = cA^{-K}$ with $c \ll 1$ and $K > (k + 10)/\kappa$, the estimates (7.19) and (7.20) yield that the map Φ_{u_0} is a contraction on the ball of radius $2A$ and centered at the origin of X_T^s . The fixed point of this contraction is a solution of (7.17). The uniqueness and the estimate on the difference of two solutions is a consequence of Proposition 7.3. Finally coming back to (7.19), we infer that the solution satisfies

$$(7.21) \quad \|u\|_{X_T^s} = \|\Phi_{u_0}(u)\|_{X_T^s} \leq A + Cc^\kappa(1 + A)^{-\kappa K} A^k \leq A + A^{-1},$$

for c small enough and by possibly taking K slightly larger (replacing K by $K + 1/\kappa$ for instance). Let us now prove (7.18). Using Lemma 7.2 (η in the scope of its applicability), Proposition 7.3 and (7.21), we obtain from Lemma 7.2,

$$\begin{aligned} \|u(t)\|_{Y^s} &\leq \|u_0\|_{Y^s} + CT^\kappa \| |u|^{k-1}u \|_{L_T^1 \mathcal{H}^{s-\eta}} \\ &\leq A + CT^\kappa \|u\|_{X_T^s}^k \\ &\leq A + A^{-1}. \end{aligned}$$

This completes the proof of Proposition 7.10. \square

Let us remark that the existence statement in Proposition 7.10 is not of importance (indeed see the next section for a global existence statement). The important point is the uniformness with respect to N of the bounds obtained. Similarly, we can also obtain a well-posedness result for the original problem

$$(7.22) \quad (i\partial_t - H)u = \kappa_0 |u|^{k-1}u, \quad u(0, x) = u_0(x) \in Y^s.$$

Proposition 7.11. — *Then there exist $C > 0$, $c \in (0, 1)$, $\gamma > 0$ such that for every $A \geq 1$ if we set $T = cA^{-\gamma}$ for every $u_0 \in Y^s$ satisfying $\|u_0\|_{Y^s} \leq A$ there exists a unique solution of (7.22) on $[-T, T]$ such that $\|u\|_{X_T^s} \leq A + A^{-1}$. In addition for $t \in [0, T]$,*

$$\|u(t)\|_{Y^s} \leq A + A^{-1}.$$

Moreover, if u and v are two solutions with data u_0 and v_0 respectively, satisfying $\|u_0\|_{Y^s} \leq A$, $\|v_0\|_{Y^s} \leq A$ then $\|u - v\|_{X_T^s} \leq C\|u_0 - v_0\|_{Y^s}$.

The proof of Proposition 7.11 is essentially the same as that of Proposition 7.10 and hence will be omitted.

8. Global well-posedness

In this section, we prove the global existence results for a full measure set for (1.1). Moreover this set will be reproduced by the flow which is a key element in the measure invariance argument of the next section.

8.1. Hamiltonian structure of the approximate problem. — Here we consider again the problem

$$(8.1) \quad (i\partial_t - H)u = \kappa_0 S_N(|S_N u|^{k-1} S_N u), \quad u(0, x) = \Pi_N(u(0, x)) \in E_N,$$

with $\kappa_0 = \pm 1$ if $k = 3$ and $\kappa_0 = 1$ if $k \geq 5$.

For $u \in E_N$, write

$$u = \sum_{n=0}^N c_n h_n = \sum_{n=0}^N (a_n + ib_n) h_n, \quad a_n, b_n \in \mathbb{R}.$$

Then we have the following result.

Lemma 8.1. — *Set*

$$J(a_0, \dots, a_N, b_0, \dots, b_N) = \frac{1}{2} \sum_{n=0}^N \lambda_n^2 (a_n^2 + b_n^2) + \frac{\kappa_0}{k+1} \|S_N(\sum_{n=0}^N (a_n + ib_n) h_n)\|_{L^{k+1}(\mathbb{R})}^{k+1}.$$

The equation (8.1) is a Hamiltonian ODE of the form

$$(8.2) \quad \dot{a}_n = \frac{\partial J}{\partial b_n}, \quad \dot{b}_n = -\frac{\partial J}{\partial a_n}, \quad 0 \leq n \leq N.$$

In particular J is conserved by the flow. Moreover the mass

$$(8.3) \quad \|u\|_{L^2(\mathbb{R})}^2 = \sum_{n=0}^N (a_n^2 + b_n^2)$$

is conserved under the flow of (8.1). As a consequence, (8.1) has a well-defined global flow $\tilde{\Phi}_N$.

Proof. — The proof of (8.2) is straightforward. Let us next show the L^2 conservation. Multiply the equation (8.1) with \bar{u} and integrate over \mathbb{R} . First, by an integration by parts, we have

$$(8.4) \quad - \int_{\mathbb{R}} \bar{u} H u = \int_{\mathbb{R}} |H^{1/2} u|^2 \in \mathbb{R}.$$

Then by (2.4), we deduce that

$$(8.5) \quad \begin{aligned} \int_{\mathbb{R}} S_N(|S_N u|^{k-1} S_N u) \bar{u} &= \int_{\mathbb{R}} S_N(|S_N u|^{k-1} S_N u) S_N \bar{u} \\ &= \int_{\mathbb{R}} (|S_N u|^{k-1} S_N u) S_N \bar{u} \in \mathbb{R}. \end{aligned}$$

Hence, from (8.4) and (8.5) we infer that

$$\frac{d}{dt} \|u\|_{L^2(\mathbb{R})}^2 = 0.$$

This completes the proof of Lemma 8.1. \square

Denote by $\tilde{\Phi}_N(t) : E_N \rightarrow E_N$ the flow of the ODE (8.1). We now state an invariance result which holds both in the defocusing and in the focusing cases.

Proposition 8.2. — *The measure $\tilde{\rho}_N$ defined by (2.10) (or (2.12)) is invariant under the flow $\tilde{\Phi}_N$ of (8.1).*

Proof. — The proof is based on the Liouville theorem which we recall below. See e.g. [2, page 528].

Lemma 8.3. — *Consider the ODE $\dot{x}(t) = F(t, x(t))$, $x(t) \in \mathbb{R}^n$ with a local flow $\Phi(t)$. Suppose also that F is divergence free, i.e. $\sum_{j=1}^n \partial_j F_j = 0$ (∂_j being the derivative with respect to the j 'th variable). Then the Lebesgue measure of \mathbb{R}^n is invariant under the flow $\Phi(t)$.*

Observe that the ODE's in the scope of applicability of the Liouville theorem are not necessarily autonomous. Let us now return to the proof of Proposition 8.2. By Lemma 8.3, the measure $dadb = \prod_{n=0}^N a_n b_n$ is invariant under Φ_N . Then, as the Hamiltonian J is conserved, the measure $d_N^{-1} e^{-J} \prod_{n=0}^N da_n db_n$ is also invariant by the flow of (8.1). This completes the proof in the defocusing case. A similar argument applies in the focusing case by invoking the L^2 conservation. This completes the proof of Proposition 8.2. \square

Let us now decompose the space $\mathcal{H}^{-\sigma}(\mathbb{R}) = E_N^\perp \oplus E_N$, and denote by $\Phi_N(t) = (e^{itH}, \tilde{\Phi}_N(t))$ the flow of the equation

$$(8.6) \quad (i\partial_t - H)u = \kappa_0 S_N(|S_N u|^{k-1} S_N u), \quad u(0, x) = (u_0^N, u_{0,N}) \in E_N^\perp \oplus E_N.$$

Corollary 8.4. — *The measure ρ_N is invariant under the flow $\Phi_N(t)$.*

Indeed, it is clear for product sets $A = A^N \times A_N$, $A^N \subset E_N^\perp$, $A_N \subset E_N$ and these sets generate the Borelian σ -algebra.

8.2. Global existence. — Here we show that the problem (1.1) is globally well-posed on a set of full ρ measure. Our first result gives bounds (independent of N) on the solution of the approximate equation (8.6).

Proposition 8.5. — *There exists a constant $C > 0$ such that for all $m, N \in \mathbb{N}^*$, there exists a $\tilde{\rho}_N$ measurable set $\tilde{\Sigma}_N^m \subset E_N$ so that*

i) The following estimate holds true

$$\tilde{\rho}_N(E_N \setminus \tilde{\Sigma}_N^m) \leq 2^{-m};$$

ii) For all $f \in \tilde{\Sigma}_N^m$ and $t \in \mathbb{R}$

$$(8.7) \quad \|\tilde{\Phi}_N(t)f\|_{Y^s} \leq C(m + \log(1 + |t|))^{\frac{1}{2}};$$

iii) There exists $c > 0$ such that for every t_0 , every $m \geq 1$ and $N \geq 1$,

$$(8.8) \quad \tilde{\Phi}_N(t_0)(\tilde{\Sigma}_N^m) \subset \tilde{\Sigma}_N^{m+[c \log(|t_0|+1)]+3}.$$

The property (8.8) allows to simplify the construction of a set invariant by the limit flow, compared to a similar situation in [38, 7].

Proof. — We set, for m, j integers ≥ 1 ,

$$\tilde{B}_N^{m,j}(D) \equiv \{u \in E_N : \|u\|_{Y^s} \leq D(m+j)^{\frac{1}{2}}\},$$

where the number $D \gg 1$ (independent of m, j, N) will be fixed later. Thanks to Proposition 7.10, there exist $c > 0$, $\gamma > 0$ only depending on s such that if we set $\tau \equiv cD^{-\gamma}(m+j)^{-\gamma/2}$ then for every $t \in [-\tau, \tau]$,

$$(8.9) \quad \tilde{\Phi}_N(t)(\tilde{B}_N^{m,j}(D)) \subset \{u \in E_N : \|u\|_{Y^s} \leq D(m+j)^{\frac{1}{2}} + D^{-1}(m+j)^{-\frac{1}{2}} \leq D(m+j+1)^{\frac{1}{2}}\},$$

provided $D \gg 1$, independently of m, j . Following [4], we set

$$\tilde{\Sigma}_N^{m,j}(D) \equiv \bigcap_{k=-[2^j/\tau]}^{[2^j/\tau]} \tilde{\Phi}_N(-k\tau)(\tilde{B}_N^{m,j}(D)),$$

where $[2^j/\tau]$ stands for the integer part of $2^j/\tau$. Notice that thanks to (8.9), we obtain that the solution of (8.6) with data $f \in \tilde{\Sigma}_N^{m,j}(D)$ satisfies

$$(8.10) \quad \|\tilde{\Phi}_N(t)(f)\|_{Y^s} \leq D(m+j+1)^{\frac{1}{2}}, \quad |t| \leq 2^j.$$

Indeed, for $|t| \leq 2^j$, we can find an integer $k \in [-[2^j/\tau], [2^j/\tau]]$ and $\tau_1 \in [-\tau, \tau]$ so that $t = k\tau + \tau_1$ and thus $u(t) = \tilde{\Phi}_N(\tau_1)(\tilde{\Phi}_N(k\tau)(f))$. Since $f \in \tilde{\Sigma}_N^{m,j}(D)$ implies that $\tilde{\Phi}_N(k\tau)(f) \in \tilde{B}_N^{m,j}(D)$, we can apply (8.9) and arrive at (8.10).

By Proposition 8.2, the measure $\tilde{\rho}_N$ is invariant by the flow $\tilde{\Phi}_N$. Hence

$$\begin{aligned} \tilde{\rho}_N(E_N \setminus \tilde{\Sigma}_N^{m,j}(D)) &\leq (2[2^j/\tau] + 1)\tilde{\rho}_N(E_N \setminus \tilde{B}_N^{m,j}(D)) \\ &\leq C2^j D^\gamma (m+j)^{\gamma/2} \tilde{\rho}_N(E_N \setminus \tilde{B}_N^{m,j}(D)). \end{aligned}$$

Now, by the large deviation bounds of Lemma 7.1

$$(8.11) \quad \tilde{\rho}_N(E_N \setminus \tilde{\Sigma}_N^{m,j}(D)) \leq C2^j D^\gamma (m+j)^{\gamma/2} e^{-cD^2(m+j)} \leq 2^{-(m+j)},$$

provided $D \gg 1$, independently of m, j, N .

Next, we set

$$\tilde{\Sigma}_N^m = \bigcap_{j=1}^{\infty} \tilde{\Sigma}_N^{m,j}(D).$$

Thanks to (8.11), $\tilde{\rho}_N(E_N \setminus \tilde{\Sigma}_N^m) \leq 2^{-m}$. In addition, using (8.10), we get that there exists C such that for every m , every N , every $f \in \tilde{\Sigma}_N^m$, every $t \in \mathbb{R}$,

$$\|\tilde{\Phi}_N(t)(f)\|_{Y^s} \leq C(m+2+\log(1+|t|))^{\frac{1}{2}}.$$

Indeed for $t \in \mathbb{R}$ there exists $j \in \mathbb{N}$ such that $2^{j-1} \leq 1+|t| \leq 2^j$ and we apply (8.10) with this j . This proves (8.7).

Let us now turn to the proof of (8.8). Consider $f \in \tilde{\Sigma}_N^m$. Denote by j_0 the integer part of $2 + \frac{\log(1+|t_0|)}{\log(2)}$. According to (8.10), as soon as $j \geq j_0$, we have $|t_0| \leq 2^{j-1}$, and for any $|t| \leq 2^{j-1}$

$$(8.12) \quad \tilde{\Phi}_N(t)\tilde{\Phi}_N(t_0)f \in \tilde{B}_N^{m+2,j-1}(D) \subset \tilde{B}_N^{m+j_0+2,j-1}(D),$$

which implies that

$$\tilde{\Phi}_N(t_0)f \in \tilde{\Sigma}_N^{m+j_0+2,j-1}(D), \quad \forall j \geq j_0.$$

On the other hand, the trivial relation (for $j_0 - k > 0$, *i.e.* $k = 1, 2, \dots, j_0 - 1$)

$$\tilde{B}_N^{m+2,j_0}(D) = \tilde{B}_N^{m+2+k,j_0-k}(D) \subset \tilde{B}_N^{m+j_0+2,j_0-k}(D),$$

and (8.12) (applied with $j = j_0 + 1$) implies that for $j < j_0$ and $|t| \leq 2^j \leq 2^{j_0-1}$,

$$\tilde{\Phi}_N(t)\tilde{\Phi}_N(t_0)f \in \tilde{B}_N^{m+j_0+2,j}(D), \quad \forall 1 \leq j < j_0,$$

and consequently

$$\tilde{\Phi}_N(t_0)f \in \tilde{\Sigma}_N^{m+j_0+2,j}(D), \quad \forall j \geq 1.$$

This proves (8.8) and therefore the proof of Proposition 8.5 is completed. \square

For integers $m \geq 1$ and $N \geq 1$, we define the cylindrical sets

$$\Sigma_N^m \equiv \{u \in \mathcal{H}^{-\sigma} : \Pi_N(u) \in \tilde{\Sigma}_N^m\}.$$

Next, for $m \geq 1$, we set

$$\Sigma^m = \{u \in \mathcal{H}^{-\sigma} : \exists N_k, \lim_{k \rightarrow +\infty} N_k = \infty, \exists u_{N_k} \in \Sigma_{N_k}^m, \lim_{k \rightarrow +\infty} \|S_{N_k} u_{N_k} - u\|_{Y^s} = 0\}.$$

Observe that Σ^m is a closed subset of Y^s . Indeed, assume that there exists $u_{N_k} \in \Sigma_{N_k}^m$ such that $\lim_{k \rightarrow +\infty} \|S_{N_k} u_{N_k} - u\|_{Y^s} = 0$. Then for any $P \in \mathbb{N}$, as soon as $N_k \gg P$, we have

$$\|S_P(u_{N_k} - u)\|_{Y^s} = \|S_P(S_{N_k} u_{N_k} - u)\|_{Y^s} \leq C \|S_{N_k} u_{N_k} - u\|_{Y^s} \rightarrow 0.$$

As a consequence, using (8.7) (with $t = 0$), we deduce

$$\|S_P(u)\|_{Y^s} \leq \limsup_{k \rightarrow +\infty} \|S_P(u_{N_k})\|_{Y^s} = \limsup_{k \rightarrow +\infty} \|S_P(u_{N_k})\|_{Y^s} \leq C \sup_Q \|S_Q\|_{\mathcal{L}(L^r(\mathbb{R}))} m^{1/2}$$

and passing to the limit $P \rightarrow +\infty$, we deduce

$$u \in Y^s, \|u\|_{Y^s} \leq C' m^{1/2}.$$

The closeness property is clear. Notice also that we have the following inclusions

$$(8.13) \quad \limsup_{N \rightarrow +\infty} \Sigma_N^m = \bigcap_{N=1}^{\infty} \bigcup_{N_1=N}^{\infty} \Sigma_{N_1}^m \subset \Sigma^m.$$

Indeed, if $u \in \limsup_{N \rightarrow +\infty} \Sigma_N^m$, there exists $N_k \rightarrow +\infty$ such that

$$\Pi_{N_k}(u) \in \tilde{\Sigma}_{N_k}^m,$$

and the same proof as above shows that

$$u \in Y^s, \|u\|_{Y^s} \leq C' m^{1/2}.$$

Now, we clearly have

$$\|S_n u - u\|_{Y^s} = o(1)_{n \rightarrow +\infty},$$

and since $S_n(\Pi_n(u)) = S_n(u)$, the sequence $u_{N_k} \equiv \Pi_{N_k}(u)$ is the one ensuring that $u \in \Sigma^m$. This proves (8.13). As a consequence of (8.13), we get

$$(8.14) \quad \rho(\Sigma^m) \geq \rho(\limsup_{N \rightarrow +\infty} \Sigma_N^m).$$

Using Fatou's lemma, we get

$$(8.15) \quad \rho(\limsup_{N \rightarrow \infty} \Sigma_N^m) \geq \limsup_{N \rightarrow \infty} \rho(\Sigma_N^m).$$

In the defocusing case, consider $G_N(u) = \exp(-\frac{1}{k+1} \|S_N u\|_{L^{k+1}(\mathbb{R})}^{k+1})$ and $G(u) = \exp(-\frac{1}{k+1} \|u\|_{L^{k+1}(\mathbb{R})}^{k+1})$. In the focusing case, let G_N be defined by (2.13) and G by Theorem 2.3. We have that

$$\rho(\Sigma_N^m) = \int_{\Sigma_N^m} G(u) d\mu(u),$$

and

$$\rho_N(\Sigma_N^m) = \int_{\Sigma_N^m} G_N(u) d\mu_N(u) = \int_{\tilde{\Sigma}_N^m} G_N(u) d\tilde{\mu}(u) = \tilde{\rho}_N(\tilde{\Sigma}_N^m).$$

Therefore, thanks to (3.15), we get

$$\lim_{N \rightarrow \infty} (\rho(\Sigma_N^m) - \rho_N(\Sigma_N^m)) = 0.$$

Therefore, using Proposition 8.5 and (3.16), we obtain

$$\begin{aligned} \limsup_{N \rightarrow \infty} \rho(\Sigma_N^m) &= \limsup_{N \rightarrow \infty} \rho_N(\Sigma_N^m) = \limsup_{N \rightarrow \infty} \tilde{\rho}_N(\tilde{\Sigma}_N^m) \\ &\geq \limsup_{N \rightarrow \infty} (\rho_N(Y^s) - 2^{-m}) = \rho(Y^s) - 2^{-m}. \end{aligned}$$

Collecting the last estimate and (8.14), (8.15), we obtain that

$$(8.16) \quad \rho(\Sigma^m) \geq \rho(Y^s) - 2^{-m}.$$

Now, we set

$$(8.17) \quad \Sigma \equiv \bigcup_{i=1}^{\infty} \Sigma^m.$$

Then, by (8.16), the set Σ is of full ρ measure. It turns out that one has global existence for any initial condition $f \in \Sigma$.

Proposition 8.6. — *For every integer $m \geq 1$ the local solution u of (7.22) with initial condition $f \in \Sigma^m$ is globally defined and we shall denote it by $u = \Phi(t)f$. Moreover, there exists $C > 0$ such that for every $f \in \Sigma^m$ and every $t \in \mathbb{R}$,*

$$\|u(t)\|_{Y^s} \leq C(m + \log(1 + |t|))^{\frac{1}{2}}.$$

Furthermore, if $f_p \in \Sigma_{N_p}^m$ and $N_p \rightarrow +\infty$ are so that

$$\lim_{p \rightarrow +\infty} \|S_{N_p} f_p - f\|_{Y^s} = 0,$$

then

$$(8.18) \quad \lim_{p \rightarrow +\infty} \|u(t) - S_{N_p}(\Phi_{N_p}(t)(f_p))\|_{Y^s} = 0.$$

Finally, for every $t \in \mathbb{R}$, $\Phi(t)(\Sigma) = \Sigma$.

Proof. — The key point is now the following lemma.

Lemma 8.7. — *There exist $\Lambda_0 > 0$, $C > 0$, $K > 0$ such that the following holds true. Consider a sequence $u_{0,N_p} \in E_{N_p}$ and $u_0 \in Y^s$. Assume that there exists $\Lambda > \Lambda_0$ such that*

$$\|u_{0,N_p}\|_{Y^s} \leq \Lambda, \quad \|u_0\|_{Y^s} \leq \Lambda, \quad \lim_{p \rightarrow +\infty} \|S_{N_p}u_{0,N_p} - u_0\|_{Y^s} = 0.$$

Then if we set $\tau = C\Lambda^{-K}$, $\Phi_{N_p}(t)(u_{0,N_p})$ and $\Phi(t)(u_0)$ exist for $t \in [0, \tau]$ and satisfy

$$\|\Phi_{N_p}(t)(u_{0,N_p})\|_{L^\infty Y^s \cap X^s_\tau} \leq \Lambda + 1, \quad \|\Phi(t)(u_0)\|_{L^\infty Y^s \cap X^s_\tau} \leq \Lambda + 1.$$

Furthermore

$$\lim_{p \rightarrow +\infty} \|S_{N_p}\Phi_{N_p}(t)(u_{0,N_p}) - \Phi(t)(u_0)\|_{L^\infty((0,\tau);Y^s)} = 0.$$

Proof. — The first part of this lemma is a direct consequence of our local well posedness results of Propositions 7.10, 7.11. For the second part, let us write

$$\Phi(t)(u_0) \equiv u = e^{-itH}(u_0) + v, \quad \Phi_{N_p}(t)(u_{0,N_p}) \equiv u_p = e^{-itH}(u_{0,N_p}) + v_p,$$

and $w_p = v - S_{N_p}v_p$. We have

$$u - u_p = e^{-itH}(u_0 - S_{N_p}u_{0,N_p}) + w_p$$

and by assumption,

$$\|e^{-itH}(u_0 - S_{N_p}u_{0,N_p})\|_{Y^s} = \|u_0 - S_{N_p}u_{0,N_p}\|_{Y^s} = o(1)_{p \rightarrow +\infty}.$$

Therefore it remains to show that $\|w_p\|_{L^\infty Y^s} = o(1)_{p \rightarrow +\infty}$, for τ chosen as in the statement of the lemma. Observe that w_p solves the problem

$$(8.19) \quad \begin{aligned} (i\partial_t - H)w_p &= \kappa_0|u|^{k-1}u - \kappa_0 S_{N_p}^2(|S_{N_p}u_p|^{k-1}S_{N_p}u_p) \\ &= \kappa_0(\text{Id} - S_{N_p}^2)(|u|^{k-1}u) + \kappa_0 S_{N_p}^2(|u|^{k-1}u - |S_{N_p}u_p|^{k-1}S_{N_p}u_p) \end{aligned}$$

with initial condition $w_p|_{t=0} = 0$. Using Proposition 7.3 and Proposition 7.11, we obtain that for $\eta > 0$

$$\||u|^{k-1}u\|_{L^1((0,\tau);H^{s-\eta})} \leq C\tau^\kappa \|u\|_{X^s_\tau}^k \leq C\tau^\kappa (\Lambda + 1)^k$$

and consequently,

$$(8.20) \quad \|(\text{Id} - S_{N_p}^2)(|u|^{k-1}u)\|_{L^1((0,\tau);H^{s-\eta})} \rightarrow 0 \text{ as } p \rightarrow +\infty.$$

We estimate the second term in the r.h.s. of (8.19) by using a direct manipulation on the expression $|z_1|^{k-1}z_1 - |z_2|^{k-1}z_2$ and invoking Proposition 7.3. This yields

$$\begin{aligned}
(8.21) \quad & \| |u|^{k-1}u - |S_{N_p}u_p|^{k-1}S_{N_p}u_p \|_{L^1((0,\tau);H^{s-\eta})} \leq \\
& \leq C\tau^\kappa \|u - S_{N_p}u_p\|_{X_\tau^s} (\|u\|_{X_\tau^s}^{k-1} + \|S_{N_p}u_p\|_{X_\tau^s}^{k-1}) \\
& \leq C\tau^\kappa (\Lambda + 1)^{k-1} (\|e^{-itH}(u_0 - S_{N_p}(u_{0,N_p}))\|_{Y^s} + \|w_p\|_{X_\tau^s}) \\
& \leq o(1)_{p \rightarrow +\infty} + C\tau^\kappa (\Lambda + 1)^{k-1} \|w_p\|_{X_\tau^s}.
\end{aligned}$$

We deduce from (8.20), (8.21), (8.19) and Lemma 7.2 that if $\eta \ll 1$,

$$\|w_p\|_{X_\tau^s} \leq C\tau^\kappa (\Lambda + 1)^{k-1} \|w_p\|_{X_\tau^s} + o(1)_{p \rightarrow +\infty}.$$

By taking $C\tau^\kappa (\Lambda + 1)^{k-1} < 1/2$, we infer that $\|w_p\|_{X_\tau^s} = o(1)_{p \rightarrow +\infty}$. Next using (7.3) of Lemma 7.2, we obtain that $\|w_p\|_{L_\tau^\infty Y^s} = o(1)_{p \rightarrow +\infty}$. This completes the proof of Lemma 8.7. \square

Let us now finish the proof of Proposition 8.6. By assumption, we know that there exist sequences $N_p \in \mathbb{N}$, $u_{N_p} \in \widetilde{\Sigma}_{N_p}^m$ (i.e. $\Pi_{N_p}(u_{N_p}) \in \Sigma_{N_k}^m$) such that

$$\lim_{p \rightarrow +\infty} \|S_{N_p}u_{N_p} - u_0\|_{Y^s} = 0.$$

Consequently, by Proposition 8.5, we know that

$$(8.22) \quad \|\tilde{\Phi}_{N_p}(t)(\Pi_{N_p}u_{N_p})\|_{Y^s} \leq C(m + \log(1 + |t|))^{\frac{1}{2}}.$$

The strategy of proof consists in proving that as long as the solution to (7.22) exists, we can pass to the limit in (8.22) and there exists a constant C' ($= \sup_Q \|S_Q\|_{\mathcal{L}(Y^s)}$) such that

$$(8.23) \quad \|\Phi(t)(u)\|_{Y^s} \leq C' C(m + \log(1 + |t|))^{\frac{1}{2}}$$

which (taking into account that the norm in Y^s controls the local existence time), implies that the solution is global and satisfies (8.23) for all times.

Equivalently, let us fix $T > 0$ and $\Lambda > \Lambda_0$ (the number Λ_0 being fixed in Lemma 8.7). We assume

$$(8.24) \quad \|\Phi_{N_p}(t)(\Pi_{N_p}u_{N_p})\|_{Y^s} \leq \Lambda, \text{ for } |t| \leq T$$

and we want to show

$$(8.25) \quad \|\Phi(t)(u_0)\|_{Y^s} \leq C'\Lambda, \text{ for } |t| \leq T.$$

As a first step, let us fix $t = 0$. For $Q \in \mathbb{N}$, if $N_p \geq Q$, $\Pi_{N_p} \circ S_Q = S_Q$ and consequently, using Proposition 4.1 and the definition of Σ^m , we obtain

$$\|S_Q(u_0)\|_{Y^s} = \lim_{p \rightarrow +\infty} \|S_Q \Pi_{N_p}(u_{N_p})\|_{Y^s} \leq C'\Lambda$$

and passing to the limit $Q \rightarrow +\infty$, we deduce

$$\|u_0\|_{Y^s} = \lim_{Q \rightarrow +\infty} \|S_Q(u_0)\|_{Y^s} \leq C'\Lambda.$$

This implies that the sequences $\Pi_{N_p}u_{N_p} \equiv u_{0,p}$ and u_0 satisfy the assumptions of Lemma 8.7 (with Λ replaced by $C'\Lambda$). As a consequence, we know that

$$\lim_{p \rightarrow +\infty} \|\Phi_{N_p}(t)(\Pi_{N_p}(u_{N_p})) - \Phi(t)(u_0)\|_{L^\infty((0,\tau);Y^s)} = 0$$

for $\tau \equiv c\Lambda^{-K}$. This convergence allows to pass to the limit in (8.24) for $t = \tau$, using again Proposition 4.1. Indeed, fix Q , then for $N_p \gg 2Q$,

$$\|S_Q \Phi(\tau)(u_0)\|_{Y^s} = \lim_{p \rightarrow \infty} \|S_Q \Phi_{N_p}(\tau) \Pi_{N_p}(u_{N_p})\|_{Y^s},$$

and using first (8.24) and passing to the limit $Q \rightarrow +\infty$, we deduce

$$\|\Phi(\tau)(u_0)\|_{Y^s} = \lim_{Q \rightarrow +\infty} \|S_Q \Phi(\tau)(u_0)\|_{Y^s} \leq \sup_Q \|S_Q\|_{\mathcal{L}(Y^s)} \Lambda.$$

Now, we can apply the results in Lemma 8.7, with the same Λ as in the previous step, which implies that (8.25) holds for $t \in [0, 2\tau]$, and so on and so forth.

Notice here that at each step the a priori bound does not get worse, because we only use the results in Lemma 8.7 to obtain the convergence of $\|\Phi_{N_p}(t)(\Pi_{N_p}(u_{N_p})) - \Phi(t)u_0\|_{Y^s}$ to 0, and then obtain the estimates on the norm $\|\Phi(t)(u_0)\|_{Y^s}$ by passing to the limit in (8.24) (applying first S_Q , passing to the limit $p \rightarrow +\infty$, then to the limit $Q \rightarrow +\infty$). A completely analogous argument holds for the negative times t .

In order to prove the last statement in Proposition 8.6 we observe that, according to (8.8) there exists $c > 0$ such that for any $t \in \mathbb{R}$,

$$\Phi(t)(\Sigma^m) \subset \Sigma^{m+[c \log(|t|+1)]+3}$$

which is a straightforward consequence of (8.18) and (8.8). As consequence we have $\Phi(t)(\Sigma) \subset \Sigma$ and thanks to the reversibility of the flow $\Phi(t)$, we infer that $\Phi(\Sigma) = \Sigma$. This completes the proof of Proposition 8.6. \square

9. Measure invariance

In this section, we prove the last part of Theorem 2.4. Recall that we see ρ as a finite Borel measure on Y^s . Let Σ be the set of full ρ measure constructed in the previous section. This is the set involved in the statement of Theorem 2.4. Recall that thanks to the last claim in Proposition 8.6, $\Phi(t)(\Sigma) = \Sigma$ and thanks to the reversibility of the flow $\Phi(t)$, it suffices to prove that for every ρ measurable set $A \subset \Sigma$ and every $t \in \mathbb{R}$, $\rho(A) \leq \rho(\Phi(t)(A))$. We perform several reductions which will allow us to reduce the matters to compact sets A and small times t . First by the regularity properties of ρ , we may assume that A is a closed set of Y^s . Then thanks to Lemma 7.9, it suffices to prove $\rho(K) \leq \rho(\Phi(t)(K))$ for K a compact set of Y^s . Let us fix a compact K of Y^s and a time $t > 0$ (the case $t < 0$ is analogous). Thanks to Proposition 8.6, there exists $R > 1$ such that $\{\Phi(\tau)(K), 0 \leq \tau \leq t\} \subset B_R$, where here and for future references B_R denotes the open ball of Y^s centered at the origin and of radius R . We have the following statement comparing $\Phi(t)$ and $\Phi_N(t)$ for small (but uniform) times and compacts contained in B_R .

Lemma 9.1. — *There exist $c > 0$ and $\gamma > 0$ such that the following holds true. For every $R > 1$, every compact K of B_R and every $\varepsilon > 0$ there exists $N_0 \geq 1$ such that for every $N \geq N_0$, every $u_0 \in K$, every $\tau \in [0, cR^{-\gamma}]$, $\|\Phi(\tau)(u_0) - \Phi_N(\tau)(u_0)\|_{Y^s} < \varepsilon$.*

Proof. — To prove this lemma, take two new cut off $S_{N,i} = \chi_i(\frac{H}{2N+1})$, $i = 1; 2$, with $\chi_1\chi = \chi$, $\chi_2\chi_1 = \chi_1$ so that $S_{N,1}S_N = S_N$, $S_{N,2}S_{N,1} = S_{N,1}$. Notice first that

$$\begin{aligned} \|\Phi(\tau)(u_0) - \Phi_N(\tau)(u_0)\|_{Y^s} &\leq \|(1 - S_{N,1})(\Phi(\tau)(u_0) - \Phi_N(\tau)(u_0))\|_{Y^s} \\ &\quad + \|S_{N,1}(\Phi(\tau)(u_0) - \Phi_N(\tau)(u_0))\|_{Y^s}. \end{aligned}$$

To bound the first term, we notice that

$$\|(1 - S_{N,1})(\Phi_N(\tau)(u_0))\|_{Y^s} = \|(1 - S_{N,1})(e^{itH}u_0)\|_{Y^s} = \|(1 - S_{N,1})(u_0)\|_{Y^s},$$

and

$$\lim_{N \rightarrow +\infty} \|(1 - S_{N,1})(\Phi(\tau)(u_0))\|_{Y^s} = 0$$

uniformly with respect to u_0 in a compact set of Y^s . To bound the second term, we notice that

$$\|S_{N,1}(\Phi(\tau)(u_0) - \Phi_N(\tau)(u_0))\|_{Y^s} = \|S_{N,1}(\Phi(\tau)(u_0) - S_{N,2}\Phi_N(\tau)(u_0))\|_{Y^s}.$$

Now to estimate this term we proceed as in the proof of Lemma 8.7, the only additional point being the observation that $S_{N,2}(u)$ converges to u in Y^s , uniformly with respect to u in a compact of Y^s . \square

We next observe that we only need to prove $\rho(K) \leq \rho(\Phi(\tau)(K))$ for $\tau \in [0, cR^{-\gamma}]$, where R and γ are fixed by Lemma 9.1. Then we can iterate the inequality on the same time intervals since we know that $\Phi(\tau)(K)$ remains included in B_R as far as $\tau \in [0, t]$. Using (2.14), Lemma 9.1 and the well-posedness result of Proposition 7.10 (notice that, though only stated for the flow $\tilde{\Phi}_N(t)$ on E_N , the result holds clearly for the flow $\Phi_N = (e^{itH}, \tilde{\Phi}_N(t))$ on $E_N^\perp \times E_N$), we can write

$$\begin{aligned} \rho(\Phi(\tau)(K) + B_{2\varepsilon}) &= \lim_{N \rightarrow \infty} \rho_N(\Phi(\tau)(K) + B_{2\varepsilon}) \\ &\geq \limsup_{N \rightarrow \infty} \rho_N(\Phi_N(\tau)(K) + B_\varepsilon) \geq \limsup_{N \rightarrow \infty} \rho_N(\Phi_N(\tau)(K + B_{\alpha\varepsilon})), \end{aligned}$$

where α is a fixed constant depending on R but independent of ε . Next, using the invariance of the measure ρ_N under the flow $\Phi_N(t)$ and using once again (2.14), we can write

$$\rho(\Phi(\tau)(K) + B_{2\varepsilon}) \geq \limsup_{N \rightarrow \infty} \rho_N(K + B_{\alpha\varepsilon}) = \rho(K + B_{\alpha\varepsilon}) \geq \rho(K).$$

Using that $\Phi(t)(K)$ is closed and letting ε to zero, the dominated convergence theorem implies that $\rho(\Phi(\tau)(K)) \geq \rho(K)$. This proves the measure invariance. The proof of Theorem 2.4 is therefore completed.

10. Proof of Theorem 1.2

We are looking for a (global in time) solution $v(s, y)$ of the system

$$(10.1) \quad i\partial_s v + \partial_y^2 v = |v|^{k-1}v, \quad s \in \mathbb{R}, \quad y \in \mathbb{R}, \quad v|_{t=0} = u_0.$$

We define $u(t, x)$ for $|t| < \frac{\pi}{4}$, $x \in \mathbb{R}$ by

$$(10.2) \quad u(t, x) = \frac{1}{\cos^{\frac{1}{2}}(2t)} v\left(\frac{\tan(2t)}{2}, \frac{x}{\cos(2t)}\right) e^{-\frac{ix^2 \tan(2t)}{2}}.$$

We then can check that v is a solution on \mathbb{R}_t of (10.1) if and only if u is a solution on $(-\frac{\pi}{4}, \frac{\pi}{4})_s$ of the system

$$(10.3) \quad i\partial_t u - Hu = \cos^{\frac{k-5}{2}}(2t)|u|^{k-1}u, \quad |t| < \frac{\pi}{4}, \quad x \in \mathbb{R}, \quad u|_{t=0} = u_0.$$

One also has that the map (10.2) sends solutions of the linear Schrödinger equation without harmonic potential to solutions of the linear Schrödinger equation with harmonic potential. We refer to [12] for a use of (10.2) in the context of scattering for L^2 critical problems, *i.e.* quintic nonlinearities in 1d. The problem (10.3) has also the following Duhamel formulation

$$(10.4) \quad u(t) = e^{-i(t-t_0)H}(u(t_0)) - i \int_{t_0}^t e^{-i(t-\tau)H} (\cos^{\frac{k-5}{2}}(2\tau)|u(\tau)|^{k-1}u(\tau))d\tau$$

with $t_0, t \in (-\frac{\pi}{4}, \frac{\pi}{4})$. The local analysis of (10.3) will be applied to (10.4) which fits well in the framework of Propositions 7.10, 7.11. By the transformation (10.2) we may link the solutions of (10.1) on $\mathbb{R} \times \mathbb{R}$ to the solutions of (10.3) on $(-\pi/4, \pi/4) \times \mathbb{R}$.

Remark 10.1. — Notice that the lens transform reduced the proof of global existence for the non linear Schrödinger equation without potential to the proof of a local existence result ($s \in (-\frac{\pi}{4}, \frac{\pi}{4})$) for an harmonic non linear Schrödinger equation. However, this fact that we are only interested in proving existence on a bounded time interval do *not* lead to any substantial simplification in the proof: indeed, in the previous section, we had an invariant measure, ρ . As a consequence, any local well posedness on any set of ρ measure 1, S (with time existence T independent of the initial data), implies automatically global existence on the set

$$\tilde{S} = \cap_{k \in \mathbb{Z}} \Phi(T)^{-k}(S),$$

which, of course has also ρ measure 1. As a consequence, in our framework, local and global existence results are not essentially different, as long as the time existence is uniform.

Let us now remark that thanks to (10.2) the \mathcal{H}^s convergence in the context of (10.3) implies the \mathcal{H}^s convergence for the original problem (10.1), as shown in the following Lemma.

Lemma 10.2. — *Let u and v be related together by the relation*

$$u(x) = \frac{1}{\cos^{1/2}(\tau)} v\left(\frac{x}{\cos(\tau)}\right) e^{-\frac{ix^2 \tan(\tau)}{2}}.$$

Then for any $\tau_0 > 0$ there exists $C > 0$ such that for any $s \in [0, 1]$ and any $\tau_0 \leq \tau \leq \frac{\pi}{2}$,

$$\|v\|_{\mathcal{H}^s} \leq C \|u\|_{\mathcal{H}^s}.$$

Proof. — For $s = 0$, the inequality above is an equality with $C = 1$. As a consequence, it suffices to prove the estimate for $s = 1$, the general case follows by interpolation.

$$(10.5) \quad \|u\|_{\mathcal{H}^1}^2 = \int \frac{1}{\cos^3(\tau)} |\partial_x v|^2 \left(\frac{x}{\cos(\tau)}\right) + \frac{1}{\cos(\tau)} \tan^2(\tau) x^2 |v|^2 \left(\frac{x}{\cos(\tau)}\right) \\ + \frac{1}{\cos(\tau)} x^2 |v|^2 \left(\frac{x}{\cos(\tau)}\right) + 2 \operatorname{Re} \frac{i}{\cos^2(\tau)} (\partial_x v) \left(\frac{x}{\cos(\tau)}\right) \times ix \tan(\tau) \bar{v} \left(\frac{x}{\cos(\tau)}\right) dx$$

The three first terms in the equation above sum up to

$$\int \frac{1}{\cos^2(\tau)} |\partial_x v(x)|^2 + x^2 |v|^2(x) dx,$$

while the last term is bounded (in absolute value) using Cauchy Schwarz by

$$\int \frac{1}{\cos^2(\tau/2)} |\partial_x v(x)|^2 + \cos^2(\tau/2) x^2 |v|^2(x) dx.$$

□

The results of Theorem 1.2 will therefore be a consequence of the following result and the fact that the lens transform (10.2) maps the solution to the linear Harmonic Schrödinger equation, $e^{-itH}(f^\pm)$ to the solution to the linear Schrödinger equation $e^{it\partial_y^2}(f^\pm)$.

Proposition 10.3. — *The equation (10.3) has μ almost surely a unique solution in $C([-\frac{\pi}{4}, \frac{\pi}{4}]; Y^s)$. Moreover for any $0 < s < \frac{1}{2}$, we can write the solution as*

$$u(t) = e^{-itH}(u|_{t=0} + f^\pm) + w^\pm(t),$$

with $f^\pm \in Y^s$ and where w^\pm are such that

$$\lim_{t \rightarrow \pm\pi/4} \|w^\pm(t)\|_{\mathcal{H}^s} = 0.$$

Proof of Proposition 10.3. — The proof of this proposition is very similar in spirit to the proof of Theorem 2.4. The local analysis is essentially the same, if one is interested only in proving that the solution exists, with a time existence depending on the initial data (see Proposition 10.6). There is however a nontrivial modification in the globalization arguments (i.e. in the proof of the fact that the time interval of existence is $[-\frac{\pi}{4}, \frac{\pi}{4}]$) because of the lack of energy conservation of (10.3), which implies that there is no more any invariant measure. We consider the ODE

$$(10.6) \quad i\partial_t u - Hu = \cos^{\frac{k-5}{2}}(2t) S_N(|S_N u|^{k-1} S_N u), \quad u(0) \in E_N.$$

One may multiply (10.6) by \bar{u} and integrate over \mathbb{R} to obtain that the L^2 norm is conserved by the flow and combining this fact with the local existence theory of ODE's, we obtain that the ODE (10.6) with phase space E_N has a unique global in time solution. For two real numbers t_1, t_2 let us denote by $\tilde{\Phi}_N(t_1, t_2)$ the flow of (10.6) from t_1 to t_2 . We have the following monotonicity property for the solutions of (10.6).

Lemma 10.4. — *Set*

$$\mathcal{E}_N(t, u(t)) = \frac{1}{2} \|\sqrt{H} u(t)\|_{L^2(\mathbb{R})}^2 + \frac{\cos^{\frac{k-5}{2}}(2t)}{k+1} \|S_N u(t)\|_{L^{k+1}(\mathbb{R})}^{k+1}.$$

Then the solution of (10.6) satisfies

$$\mathcal{E}_N(t, u(t)) \leq \mathcal{E}_N(0, u(0)), \quad |t| \leq \frac{\pi}{4}.$$

Proof. — A direct computation shows that along the flow of (10.6) one has

$$\frac{d}{dt}(\mathcal{E}_N(t, u(t))) = -\frac{(k-5)\sin(2t)\cos^{\frac{k-5}{2}}(2t)}{k+1}\|S_N u(t)\|_{L^{k+1}(\mathbb{R})}^{k+1}.$$

Therefore the function $\mathcal{E}_N(t, u(t))$ increases on the interval $[-\pi/4, 0]$ and decreases on the interval $[0, \pi/4]$, and attains its maximum at 0. This completes the proof of Lemma 10.4. \square

We shall prove that (10.3) is well-posed on $[-\pi/4, \pi/4]$ ρ -almost surely which in turn will imply the claimed well-posedness μ . The result of Lemma 10.4 implies the following key measure monotonicity property, which is our substitute for the fact that we do not have an invariant measure any more.

Lemma 10.5. — *For every Borel set A of E_N and every $|t| \leq \frac{\pi}{4}$,*

$$\tilde{\mu}_N(\tilde{\Phi}_N(t, 0)(A)) \geq \tilde{\rho}_N(A).$$

Proof. — By definition

$$\tilde{\mu}_N(\tilde{\Phi}_N(t, 0)(A)) = d_N \int_{\tilde{\Phi}_N(t, 0)(A)} e^{-\frac{1}{2}\|\sqrt{H}u\|_{L^2(\mathbb{R})}^2} du,$$

where du is the Lebesgue measure on E_N induced by $\mathbb{C}^{(N+1)}$ by the map (2.5). Let us perform the variable change $u \mapsto \tilde{\Phi}_N(t, 0)(u)$. We can apply the result of Lemma 8.3 to obtain that the Jacobian of this variable change is one (the divergence free assumption can be readily checked by expressing $\tilde{\Phi}_N(t, 0)(u)$ in terms of its decomposition with respect to h_0, \dots, h_N). Thus we get

$$\begin{aligned} \tilde{\mu}_N(\tilde{\Phi}_N(t, 0)(A)) &= d_N \int_A e^{-\frac{1}{2}\|\sqrt{H}\tilde{\Phi}_N(t, 0)(u)\|_{L^2(\mathbb{R})}^2} du \\ &\geq d_N \int_A e^{-\frac{1}{2}\|\sqrt{H}\tilde{\Phi}_N(t, 0)(u)\|_{L^2(\mathbb{R})}^2 - \frac{\cos^{\frac{k-5}{2}}(2t)}{k+1}\|S_N \tilde{\Phi}_N(t, 0)(u)\|_{L^{k+1}(\mathbb{R})}^{k+1}} du. \end{aligned}$$

Using Lemma 10.4 we hence obtain

$$\tilde{\mu}_N(\tilde{\Phi}_N(t, 0)(A)) \geq d_N \int_A e^{-\frac{1}{2}\|\sqrt{H}u\|_{L^2(\mathbb{R})}^2 - \frac{1}{k+1}\|S_N u\|_{L^{k+1}(\mathbb{R})}^{k+1}} du = \tilde{\rho}_N(A).$$

This completes the proof of Lemma 10.5. \square

For I an interval, we can define the spaces X_I^s similarly to the spaces X_T^s by replacing $[-T, T]$ by I . We have the following well-posedness result concerning (10.6).

Proposition 10.6. — *There exist $C > 0$, $c \in (0, 1)$, $\gamma > 0$, $\kappa > 0$ such that for every $A \geq 1$ and for every $N \geq 1$, every $t_0 \in [-\frac{\pi}{4}, \frac{\pi}{4}]$, every $u_0 \in E_N$ satisfying $\|u_0\|_{Y^s} \leq A$ there exists a unique solution of (10.6) with data $u(t_0) = u_0$ on the interval $I = [t_0 - cA^{-\gamma}, t_0 + cA^{-\gamma}]$ such that $\|u\|_{X_I^s} \leq A + A^{-1}$. In addition for $t \in I$,*

$$(10.7) \quad \|u(t)\|_{Y^s} \leq A + A^{-1}.$$

Moreover, if u and v are two solutions with data u_0 and v_0 respectively, satisfying $\|u_0\|_{Y^s} \leq A$, $\|v_0\|_{Y^s} \leq A$ then $\|u - v\|_{X_I^s} \leq C\|u_0 - v_0\|_{Y^s}$ and for $t \in I$,

$$\|u(t) - v(t)\|_{Y^s} \leq C\|u_0 - v_0\|_{Y^s}.$$

Finally, if $J \subset I$ is an interval, then for $\eta > 0$,

$$(10.8) \quad \left\| \int_J e^{-i(t-\tau)H} \left(\cos^{\frac{k-5}{2}}(2\tau) |u(\tau)|^{k-1} u(\tau) \right) d\tau \right\|_{\mathcal{H}^{s-\eta}} \leq C |J|^\kappa A.$$

Proof. — The proof of this statement is completely analogous to that of Proposition 7.10, one needs to observe that in Lemma 7.2 and Proposition 7.3 one may replace $[-T, T]$ by an arbitrary interval, T by the size of this interval and one may add the factor $\cos^{\frac{k-5}{2}}(2\tau)$ with the same conclusion. The only additional point is the estimate (10.8). To prove estimates (10.8), we use that

$$\left\| \int_J e^{-i(t-\tau)H} \left(\cos^{\frac{k-5}{2}}(2\tau) F(\tau) \right) d\tau \right\|_{\mathcal{H}^{s-\eta}} \leq C \|F\|_{L^1_J \mathcal{H}^{s-\eta}}$$

and apply the estimates of Proposition 7.3. \square

The rest of the proof of Proposition 10.3 is very similar to the existence part of Theorem 2.4. We start by the counterpart of Proposition 8.5.

Proposition 10.7. — *There exists a constant $C > 0$ such that for all $m, N \in \mathbb{N}^*$, there exists a ρ_N measurable set $\tilde{\Sigma}_N^m \subset E_N$ so that for all $m, N \in \mathbb{N}^*$*

$$\tilde{\rho}_N(E_N \setminus \tilde{\Sigma}_N^m) \leq 2^{-m}.$$

For all $f \in \tilde{\Sigma}_N^m$ and $t \in [-\frac{\pi}{4}, \frac{\pi}{4}]$

$$\|\tilde{\Phi}_N(t, 0)f\|_{Y^s} \leq Cm^{\frac{1}{2}}.$$

Proof. — We set, for m an integer ≥ 1 , $\tilde{B}_N^m(D) \equiv \{u \in E_N : \|u\|_{Y^s} \leq Dm^{\frac{1}{2}}\}$, where the number $D \gg 1$ (independent of m, N) will be fixed later. Thanks to Proposition 10.6, there exist $c > 0, \gamma > 0$ only depending on s such that if we set $\tau \equiv cD^{-\gamma}m^{-\gamma/2}$ then for every t_1, t_2 , such that $|t_1 - t_2| \leq \tau$,

$$(10.9) \quad \tilde{\Phi}_N(t_1, t_2)(\tilde{B}_N^m(D)) \subset \{u \in E_N : \|u\|_{Y^s} \leq D(m+1)^{\frac{1}{2}}\},$$

provided $D \gg 1$, independently of m . Set

$$\tilde{\Sigma}_N^m(D) \equiv \bigcap_{k=-[\pi/4\tau]}^{[\pi/4\tau]} \tilde{\Phi}_N(k\tau, 0)^{-1}(\tilde{B}_N^m(D)).$$

Notice that thanks to (10.9), we obtain that the solution of (10.3) with data $f \in \tilde{\Sigma}_N^m(D)$ satisfies

$$(10.10) \quad \|\tilde{\Phi}_N(t, 0)(f)\|_{Y^s} \leq D(m+1)^{\frac{1}{2}}, \quad |t| \leq \frac{\pi}{4}.$$

Indeed, for $|t| \leq \frac{\pi}{4}$, we can find an integer $k \in [-[\pi/4\tau], [\pi/4\tau]]$ and $\tau_1 \in [-\tau, \tau]$ so that $t = k\tau + \tau_1$ and thus

$$\tilde{\Phi}_N(t, 0)(f) = \tilde{\Phi}_N(t, k\tau)\tilde{\Phi}_N(k\tau, 0)(f).$$

Since $f \in \tilde{\Sigma}_N^m(D)$ implies that $\tilde{\Phi}_N(k\tau, 0)(f) \in \tilde{B}_N^m(D)$, we can apply (10.9) and arrive at (10.10). It remains to evaluate the $\tilde{\rho}_N$ complementary measure of the set $\tilde{\Sigma}_N^m(D)$. Using Lemma 10.5, we can write

$$\begin{aligned} \tilde{\rho}_N(E_N \setminus \tilde{\Sigma}_N^m(D)) &\leq (2[\pi/4\tau] + 1)\rho_N(\tilde{\Phi}_N(k\tau, 0)^{-1}(E_N \setminus \tilde{B}_N^m(D))) \\ &\leq CD^\gamma m^{\gamma/2} \tilde{\mu}_N(E_N \setminus \tilde{B}_N^m(D)). \end{aligned}$$

By the large deviation bounds of Lemma 7.1, we get

$$\tilde{\rho}_N(E_N \setminus \tilde{\Sigma}_N^m(D)) \leq CD^\gamma m^{\gamma/2} e^{-cD^2 m} \leq 2^{-m},$$

provided $D \gg 1$, independently of m, N . This completes the proof of Proposition 10.7. \square

Since we are only concerned with a well-posedness statement, we need to prove less compared with Theorem 2.4 (we do not need to prove that the statistical ensemble is a set reproduced by the flow). For integers $m \geq 1$ and $N \geq 1$, we define the cylindrical sets

$$\Sigma_N^m \equiv \{u \in Y^s : \Pi_N(u) \in \tilde{\Sigma}_N^m\}.$$

For $m \geq 1$, we set

$$\Sigma^m = \{u \in Y^s : \exists N_k \in \mathbb{N}, N_k \rightarrow +\infty, \exists u_{N_k} \in \Sigma_{N_k}^m, S_{N_k}(u_{N_k}) \rightarrow u \text{ in } Y^s\}.$$

As in the proof of Theorem 2.4, we obtain the bound

$$(10.11) \quad \rho(\Sigma^m) \geq \rho(Y^s) - 2^{-m}.$$

Next, we set

$$(10.12) \quad \Sigma \equiv \bigcup_{i=1}^{\infty} \Sigma^m.$$

and by (10.11), the set Σ is of full ρ measure. We now state a proposition yielding the existence part of Proposition 10.3.

Proposition 10.8. — *For every integer $m \geq 1$, every $f \in \Sigma^m$, the problem (10.3) with initial condition f has a unique solution in $C([- \frac{\pi}{4}, \frac{\pi}{4}]; Y^s)$.*

The proof of Proposition 10.8 is very similar (simpler) to that of Proposition 8.6, by invoking the counterpart of the approximation statement of Lemma 8.7. This implies the existence part of Proposition 10.3. Namely we proved the well-posedness for data in Σ (defined by (10.12)) and since Σ is of full ρ measure it is of full μ measure too.

To prove the last statement of Proposition 10.3, we write the obtained solution as

$$\begin{aligned} u(t) = e^{-itH} & \left(u(0) - 2i \int_0^{\pi/4} e^{i\tau H} \left(\cos^{\frac{k-5}{2}}(2\tau) |u(\tau)|^{k-1} u(\tau) \right) d\tau \right) \\ & + 2i \int_t^{\pi/4} e^{-i(t-\tau)H} \left(\cos^{\frac{k-5}{2}}(2\tau) |u(\tau)|^{k-1} u(\tau) \right) d\tau \end{aligned}$$

and we apply estimate (10.8). A similar argument applies near $-\pi/4$. This completes the proof of Proposition 10.3, and hence of Theorem 1.2. \square

A

Typical properties on the support of the measure

In this section, we give some additional properties of the stochastic series

$$\varphi(\omega, x) = \sum_{n=0}^{\infty} \frac{\sqrt{2}}{\lambda_n} g_n(\omega) h_n(x).$$

A.1. Mean and pointwise properties. —

Proposition A.1 (L^p regularisation). — Let $2 \leq p < +\infty$ and denote by

$$\theta(p) = \begin{cases} \frac{1}{2} - \frac{1}{p} & \text{if } 2 \leq p \leq 4, \\ \frac{1}{3}(\frac{1}{2} + \frac{1}{p}) & \text{if } 4 \leq p < \infty. \end{cases}$$

Then for all $s < \theta(p)$, there exist $C, c > 0$ so that

$$\mathbf{p}(\omega \in \Omega : \|\varphi(\omega, \cdot)\|_{\mathcal{W}^{s,p}(\mathbb{R})} > \lambda) \leq C e^{-c\lambda^2}.$$

In particular $\|\varphi(\omega, \cdot)\|_{\mathcal{W}^{s,p}(\mathbb{R})} < +\infty$, \mathbf{p} a.s.

Proof. — The proof is essentially the same as the proof of Lemma 3.3, using the precise L^p bounds on the Hermite functions h_n (see [40] or [35, Theorem 2.1]). \square

Corollary A.2 (Decay). — Let $\alpha < \frac{1}{6}$. Then there exist $C, c > 0$ so that for all $x \in \mathbb{R}$

$$\mathbf{p}(\omega \in \Omega : |\varphi(\omega, x)| > \frac{\lambda}{\langle x \rangle^\alpha}) \leq C e^{-c\lambda^2}.$$

In particular, for almost all $\omega \in \Omega$,

$$\varphi(\omega, x) \longrightarrow 0 \quad \text{when } x \longrightarrow \pm\infty.$$

Proof. — Let $\alpha < \frac{1}{6}$. Then choose $s > 0$ so that $s + \alpha < \frac{1}{6}$ and $p \geq 4$ so that $s > \frac{1}{p}$. Then by Sobolev, there exists $C > 0$ so that for all $\omega \in \Omega$

$$\|\langle x \rangle^\alpha \varphi(\omega, \cdot)\|_{L^\infty(\mathbb{R})} \leq C \|\langle x \rangle^\alpha \varphi(\omega, \cdot)\|_{\mathcal{W}^{s,p}(\mathbb{R})}.$$

Now by [41, Lemma 2.4],

$$\|\langle x \rangle^\alpha \varphi(\omega, \cdot)\|_{\mathcal{W}^{s,p}(\mathbb{R})} \leq C \|\varphi(\omega, \cdot)\|_{\mathcal{W}^{s+\alpha,p}(\mathbb{R})},$$

thus

$$\{\omega \in \Omega : \langle x \rangle^\alpha |\varphi(\omega, x)| > \lambda\} \subset \{\omega \in \Omega : \|\varphi(\omega, \cdot)\|_{\mathcal{W}^{s+\alpha,p}(\mathbb{R})} > \frac{\lambda}{C}\}.$$

and we can conclude with the Proposition A.1, as $s + \alpha < \theta(p)$. \square

Proposition A.3 (Hölderian regularity). — Let $\alpha < \frac{1}{6}$. There exist $C, c > 0$ so that for all $x, y \in \mathbb{R}$

$$\mathbf{p}(\omega \in \Omega : |\varphi(\omega, x) - \varphi(\omega, y)| > \lambda |x - y|^\alpha) \leq C e^{-c\lambda^2}.$$

In particular, for almost all $\omega \in \Omega$, the function $x \mapsto \varphi(\omega, x)$ is α -Hölderian on \mathbb{R} .

Proof. — By Lemma 3.2, for all $x, y \in \mathbb{R}$ we have

$$|h_n(x) - h_n(y)| \leq C\lambda_n^{-\frac{1}{6}}.$$

By Lemma 3.2 again, we also have the bound (see (7.10))

$$|h_n(x) - h_n(y)| \leq \|h_n\|_{\mathcal{W}^{1,\infty}(\mathbb{R})} |x - y| \leq C\lambda_n^{\frac{5}{6}} |x - y|,$$

and we can deduce by interpolation that for all $0 \leq \alpha \leq 1$,

$$|h_n(x) - h_n(y)| \leq C\lambda_n^{\alpha - \frac{1}{6}} |x - y|^\alpha.$$

Now, by Lemma 3.1, for all $r \geq 2$

$$\begin{aligned} \|\varphi(\omega, x) - \varphi(\omega, y)\|_{L^r(\Omega)} &\leq C\sqrt{r} \left(\sum_{n=0}^{\infty} \frac{1}{\lambda_n^2} |h_n(x) - h_n(y)|^2 \right)^{\frac{1}{2}} \\ &\leq C\sqrt{r} |x - y|^\alpha \left(\sum_{n=0}^{\infty} \frac{1}{\lambda_n^{2(1-\alpha+\frac{1}{6})}} \right)^{\frac{1}{2}} \\ &\leq C\sqrt{r} |x - y|^\alpha, \end{aligned}$$

for all $0 \leq \alpha < \frac{1}{6}$. We conclude with the Tchebychev inequality. \square

The Proposition A.1 does not yield a gain of derivatives in \mathcal{H}^s spaces for the random variables $(g_n(\omega))_{n \geq 0}$ (and it can actually be shown that no such gain is indeed true, see [8, Appendix B]), however we can prove a local gain of regularity.

Proposition A.4 (Local smoothing). — *Let $\nu > 0$ and define $\Psi(x) = \langle x \rangle^{-\frac{1}{2}-\nu}$. Then for all $s < \frac{1}{2}$, there exist $C, c > 0$ so that*

$$\mathbf{p}(\omega \in \Omega : \|\Psi \varphi(\omega, \cdot)\|_{\mathcal{H}^s(\mathbb{R})} > \lambda) \leq C e^{-c\lambda^2}.$$

In particular $\|\Psi \varphi(\omega, \cdot)\|_{\mathcal{H}^s(\mathbb{R})} < +\infty$, \mathbf{p} a.s.

Proof. — By [36, Corollary 1.2] the following bound holds

$$\|\Psi h_n\|_{L^2(\mathbb{R})} \leq \lambda_n^{-\frac{1}{2}}.$$

Then we can perform the same argument as in the proof of Lemma 3.3. \square

A.2. Spatial decorrelation. — Define the function E for $(x, y, \alpha) \in \mathbb{R} \times \mathbb{R} \times [0, 1[$ by

$$(A.1) \quad E(x, y, \alpha) = \sum_{n \geq 0} \alpha^n h_n(x) h_n(y).$$

Then we have an explicit formula for E .

Lemma A.5. — *For all $(x, y, \alpha) \in \mathbb{R} \times \mathbb{R} \times [0, 1[$*

$$(A.2) \quad E(x, y, \alpha) = \frac{1}{\sqrt{\pi(1-\alpha^2)}} \exp\left(-\frac{1-\alpha}{1+\alpha} \frac{(x+y)^2}{4} - \frac{1+\alpha}{1-\alpha} \frac{(x-y)^2}{4}\right).$$

Remark A.6. — Notice that by taking $\alpha = e^{2it}$, one can see that Lemma A.5 is equivalent to Mehler formula (4.1), which in turn implies that the function defined by (10.2) satisfies (10.3). Actually, one could probably extend Lemma A.5 to more general potential (with quadratic growth) by precisely writing down a parametrix for $e^{it(-\partial_x^2 + V(x))}$, or for the heat kernel $e^{-t(-\partial_x^2 + V(x))}$.

Proof. — First we recall that the Fourier transform of the Gaussian reads

$$(A.3) \quad e^{-\sigma^2 x^2} = \frac{1}{2\sigma\sqrt{\pi}} \int_{\mathbb{R}} e^{ix\xi - \frac{\xi^2}{4\sigma^2}} d\xi,$$

thus, for all $n \geq 1$,

$$(A.4) \quad \frac{d^n}{dx^n} (e^{-x^2}) = \frac{1}{2\sqrt{\pi}} \int_{\mathbb{R}} (i\xi)^n e^{ix\xi - \xi^2/4} d\xi.$$

With (2.1) and (A.4), we deduce that

$$\begin{aligned} E(x, y, \alpha) &= \frac{1}{4\pi^{3/2}} e^{(x^2+y^2)/2} \sum_{n \geq 0} \frac{\alpha^n}{2^n n!} \int_{\mathbb{R}} (i\xi)^n e^{ix\xi - \xi^2/4} d\xi \int_{\mathbb{R}} (i\eta)^n e^{iy\eta - \eta^2/4} d\eta \\ &= \frac{1}{4\pi^{3/2}} e^{(x^2+y^2)/2} \int_{\mathbb{R}^2} \sum_{n \geq 0} \frac{1}{n!} \left(-\frac{\alpha\xi\eta}{2}\right)^n e^{i(x\xi+y\eta) - \xi^2/4 - \eta^2/4} d\xi d\eta \\ &= \frac{1}{4\pi^{3/2}} e^{(x^2+y^2)/2} \int_{\mathbb{R}^2} e^{-\alpha\xi\eta/2 + ix(\xi+\eta) - \xi^2/4 - \eta^2/4} d\xi d\eta. \end{aligned}$$

To compute the last integral, we make the change of variables $(\xi', \eta') = \frac{1}{\sqrt{2}}(\xi + \eta, \xi - \eta)$ and use (A.3). This completes the proof. \square

Proposition A.7 (Spatial decorrelation). — *There exists $C > 0$ so that for all $x, y \in \mathbb{R}$,*

$$(A.5) \quad |\mathbb{E}[\varphi(x, \omega) \overline{\varphi(y, \omega)}]| \leq C e^{-\frac{(x-y)^2}{4}}.$$

Proof. — Consider the function F defined by

$$F(x, y, \alpha) = 2 \sum_{n \geq 0} \frac{\alpha^{2n+1}}{\lambda_n^2} h_n(x) h_n(y) = 2 \sum_{n \geq 0} \frac{\alpha^{2n+1}}{2n+1} h_n(x) h_n(y),$$

for $(x, y, \alpha) \in \mathbb{R} \times \mathbb{R} \times [0, 1]$. Thanks to the bound (3.2) we have

$$\left| \frac{\alpha^{2n+1}}{\lambda_n^2} h_n(x) h_n(y) \right| \leq C \frac{1}{\langle n \rangle} \|h_n\|_{L^\infty(\mathbb{R})}^2 \leq C \frac{1}{\langle n \rangle^{1+\frac{1}{6}}},$$

hence $F \in \mathcal{C}(\mathbb{R} \times \mathbb{R} \times [0, 1]; \mathbb{R})$. Therefore,

$$(A.6) \quad F(x, y, \alpha) \longrightarrow \sum_{n \geq 0} \frac{2}{\lambda_n^2} h_n(x) h_n(y) = \mathbb{E}[\varphi(x, \omega) \overline{\varphi(y, \omega)}],$$

when $\alpha \longrightarrow 1$.

Now observe that F is smooth in $\alpha \in [0, 1[$. Thus (as $F(x, y, 0) = 0$)

$$(A.7) \quad F(x, y, \alpha) = \int_0^\alpha \partial_\alpha F(x, y, \beta) d\beta.$$

By (A.2) we have

$$\begin{aligned}\partial_\alpha F(x, y, \beta) &= 2 \sum_{n \geq 0} \beta^{2n} h_n(x) h_n(y) \\ &= \frac{2}{\sqrt{\pi(1-\beta^4)}} \exp\left(-\frac{1-\beta^2}{1+\beta^2} \frac{(x+y)^2}{4} - \frac{1+\beta^2}{1-\beta^2} \frac{(x-y)^2}{4}\right).\end{aligned}$$

Hence there exists $C > 0$ so that for all $x, y \in \mathbb{R}$ and $\beta \in [0, 1[$

$$|\partial_\alpha F(x, y, \beta)| \leq \frac{C}{\sqrt{1-\beta}} e^{-\frac{(x-y)^2}{4}},$$

and this, together with (A.6) and (A.7) yields the estimate (A.5). \square

A.3. Bilinear estimates. — In this section we give a proof of (1.2). Observe that (1.2), applied with $t = 0$ implies that $\varphi^2(\omega, x)$ is a.s. in \mathcal{H}^θ for every $\theta < 1/2$ which is a remarkable smoothing property satisfied by the random series $\varphi(\omega, x)$. The key point in the proof of (1.2) is the following bilinear estimate for Hermite functions.

Lemma A.8. — *There exists $C > 0$ so that for all $0 \leq \theta \leq 1$ and $n, m \in \mathbb{N}$*

$$(A.8) \quad \|h_n h_m\|_{\mathcal{H}^\theta(\mathbb{R})} \leq C \max(n, m)^{-\frac{1}{4} + \frac{\theta}{2}} (\log(\min(n, m) + 1))^{\frac{1}{2}}.$$

Proof. — We give an argument we learned from Patrick Gérard. It suffices to prove (A.8) for $\theta = 0$ and $\theta = 1$ (the general case then follows by interpolation). The case $\theta = 1$ can be directly reduced to the case $\theta = 0$ thanks to (7.10). Let us now give the proof of (A.8) in the case $\theta = 0$. Consider again the function E defined by (A.1) which can also be expressed by (A.2). Let $0 \leq \alpha, \beta < 1$ and $x \in \mathbb{R}$. By (A.2) we have

$$E(x, x, \alpha) = \frac{1}{\sqrt{\pi}} (1 - \alpha^2)^{-\frac{1}{2}} e^{-\frac{1-\alpha}{1+\alpha} x^2}.$$

Therefore, if we set

$$I(\alpha, \beta) \equiv \int_{\mathbb{R}} E(x, x, \alpha) E(x, x, \beta) dx,$$

then we get

$$\begin{aligned}(A.9) \quad I(\alpha, \beta) &= \frac{1}{\pi} (1 - \alpha^2)^{-\frac{1}{2}} (1 - \beta^2)^{-\frac{1}{2}} \int_{\mathbb{R}} e^{-\left(\frac{1-\alpha}{1+\alpha} + \frac{1-\beta}{1+\beta}\right) x^2} dx \\ &= \frac{1}{\sqrt{2\pi}} (1 - \alpha)^{-\frac{1}{2}} (1 - \beta)^{-\frac{1}{2}} (1 - \alpha\beta)^{-\frac{1}{2}}.\end{aligned}$$

On the other hand, coming back to the definition

$$I(\alpha, \beta) = \sum_{n, m \geq 0} \alpha^n \beta^m \int_{\mathbb{R}} h_n^2(x) h_m^2(x) dx.$$

Hence to get a useful expression for the L^2 norm of the product of two Hermite functions, it suffices to expand (A.9) in entire series in α and β . Write

$$(1-x)^{-\frac{1}{2}} = \sum_{p \geq 0} c_p x^p, \quad c_0 = 1, \quad c_p = \frac{(2p-1)!}{2^{2p-1} p! (p-1)!}, \quad p \geq 1.$$

Therefore, by the Stirling formula, there exists $C > 0$ so that $|c_p| \leq \frac{C}{\sqrt{p+1}}$ for all $p \geq 0$. Now by (A.9) and the previous estimate

$$\begin{aligned} \int_{\mathbb{R}} h_n^2(x) h_m^2(x) dx &= \frac{1}{\sqrt{2\pi}} \sum_{\substack{p,q,r \geq 0 \\ p+r=n, q+r=m}} c_p c_q c_r \\ &\leq C \sum_{0 \leq r \leq \min(n,m)} (n-r+1)^{-\frac{1}{2}} (m-r+1)^{-\frac{1}{2}} (r+1)^{-\frac{1}{2}}. \end{aligned}$$

Without restricting the generality we may suppose that $m \geq n$. If $m \leq 2n$ then we obtain the needed bound by considering separately the cases when the sum runs over $r < m/2$ and $r \geq m/2$. If $m > 2n$, then we can write $(m-r+1)^{-\frac{1}{2}} \leq c(1+m)^{-\frac{1}{2}}$ and the needed bound follows directly. Therefore we get (A.8) in the case $\theta = 0$. This completes the proof of Lemma A.8. \square

Denote by $u(\omega, t, x)$ the free Schrödinger solution with initial condition $\varphi(\omega, x)$, *i.e.*

$$u(\omega, t, x) = e^{-itH} \varphi(x, \omega) = \sum_{n \geq 0} \frac{\sqrt{2}}{\lambda_n} e^{-it\lambda_n^2} g_n(\omega) h_n(x).$$

Write the decomposition $u = u_0 + \sum_N u_N$, where the summation is taken over the dyadic integers and for N a dyadic integer

$$u_N(\omega, t, x) = \sum_{N \leq n < 2N} \alpha_n(t) h_n(x) g_n(\omega), \quad \alpha_n(t) = \sqrt{\frac{2}{2n+1}} e^{-i(2n+1)t}.$$

Let us fix $t \in \mathbb{R}$ and $0 \leq \theta < \frac{1}{2}$. To prove (1.2), it suffices to show that the expression

$$J(t, x, \omega) \equiv \left| \sum_M \sum_N H^{\theta/2}(u_N u_M) \right|$$

belongs to $L^2(\mathbb{R} \times \Omega)$ (here the summation is again taken over the dyadic values of M, N). Using the Cauchy-Schwarz inequality, a symmetry argument and summing geometric series, for all $\varepsilon > 0$ we can write

$$(A.10) \quad J(t, x, \omega) \leq C \left(\sum_{N \leq M} M^\varepsilon |H^{\theta/2}(u_N u_M)|^2 \right)^{\frac{1}{2}}.$$

Coming back to the definition we can write

$$H^{\theta/2}(u_N u_M) = \sum_{\substack{N \leq n \leq 2N \\ M \leq m \leq 2M}} \alpha_n \alpha_m g_n g_m H^{\theta/2}(h_n h_m).$$

We now estimate $\|H^{\theta/2}(u_N u_M)\|_{L^2(\Omega)}$. We make the expansion

$$|H^{\theta/2}(u_N u_M)|^2 = \sum_{\substack{N \leq n_1, n_2 \leq 2N \\ M \leq m_1, m_2 \leq 2M}} \alpha_{n_1} \bar{\alpha}_{n_2} \alpha_{m_1} \bar{\alpha}_{m_2} g_{n_1} \bar{g}_{n_2} g_{m_1} \bar{g}_{m_2} H^{\theta/2}(h_{n_1} h_{m_1}) \overline{H^{\theta/2}(h_{n_2} h_{m_2})}.$$

The random variables g_n are centered and independent, and consequently, $\mathbb{E}[g_{n_1} \bar{g}_{n_2} g_{m_1} \bar{g}_{m_2}] = 0$, unless the indexes are pairwise equal (*i.e.* $(n_1 = n_2$ and $m_1 = m_2)$, or $(n_1 = m_2$ and $n_2 = m_1)$, or $(n_1 = m_1$ and $n_2 = m_2)$). This implies that

$$(A.11) \quad \int_{\Omega} |H^{\theta/2}(u_N u_M)|^2 \leq C \sum_{\substack{N \leq n \leq 2N \\ M \leq m \leq 2M}} |\alpha_n|^2 |\alpha_m|^2 |H^{\theta/2}(h_n h_m)|^2.$$

We integrate (A.11) in x and by (A.8) we deduce that for all $\varepsilon > 0$

$$\begin{aligned} \int_{\Omega \times \mathbb{R}} |H^{\theta/2}(u_N u_M)|^2 &\leq C \sum_{\substack{N \leq n \leq 2N \\ M \leq m \leq 2M}} |\alpha_n|^2 |\alpha_m|^2 \int_{\mathbb{R}} |H^{\theta/2}(h_n h_m)|^2 dx \\ &\leq C \sum_{\substack{N \leq n \leq 2N \\ M \leq m \leq 2M}} (\max(M, N))^{-\frac{1}{2} + \theta + \varepsilon} |\alpha_n|^2 |\alpha_m|^2. \end{aligned}$$

Therefore using that $|\alpha_n| \leq \langle n \rangle^{-\frac{1}{2}}$, we get

$$\begin{aligned} \int_{\Omega \times \mathbb{R}} (J(t, x, \omega))^2 &\leq C \sum_{N \leq M} \sum_{\substack{N \leq n \leq 2N \\ M \leq m \leq 2M}} M^{-\frac{1}{2} + \theta + 2\varepsilon} |\alpha_n|^2 |\alpha_m|^2 \\ &\leq C \sum_{N \leq M} \sum_{\substack{N \leq n \leq 2N \\ M \leq m \leq 2M}} M^{-\frac{1}{2} + \theta + 2\varepsilon} (MN)^{-1} < \infty, \end{aligned}$$

provided ε is small enough, namely ε such that $-\frac{1}{2} + \theta + 2\varepsilon < 0$. This completes the proof of (1.2).

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