# THE WKB METHOD AND GEOMETRIC INSTABILITY FOR NON LINEAR SCHRÖDINGER EQUATIONS ON SURFACES

*by* Laurent Thomann

ABSTRACT. — In this paper we are interested in constructing WKB approximations for the non linear cubic Schrödinger equation on a Riemannian surface which has a stable geodesic. These approximate solutions will lead to some instability properties of the equation.

RÉSUMÉ (*Méthode WKB et instabilité géométrique pour Schrödinger non linéaire sur des surfaces*)

À l'aide de la méthode WKB nous construisons des solutions approchées à l'équation de Schrödinger cubique sur une variété qui possède une géodésique stable. Cette construction permet d'obtenir des résultats d'instabilités dans des espaces de Sobolev.

## 1. Introduction

Let $(M, g)$ be a Riemannian surface (i.e. a Riemannian manifold of dimension 2), orientable or not. We assume that $M$ is either compact or a compact perturbation of the euclidian space, so that the Sobolev embeddings are true. Consider $\Delta = \Delta_g$ the Laplace-Beltrami operator. In this paper we are interested in constructing WKB approximations for the non linear cubic Schrödinger equation

$$(1) \qquad \begin{cases} i\partial_t u(t,x) + \Delta u(t,x) = \varepsilon |u|^2 u(t,x), \quad \varepsilon = \pm 1, \\ u(0,x) = u_0(x) \in H^\sigma(M), \end{cases}$$

LAURENT THOMANN, Université Paris-Sud, Mathématiques, Bât 425, Tel 0169155785, 91405 Orsay Cedex. • *E-mail :* laurent.thomann@math.u-psud.fr
*Url :* http://www.math.u-psud.fr/~thomann

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that is, given a small parameter $0 < h < 1$ and an integer $N$, functions $u_N(h)$ satisfying

$$i\partial_t u_N(h) + \Delta u_N(h) = \varepsilon |u_N(h)|^2 u_N(h) + R_N(h), \tag{2}$$

with $\|u_N(h)\|_{H^\sigma} \sim 1$ and $\|R_N(h)\|_{H^\sigma} \leq C_N h^N$.
Here $h$ is introduced so that $u_N(h)$ oscillates with frequency $\sim \frac{1}{h}$.
These approximate solutions to (1) will lead to some instability properties in the following sense (where $h^{-1}$ will play the role of $n$):

Definition 1.1. — We say that the Cauchy problem (1) is unstable near 0 in $H^\sigma(M)$, if for all $C > 0$ there exist times $t_n \longrightarrow 0$ and $u_{1,n}, u_{2,n} \in H^\sigma(M)$ solutions of (1) so that

$$\begin{aligned} \|u_{1,n}(0)\|_{H^\sigma(M)},\ \|u_{2,n}(0)\|_{H^\sigma(M)} &\leq C, \\ \|u_{1,n}(0) - u_{2,n}(0)\|_{H^\sigma(M)} &\longrightarrow 0, \\ \limsup \|u_{1,n}(t_n) - u_{2,n}(t_n)\|_{H^\sigma(M)} &\geq \frac{1}{2}C, \end{aligned}$$

when $n \longrightarrow +\infty$.

This means that the problem is not uniformly well-posed, if we refer to the following definition:

Definition 1.2. — Let $\sigma \in \mathbb{R}$. Denote by $B_{R,\sigma}$ the ball of radius $R$ in $H^\sigma$. We say that the Cauchy problem (1) is uniformly well-posed in $H^\sigma$ if the flow map

$$u_0 \in B_{R,\sigma} \cap H^1(M) \longmapsto \Phi_t(u_0) \in H^\sigma(M),$$

is uniformly continuous for any $t$.

We now state our instability result:

Proposition 1.3. — *Let $0 < \sigma < \frac{1}{4}$, and assume that $M$ has a stable and non degenerated periodic geodesic (see Assumptions 1 and 2 ), then the Cauchy problem* (1) *is not uniformly well-posed.*

This problem is motivated by the following results: Let $(M, g)$ be a riemannian compact surface, then in [5], N. Burq, P. Gérard and N. Tzvetkov prove that (1) is uniformly well-posed in $H^\sigma(M)$ for $\sigma > \frac{1}{2}$. Whereas, in [4], they show that (1) is unstable on the sphere $\mathbb{S}^2$ for $0 < \sigma < \frac{1}{4}$. In fact they construct solutions of (1) of the form

$$u_n^\kappa(t,x) = \kappa \mathrm{e}^{i\lambda_n^\kappa t}(n^{\frac{1}{4}-\sigma}\psi_n(x) + r_n(t,x)), \tag{3}$$

where $0 < \kappa < 1$, $\psi_n = (x_1 + ix_2)^n$ is a spherical harmonic which concentrates on the equator of the sphere when $n \longrightarrow +\infty$ and where $r_n$ is an error term which is small. To obtain instability, they consider $\kappa_n \longrightarrow \kappa$, then

$$\|u_n^\kappa(0) - u_n^{\kappa_n}(0)\|_{H^\sigma(\mathbb{S}^2)} \lesssim |\kappa - \kappa_n| \longrightarrow 0,$$

but

$$\|u_n^\kappa(t_n) - u_n^{\kappa_n}(t_n)\|_{H^\sigma(\mathbb{S}^2)} \gtrsim \kappa|\mathrm{e}^{i\lambda_n^\kappa t_n} - \mathrm{e}^{i\lambda_n^{\kappa_n} t_n}| \longrightarrow 2\kappa,$$

with a suitable choice of $t_n \longrightarrow 0$.
We follow this strategy but as the surface is not rotation invariant, the ansatz will be more complicated than (3).
This result is sharp, because in [6] they show that (1) is uniformly well-posed on $\mathbb{S}^2$ when $\sigma > \frac{1}{4}$.
On the other hand, in [3] J. Bourgain shows that (1) is uniformly well-posed on the rational torus $\mathbb{T}^2$ when $\sigma > 0$.
These results show how the geometry of $M$ can lead to instability for the equation (1). Therefore it seems reasonable to obtain a result like Proposition 1.3 with purely geometric assumptions.

We first make the following assumption on $M$:

Assumption 1. — *The manifold $M$ has a periodic geodesic.*

Denote by $\gamma$ such a geodesic, then there exists a system of coordinates $(s, r)$ near $\gamma$, say for $(s, r) \in \mathbb{S}^1\times ]-r_0, r_0[$, called Fermi coordinates such that (see [12], p. 80)

1. The curve $r = 0$ is the geodesic $\gamma$ parametrized by arclength and
2. The curves $s =$ constant are geodesics parametrized by arclength. The curves $r =$ constant meet these curves perpendicularly.
3. In this system the metric writes
$$g = \begin{pmatrix} 1 & 0 \\ 0 & a^2(s, r) \end{pmatrix}.$$

We set the length of $\gamma$ equal to $2\pi$. Denote by $R(s, r)$ the Gauss curvature at $(s, r)$, then $a$ is the unique solution of

$$\begin{cases} \dfrac{\partial^2 a}{\partial r^2} + R(s, r)a = 0, \\ a(s, 0) = 1, \ \dfrac{\partial a}{\partial r}(s, 0) = 0. \end{cases} \tag{4}$$

The initial conditions traduce the fact that the curve $r = 0$ is a unit-speed geodesic. In these coordinates the Laplace-Beltrami operator is

$$\Delta := \frac{1}{\sqrt{\mathrm{det} g}}\mathrm{div}(\sqrt{\mathrm{det} g}\; g^{-1}\nabla) = \frac{1}{a}\partial_s(\frac{1}{a}\partial_s) + \frac{1}{a}\partial_r(a\partial_r).$$

A function on $M$, defined locally near $\gamma$, can be identified with a function of $[0, 2\pi]\times ]-r_0, r_0[$ such that

$$\forall (s, r) \in [0, 2\pi]\times ]-r_0, r_0[ \quad f(s + 2\pi, r) = f(s, \omega r)$$

where $\omega = 1$ if $M$ is orientable and $\omega = -1$ if $M$ is not. Define

$$\omega_1 = \frac{1}{2}(\omega - 1) \in \{-1, 0\}. \tag{5}$$

From (4) we deduce that $a$ admits the Taylor expansion

$$a = 1 - \frac{1}{2}R(s)r^2 + R_3(s)r^3 + \cdots + R_p(s)r^p + o(r^p), \tag{6}$$

with $R(s) = R(s, 0)$ and

$$R_k(s) = \frac{1}{k!}\frac{\partial^k a}{\partial r^k}(s, 0), \tag{7}$$

for $k \geq 3$.
As $a(s + 2\pi, r) = a(s, \omega r)$, we deduce $R(s + 2\pi) = R(s)$ and for all $j \geq 3$, $R_j(s + 2\pi) = \omega^j R_j(s)$.
Let $p_2 = \frac{1}{a^2}\sigma^2 + \rho^2$ be the principal symbol of $\Delta$, and

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}s(t) = \dfrac{\partial p_2}{\partial \sigma} = \dfrac{2\sigma}{a^2}, \ \dfrac{\mathrm{d}}{\mathrm{d}t}\sigma(t) = -\dfrac{\partial p_2}{\partial s} = -\partial_s(\dfrac{1}{a^2})\sigma^2, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}r(t) = \dfrac{\partial p_2}{\partial \rho} = 2\rho, \ \dfrac{\mathrm{d}}{\mathrm{d}t}\rho(t) = -\dfrac{\partial p_2}{\partial r} = -\partial_r(\dfrac{1}{a^2})\sigma^2, \\ s(0) = s_0, \ \sigma(0) = \sigma_0, r(0) = r_0, \ \rho(0) = \rho_0, \end{cases} \tag{8}$$

its associated hamiltonian system, where $p_2 = p_2(s(t), r(t), \sigma(t), \rho(t))$. The system (8) admits a unique solution and defines the hamiltonian flow

$$\Phi_t : (s_0, \sigma_0, r_0, \rho_0) \longmapsto (s(t), \sigma(t), r(t), \rho(t)).$$

The curve $\Gamma = \{(s(t) = t, \sigma(t) = 1/2, r(t) = 0, \rho(t) = 0), t \in [0, 2\pi]\}$ is solution of (8) and its projection in the $(s, r)$ space is the curve $\gamma$. Now denote by $\phi$ the Poincaré map associated to the trajectory $\Gamma$ and to the hyperplane $\Sigma = \{s = 0\}$. There exists a neighborhood $\mathcal{N}$ of $(\sigma = 1/2, r = 0, \rho = 0)$ such that the following makes sense: solve the system (8) with the initial conditions $(0, \sigma_0, r_0, \rho_0) \in \{0\} \times \mathcal{N}$ and let $T$ be such that $s(T) = 2\pi$, then $\phi$ is the application

$$\phi : (r_0, \rho_0) \longmapsto (r(T), \rho(T)).$$

Moreover, the Poincaré map is continuously differentiable (see [13] p. 193). To obtain its differential $\mathrm{d}\phi(0, 0)$ at $(0, 0)$, we linearize the system (8) about the orbit $\Gamma$, i.e.

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}s(t) = 2\sigma, \ \dfrac{\mathrm{d}}{\mathrm{d}t}\sigma(t) = 0, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}r(t) = 2\rho, \ \dfrac{\mathrm{d}}{\mathrm{d}t}\rho(t) = -\dfrac{1}{2}R(s(t))r, \end{cases} \tag{9}$$

then $\sigma = \frac{1}{2}$, $s(t) = t$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} r \\ \rho \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -R/2 & 0 \end{pmatrix}\begin{pmatrix} r \\ \rho \end{pmatrix}. \tag{10}$$

Hence the application $\mathrm{d}\phi(0,0)$ is

$$\mathrm{d}\phi(0,0) : (r_0, \rho_0) \longmapsto \big(r(2\pi), \rho(2\pi)\big), \tag{11}$$

where $(r, \rho)$ solves (10). As $\mathrm{d}\phi(0,0)$ is symplectic, it admits two eigenvalues $\Lambda$ and $\Lambda^{-1}$ that are called the characteristic multipliers of the system (10). We add the following assumption on $\gamma$, which can be formulated in terms of the eigenvalues of $\mathrm{d}\phi(0,0)$:

Assumption 2. — *The geodesic $\gamma$ is stable, i.e. $d\phi(0,0)$ is a rotation. Then the multipliers take the form $\Lambda = e^{i\lambda}$ and $\Lambda^{-1} = e^{-i\lambda}$ with $\lambda \in \mathbb{R}$. We assume moreover that there exist $\tau, \mu > 0$ such that*

$$\forall (p,q) \in \mathbb{Z} \times \mathbb{N} \quad |p - q\frac{\lambda}{\pi}| \geq \frac{\mu}{|(p,q)|^{\tau}}, \tag{12}$$

*where $|(p,q)| = |p| + |q|$. When this condition is fulfilled, we say that $\gamma$ is non degenerated.*

Remark 1.4. — Almost every $\lambda \in \mathbb{R}$ satisfies (12) with $\tau > 1$. This is an easy consequence of [1] p. 159, e.g.

Examples 1. — *Let $M$ be a surface which has a periodic geodesic $\gamma$. In the general case, the eigenvalues of $d\phi(0,0)$ defined by (11) are $\Lambda = \rho e^{i\lambda}$ and $\Lambda^{-1} = \rho^{-1}e^{-i\lambda}$, with $\Lambda + \Lambda^{-1} \in \mathbb{R}_+$, i.e.*

$$(\rho - \rho^{-1}) \sin\lambda = 0. \tag{13}$$

*Assume that $M$ is a surface of revolution and that $R > 0$ on $\gamma$. Then the characteristic multipliers are*

$$\Lambda = \rho e^{2\pi i\sqrt{R}} \quad \textit{and} \quad \Lambda^{-1} = \rho^{-1}e^{-2\pi i\sqrt{R}}.$$

*i) If $\lambda = 2\pi\sqrt{R}$ satisfies (12) then $\rho = 1$ and $M$ satisfies the assumptions.*
*ii) Let $2\sqrt{R} \notin \mathbb{N}$. Let $\tilde{M}$ be a perturbation of $M$, and denote by*

$$\tilde{\Lambda} = \tilde{\rho}e^{i\tilde{\lambda}} \quad \textit{and} \quad \tilde{\Lambda}^{-1} = \tilde{\rho}^{-1}e^{-i\tilde{\lambda}},$$

*the new characteristic multipliers.*
*By (13), $\tilde{\rho} = 1$, and Assumption 2 is satisfied almost surely.*
*iii) Let $a > 0$, then the torus $M = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/a\mathbb{Z}$ is not under the hypotheses : in this case $d\phi(0,0)$ is not diagonalizable.*

Notice that the function $r$ which satisfies (10) is solution of

$$\ddot{y}(s) + R(s)y(s) = 0. \tag{14}$$

Consider $a_0$ the solution of (14) with initial conditions $a_0(0) = 1$ and $\dot{a}_0(0) = i$. Then, from the Floquet theory, there exists a $2\pi$-periodic function $P$ so that

$$a_0(s) = \mathrm{e}^{i\frac{\lambda}{2\pi}s}P(s)$$

(or $a_0(s) = \exp(-i\frac{\lambda}{2\pi}s)P(s)$, but $\lambda$ can be replaced with $-\lambda$).
Here, and in all the paper we denote by $\dot{f} = \frac{\mathrm{d}}{\mathrm{d}s}f$ if $f$ is differentiable. This notation is motived by the fact that $s$ will play the role of a time variable (see section 2).

In order to prove Proposition 1.3, we construct stationnary approximate solutions of (1), as stated in the following theorem

Theorem 1.5. — *Assume* 1 *and* 2. *Let $h \in ]0,1]$ such that $\frac{1}{h} \in \mathbb{N}$, let $\kappa, \sigma > 0$ and $k \in \mathbb{N}$. Let $\lambda$ be given by Assumption 2 and $\omega_1$ by* (5).
*Define $E_0(k) = -\frac{1}{4\pi}\lambda + \frac{1}{2}k(\omega_1 - \frac{\lambda}{\pi})$.*
*Then for all $N \in \mathbb{N}$, there exist $\lambda_N(k) \in \mathbb{R}$ and a family $u_N(h)$ such that $C_1 h^\sigma \leq \|u_N(h)\|_{L^2(M)} \leq C_2 h^\sigma$ with $C_1, C_2 > 0$ independent of $N$ and $h$, and*

$$-\Delta u_N(h) = \lambda_N(k)u_N(h) - \varepsilon|u_N(h)|^2 u_N(h) + h^N g_N(h) \tag{15}$$

*with for all $N \in \mathbb{N}$*

$$\|h^N g_N(h)\|_{H^n(M)} \lesssim h^{N-n}.$$

*Moreover*

$$\lambda_N(k) = \frac{1}{h^2} - \frac{2}{h}E_0(k) + \frac{1}{\sqrt{h}}\varepsilon\kappa^2 h^{2\sigma}C_0 + \mathcal{O}(1),$$

*where $C_0 > 0$ is independent of $\varepsilon$, $\kappa$ and $\sigma$.*

Remark 1.6. — The analog of Theorem 1.5 was proved by J. Ralston in [14] for the linear case ($\varepsilon = 0$), with the same type of assumptions.

Remark 1.7. — Consider the more general equations

$$i\partial_t u + \Delta u = F(u), \tag{16}$$

where $F : \mathbb{C} \longrightarrow \mathbb{C}$ is a $\mathcal{C}^\infty$ function. The result of Theorem 1.5 is likely to hold with other nonlinearities $F(u)$, for example for $F(z) = z^3$, $F(z) = z^4$ or $F(z) = (1+|z|^2)^\alpha z$ with $\alpha < 1$. However, the instability phenomenon is strongly related to the gauge invariance of the equation (16).

The scheme of the paper is the following: Thanks to a scaling, we reduce the problem (15) to the resolution of linear Schrödinger equations with a harmonic time dependent potential, and we will see, using Assumption 2, that these equations have periodic solutions. To prove Proposition 1.3 we show that the family $u_N(h)$ provides good approximations of (1) in times where instability occurs.

NOTATIONS 1.8. — *In this paper c, C denote constants the value of which may change from line to line. We use the notations* $a \sim b$, $a \lesssim b$ *if* $\frac{1}{C}b \leq a \leq Cb$, $a \leq Cb$ *respectively. By* $\delta_{i,j}$ *we mean the Kronecker symbol, i.e.* $\delta_{i,j} = 0$ *for* $i \neq j$ *and* $\delta_{i,i} = 1$.

REMARK 1.9. — In the sequel we do not always mention the dependence on $h$ of the functions: we will write $u$, $f$, $r_i$, ... instead of $u_h$, $f_h$, $r_{i,h}$, ...

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## 2. The WKB construction

Consider the equation

$$-\Delta u = \lambda u - \varepsilon |u|^2 u. \tag{17}$$

Given $h > 0$, we are looking for a solution of the form

$$u = \delta h^{-\frac{1}{4}} \mathrm{e}^{i\frac{s}{h}} f(s, r, h), \tag{18}$$

where $\delta = \kappa h^{\sigma}$, with $\kappa > 0$ and $0 \leq \sigma \leq \frac{1}{4}$. In all this section, $\delta$ will play the role of a parameter.

We try to find a solution $(u, \lambda)$ of (17) of the form

$$u \sim \sum_{j \geq 0} h^{j/2} u_j, \quad \lambda \sim h^{-2} \sum_{j \geq 0} h^{j/2} \lambda_j.$$

As we will see, identifying each power of $h$ will lead to a linear equation which can be solved with a suitable choice of $\lambda_j$.

Choose $h$ such that $h^{-1} \in \mathbb{N}$, this ensures that $\exp i\frac{s}{h}$ is $2\pi-$periodic. Such a condition on $h$ is natural and is known as a Bohr-Sommerfeld quantification condition.

With the ansatz (18), equation (17) becomes

$$\begin{aligned} &-\frac{1}{a^2}\Big(\frac{2i}{h}\partial_s f + \partial_s^2 f - \frac{1}{h^2} f\Big) - \frac{1}{a}\partial_s\Big(\frac{1}{a}\Big)\Big(\frac{i}{h} f + \partial_s f\Big) \\ &-\partial_r^2 f - \frac{\partial_r a}{a}\partial_r f = \lambda f - \varepsilon \delta^2 h^{-\frac{1}{2}} |f|^2 f. \end{aligned} \tag{19}$$

We make the change of variables $x = \frac{r}{\sqrt{h}}$ and set $v(s, x, h) = f(s, \sqrt{h}x, h)$. Thus $\partial_r f = \frac{1}{\sqrt{h}}\partial_x v$ and $\partial_r^2 f = \frac{1}{h}\partial_x^2 v$.

Therefore we now have to find $v \sim \sum_{j\geq 0} h^{j/2} v_j$.
Using (6) we obtain the following Taylor expansions in $h$

$$\frac{1}{a^2} = 1 + hRx^2 - 2h^{\frac{3}{2}} R_3 x^3 + \mathcal{O}(h^2),$$

$$a^{-1}\partial_s(a^{-1}) = \mathcal{O}(h) \quad \text{and} \quad a^{-1}\partial_r a = \mathcal{O}(h^{\frac{1}{2}}).$$

Equation (19) can therefore be written, after multiplication by $\frac{1}{2}h$

$$\begin{aligned} i\partial_s v + \frac{1}{2}\partial_x^2 v - \frac{1}{2}Rx^2 v \\ = \frac{1-\lambda h^2}{2h} v + h^{\frac{1}{2}} R_3 x^3 v + \frac{1}{2}\varepsilon\delta^2 h^{\frac{1}{2}} |v|^2 v + hPv, \end{aligned} \tag{20}$$

where

$$P = A_1 \partial_s^2 + A_2 \partial_s + A_3 \partial_x + A_4 \tag{21}$$

is a second order differential operator with coefficients $A_j = A_j(s,x,h)$ satisfying $A_j(s+2\pi, x, h) = A_j(s, \omega x, h)$ for $0 \leq j \leq 4$.
Denote by $E = \frac{1-\lambda h^2}{2h} = E_0 + h^{\frac{1}{2}} E_1 + \cdots + h^{\frac{p}{2}} E_p + o(h^{\frac{p}{2}})$ and write $v = v_0 + h^{\frac{1}{2}} v_1 + \cdots + h^{\frac{p}{2}} v_p + o(h^{\frac{p}{2}})$ and by identifying the powers of $h$ we obtain the system of equations:

$$\big(i\partial_s + \frac{1}{2}\partial_x^2 - \frac{1}{2}Rx^2 - E_0\big) v_0 = 0, \tag{22}$$

$$\big(i\partial_s + \frac{1}{2}\partial_x^2 - \frac{1}{2}Rx^2 - E_0\big) v_1 = E_1 v_0 + R_3 x^3 v_0 + \frac{1}{2}\varepsilon\delta^2 |v_0|^2 v_0, \tag{23}$$

$$\cdots = \cdots$$

$$\big(i\partial_s + \frac{1}{2}\partial_x^2 - \frac{1}{2}Rx^2 - E_0\big) v_p = E_p v_0 + Q_p. \tag{24}$$

$$\cdots = \cdots$$

so that the $(j+1)$th equation of unknown $(v_j, E_j)$ corresponds to the annihilation of the coefficient of $h^{\frac{j}{2}}$ in (20).
Here $Q_p$ is a function which only depends on $x, s$, $(v_j)_{j\leq p-1}$ and $(E_j)_{j\leq p-1}$.

Remark 2.1. — Notice that thanks to the scaling, we have reduced the problem (17) to the resolution of linear equations. However we have to solve them exactly; no smallness assumption on $x$ is possible, as $x$ can be of size $\sim \frac{1}{\sqrt{h}}$.

In this section we will show

Proposition 2.2. — *For all $p \in \mathbb{N}$, there exist $(E_0, \cdots, E_p) \in \mathbb{R}^{p+1}$ and $(v_0, \cdots, v_p) \in \big(\mathcal{C}^\infty([0,2\pi], \mathcal{S}(\mathbb{R}))\big)^{p+1}$ with $v_0 \neq 0$, which solve the system* (22)-(24).

This permits us to construct approximate solutions of (17); more precisely, we will obtain the following proposition, which is the main result of this section.

PROPOSITION 2.3. — *Let* $\chi \in \mathcal{C}_0^\infty(]-r_0,r_0[)$ *be such that* $0 \le \chi \le 1$, $\chi = 1$ *on* $[-r_0/2, r_0/2]$ *and suppose moreover that* $\chi$ *is an even function. Let* $\delta > 0$. *Denote by*

$$u_p(s,r) = \delta h^{-\frac{1}{4}} \chi(r) e^{i\frac{s}{h}} (v_0 + h^{\frac{1}{2}} v_1 + \cdots + h^{\frac{p}{2}} v_p)(s, \frac{r}{\sqrt{h}}) \tag{25}$$

*and by*

$$\lambda_p = \frac{1}{h^2} - \frac{2}{h}(E_0 + h^{\frac{1}{2}} E_1 + \cdots + h^{\frac{p}{2}} E_p). \tag{26}$$

*Then* $u_p$ *satisfies* $\|u_p\|_{L^2(M)} \sim \delta$ *and*

$$-\Delta u_p = \lambda_p u_p - \varepsilon |u_p|^2 u_p + h^{\frac{p-1}{2}} g_p(h) \tag{27}$$

*with*

$$\forall h \in ]0,1], \ \forall n \in \mathbb{N}, \quad \|h^{\frac{p-1}{2}} g_p(h)\|_{H^n([0,2\pi]\times\mathbb{R})} \lesssim \delta h^{\frac{p-1}{2}-n}.$$

**2.1. Preliminaries: the analysis of the linear equations.** — We will solve the system (22)-(24) for $x \in \mathbb{R}$. Notice that the Fermi coordinates are only defined for $|r| \le r_0$ i.e. for $x \le \frac{r_0}{\sqrt{h}}$. That's the reason why we need the cutoff which appears in the Proposition 2.3.

We first give an expansion of the operator $P$ defined by (21).

LEMMA 2.4. — *Let*

$$P(s,x,h) = A_1(s,x,h)\partial_s^2 + A_2(s,x,h)\partial_s + A_3(s,x,h)\partial_x + A_4(s,x,h),$$

*be the differential operator defined by* (21). *Then for all* $p \ge 2$, $P$ *can be written*

$$P(s,x,h) = \sum_{k=0}^{p-1} h^{\frac{k}{2}} P_k(s,x) + h^{\frac{p}{2}} \tilde{P}_p(s,x,h), \tag{28}$$

*so that*
*i) For all* $0 \le k \le p-1$,

$$P_k(s,x) = A_1^k(s,x)\partial_s^2 + A_2^k(s,x)\partial_s + A_3^k(s,x)\partial_x + A_4^k(s,x),$$

*where* $A_j^k \in \mathcal{C}^\infty\big([0,2\pi]\times\mathbb{R}\big)$, *for all* $s \in [0,2\pi]$ *the function* $x \mapsto A_j^k(s,x)$ *is a polynomial and* $A_j^k(s+2\pi, x) = A_j^k(s, \omega x)$.
*ii) Let* $\chi \in \mathcal{C}_0^\infty\big(]-r_0,r_0[\big)$ *and* $v \in \mathcal{C}^\infty\big([0,2\pi], \mathcal{S}(\mathbb{R})\big)$, *then for all* $n \in \mathbb{N}$, *there exists* $C = C(p,n)$ *independent of* $h \in ]0,1]$ *so that*

$$\|\chi(h^{\frac{1}{2}}x)\tilde{P}_p v(s,x)\|_{H^n([0,2\pi]\times\mathbb{R})} \le C. \tag{29}$$

*Proof.* — We first compute the coefficients of $P$.
By the Taylor formula near $r = 0$ we have

$$\frac{1}{a^2}(s,r) = 1 + R(s)r^2 - 2R_3(s)r^3 + \sum_{k=4}^{p+3} r^k R_k(s) + \frac{r^{p+4}}{(p+3)!}\int_0^1 (1-t)^{p+3}\frac{\partial^{p+4}}{\partial r^{p+4}}\big(\frac{1}{a^2}\big)(s,tr)\mathrm{d}t,$$

where $R_k$ is given by (7).
Now write $r = \sqrt{h}x$ and obtain

$$\frac{1}{a^2}(s,\sqrt{h}x) = 1 + hR(s)x^2 - 2h^{\frac{3}{2}}R_3(s)x^3 + h^2 I_1(s,x,h), \tag{30}$$

where

$$I_1(s,x,h) = \sum_{k=4}^{p+3} h^{\frac{k-4}{2}} x^k R_k(s) + h^{\frac{p}{2}}\frac{x^{p+4}}{(p+3)!}\int_0^1 (1-t)^{p+3}\frac{\partial^{p+4}}{\partial r^{p+4}}\big(\frac{1}{a^2}\big)(s,\sqrt{h}xt)\mathrm{d}t. \tag{31}$$

Similarly

$$\frac{1}{a}\partial_s\big(\frac{1}{a}\big)(s,\sqrt{h}x) = hI_2(s,x,h), \tag{32}$$

with

$$I_2(s,x,h) = \sum_{k=2}^{p+1} h^{\frac{k-2}{2}}\frac{x^k}{k!}\frac{1}{a}\partial_s\big(\frac{1}{a}\big)(s,0) + h^{\frac{p}{2}}\frac{x^{p+2}}{(p+1)!}\int_0^1 (1-t)^{p+1}\frac{\partial^{p+2}}{\partial r^{p+2}}\big(\frac{1}{a}\partial_s\big(\frac{1}{a}\big)\big)(s,\sqrt{h}xt)\mathrm{d}t, \tag{33}$$

and

$$\frac{\partial_r a}{a}(s,\sqrt{h}x) = h^{\frac{1}{2}} I_3(s,x,h), \tag{34}$$

where

$$I_3(s,x,h) = \sum_{k=1}^{p} h^{\frac{k-1}{2}}\frac{x^k}{k!}\frac{\partial^k}{\partial r^k}\big(\frac{\partial_r a}{a}\big)(s,0) + h^{\frac{p}{2}}\frac{x^{p+1}}{p!}\int_0^1 (1-t)^p\frac{\partial^{p+1}}{\partial r^{p+1}}\big(\frac{\partial_r a}{a}\big)(s,\sqrt{h}xt)\mathrm{d}t. \tag{35}$$

Plug the expressions (30), (32) and (34) in equation (20), and deduce that coefficients $A_j$ are

$$
\begin{aligned}
A_1 &= \frac{1}{2}(-1 - hRx^2 + 2h^{\frac{3}{2}}R_3x^3 - h^2 I_1),\\
A_2 &= -iRx^2 + 2ih^{\frac{1}{2}}R_3x^3 - ihI_1 - \frac{1}{2}I_2,\\
A_3 &= -\frac{1}{2}I_3,\\
A_4 &= \frac{1}{2}(I_1 - iI_2).
\end{aligned}
$$

Then with the developments (31), (33) and (35), we see that for all $1 \leq j \leq 4$ and $0 \leq k \leq p-1$, $x \longmapsto A_j^k(s,x)$ is a polynomial. Moreover as $a(s+2\pi, x) = a(s, \omega x)$, we also have $A_j^k(s+2\pi, x) = A_j^k(s, \omega x)$.

To obtain the bound (29), we now have to control the integral rests which appear in (31),(33) and (35).

Let $q \in \mathbb{N}^*$ and let $(s,r) \longmapsto f(s,r)$ be one of the functions $a^{-2}$, $a^{-1}\partial_s(a^{-1})$ or $a^{-1}\partial_r$. Let $\chi \in \mathcal{C}^\infty\big(]-r_0, r_0[\big)$ and define $F_q$ by

$$F_q(s,x) = \chi(\sqrt{h}x)\int_0^1 (1-t)^{q-1}\frac{\partial^q}{\partial r^q} f(s, \sqrt{h}xt)\mathrm{d}t.$$

As $f \in \mathcal{C}^\infty\big([0,2\pi]\times]-r_0, r_0[\big)$, we deduce that for all $n_1, n_2 \in \mathbb{N}$ there exists $C = C(q, n_1, n_2)$, independent of $h \in ]0,1]$ so that

$$\forall (s,x) \in [0,2\pi]\times\mathbb{R}, \qquad |\partial_s^{n_1}\partial_x^{n_2} F_q(s,x)| \leq C. \tag{36}$$

Now let $v \in \mathcal{C}^\infty\big([0,2\pi], \mathcal{S}(\mathbb{R})\big)$ and $n \in \mathbb{N}$. We can assume that $n \geq 2$, so that $H^n$ is an algebra. Then by (36)

$$\|x^q\, F_q\, v\|_{H^n([0,2\pi]\times\mathbb{R})} \leq C\|F_q\|_{H^n([0,2\pi]\times\mathbb{R})}\|x^q\, v\|_{H^n([0,2\pi]\times\mathbb{R})} \leq C, \tag{37}$$

and this yields ii). □

Consider the Hilbertian basis of $L^2(\mathbb{R})$ composed of the Hermite functions $(\varphi_k)_{k\geq 0}$ which are the eigenfunctions of the harmonic oscillator $H = -\frac{1}{2}\partial_x^2 + \frac{1}{2}x^2$, i.e. $H\varphi_k = (k+\frac{1}{2})\varphi_k$. Moreover $\varphi_k(x) = P_k(x)\mathrm{e}^{-x^2/2}$ where $P_k$ is a polynomial of degree $k$ with $P_k(-x) = (-1)^k P_k(x)$. The link between the $s$-dependent operator $-\frac{1}{2}\partial_x^2 + \frac{1}{2}R(s)x^2$ and $H$ is given by the following result proved by M. Combescure in [10].

Theorem 2.5. — *Let $a_0 : \mathbb{R} \longrightarrow \mathbb{C}$ be the solution of* (14) *with* $a_0(0) = 1$, $\dot{a}_0(0) = i$. *Define*

$$\alpha = \log|a_0|,\ \beta = \frac{1}{2i}\log\frac{a_0}{\overline{a_0}},$$

*let the unitary transform $T(s)$ be defined by*

$$T(s) = e^{i\dot{\alpha}(s)x^2/2}e^{-i\alpha(s)D},\ \textit{where}\ D = -\frac{i}{2}(x\cdot\nabla + \nabla\cdot x),$$

*and let $U(s,\tau)$ be the unitary evolution operator for $-\frac{1}{2}\partial_x^2+\frac{1}{2}R(s)x^2$, i.e. $U(s,\tau)\varphi$ is the unique solution of the problem*

$$\begin{cases} \big(i\partial_s+\dfrac{1}{2}\partial_x^2-\dfrac{1}{2}R(s)x^2\big)u=0,\\ u(\tau,x)=\varphi(x)\in L^2(\mathbb{R}). \end{cases}$$

*Then we have for any $s,\tau\in\mathbb{R}$*

$$U(s,\tau)=T(s)e^{-i(\beta(s)-(\beta(\tau))H}T(\tau)^{-1}.$$

Remark 2.6. — The functions $\alpha$ and $\beta$ are well defined: suppose that there exists $s_0$ such that $a_0(s_0)=0$, then $\operatorname{Re}a_0$ and $\operatorname{Im}a_0$ are linearly dependent, which is impossible with this choice of the initial conditions.

Remark 2.7. — Define $\theta(s)=\beta(s)-\frac{\lambda}{2\pi}s$ where $\lambda$ is given by Assumption 2. Then $\alpha$ and $\theta$ are $2\pi$-periodic real functions. Moreover $\alpha(0)=\dot\alpha(0)=\beta(0)=\theta(0)=0$.

Denote by $\mathcal{S}(\mathbb{R})$ the Schwartz space, i.e. the space of smooth functions which are fast decreasing and their derivatives too.

Proposition 2.8. — *Let $\psi_0\in\mathcal{S}(\mathbb{R})$ and $E\in\mathbb{C}$. Let $f\in\mathcal{C}^\infty([0,2\pi]\times\mathbb{R},\mathbb{R})$ be such that*

$$\forall n\in\mathbb{N},\ \forall s\in[0,2\pi],\quad \partial_s^n f(s,\cdot)\in\mathcal{S}(\mathbb{R}),$$

*in other words $f\in\mathcal{C}^\infty\big([0,2\pi],\mathcal{S}(\mathbb{R})\big)$.*
*Let $\psi\in\mathcal{C}^1\left([0,2\pi],L^2(\mathbb{R})\right)\cap\mathcal{C}^0\left([0,2\pi],H^2(\mathbb{R})\right)$ be the solution of*

$$\begin{cases} i\partial_s\psi+\dfrac{1}{2}\partial_x^2\psi-\dfrac{1}{2}R(s)x^2\psi-E\psi=f,\\ \psi(0,x)=\psi_0(x). \end{cases}\tag{38}$$

*Then $\psi\in\mathcal{C}^\infty\left([0,2\pi],\mathcal{S}(\mathbb{R})\right)$.*

*Proof.* — By replacing $\psi$ with $\mathrm{e}^{iEt}\psi$, we can assume that $E=0$. The solution of equation (38) is given by

$$\begin{aligned}\psi(s,\cdot)&=U(s,0)\psi_0-i\int_0^s U(s,\tau)f(\tau,\cdot)\mathrm{d}\tau\\ &=T(s)\mathrm{e}^{-i\beta(s)H}\left(\psi_0-i\int_0^s e^{i\beta(\tau)H}T(\tau)^{-1}f(\tau,\cdot)\mathrm{d}\tau\right).\end{aligned}\tag{39}$$

As $D$ is a transport operator, we have

$$T,T^{-1}:\mathcal{C}^\infty\left([0,2\pi],\mathcal{S}(\mathbb{R})\right)\longrightarrow\mathcal{C}^\infty\left([0,2\pi],\mathcal{S}(\mathbb{R})\right),$$

we only have to show that

$$\mathrm{e}^{i\beta H}:\mathcal{C}^\infty\left([0,2\pi],\mathcal{S}(\mathbb{R})\right)\longrightarrow\mathcal{C}^\infty\left([0,2\pi],\mathcal{S}(\mathbb{R})\right).$$

This follows from the fact that $\beta$ is regular and $\mathrm{e}^{iH}:\mathcal{S}(\mathbb{R})\longrightarrow\mathcal{S}(\mathbb{R})$. □

The description of $U$ given in Theorem 2.5 yields the following representation of $U(s,0)\varphi_k$:

PROPOSITION 2.9. — *For all $k \in \mathbb{N}$ and $s, x \in \mathbb{R}$ we have*

$$U(s,0)\varphi_k(x) = e^{i\dot\alpha(s)x^2/2} e^{-i(\frac{1}{2}+k)\beta(s)} e^{-\frac{1}{2}\alpha(s)} \varphi_k\big(x e^{-\alpha(s)}\big). \tag{40}$$

*Proof.* — According to Theorem 2.5, and as $H\varphi_k = (k+\frac{1}{2})\varphi_k$,

$$U(s,0)\varphi_k = \mathrm{e}^{i\dot\alpha(s)x^2/2}\mathrm{e}^{-i(k+\frac{1}{2})\beta(s)}\mathrm{e}^{-i\alpha(s)D}\varphi_k.$$

Denote by $f(s) = \mathrm{e}^{-i\alpha(s)D}\varphi_k$. Then $f$ is solution of the transport equation

$$\partial_s f = -\frac{1}{2}\dot\alpha(s)\big(x\partial_x f + \partial_x(xf)\big) = -\frac{1}{2}\dot\alpha(s)\big(f + 2x\partial_x f\big)$$

with Cauchy data $f(0,x) = \varphi_k(x)$. Make the change of variables $\sigma = \alpha(s)$ and set $g(\sigma) = f(s)$. Therefore $g$ satisfies $\partial_\sigma g = -\frac{1}{2}(g + 2x\partial_x g)$. The equation $x = \dot x$, $x(0) = x_0$ admits the solution $x(\tau) = x_0\mathrm{e}^{\tau}$ and the characteristics method gives $g(\tau, x(\tau)) = \mathrm{e}^{-\frac{1}{2}\tau}\varphi_k(x_0) = \mathrm{e}^{-\frac{1}{2}\tau}\varphi_k(x(\tau)\mathrm{e}^{-\tau})$, hence

$$f(s) = \mathrm{e}^{-\frac{1}{2}\alpha(s)}\varphi_k(x\mathrm{e}^{-\alpha(s)}).$$

□

COROLLARY 2.10. — *Let $k \in \mathbb{N}$, define $\omega_1 = \frac{1}{2}(\omega - 1)$ and $E_0(k) = -\frac{1}{4\pi}\lambda + \frac{1}{2}k(\omega_1 - \frac{\lambda}{\pi})$. Then*

$$\begin{aligned} w_k &= e^{-isE_0(k)} U(s,0)\varphi_k \\ &= e^{-isE_0(k)} e^{i\dot\alpha(s)x^2/2} e^{-i(\frac{1}{2}+k)\beta(s)} e^{-\frac{1}{2}\alpha(s)} \varphi_k\big(x e^{-\alpha(s)}\big) \end{aligned} \tag{41}$$

*is solution of the equation*

$$\big(i\partial_s + \frac{1}{2}\partial_x^2 - \frac{1}{2}R(s)x^2 - E_0(k)\big)w_k(s,x) = 0.$$

*Proof.* — On the one hand, from Proposition 2.9 we deduce

$$\begin{aligned} w_k(s+2\pi, x) &= \mathrm{e}^{-2i\pi E_0(k)}\mathrm{e}^{-i\lambda(\frac{1}{2}+k)} w_k(s,x) = \mathrm{e}^{-ik\omega_1\pi} w_k(s,x) \\ &= (-1)^{k\omega_1} w_k(s,x) = w_k(s,\omega x). \end{aligned}$$

On the other hand, $w_k$ satisfies (22) because of the definition of $U(s,0)$. □

Fix $k_0 \in \mathbb{N}$ and take $v_0 = w_{k_0}$ with the previous choice of $E_0(k_0)$. This choice corresponds to the $k_0$th level of energy for the harmonic oscillator.

REMARK 2.11. — Until now we didn't use the restriction (12), but it will be crucial in the following.

PROPOSITION 2.12. — *For all $p \geq 0$, there exist $E_p \in \mathbb{C}$ and $v_p \in \mathcal{C}^\infty\big([0,2\pi], \mathcal{S}(\mathbb{R})\big)$ which solve (24).*

Remark 2.13. — As stated in Theorem 1.5, the $E_j$'s are in fact real numbers. This will be proved in Lemma 2.17.

*Proof.* — We proceed by induction on $p \in \mathbb{N}$.
For $p = 0$ the result was proved in Corollary 2.10.
Let $p \geq 1$, and suppose that for all $j \leq p-1$ there exist $E_j \in \mathbb{C}$ and $v_j \in \mathcal{C}^\infty\big([0, 2\pi], \mathcal{S}(\mathbb{R})\big)$ which solve the $(j+1)$th equation of (22). When $p \geq 2$, set

$$\tilde{v}_{p-1} = h^{\frac{1}{2}} v_1 + \cdots + h^{\frac{p-1}{2}} v_{p-1},$$
$$\tilde{E}_{p-1} = h^{\frac{1}{2}} E_1 + \cdots + h^{\frac{p-1}{2}} E_{p-1}$$

and $\tilde{v}_0 = \tilde{E}_0 = 0$. By (28), the function $Q_p$ given by (24) is the coefficient of $h^{\frac{p}{2}}$ in the expansion in h of

$$\tilde{E}_{p-1}\tilde{v}_{p-1} + \frac{1}{2}\varepsilon\delta^2 |v_0 + \tilde{v}_{p-1}|^2 (v_0 + \tilde{v}_{p-1}) + h\big(\sum_{k=0}^{p-1} h^{\frac{k}{2}} P_k\big)(v_0 + \tilde{v}_{p-1}).$$

Now using the regularity of the $v_j$'s and the fact that for all $0 \leq k \leq p-1$, $P_k$ is an operator

$$P_k : \mathcal{C}^\infty\big([0, 2\pi], \mathcal{S}(\mathbb{R})\big) \longrightarrow \mathcal{C}^\infty\big([0, 2\pi], \mathcal{S}(\mathbb{R})\big),$$

we obtain $Q_p \in \mathcal{C}^\infty\big([0, 2\pi], \mathcal{S}(\mathbb{R})\big)$.
Moreover $Q_p$ satisfies, $\forall (s, x) \in [0, 2\pi] \times \mathbb{R}$

$$Q_p(s + 2\pi, x) = Q_p(s, \omega x)$$

because this property holds for the $v_j$'s, and $a$.
Define $F_p(s, x) = \mathrm{e}^{-i\dot{\alpha}(s)\mathrm{e}^{2\alpha(s)}x^2/2} Q_p(s, x e^{\alpha(s)})$, then $F_p \in \mathcal{C}^\infty\big([0, 2\pi], \mathcal{S}(\mathbb{R})\big)$ and satisfies $Q_p(s, x) = \mathrm{e}^{i\dot{\alpha}(s)x^2/2} F_p(s, x\mathrm{e}^{-\alpha(s)})$ and $F_p(s + 2\pi, x) = F_p(s, \omega x)$.
Let us decompose $F_p$ on the basis $(\varphi_j)_{j\geq 0}$: there exists a unique family of smooth functions $(g_j^p(s))_{j\geq 0} \in l^2(\mathbb{N})$ so that

$$F_p(s, y) = \sum_{j\geq 0} g_j^p(s)\varphi_j(y). \tag{42}$$

Then

$$Q_p(s, x) = \sum_{j\geq 0} g_j^p(s)\mathrm{e}^{i\dot{\alpha}(s)x^2/2}\varphi_j(x\mathrm{e}^{-\alpha(s)}) = \sum_{j\geq 0} h_j^p(s) w_j(s, x), \tag{43}$$

where according to (41)

$$h_j^p(s) = \mathrm{e}^{isE_0(j)}\mathrm{e}^{i(\frac{1}{2}+j)\beta(s)}\mathrm{e}^{\frac{1}{2}\alpha(s)} g_j^p(s). \tag{44}$$

We have

$$Q_p(s, \omega x) = \sum_{j\geq 0} h_j^p(s) w_j(s, \omega x),$$

but also

$$Q_p(s, \omega x) = Q_p(s+2\pi, x) = \sum_{j\geq 0} h_j^p(s+2\pi) w_j(s+2\pi, x)$$
$$= \sum_{j\geq 0} h_j^p(s+2\pi) w_j(s, \omega x),$$

and from the uniqueness of the $h_j^p$'s we deduce $h_j^p(s+2\pi) = h_j^p(s)$.
We are now looking for a solution of (24) of the form

$$v_p(s,x) = \sum_{j\geq 0} e_j^p(s) w_j(s,x) \tag{45}$$

where the $e_j^p$'s are $2\pi$-periodic functions. For all $j \geq 0$, by Corollary 2.10 we have

$$\big(i\partial_s + \frac{1}{2}\partial_x^2 - \frac{1}{2}Rx^2\big)\big(e_j^p w_j\big) = i\dot{e}_j^p w_j + \big(E_0(k_0) - E_0(j)\big) e_j^p w_j,$$

hence we have to solve the equations

$$i\dot{e}_j^p + \big(E_0(k_0) - E_0(j)\big) e_j^p = h_j^p + \delta_{j,k_0} E_p. \tag{46}$$

As $E_0(k_0) - E_0(j) = \frac{1}{2}(k_0 - j)(\omega_1 - \frac{\lambda}{\pi})$, the solutions of (46) take the form

$$e_j^p(s) = \mathrm{e}^{\frac{1}{2}i(k_0-j)(\omega_1-\frac{\lambda}{\pi})s}\big(C_j^p - i\int_0^s h_j^p(\tau)\mathrm{e}^{-\frac{1}{2}i(k_0-j)(\omega_1-\frac{\lambda}{\pi})\tau}\mathrm{d}\tau\big) \tag{47}$$

for $j \neq k_0$, and

$$e_{k_0}^p(s) = C_{k_0}^p - i\int_0^s h_{k_0}^p(\tau)\mathrm{d}\tau - iE_p s.$$

The constants $C_j^p \in \mathbb{C}$ and $E_p \in \mathbb{C}$ have to be determined such that $e_j^p(s+2\pi) = e_j^p(s)$.

• Case $j = k_0$:

$$e_{k_0}^p(s+2\pi) = -i\int_0^{2\pi} h_{k_0}^p(\tau)\mathrm{d}\tau - 2\pi i E_p + e_{k_0}^p(s),$$

thus $e_{k_0}^p$ is $2\pi$-periodic iff

$$E_p = -\frac{1}{2\pi}\int_0^{2\pi} h_{k_0}^p(\tau)\mathrm{d}\tau. \tag{48}$$

• Case $j \neq k_0$:
Denote by $\tilde{h}_j^p : \tau \longmapsto h_j^p(\tau)\mathrm{e}^{-i\frac{1}{2}(k_0-j)(\omega_1-\frac{\lambda}{\pi})\tau}$ and by $K = \mathrm{e}^{i(k_0-j)(\pi\omega_1-\lambda)}$. Then

$$\int_0^{s+2\pi} \tilde{h}_j^p(\tau)\mathrm{d}\tau = \int_0^{2\pi} \tilde{h}_j^p(\tau)\mathrm{d}\tau + \int_{2\pi}^{s+2\pi} \tilde{h}_j^p(\tau)\mathrm{d}\tau$$
$$= \int_0^{2\pi} \tilde{h}_j^p(\tau)\mathrm{d}\tau + K^{-1}\int_0^{s} \tilde{h}_j^p(\tau)\mathrm{d}\tau,$$

and by (47)

$$e_j^p(s+2\pi) = K\mathrm{e}^{i\frac{1}{2}(k_0-j)(\omega_1-\frac{\lambda}{\pi})s}\big(C_j^p - i\int_0^{s+2\pi}\tilde{h}_j^p(\tau)\mathrm{d}\tau\big)$$

$$= \mathrm{e}^{i\frac{1}{2}(k_0-j)(\omega_1-\frac{\lambda}{\pi})s}\big(KC_j^p - iK\int_0^{2\pi}\tilde{h}_j^p(\tau)\mathrm{d}\tau - i\int_0^{s}\tilde{h}_j^p(\tau)\mathrm{d}\tau\big). \tag{49}$$

Notice that $K \neq 1$, as $\lambda \notin \pi\mathbb{Q}$ and choose

$$C_j^p = \frac{iK}{K-1}\int_0^{2\pi}\tilde{h}_j^p(\tau)\mathrm{d}\tau,$$

then according to (47)and (49), the function $e_j^p$ is $2\pi-$periodic.
Now, we show that the constants $C_j^p$ are uniformly bounded in $j \geq 0$, so that the function $v_p$ given by (45) is well defined. We first need the

Lemma 2.14. — *Let $(h_j^p)_{j\geq 0} \in l^2(\mathbb{N})$ be the family of $2\pi-$periodic functions defined by* (44) *and $h_j^p(s) = \sum_{n\in\mathbb{Z}} c_{l,j}^p e^{ils}$ its Fourier decomposition. Then for all $n_1, n_2 \in \mathbb{N}$ there exists $C^p > 0$ such that for all $j \in \mathbb{N}$*

$$\sum_{l\in\mathbb{Z}} j^{2n_1} l^{2n_2} |c_{l,j}^p|^2 \leq C^p.$$

*Proof.* — Consider the function $F_p \in \mathcal{C}^\infty\big([0,2\pi],\mathcal{S}(\mathbb{R})\big)$ which defines the family $(g_j^p(s))_{j\geq 0} \in l^2(\mathbb{N})$ with (42). Denote by $H = -\frac{1}{2}\partial_x^2 + \frac{1}{2}x^2$. Let $n_1, n_2 \in \mathbb{N}$ and decompose the function $\partial_s^{n_2} H^{n_1} F_p$ on the basis $(\varphi_j)_{j\geq 0}$

$$\partial_s^{n_2} H^{n_1} F_p(s,y) = \sum_{j\geq 0} \tilde{g}_j^p(s)\varphi_j(y)$$

where $(\tilde{g}_j^p)_{j\geq 0}$ is a smooth family of functions in $l^2(\mathbb{N})$.
Using that $H\varphi_j = (j+\frac{1}{2})\varphi_j$ and that $F_p \in \mathcal{C}^\infty\big([0,2\pi],\mathcal{S}(\mathbb{R})\big)$, we have for all $n_1, n_2 \in \mathbb{N}$

$$\partial_s^{n_2} H^{n_1} F_p(s,y) = \sum_{j\geq 0}(j+\frac{1}{2})^{n_1}(g_j^p)^{(n_2)}(s)\varphi_j(y).$$

By uniqueness of such a decomposition,

$$\Big((j+\frac{1}{2})^{n_1}(g_j^p)^{(n_2)}\Big)_{j\geq 0} = (\tilde{g}_j^p)_{j\geq 0} \in l^2(\mathbb{N}).$$

Then by the definition (44) of $h_j^p$, an easy induction on $n_1, n_2 \in \mathbb{N}$ shows that $\big(j^{n_1}(h_j^p)^{(n_2)}\big)_{j\geq 0} \in l^2(\mathbb{N})$. Write the Fourier decomposition of $h_j^p$

$$h_j^p(s) = \sum_{n\in\mathbb{Z}} c_{l,j}^p \mathrm{e}^{ils}$$

and by Parseval

$$\sum_{j\geq 0}\sum_{l\in\mathbb{Z}} j^{2n_1} l^{2n_2}(s)|c_{l,j}^p|^2 = \sum_{j\geq 0} j^{2n_1}\int_0^{2\pi} |(h_j^p)^{(n_2)}(s)|^2 \mathrm{d}s \leq C^p.$$

In particular, for all $j\in\mathbb{N}$

$$\sum_{l\in\mathbb{Z}} j^{2n_1} l^{2n_2}|c_{l,j}^p|^2 \leq C^p,$$

hence the result. □

*End of the proof of Proposition 2.12:* Using the Fourier decomposition of $h_j$ we obtain

$$\begin{aligned} C_j^p &= \frac{iK}{K-1}\int_0^{2\pi} \tilde{h}_j^p(\tau)\mathrm{d}\tau \\ &= \frac{iK}{K-1}\sum_{l\in\mathbb{Z}} c_{l,j}^p \int_0^{2\pi} \mathrm{e}^{i(l-\frac{1}{2}(k_0-j)(\omega_1-\frac{\lambda}{\pi}))\tau} \\ &= -i\sum_{l\in\mathbb{Z}} \frac{c_{l,j}^p}{l-\frac{1}{2}(k_0-j)(\omega_1-\frac{\lambda}{\pi})}. \end{aligned} \tag{50}$$

With Assumption 2 we have

$$\begin{aligned} \big|l-\frac{1}{2}(k_0-j)(\omega_1-\frac{\lambda}{\pi})\big| &= \frac{1}{2}|(2l-(k_0-j)\omega_1)+(k_0-j)\frac{\lambda}{\pi}| \\ &\geq \frac{1}{2}\frac{\mu}{|(2l-(k_0-j)\omega_1, k_0-j)|^\tau}, \end{aligned}$$

and for $j\geq k_0$, $|2l-(k_0-j)\omega_1|+|k_0-j|\leq 2(|l|+|j|)$, then

$$\big|l-\frac{1}{2}(k_0-j)(\omega_1-\frac{\lambda}{\pi})\big| \geq \frac{c\mu}{(|l|+|j|)^\tau}. \tag{51}$$

Hence, from (50) and (51) we deduce

$$|C_j^p| \lesssim \sum_{l\in\mathbb{Z}} |c_{l,j}^p|(|j|+|l|)^\tau \lesssim \sum_{l\in\mathbb{Z}} |c_{l,j}^p|(|j|^\tau+|l|^\tau). \tag{52}$$

By Cauchy-Schwarz and Lemma 2.14, from (52) we obtain

$$\begin{aligned} |C_j^p| &\lesssim \sum_{l\in\mathbb{Z}} \frac{1+|l|}{1+|l|}|c_{l,j}^p|(|j|^\tau+|l|^\tau) \\ &\lesssim \big(\sum_{l\in\mathbb{Z}} \frac{1}{(1+|l|)^2}\big)^{\frac{1}{2}}\big(\sum_{l\in\mathbb{Z}} |c_{l,j}^p|^2(1+|l|)^2(|j|^{2\tau}+|l|^{2\tau})\big)^{\frac{1}{2}} \\ &\leq C^p. \end{aligned} \tag{53}$$

Set

$$v_p(s,x) = \sum_{j\geq 0} e_j^p(s) w_j(s,x).$$

For all $j \in \mathbb{N}$, $s \longmapsto e_j^p(s)w_j(s,x)$ is continuous and there exists $c > 0$ such that for all $j > k_0$, and for all $s \in [0, 2\pi]$

$$|e_j^p(s)w_j(s,x)| \lesssim |g_j^p(s)||\varphi_j(cx)|$$

and this shows that $v_p \in C\big([0,2\pi], L^2(\mathbb{R})\big)$. Now using Proposition 2.8 we conclude, by uniqueness of such a solution, that $v_p \in \mathcal{C}^\infty\big([0,2\pi], \mathcal{S}(\mathbb{R})\big)$. □

### 2.2. The nonlinear analysis and proof of Proposition 2.3. —

LEMMA 2.15. — *The constant $E_1$ given by Proposition 2.12 writes $E_1 = -\varepsilon\delta^2 C_0$ where $C_0 > 0$ is independent of $\varepsilon$ and $\delta$.*

*Proof.* — Consider the equation

$$\big(i\partial_s + \frac{1}{2}\partial_x^2 - \frac{1}{2}Rx^2 - E_0\big)v_1 = E_1 v_0 + R_3 x^3 v_0 + \frac{1}{2}\varepsilon\delta^2|v_0|^2 v_0,$$

with

$$v_0(s,x) = \mathrm{e}^{-isE_0(k_0)}\mathrm{e}^{i\dot\alpha(s)x^2/2}\mathrm{e}^{-i(\frac{1}{2}+k_0)\beta(s)}\mathrm{e}^{-\frac{1}{2}\alpha(s)}\varphi_{k_0}\big(x\mathrm{e}^{-\alpha(s)}\big).$$

By the definition of $Q_p$ (see (24)),

$$Q_1(s,x) = R_3(s)x^3 v_0(s,x) + \frac{1}{2}\varepsilon\delta^2|v_0|^2 v_0(s,x),$$

and by (43), $Q_1$ can be written

$$Q_1(s,x) = \sum_{j\geq 0} h_j^1(s) w_j(s,x).$$

According to formula (48), we only have to compute the term $h_{k_0}^1$ in the previous expansion.

Write the expansion of $|\varphi_{k_0}|^2\varphi_{k_0}$ on the basis $(\varphi_j)_{j\geq 0}$:

$$|\varphi_{k_0}|^2\varphi_{k_0} = \sum_{j\geq 0} p_j\varphi_j, \tag{54}$$

with $p_j \in \mathbb{R}$ and $p_j = 0$ for $j - k_0 = 1 \ mod\ 2$ as $\varphi_k(-x) = (-1)^k\varphi_k(x)$.
Then by (54) and the expression (41) of $w_j$

$$\begin{aligned}|v_0|^2 v_0(s,x) &= \mathrm{e}^{-isE_0(k_0)}\mathrm{e}^{i\dot\alpha(s)x^2/2}\mathrm{e}^{-i(\frac{1}{2}+k_0)\beta(s)}\mathrm{e}^{-\frac{3}{2}\alpha(s)}|\varphi_{k_0}|^2\varphi_{k_0}\big(x\mathrm{e}^{-\alpha(s)}\big)\\ &= \sum_{j\geq 0} p_j \mathrm{e}^{-isE_0(k_0)}\mathrm{e}^{i\dot\alpha(s)x^2/2}\mathrm{e}^{-i(\frac{1}{2}+k_0)\beta(s)}\mathrm{e}^{-\frac{3}{2}\alpha(s)}\varphi_j\big(x\mathrm{e}^{-\alpha(s)}\big)\\ &= \sum_{j\geq 0} f_j(s) w_j(s,x)\end{aligned}$$

where

$$\begin{aligned} f_j(s) &= p_j \mathrm{e}^{-is(E_0(k_0)-E_0(j))}\mathrm{e}^{-i(k_0-j)\beta(s)}\mathrm{e}^{-\alpha(s)} \\ &= p_j \mathrm{e}^{-i(k_0-j)(\theta(s)+\frac{s}{2}\omega_1)}\mathrm{e}^{-\alpha(s)}. \end{aligned}$$

Therefore $f_{k_0}(s) = p_{k_0}\mathrm{e}^{-\alpha(s)}$ with, using (54), $p_{k_0} = \int_{\mathbb{R}} |\phi_{k_0}|^4 > 0$.
In the same manner we write

$$x^3 \varphi_{k_0}(x) = \sum_{j\geq 0} q_j \varphi_j(x), \tag{55}$$

with $q_j = 0$ when $j - k_0 = 0 \ mod\ 2$ and by (55) we have

$$\begin{aligned} & R_3(s)x^3 v_0(s,x) \\ &\quad = R_3(s)\mathrm{e}^{-isE_0(k_0)}\mathrm{e}^{i\dot{\alpha}(s)x^2/2}\mathrm{e}^{-i(\frac{1}{2}+k_0)\beta(s)}\mathrm{e}^{\frac{5}{2}\alpha(s)}\big(x\mathrm{e}^{-\alpha(s)}\big)^3\varphi_{k_0}\big(x\mathrm{e}^{-\alpha(s)}\big) \\ &\quad = \sum_{j\geq 0} q_j R_3(s)\mathrm{e}^{-isE_0(k_0)}\mathrm{e}^{-i(\frac{1}{2}+k_0)\beta(s)}\mathrm{e}^{\frac{5}{2}\alpha(s)}\mathrm{e}^{i\dot{\alpha}(s)x^2/2}\varphi_j\big(x\mathrm{e}^{-\alpha(s)}\big). \end{aligned}$$

By (41) we have

$$\mathrm{e}^{i\dot{\alpha}(s)x^2/2}\varphi_{k_0}\big(x\mathrm{e}^{-\alpha(s)}\big) = \mathrm{e}^{isE_0(j)}\mathrm{e}^{i(\frac{1}{2}+j)\beta(s)}\mathrm{e}^{\frac{1}{2}\alpha(s)}.$$

Then

$$R_3(s)x^3 v_0(s,x) = \sum_{j\geq 0} \tilde{f}_j(s) w_j(s,x),$$

where

$$\begin{aligned} \tilde{f}_j(s) &= q_j R_3(s)\mathrm{e}^{-is(E_0(k_0)-E_0(j))}\mathrm{e}^{-i(k_0-j)\beta(s)}\mathrm{e}^{3\alpha(s)} \\ &= q_j R_3(s)\mathrm{e}^{-i(k_0-j)(\theta(s)+\frac{s}{2}\omega_1)}\mathrm{e}^{3\alpha(s)}. \end{aligned}$$

Then $\tilde{f}_{k_0} = 0$ as $q_j = 0$ when $j - k_0 = 0 \ mod\ 2$. Thus

$$h^1_{k_0}(s) = \frac{1}{2}\varepsilon\delta^2 f_{k_0}(s) = \frac{1}{2}\varepsilon\delta^2 p_{k_0}\mathrm{e}^{-\alpha(s)}.$$

Finally, from (48) we deduce

$$E_1 = -\frac{1}{4\pi}\varepsilon\delta^2 p_{k_0}\int_0^{2\pi} \mathrm{e}^{-\alpha(\tau)}\mathrm{d}\tau = -\varepsilon\delta^2 C_0,$$

where $C_0 > 0$ as $p_{k_0} > 0$. □

LEMMA 2.16. — *Let $\psi \in \mathcal{C}_0^\infty(\mathbb{R})$ such that $\psi = 0$ near $0$, and let $f \in \mathcal{S}(\mathbb{R})$. Then for all $n, N \in \mathbb{N}$, there exists $C = C(n,N)$ so that*

$$\|\psi(h^{\frac{1}{2}}\cdot)f\|_{H^n(\mathbb{R})} \leq Ch^N. \tag{56}$$

*Proof.* — We only show (56) for $n = 0$, the general case follows from the Leibniz rule. We can assume that $\operatorname{supp}\psi \subset [a, b]$ with $a > 0$. Then as $f \in \mathcal{S}(\mathbb{R})$, for all $N \in \mathbb{N}$, there exists $C_N > 0$ so that

$$|f(x)| \le C_N \frac{1}{1 + |x|^N}.$$

Thus

$$\begin{aligned}\int_{\mathbb{R}} |\psi(h^{\frac{1}{2}}x)|^2 |f(x)|^2 \mathrm{d}x &= h^{-\frac{1}{2}} \int_a^b |\psi(x)|^2 |f(h^{-\frac{1}{2}}x)|^2 \mathrm{d}x \\ &\le C_N h^{N-\frac{1}{2}} \int_a^b |\psi(x)|^2 \frac{1}{h^N + x^{2N}} \mathrm{d}x \\ &\le C_N h^{N-\frac{1}{2}},\end{aligned}$$

hence the result. □

*Proof of Proposition 2.3.* — Let $p \ge 1$, and consider

$$V_p(s, x) = \big(v_0 + h^{\frac{1}{2}} v_1 + \dots + h^{\frac{p}{2}} v_p\big)(s, x),$$

and

$$\tilde{E}_p = E_0 + h^{\frac{1}{2}} E_1 + \dots + h^{\frac{p}{2}} E_p,$$

where the $v_j$'s and the $E_j$'s are given by Proposition 2.12.
Let $\chi \in \mathcal{C}_0^\infty(]-r_0, r_0[)$ be an even function such that $0 \le \chi \le 1$ and $\chi = 1$ on $[-r_0/2, r_0/2]$.
We claim that there exists $G_p(h) \in \mathcal{C}^\infty\big([0, 2\pi], \mathcal{S}(\mathbb{R})\big)$, so that

$$\forall n \in \mathbb{N}, \quad \|G_p(h)\|_{H^n([0,2\pi]\times\mathbb{R})} \le C_{n,p}, \tag{57}$$

where $C_{n,p}$ is independent of $h \in ]0, 1]$, and such that $G_p(h)$ satisfies

$$\begin{aligned}&\chi(h^{\frac{1}{2}}x)\, \Big((i\partial_s + \frac{1}{2}\partial_x^2 - \frac{1}{2}Rx^2 - \tilde{E}_p)V_p \\ &\quad -h^{\frac{1}{2}} R_3 x^3 V_p - \frac{1}{2}\varepsilon\delta^2 h^{\frac{1}{2}} |V_p|^2 V_p - hPV_p\Big) = h^{\frac{p+1}{2}} G_p(h).\end{aligned} \tag{58}$$

By construction of the $v_j$'s and the $E_j$'s, in the l.h.s. of (58), the coefficient of $h^j$ cancels for $0 \le j \le p$.
Then write the expansion in powers of $h$

$$\frac{1}{2}\varepsilon\delta^2 |V_p|^2 V_p = \sum_{k=0}^{3p+1} h^{\frac{k}{2}} V_p^k,$$

and use (28) to obtain

$$hPV_p = h\big(\sum_{k=0}^{p-1} h^{\frac{k}{2}} P_k + h^{\frac{p}{2}} \tilde{P}_p\big)\big(\sum_{k=0}^{p} h^{\frac{k}{2}} v_k\big) := \sum_{k=0}^{2p+2} h^{\frac{k}{2}} W_p^k$$

We therefore obtain the explicit formula of $G_p(h)$

$$\begin{aligned}h^{\frac{p+1}{2}}G_p(h) &:= -\chi(h^{\frac12}x)\sum_{k=p+1}^{2p+2}h^{\frac{k}{2}}W_p^k - \chi(h^{\frac12}x)\sum_{k=p+1}^{3p+1}h^{\frac{k}{2}}V_p^k - \chi(h^{\frac12}x)h^{\frac{p+1}{2}}R_3x^3v_p\\ &= -h^{\frac{p+1}{2}}\chi(h^{\frac12}x)\Big(\sum_{l=0}^{p+1}h^{\frac{l}{2}}W_p^{l+p+1}\sum_{l=0}^{2p}h^{\frac{l}{2}}V_p^{l+p+1}+R_3x^3v_p\Big).\end{aligned}$$

The bound (57) then follows from an application of Lemma 2.4.

Denote by $\tilde V_p = \chi(h^{\frac12}x)V_p$, and write

$$\begin{aligned}P\tilde V_p &= (A_1\partial_s^2 + A_2\partial_s + A_3\partial_x + A_4)(\chi(h^{\frac12}x)V_p)\\ &= \chi(h^{\frac12}x)PV_p + h^{\frac12}\chi'(h^{\frac12}x)A_3V_p.\end{aligned}$$

By (58) we deduce that

$$\begin{aligned}&(i\partial_s + \frac12\partial_x^2 - \frac12 Rx^2 - \tilde E_p)\tilde V_p - h^{\frac12}R_3x^3\tilde V_p - \frac12\varepsilon\delta^2h^{\frac12}|\tilde V_p|^2\tilde V_p - hP\tilde V_p\\ &= h^{\frac{p+1}{2}}G_h^p + h^{\frac12}\chi'(h^{\frac12}x)\partial_xV_p + \frac12 h\chi''(h^{\frac12}x)V_p\\ &\quad + \frac12\varepsilon\delta^2h^{\frac12}\chi(1-\chi^2)(h^{\frac12}x)|V_p|^2V_p - h^{\frac32}\chi'(h^{\frac12}x)A_3V_p\\ &:= h^{\frac{p+1}{2}}\tilde G_p(h).\end{aligned}$$

Each of the functions $\chi'$, $\chi''$ and $\chi(1-\chi^2)$ vanishes near 0, hence by Lemma 2.16 and (57)

$$\forall n\in\mathbb{N},\quad \|\tilde G_p(h)\|_{H^n([0,2\pi]\times\mathbb{R})} \le C_{n,p}. \tag{59}$$

Finally, set

$$u_p = \delta h^{-\frac14}\mathrm{e}^{i\frac{s}{h}}V_p(s,\frac{r}{\sqrt h}),$$

then

$$-\Delta u_p - \lambda_pu_p + \varepsilon|u_p|^2u_p = \frac2h\mathrm{e}^{i\frac{s}{h}}h^{\frac{p+1}{2}}\tilde G_p(h),$$

and $g_p(h) = 2\mathrm{e}^{i\frac{s}{h}}\tilde G_p(h)$ satisfies the conclusion of Proposition 2.3 by (59).

□

Lemma 2.17. — *Let $p\ge 1$ and $E_p$ given by Proposition* (2.12)*. Then $E_p\in\mathbb{R}$.*

*Proof.* — We already know that $E_0, E_1\in\mathbb{R}$. Let $p\ge 3$. Multiply (27) by $\overline{u}_p$, integrate on $M$ and take the imaginary part

$$0 = \|u_p\|_{L^2}^2\operatorname{Im}\lambda_p + h^{\frac{p-1}{2}}\operatorname{Im}\int g_p(h)\overline{u}_p.$$

As $\|u_p\|_{L^2}\sim 1$ and $\|g_p\|_{L^2}\lesssim 1$, we obtain the estimate

$$|\operatorname{Im}\lambda_p| \lesssim h^{\frac{p-1}{2}}\|g_p\|_{L^2}\|u_p\|_{L^2} \lesssim h^{\frac{p-1}{2}}$$

and as

$$\operatorname{Im}\lambda_p = -2(\operatorname{Im}E_2 + h^{\frac{1}{2}}\operatorname{Im}E_3 + \cdots + h^{\frac{p-1}{2}}\operatorname{Im}E_p)$$

it follows that for all $0 \leq j \leq p-1$, $\operatorname{Im}E_j = 0$, i.e. $E_j \in \mathbb{R}$. □

## 3. The instability for the nonlinear Schrödinger equation

### 3.1. The error estimate. —

Proposition 3.1. — *Let $\alpha > 0$, $\sigma \in ]0, \frac{1}{4}]$ and let $v \in H^2(M)$ be such that*

$$\|v\|_{L^2} \lesssim 1, \quad \|v\|_{L^\infty} \lesssim h^{-\frac{1}{4}+\sigma}, \quad \|\Delta v\|_{L^\infty} \lesssim h^{-\frac{9}{4}+\sigma},$$

*and suppose that $v$ satisfies*

$$i\partial_t v + \Delta v = \varepsilon|v|^2 v + h^\alpha R(h),$$

*with for all $\beta \in [0,2]$, $\|R(h)\|_{H^\beta} \lesssim h^{-\beta}$. Let $u$ be solution of*

$$\begin{cases} i\partial_t u + \Delta u = \varepsilon|u|^2 u, \\ u(0,x) = v(0,x). \end{cases}$$

*Then, if $\alpha > \frac{1}{4} + 3\sigma$ we have*

$$\|(u-v)(t_h)\|_{H^\sigma} \longrightarrow 0 \quad \text{when} \quad h \longrightarrow 0,$$

*where $t_h \sim h^{\frac{1}{2}-2\sigma}\log(\frac{1}{h})$.*

*Proof.* — Define $w = u - v$ and

$$E(t) = \|w\|_{L^2}^2 + \|h^2\Delta w\|_{L^2}^2.$$

We have $E(0) = 0$ and the following estimates:

$$\|w\|_{L^2} \leq E^{\frac{1}{2}}, \quad \|\Delta w\|_{L^2} \leq h^{-2}E^{\frac{1}{2}}, \quad \|\nabla w\|_{L^2} \leq h^{-1}E^{\frac{1}{2}}. \tag{60}$$

The function $w$ satisfies the equation

$$i\partial_t w + \Delta w = \varepsilon(|w+v|^2(w+v) - |v|^2 v) - h^\alpha R(h). \tag{61}$$

The energy method gives

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{L^2}^2 &= \operatorname{Im}\int \overline{w}\big(\varepsilon(|w+v|^2(w+v) - |v|^2 v) - h^\alpha R(h)\big) \\ &\lesssim h^\alpha\|w\|_{L^2} + \|w\|_{L^4}^4 + \|w\|_{L^2}^2\|v\|_{L^\infty}^2. \end{aligned}$$

The Gagliardo-Nirenberg inequality gives

$$\|w\|_{L^4}^4 \lesssim \|w\|_{L^2}^2\|\nabla w\|_{L^2}^2 \lesssim h^{-2}E^2,$$

and as $\|v\|_{L^\infty} \lesssim h^{-\frac{1}{4}+\sigma}$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{L^2}^2 \lesssim h^\alpha E^{\frac{1}{2}} + h^{-\frac{1}{2}+2\sigma}E + h^{-2}E^2. \tag{62}$$

Now, apply $\Delta$ to (61)

$$i\partial_t \Delta w + \Delta^2 w = \varepsilon \Delta A - h^\alpha \Delta R(h), \tag{63}$$

with

$$\begin{aligned} A &= |w+v|^2(w+v) - |v|^2 v \\ &= 2w|v|^2 + \overline{w}v^2 + w^2\overline{v} + 2|w|^2 v + |w|^2 w, \end{aligned}$$

then

$$\begin{aligned} |\Delta A| \lesssim\ & |v|^2|\Delta w| + |v||\nabla v||\nabla w| + |\nabla v|^2|w| + |v||\Delta v||w| \\ & + |\Delta v||w|^2 + |w|^2|\Delta w| + |w||\nabla w|^2, \end{aligned}$$

hence

$$\begin{aligned} \|\Delta A\|_{L^2} \lesssim\ & \|v\|_{L^\infty}^2\|\Delta w\|_{L^2} + \|v\|_{L^\infty}\|\nabla v\|_{L^\infty}\|\nabla w\|_{L^2} + \|\nabla v\|_{L^\infty}^2\|w\|_{L^2} \\ & + \|v\|_{L^\infty}\|\Delta v\|_{L^\infty}\|w\|_{L^2} + \|\Delta v\|_{L^\infty}\|w\|_{L^4}^2 \\ & + \|w\|_{L^\infty}^2\|\Delta w\|_{L^2} + \|w\|_{L^2}\|\nabla w\|_{L^4}^2. \end{aligned} \tag{64}$$

The following inequality holds in dimension 2

$$\|w\|_{L^\infty} \lesssim \|w\|_{L^2}^{\frac12}\|\Delta w\|_{L^2}^{\frac12} \lesssim h^{-1}E^{\frac12},$$

and with (60) and (64) we deduce

$$\|\Delta A\|_{L^2} \lesssim h^{-\frac52+2\sigma}E^{\frac12} + h^{-\frac{13}{4}+\sigma}E + h^{-4}E^{\frac32}.$$

But

$$h^{-\frac{13}{4}+\sigma}E = h^{-\frac54+\sigma}E^{\frac14}h^{-2}E^{\frac34} \lesssim h^{-\frac52+2\sigma}E^{\frac12} + h^{-4}E^{\frac32},$$

and we obtain

$$\|\Delta(A)\|_{L^2} \lesssim h^{-\frac52+2\sigma}E^{\frac12} + h^{-4}E^{\frac32}. \tag{65}$$

Now, using (65) and $\|\Delta(R(h))\|_{L^2} \lesssim h^{-2}$, the energy method and the Cauchy-Schwarz inequality gives

$$\begin{aligned} \frac12\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta w\|_{L^2}^2 &= \operatorname{Im}\int \Delta\overline{w}\big(\Delta A - h^\alpha \Delta R(h)\big) \\ &\lesssim h^{-2}E^{\frac12}(h^{\alpha-2} + h^{-\frac52+2\sigma}E^{\frac12} + h^{-4}E^{\frac32}), \end{aligned} \tag{66}$$

therefore from (62) and (66) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}E \lesssim h^\alpha E^{\frac12} + h^{-\frac12+2\sigma}E + h^{-2}E^2.$$

Interpolation gives

$$\|w\|_{H^\sigma} \lesssim \|w\|_{L^2} + \|w\|_{\dot H^\sigma} \lesssim \|w\|_{L^2} + \|w\|_{L^2}^{1-\frac{\sigma}{2}}\|\Delta w\|_{L^2}^{\frac{\sigma}{2}} \lesssim h^{-\sigma}E^{\frac12} := F.$$

The function $F$ satisfies $F(0) = 0$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}F \lesssim h^{-\sigma+\alpha} + h^{-\frac12+2\sigma}F + h^{-2+2\sigma}F^3. \tag{67}$$

As long as $h^{-2+2\sigma}F^3 \lesssim h^{-\frac{1}{2}+2\sigma}F$, i.e. $F \lesssim h^{\frac{3}{4}}$, we can write

$$\frac{\mathrm{d}}{\mathrm{d}t}F \lesssim h^{-\sigma+\alpha} + h^{-\frac{1}{2}+2\sigma}F,$$

and the Gronwall inequality yields

$$F \lesssim h^{\alpha+\frac{1}{2}-3\sigma}\mathrm{e}^{Ch^{-\frac{1}{2}+2\sigma}t}.$$

The non linear term in (67) can be removed with the continuity argument for times such that

$$h^{\alpha+\frac{1}{2}-3\sigma}\mathrm{e}^{Ch^{-\frac{1}{2}+2\sigma}t} \lesssim h^{\frac{3}{4}+\eta},$$

with $\eta > 0$ i.e. for $t \lesssim (\alpha - \frac{1}{4} - 3\sigma - \eta)h^{\frac{1}{2}-2\sigma}\log\frac{1}{h}$, which is possible with $\eta$ small enough as we assume $\alpha > \frac{1}{4} + 3\sigma$. □

Corollary 3.2. — *Let $\kappa > 0$, $0 < \sigma < \frac{1}{4}$ and set $\delta = \kappa h^{\sigma}$. Denote by $v = e^{-i\lambda_3 t}u_3$ where $u_3$ and $\lambda_3$ are defined by* (25) *and* (26) *respectively.*
*Let $u$ be solution of*

$$\begin{cases} i\partial_t u + \Delta u = \varepsilon |u|^2 u, \\ u(0,x) = v(0,x). \end{cases}$$

*Then $\|v\|_{H^\sigma} \sim 1$ and*

$$\|(u-v)(t_h)\|_{H^\sigma} \longrightarrow 0 \quad \textit{when} \quad h \longrightarrow 0,$$

*where $t_h \sim h^{\frac{1}{2}-2\sigma}\log(\frac{1}{h})$.*

*Proof.* — The result directly follows from Propositions 2.3 and 3.1, as for all $0 < \sigma < \frac{1}{4}$, we have $\sigma + 1 > \frac{1}{4} + 3\sigma$. □

**3.2. The instability argument.** — Let $\kappa, \kappa_h > 0$ and consider $v = v^1$ defined in Corollary 3.2 associated with $\kappa$ and $v^2$ associated with $\kappa_h$. Let $u$ be a solution of

$$\begin{cases} i\partial_t u^j + \Delta u^j = \varepsilon |u^j|^2 u^j, \\ u^j(0,x) = v^j(0,x), \end{cases}$$

and $t_h \sim h^{\frac{1}{2}-2\sigma}\log\frac{1}{h}$. Then

$$\begin{aligned} \|(u^2-u^1)(t_h)\|_{H^\sigma} \geq \|(v^2-v^1)(t_h)\|_{H^\sigma} - \|(u^2-v^2)(t_h)\|_{H^\sigma} \\ -\|(u^1-v^1)(t_h)\|_{H^\sigma}. \end{aligned} \tag{68}$$

From Corollary 3.2 we deduce that for $j = 1, 2$

$$\|(u^j - v^j)(t_h)\|_{H^\sigma} \longrightarrow 0. \tag{69}$$

Observe that

$$\|(v^2-v^1)(t_h)\|_{H^\sigma} \sim \left|\mathrm{e}^{-i\lambda_3^2 t_h} - \mathrm{e}^{-i\lambda_3^1 t_h}\right| = \left|\mathrm{e}^{i(\lambda_3^2-\lambda_3^1)t_h} - 1\right|,$$

from Lemma 2.15 we have

$$(\lambda_3^2 - \lambda_3^1)t_h \sim h^{2\sigma-1}(\kappa - \kappa_h)t_h \sim (\kappa - \kappa_h)\log\frac{1}{h}.$$

It is possible to choose $\kappa_h$ such that $\kappa_h \longrightarrow \kappa$ and $(\kappa - \kappa_h)\log\frac{1}{h} \longrightarrow \infty$. Then using (68) and (69)

$$\limsup_{h\longrightarrow 0} \|(u^2 - u^1)(t_h)\|_{H^\sigma} \geq \limsup_{h\longrightarrow 0} \|(v^2 - v^1)(t_h)\|_{H^\sigma} \geq 2,$$

even though

$$\|(u^2 - u^1)(0)\|_{H^\sigma} = \|(v^2 - v^1)(0)\|_{H^\sigma} \sim |\kappa - \kappa_h|,$$

which tends to 0 with $h$. According to Definition 1.1, we have proved Proposition 1.3.

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