LOW REGULARITY FOR A QUADRATIC SCHRÖDINGER EQUATION ON $\mathbb{T}$

by

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Abstract. — In this paper we consider a Schrödinger equation on the circle with a quadratic nonlinearity. Thanks to an explicit computation of the first Picard iterate, we give a precision on the dynamic of the solution, whose existence was proved by C. E. Kenig, G. Ponce and L. Vega [**15**]. We also show that the equation is well-posed in a space $\mathcal{H}^{s,p}(\mathbb{T})$ which contains the Sobolev space $H^s(\mathbb{T})$ when $p \geq 2$.

Résumé. — Dans cet article on s'intéresse à une équation de Schrödinger sur le cercle avec une non-linéarité quadratique. Un calcul explicite de la première itérée de Picard permet de donner une précision sur la dynamique de la solution, dont l'existence a été démontrée par C. E. Kenig, G. Ponce et L. Vega [**15**]. On montre également que l'équation est bien posée dans un espace $\mathcal{H}^{s,p}(\mathbb{T})$ qui contient l'espace de Sobolev $H^s(\mathbb{T})$ lorsque $p \geq 2$.

1. Introduction

Denote by $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ the unidimensional torus. In this paper we consider the following nonlinear Schrödinger equation

$$(1.1) \qquad \begin{cases} i\partial_t u + \Delta u = \kappa \overline{u}^2, \quad \kappa = \pm 1, \quad (t,x) \in \mathbb{R} \times \mathbb{T}, \\ u(0,x) = f(x) \in X, \end{cases}$$

where X is a Banach space (the space of the initial conditions).

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This equation has been intensively studied in the case $x \in M$ where M is a Riemannian manifold and for different nonlinearities, usually of the form

$$F(u,\overline{u}) = \pm u^{p_1}\,\overline{u}^{p_2}, \quad \text{where} \quad p_1, p_2 \in \mathbb{N}.$$

Here we mainly discuss the results in one dimension for quadratic nonlinearities. For the other cases see [**7**], [**15**], [**3**], and references therein.

1.1. Previous results on the real line. —

In the case $x \in \mathbb{R}$, J. Ginibre and G. Velo [**10**], Y. Tsutsumi [**18**], T. Cazenave and F. B. Weissler [**7**] showed that the Cauchy problem is well-posed for $f \in L^2(\mathbb{R})$, for every nonlinearity of the type (1.2) with $p_1 + q_1 \leq 5$. The proof relies on the use of Strichartz inequalities, which are of the form

$$\|\mathrm{e}^{it\Delta} f\|_{L^p(\mathbb{R},L^q(\mathbb{R}))} \leq C\|f\|_{L^2(\mathbb{R})}, \quad \text{with} \quad \frac{1}{p} + \frac{2}{q} = \frac{1}{2}. \tag{1.2}$$

In [**15**], C. E. Kenig, G. Ponce, and L. Vega show that (1.1) is well-posed in $X = H^s(\mathbb{R})$:

- for $s > -3/4$ in the case $F(u,\overline{u}) = \pm u^2$ or $F(u,\overline{u}) = \pm\overline{u}^2$;
- for $s > -1/4$ in the case $F(u,\overline{u}) = \pm|u|^2$.

To obtain these results, the authors prove some bilinear estimates in the conormal spaces $X^{s,b}$ (see Definition 1.2), and they also show that these estimates are optimal, and as a consequence it is impossible to perform a usual fixed point argument in these spaces, below the threshold $s = -3/4$ (resp. $s = -1/4$). Notice that the $X^{s,b}$ spaces distinguish the structure of the nonlinearity, which was not the case for the Strichartz spaces.

In [**1**], I. Bejenaru and T. Tao extend the well-posedness results to $s \geq -1$ in the case $F(u,\overline{u}) = u^2$, and show that the equation (1.1) is ill-posed in $H^s(\mathbb{R})$ when $s < -1$.

Recently, N. Kishimoto [**16**] extended the previous result to the case $F(u,\overline{u}) = \alpha u^2 + \beta\overline{u}^2$.

1.2. Previous results on the torus. —

In the case $x \in \mathbb{T}$, J. Bourgain [**2**] established the embedding $X^{0,3/8} \subset L^4_{x,t}$, which permitted to show that the problem (1.1) is locally well-posed in $L^2(\mathbb{T})$, for every nonlinearity (1.2) with $p_1 + p_2 \leq 3$.

Then, C. E. Kenig, G. Ponce, and L. Vega [**15**], thanks to bilinear estimates in

$X^{s,b}$ (see Theorem 1.4 below), obtained the well-posedness of (1.1) in $H^s(\mathbb{T})$ for $s > -1/2$ in the case $F(u,\overline{u}) = \pm u^2$ or $F(u,\overline{u}) = \pm \overline{u}^2$. Again, these estimates fail if $s < -1/2$.

1.3. The $\mathcal{H}^{s,p}(\mathbb{T})$ and $X^{s,b}$ spaces. —

Now we introduce the $\mathcal{H}^{s,p}(\mathbb{T})$ spaces

Definition 1.1. — ($\mathcal{H}^{s,p}$ spaces)
For $s \in \mathbb{R}$ and $p \geq 1$, denote by $\mathcal{H}^{s,p} = \mathcal{H}^{s,p}(\mathbb{T})$ the completion of $\mathcal{C}^\infty(\mathbb{T})$ with respect to the norm

$$\|f\|_{\mathcal{H}^{s,p}} = \Big(\sum_{n\in\mathbb{Z}} \langle n\rangle^{ps} \, |\breve{f}(n)|^p \Big)^{\frac{1}{p}}.$$

Here $\breve{f}(n)$ denotes the Fourier coefficient of f (see (2.6)).

These spaces were introduced by L. Hörmander (see [**14**], Section 10.1).

There are several motivations to introduce these spaces

• First notice that $\mathcal{H}^{s,2}(\mathbb{T}) = H^s(\mathbb{T})$, and for $p > 2$ we have the (strict) inclusion $H^s(\mathbb{T}) \subset \mathcal{H}^{s,p}(\mathbb{T})$.

• Let v be a smooth function. For $\lambda > 0$, we define $v_\lambda(x) = v(\lambda x)$ (observe that v_λ is defined on a torus of period $2\pi/\lambda$). We can compute, for $\lambda \to 0$, $\|v_\lambda\|_{\mathcal{H}^{s,p}} \sim \lambda^{s+1/p-1}$, therefore the space $\mathcal{H}^{s,p}$ scales like $H^{s(p)}$ where $s(p) = s + 1/p - 1/2$. Hence, if $s(p) < -1/2$, the space $\mathcal{H}^{s,p}$ contains elements f such that $|\breve{f}(n)| \longrightarrow +\infty$ when $n \longrightarrow +\infty$. Therefore we can go closer to the scaling of the equation (1.1) which is $-3/2$.

• T. Cazenave, L. Vega and M. C. Vilela [**6**] were the first authors to study nonlinear Schrödinger equations in $\mathcal{H}^{s,p}$-like spaces. In fact they show that a class of NLS equations on $\mathbb{R}^N$ is well-posed if the linear flow belongs to some weak L^p space. Moreover they prove that this condition can be ensured if the initial data f satisfies $\widehat{f} \in L^{p,\infty}(\mathbb{R}^N)$ for some $p \geq 1$. This latter space is a continuous version of the space $\mathcal{H}^{s,p}$.

• In [**12**] A. Grünrock establishes bilinear and trilinear estimates in conormal spaces $X^{s,b}_{p,q}$ (see definition below) based on L^r. This permits him to show that the cubic Schrödinger equation

$$i\partial_t u + \Delta u = \pm |u|^2 u, \;\; (t,x) \in \mathbb{R}\times\mathbb{R},$$

is well-posed for initial conditions in the corresponding continuous version of the space $\mathcal{H}^{s,p}$. He obtains analogous results for the DNLS equation [**12**] and for the mKdV equation [**11**].

In [**8**], M. Christ shows that the modified cubic problem

$$\begin{cases} i\partial_t u + \Delta u = \pm\big(|u|^2 - 2\mu(|u|^2)\big)u, \ \text{where } \mu(|v|^2) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} |v(x)|^2 \mathrm{d}x, \\ u(0,x) = f(x) \in \mathcal{H}^{s,p}(\mathbb{T}), \end{cases}$$

is well-posed in $\mathcal{H}^{s,p}(\mathbb{T})$ for any $s \geq 0$ and $p \geq 1$. See [**8**] for precise statements.

Recently, A. Grünrock and S. Herr [**13**] have shown the well-posedness in $\mathcal{H}^{s,p}$ spaces of the DNLS equation on the torus, thanks to multilinear estimates.

See [**8, 11, 12, 13**] for other features of the spaces $\mathcal{H}^{s,p}$ and more references.

• Notice that the $\mathcal{H}^{s,p}$ is preserved by the linear Schrödinger flow. Write

$$f(x) = \sum_{n\in\mathbb{Z}} \alpha_n \mathrm{e}^{inx}, \ \ \text{then} \ \ \mathrm{e}^{it\Delta} f(x) = \sum_{n\in\mathbb{Z}} \alpha_n \mathrm{e}^{-in^2 t}\mathrm{e}^{inx},$$

and for all $t \in \mathbb{R}$, $\|e^{it\Delta} f\|_{\mathcal{H}^{s,p}} = \|f\|_{\mathcal{H}^{s,p}}$.

We now define the $X^{s,b}$ spaces

Definition 1.2. — ($X^{s,b}$ spaces)
(i) For $s, b \in \mathbb{R}$, denote by $X^{s,b} = X^{s,b}(\mathbb{R}\times\mathbb{T})$ the completion of $\mathcal{C}^\infty\big(\mathbb{T}, \mathcal{S}(\mathbb{R})\big)$ with respect to the norm

$$\|F\|_{X^{s,b}} = \Big(\sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} \langle \tau + n^2\rangle^{2b} \langle n\rangle^{2s} |\widetilde{F}(\tau, n)|^2 \mathrm{d}\tau\Big)^{\frac{1}{2}}.$$

(ii) Let $T > 0$, we define the restriction spaces $X_T^{s,b} = X^{s,b}([-T,T]\times\mathbb{T})$ by

$$\|F\|_{X_T^{s,b}} = \inf\Big\{ \|G\|_{X^{s,b}}, \ G \in X^{s,b} \text{ with } \ G|_{[-T,T]} = F \Big\}.$$

Here $\widetilde{F}$ stands for the space-time Fourier transform (see (2.7)).

In the following, we will mainly use the space $X_1^{s,b} = X^{s,b}([-1,1]\times\mathbb{T})$.

We recall the key estimates which permit to perform a fixed point argument in the $X^{s,b}$ spaces, and to deduce that the equation (1.1) is well-posed in H^s for $s > -\frac{1}{2}$.

Proposition 1.3. — *Let $s \in \mathbb{R}$ and $\frac{1}{2} < b \leq 1$. Then for all $F \in X_1^{s,b-1}$, we have*

$$\| \int_0^t e^{i(t-t')\Delta} F(t', \cdot)\, dt' \|_{X_1^{s,b}} \leq C \|F\|_{X_1^{s,b-1}} .$$

See [**9**] for a proof. Notice this estimate holds in the general case of a Riemannian manifold, indeed the proof reduces to time integrations. Notice also that we always have the estimate

$$\|e^{it\Delta}f\|_{X_1^{s,b}} \leq C\|f\|_{H^s},$$

but we won't use it in this paper.

The following theorem is one of the main results of [**15**] (see Theorem 1.9. in [**15**])

Theorem 1.4. — *(Kenig-Ponce-Vega* [**15**]*) Let $-\frac{1}{2} < s \leq 0$, then there exists $b_0 > \frac{1}{2}$ such that for all $\frac{1}{2} < b \leq b_0$ and all $v, w \in X^{s,b}(\mathbb{R}\times\mathbb{T})$*

$$\|\overline{v}\,\overline{w}\|_{X^{s,b-1}} \lesssim \|v\|_{X^{s,b}}\|w\|_{X^{s,b}}. \tag{1.3}$$

Moreover, for any $s < -\frac{1}{2}$ and $b \in \mathbb{R}$, (1.3) *fails.*

We can deduce the following

Corollary 1.5. — *Let $-\frac{1}{2} < s \leq 0$, then there exists $b_0 > \frac{1}{2}$ such that for all $\frac{1}{2} < b \leq b_0$ and all $v, w \in X^{s,b}([-1,1]\times\mathbb{T})$*

$$\|\overline{v}\,\overline{w}\|_{X_1^{s,b-1}} \lesssim \|v\|_{X_1^{s,b}}\|w\|_{X_1^{s,b}}. \tag{1.4}$$

Proof. — For $N \geq 1$, let $v_N, w_N \in X^{s,b}$ be so that $w = v_N$ and $w = w_N$ for $t \in [-1,1]$. Then by (1.3) and the definition of $X_1^{s,b}$ we obtain

$$\|\overline{v}\,\overline{w}\|_{X_1^{s,b-1}} \leq \|\overline{v_N}\,\overline{w_N}\|_{X^{s,b-1}} \lesssim \|\overline{v_N}\|_{X^{s,b}}\|\overline{w_N}\|_{X^{s,b}}.$$

The result follows by choosing two appropriate sequences v_N, w_N so that $\|\overline{v_N}\|_{X^{s,b}}\|\overline{w_N}\|_{X^{s,b}} \to \|\overline{v}\|_{X_1^{s,b}}\|\overline{w}\|_{X_1^{s,b}}$ when $N \to \infty$. □

2. Main results of this paper

2.1. Local well-posedness in the Sobolev scale. —

Our first result is a precision on the dynamic of the solution of (1.1) when the initial condition f is in $H^{s_0}(\mathbb{T})$ with $-\frac{1}{2} < s_0 \leq 0$.
Let $f \in \mathcal{D}'(\mathbb{T})$. Then define

$$u_0(t,x) = e^{it\Delta}f(x) = \sum_{n\in\mathbb{Z}} \check{f}(n)e^{-in^2t}e^{inx},$$

the free Schrödinger evolution and

$$u_1(t,x) = -i\int_0^t \mathrm{e}^{i(t-t')\Delta}(\overline{u_0}^2)(t',x)\mathrm{d}t',$$

the first Picard iterate of the equation (1.1). Then we will show that there exists $b > \frac{1}{2}$ so that

$$\|u_1\|_{X^{0,b}([-1,1]\times\mathbb{T})} \lesssim \|f\|^2_{H^{s_0}(\mathbb{T})}. \tag{2.1}$$

Hence, u_1 is more regular than f : there is a gain of $|s_0|$ derivative. We will take profit of this phenomenon to prove that it is also the case for $u - \mathrm{e}^{it\Delta}f$, where u is the solution of (1.1).

Theorem 2.1. — *Let* $\kappa = \pm 1$. *Let* $-\frac{1}{2} < s_0 \leq 0$ *and* $f \in H^{s_0}(\mathbb{T})$. *Then there exist* $b > \frac{1}{2}$ *and* $T > 0$ *such that there exists a unique solution* u *to* (1.1) *in the space*

$$Y_T^{0,b} = \Big(e^{it\Delta}f + X^{0,b}\big([-T,T]\times\mathbb{T}\big)\Big). \tag{2.2}$$

This result will be obtained with a contraction argument in the space $X^{0,b}$ (thanks to the gain of regularity), and therefore we will only need the estimate (1.3) with $s = 0$.

2.2. Local well-posedness in the $\mathcal{H}^{s,p}$ scale. —

We can use the gain of regularity of the first Picard iterate to solve the Cauchy problem (1.1) for data $f \in \mathcal{H}^{s,p}(\mathbb{T})$, and this will improve slightly the result of [**15**], as we have the inclusion $H^{s_0}(\mathbb{T}) \subset \mathcal{H}^{s_0,p}(\mathbb{T})$ for $p \geq 2$.

The following condition on the real numbers s_0 and p will be needed for our result

$$\frac{3}{p} + s_0 > \frac{1}{2}. \tag{2.3}$$

Theorem 2.2. — *Let* $\kappa = \pm 1$. *Let* $s_0 > -\frac{1}{2}$ *and let* $2 < p < 4$ *be so that the condition* (2.3) *is satisfied. Let* $f \in \mathcal{H}^{s_0,p}(\mathbb{T})$. *Then for all* $s_1 < -1 + \frac{2}{p}$ *there exist* $b > \frac{1}{2}$, $s_1 < s < -1 + \frac{2}{p}$, *and* $T > 0$ *such that there exists a unique solution* u *to* (1.1) *in the space*

$$Y_T^{s,b} = \Big(e^{it\Delta}f + X^{s,b}\big([-T,T]\times\mathbb{T}\big)\Big). \tag{2.4}$$

To prove Theorem 2.2 we will use the estimate (1.3) in its full strength.

From the previous result, we can immediately deduce

Corollary 2.3. — *Let $\alpha < \frac{1}{6}$ and let $f \in \mathcal{D}'(\mathbb{T})$ be such that $|\check{f}(n)| \lesssim \langle n\rangle^{\alpha}$. Then there exist $s > -\frac{1}{3}$, $b > \frac{1}{2}$ and $T > 0$ such that there exists a unique solution to* (1.1) *in the space*

$$Y_T^{s,b} = \Big(e^{it\Delta} f + X^{s,b}\big([-T,T]\times\mathbb{T}\big)\Big).$$

For instance : Let $0 < \varepsilon < 1$ be small and $\alpha = \frac{1}{6} - \varepsilon$. Define $f \in \mathcal{D}'(\mathbb{T})$ by $\check{f}(n) = \langle n\rangle^{\alpha}$. Then $f \in H^s(\mathbb{T})$ for $s < -\frac{1}{2} - \frac{1}{6} + \varepsilon < -\frac{1}{2}$, but $f \in \mathcal{H}^{s_0,p}(\mathbb{T})$ for some (s_0, p) which satisfies the assumptions of Theorem 2.2

Remark 2.4. — The result of Theorem 2.2 is interesting when s_0 is close to $-\frac{1}{2}$, and p as big as possible, under the assumption (2.3).
Let $0 < \varepsilon < 1$ be small and set $s_0 = -\frac{1}{2} + \varepsilon$. Then $p > 2$ satisfies (2.3) iff

$$\frac{1}{3} - \frac{1}{3}\varepsilon < \frac{1}{p} < \frac{1}{2}.$$

Hence, the parameter s in Theorem 2.2 can be chosen close to $-\frac{1}{3}$. In other words there is a gain of $\sim \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$ derivative.

2.3. Notations and plan of the paper. —

For $F \in \mathcal{S}(\mathbb{R})$ we define the time-Fourier transform by

$$\widehat{F}(\tau) = \int_{\mathbb{R}} \mathrm{e}^{-i\tau t} F(t)\mathrm{d}t,$$

which has the following properties

$$\widehat{\overline{F}}(\tau) = \overline{\widehat{F}}(-\tau) \quad \text{and} \quad \widehat{F\mathrm{e}^{i\theta\cdot}}(\tau) = \widehat{F}(\tau-\theta) \ \text{ for all } \ \theta \in \mathbb{R}. \tag{2.5}$$

Each $F \in \mathcal{C}^{\infty}\big(\mathbb{T}; \mathcal{S}(\mathbb{R})\big)$ admits the Fourier expansion

$$F(t,x) = \sum_{n\in\mathbb{Z}} \check{F}(t,n)\mathrm{e}^{inx}, \ \text{ where } \ \check{F}(t,n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{-inx} F(t,x)\mathrm{d}x, \tag{2.6}$$

is the periodic Fourier coefficient of F.
Finally, we denote by

$$\widetilde{F}(\tau,n) = \frac{1}{2\pi}\int_{\mathbb{R}}\int_{-\pi}^{\pi} \mathrm{e}^{-i(\tau t+nx)} F(t,x)\mathrm{d}t\mathrm{d}x, \tag{2.7}$$

the space-time Fourier transform.

Notations. — *In this paper c, C denote constants the value of which may change from line to line. These constants will always be universal, or depending only on fixed quantities. We use the notations $a \sim b$, $a \lesssim b$ if $\frac{1}{C}b \le a \le Cb$, $a \le Cb$ respectively.*

In Section 3 we make explicit computations to estimate the first Picard iteration in $X^{s,b}$ spaces.
Then, in Section 4 we establish a bilinear estimate in $X^{s,b}$ spaces.
In Section 5, we follow an idea of N. Burq and N. Tzvetkov [**4, 5**] and look for a solution of (1.1) of the form $u = \mathrm{e}^{it\Delta} f + v$. The existence and uniqueness of v is then proved with a fixed point argument, using the estimates of the previous sections.

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3. The first Picard iteration

We will need the following result

Lemma 3.1. — *Let $\varphi \in \mathcal{S}(\mathbb{R})$. Then*
$$\int_{\mathbb{R}} \frac{1}{\langle \tau + A \rangle} |\varphi|(\tau)\, d\tau \lesssim \frac{1}{\langle A \rangle},$$
uniformly in $A \in \mathbb{R}$.

Proof. — As φ is in the Schwartz class $|\varphi|(\tau) \lesssim \langle \tau \rangle^{-3}$.
Then notice that $\langle \tau \rangle \langle \tau + A \rangle \gtrsim \langle A \rangle$, therefore
$$\int_{\mathbb{R}} \frac{\langle A \rangle}{\langle \tau + A \rangle} |\varphi|(\tau) \mathrm{d}\tau \lesssim \int_{\mathbb{R}} \frac{\langle A \rangle}{\langle \tau \rangle \langle \tau + A \rangle} \frac{1}{\langle \tau \rangle^2} \mathrm{d}\tau \lesssim 1,$$
hence the result. □

Let $f \in \mathcal{D}'(\mathbb{T})$, denote by $\alpha_n = \check{f}(n)$. Then define

(3.1) $$u_0(t,x) = \mathrm{e}^{it\Delta} f(x) = \sum_{n \in \mathbb{Z}} \alpha_n \mathrm{e}^{-in^2 t} \mathrm{e}^{inx},$$

the free Schrödinger evolution and

(3.2) $$u_1(t,x) = -i \int_0^t \mathrm{e}^{i(t-t')\Delta} (\overline{u_0}^2)(t', x) \mathrm{d}t',$$

which is the first Picard iterate of the equation (1.1).

Proposition 3.2. — *Let $s_0 > -\frac{1}{2}$ and $2 \leq p \leq 4$. Then there exists $b_1 > \frac{1}{2}$ such that for all $\frac{1}{2} < b < b_1$, all $f \in \mathcal{H}^{s_0,p}(\mathbb{T})$ and all $s < -1 + 2/p$ we have*

$$\|u_1\|_{X^{s,b}([-1,1]\times\mathbb{T})} \lesssim \|f\|^2_{\mathcal{H}^{s_0,p}(\mathbb{T})}. \tag{3.3}$$

Moreover, in the case $p = 2$, the estimate (3.3) *holds for $s = 0$.*

Remark 3.3. — The result of Proposition 3.2 shows that the first Picard iterate is more regular than the initial condition, when s_0 is close to $-\frac{1}{2}$ and $p < 4$. In this case, we can take $s > s_0$.
The result we stated is not optimal when s_0 is far from $-\frac{1}{2}$.

Proof. — Let $b > \frac{1}{2}$ to be chosen later. Denote by $\beta = 2(1-b) < 1$ and $\sigma = -s \geq 0$.
Let $\psi_0 \in \mathcal{C}_0^\infty(\mathbb{R})$ s.t. $\psi_0 = 1$ on $[-1,1]$, and $\psi \in \mathcal{C}_0^\infty(\mathbb{R})$ s.t. $\psi_0\psi = \psi_0$. Then by Definition 1.2 and Proposition 1.3 we have

$$\begin{aligned} \|u_1\|_{X^{s,b}([-1,1]\times\mathbb{T})} &\leq \|\psi_0(t)\, u_1\|_{X^{s,b}(\mathbb{R}\times\mathbb{T})} \\ &\lesssim \|\psi(t)\, \overline{u_0}^2\|_{X^{s,b-1}(\mathbb{R}\times\mathbb{T})}. \end{aligned} \tag{3.4}$$

Now by the expression (3.1), we have (with the change of variables $p = -n-m$)

$$\begin{aligned} \psi(t)(\overline{u_0}^2) &= \psi(t) \sum_{(n,m)\in\mathbb{Z}^2} \overline{\alpha_n}\,\overline{\alpha_m}\ \mathrm{e}^{i(n^2+m^2)t}\mathrm{e}^{-i(n+m)x} \\ &= \psi(t)\sum_{p\in\mathbb{Z}}\Big(\sum_{n\in\mathbb{Z}} \overline{\alpha_n}\,\overline{\alpha_{-n-p}}\ \mathrm{e}^{i(n^2+(n+p)^2)t}\Big)\mathrm{e}^{ipx}. \end{aligned}$$

Hence we deduce the Fourier coefficients of $\psi(t)(\overline{u_0}^2)$:

$$c_p(t) := \psi(t)\sum_{n\in\mathbb{Z}} \overline{\alpha_n}\,\overline{\alpha_{-n-p}}\ \mathrm{e}^{i(n^2+(n+p)^2)t} = \psi(t)(\check{\overline{u_0}^2})(p). \tag{3.5}$$

From the properties (2.5) of the time-Fourier transform, we deduce

$$\widehat{c_p}(\tau) = \sum_{n\in\mathbb{Z}} \overline{\alpha_n}\,\overline{\alpha_{-n-p}}\ \widehat{\psi}(\tau - n^2 - (n+p)^2), \tag{3.6}$$

and by Definition 1.2, we have

$$I := \|\psi(t)(\overline{u_0}^2)\|^2_{X^{s,b-1}(\mathbb{R}\times\mathbb{T})} = \sum_{p\in\mathbb{Z}}\int_{\mathbb{R}} \langle \tau + p^2\rangle^{-\beta}\langle p\rangle^{2s}|\widehat{c_p}(\tau)|^2 \mathrm{d}\tau,$$

with $\beta = 2(1-b)$. Now, we claim that

$$|\widehat{c_p}(\tau)|^2 \lesssim \sum_{n\in\mathbb{Z}} |\alpha_n|^2|\alpha_{-n-p}|^2|\widehat{\psi}|(\tau - n^2 - (n+p)^2), \tag{3.7}$$

uniformly in $(\tau,p)\in\mathbb{R}\times\mathbb{Z}$. Indeed, by Cauchy-Schwarz

$$|\widehat{c_p}(\tau)|^2\leq\sum_{n\in\mathbb{Z}}|\alpha_n|^2|\alpha_{-n-p}|^2|\widehat{\psi}|(\tau-n^2-(n+p)^2)\sum_{n\in\mathbb{Z}}|\widehat{\psi}|(\tau-n^2-(n+p)^2),$$

and the fact that $\sum_{n\in\mathbb{Z}}|\widehat{\psi}|(\tau-n^2-(n+p)^2)\lesssim 1$ uniformly in $(\tau,p)\in\mathbb{R}\times\mathbb{Z}$, follows from (4.5).

With the change of variables $m=-n-p$, $\tau'=\tau-n^2-m^2$ and (3.7) we deduce

$$\begin{aligned}
I &\lesssim \sum_{n\in\mathbb{Z}}\sum_{p\in\mathbb{Z}}\int_{\mathbb{R}}\frac{\langle p\rangle^{2s}}{\langle \tau+p^2\rangle^{\beta}}|\alpha_n|^2|\alpha_{-n-p}|^2|\widehat{\psi}|(\tau-n^2-(n+p)^2)\mathrm{d}\tau\\
&= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\int_{\mathbb{R}}\frac{\langle n+m\rangle^{2s}}{\langle \tau+(n+m)^2\rangle^{\beta}}|\alpha_n|^2|\alpha_{m}|^2|\widehat{\psi}|(\tau-n^2-m^2)\mathrm{d}\tau\\
&= \sum_{(n,m)\in\mathbb{Z}^2}\int_{\mathbb{R}}\frac{\langle n+m\rangle^{2s}}{\langle \tau+(n+m)^2+n^2+m^2\rangle^{\beta}}|\alpha_n|^2|\alpha_{m}|^2|\widehat{\psi}|(\tau)\mathrm{d}\tau. \qquad (3.8)
\end{aligned}$$

Apply Lemma 3.1 with $A=(n+m)^2+n^2+m^2$. Denote by $\sigma=-s\geq 0$. Then from (3.8) we deduce

$$I\lesssim\sum_{(n,m)\in\mathbb{Z}^2}\frac{|\alpha_n|^2|\alpha_m|^2}{\langle n+m\rangle^{2\sigma}\langle n^2+m^2\rangle^{\beta}}. \qquad (3.9)$$

• From here we assume that $\sigma>0$.

For $m\in\mathbb{Z}$, denote by

$$\gamma_m=\sum_{n\in\mathbb{Z}}\frac{|\alpha_n|^2}{\langle n+m\rangle^{2\sigma}\langle n\rangle^{\beta}},$$

thanks to the inequality $\langle n^2+m^2\rangle\geq\langle n\rangle\langle m\rangle$, from (3.9) we deduce

$$I\lesssim\sum_{m\in\mathbb{Z}}\bigg(\frac{|\alpha_m|^2}{\langle m\rangle^{\beta}}\Big(\sum_{n\in\mathbb{Z}}\frac{|\alpha_n|^2}{\langle n+m\rangle^{2\sigma}\langle n\rangle^{\beta}}\Big)\bigg)=\sum_{m\in\mathbb{Z}}\gamma_m\,\frac{|\alpha_m|^2}{\langle m\rangle^{\beta}}. \qquad (3.10)$$

Now by Hölder, for $p\geq 2$

(3.11)

$$\sum_{m\in\mathbb{Z}}\gamma_m\,\frac{|\alpha_m|^2}{\langle m\rangle^{\beta}}\lesssim\Big(\sum_{k\in\mathbb{Z}}\frac{|\alpha_k|^p}{\langle k\rangle^{\beta p/2}}\Big)^{\frac{2}{p}}\Big(\sum_{m\in\mathbb{Z}}\gamma_m^{q_1}\Big)^{\frac{1}{q_1}}=\|f\|^2_{\mathcal{H}^{-\beta/2,p}}\Big(\sum_{m\in\mathbb{Z}}\gamma_m^{q_1}\Big)^{\frac{1}{q_1}},$$

with

$$\frac{1}{q_1}=1-\frac{2}{p}. \qquad (3.12)$$

To estimate the last term in (3.11), we observe that

$$\gamma_m = \Big(\frac{|\alpha_k|^2}{\langle k\rangle^\beta} * \frac{1}{\langle j\rangle^{2\sigma}} \Big)(- m),$$

then by Young's inequality, for all $p_1, r_1 \geq 1$ so that

$$\frac{1}{q_1} = \frac{1}{p_1} + \frac{1}{r_1} - 1, \tag{3.13}$$

and so that for $2\sigma r_1 > 1$, we have

$$\Big(\sum_{m\in\mathbb{Z}} \gamma_m^{q_1} \Big)^{\frac{1}{q_1}} \lesssim \Big(\sum_{k\in\mathbb{Z}} \frac{|\alpha_k|^{2p_1}}{\langle k\rangle^{\beta p_1}} \Big)^{\frac{1}{p_1}} \Big(\sum_{j\in\mathbb{Z}} \frac{1}{\langle j\rangle^{2\sigma r_1}} \Big)^{\frac{1}{r_1}}. \tag{3.14}$$

We take $p_1 = p/2$. Therefore by (3.12) and (3.13) we deduce $r_1 = p/(2p-4)$ (observe that $r_1 \geq 1$ since $p \leq 4$). The condition $2\sigma r_1 > 1$ yields

$$\sigma > \frac{1}{2r_1} = 1 - \frac{2}{p},$$

and thus by (3.10), (3.11) and (3.14) we obtain

$$I \lesssim \|f\|^4_{\mathcal{H}^{-\beta/2,p}}.$$

Now we choose $b > \frac{1}{2}$ such that $\beta = -2s_0$, i.e. $b = 1 - \beta/2 = 1 + s_0$, and thus $\frac{1}{2} < b \leq 1$, as we assumed that $-\frac{1}{2} < s_0 \leq 0$.
Together with (3.4), this concludes the proof of the first statement of Proposition 3.2.

• Now we deal with the case $p = 2$ and $\sigma = 0$.

By (3.9) we only have to bound the term

$$J := \sum_{(n,m)\in\mathbb{Z}^2} \frac{|\alpha_n|^2 |\alpha_m|^2}{\langle n^2+m^2\rangle^\beta}.$$

Thanks to the inequality $\langle n^2+m^2\rangle \geq \langle n\rangle\langle m\rangle$, we get

$$J \leq \sum_{(n,m)\in\mathbb{Z}^2} \frac{|\alpha_n|^2 |\alpha_m|^2}{\langle n\rangle^\beta \langle m\rangle^\beta} = \|f\|^4_{H^{s_0}},$$

which was the claim. □

4. The bilinear estimate

This section is devoted to the proof of the following result

Proposition 4.1. — *Let* $-\frac{1}{2} < s_0 \leq 0$ *and* $p \geq 2$. *Then for all*

$$-\frac{1}{2} - s_0 - \frac{1}{p} < s \leq 0, \tag{4.1}$$

there exists $b_2 > \frac{1}{2}$ *such that for all* $\frac{1}{2} < b < b_2$, *all* $f \in \mathcal{H}^{s_0,p}(\mathbb{T})$ *and all* $v \in X_1^{s,b}([-1,1]\times\mathbb{T})$

$$\| \int_0^t e^{i(t-t')\Delta}\overline{u_0}\,\overline{v}(t',\cdot)\,dt' \|_{X^{s,b}([-1,1]\times\mathbb{T})} \lesssim \|f\|_{\mathcal{H}^{s_0,p}} \|v\|_{X^{s,b}([-1,1]\times\mathbb{T})}\,, \tag{4.2}$$

where $u_0(t) = e^{it\Delta}f$.

Proposition 4.1 shows that, under condition (4.1), the term

$$\int_0^t \mathrm{e}^{i(t-t')\Delta}\overline{u_0}\,\overline{v}(t',\cdot)\mathrm{d}t',$$

has the regularity of v, even if f is less regular. For instance, with $p = 2$ and $s = 0$, we obtain

$$\| \int_0^t \mathrm{e}^{i(t-t')\Delta}\overline{u_0}\,\overline{v}(t',\cdot)\mathrm{d}t' \|_{X_1^{0,b}} \lesssim \|f\|_{H^{s_0}} \|v\|_{X_1^{0,b}},$$

whenever $s_0 > -\frac{1}{2}$.

We now state a few technical results.

We will need the following lemma which is proved in [**15**].

Lemma 4.2. — *If* $\gamma > \frac{1}{2}$, *then we have*

$$\sup_{y\in\mathbb{R}} \sum_{n\in\mathbb{Z}} \frac{1}{\langle n-y\rangle^{2\gamma}} < \infty, \tag{4.3}$$

and

$$\sup_{(y,z)\in\mathbb{R}^2} \sum_{n\in\mathbb{Z}} \frac{1}{\langle z+n(n-y)\rangle^{\gamma}} < \infty. \tag{4.4}$$

Proof. — • Let $y \in \mathbb{R}$. Up to a shift in n, we can assume that $y \in [0,1[$. Then $\langle n-y\rangle \geq \frac{1}{2}\langle n\rangle$, hence the estimate (4.3).

• Denote by $r_1 = r_1(y,z)$ and $r_2 = r_2(y,z)$ the complex roots of the polynomial $z + X(X-y)$. Then

$$z + n(n-y) = (n-r_1)(n-r_2).$$

There are at most 10 indexes n such that $|n-r_1| \leq 2$ or $|n-r_2| \leq 2$. The remaining n's satisfy

$$\big\langle (n-r_1)(n-r_2)\big\rangle \geq \frac{1}{2}\langle n-r_1\rangle\langle n-r_2\rangle.$$

Hence by the Cauchy-Schwarz inequality

$$\sum_{n\in\mathbb{Z}} \frac{1}{\langle z+n(n-y)\rangle^{\gamma}} \lesssim \Big(\sum_{n\in\mathbb{Z}} \frac{1}{\langle n-r_1\rangle^{2\gamma}}\Big)^{\frac12}\Big(\sum_{n\in\mathbb{Z}} \frac{1}{\langle n-r_2\rangle^{2\gamma}}\Big)^{\frac12},$$

which yields the result by (4.3). □

Corollary 4.3. — *If $\gamma_1,\gamma_2>\frac12$, then*

$$\sup_{(k,\tau)\in\mathbb{Z}\times\mathbb{R}} \sum_{n\in\mathbb{Z}} \frac{1}{\langle -\tau+(n+k)^2+n^2\rangle^{\gamma_1}}<\infty, \tag{4.5}$$

and

$$\sup_{(m,k,\tau)\in\mathbb{Z}^2_*\times\mathbb{R}} \sum_{n\in\mathbb{Z}} \frac{1}{\langle \tau-(n+k)^2+(n+m)^2+m^2\rangle^{2\gamma_2}}<\infty, \tag{4.6}$$

where $\mathbb{Z}^2_*=\{(m,k)\in\mathbb{Z}^2,\ s.t.\ m\neq k\}$.

Proof. — • We first prove the estimate (4.5). For all τ,n,k we have

$$\langle -\tau+(n+k)^2+n^2\rangle=\big\langle -\tau+k^2+2n(n+k)\big\rangle \gtrsim \big\langle \frac{-\tau+k^2}{2}+n(n+k)\big\rangle.$$

The estimate then follows from (4.4) with $\gamma=\gamma_1>\frac12$, $y=-k$ and $z=(-\tau+k^2)/2$.

• We now turn to the proof of (4.6). If $m\neq k$ are integers, then $|m-k|\geq 1$ and thus

$$\begin{aligned}|\tau-(n+k)^2+(n+m)^2+m^2| &= 2|m-k|\Big|\frac{\tau-k^2+2m^2}{2(m-k)}+n\Big|\\ &\geq |C+n|,\end{aligned}$$

with $C=(\tau-k^2+2m^2)/(2(m-k))$. Therefore

$$\langle \tau-(n+k)^2+(n+m)^2+m^2\rangle \geq \langle n+C\rangle,$$

and the estimate follows from an application of (4.3). □

Proof of Proposition 4.1. — Let $f\in\mathcal{H}^{s_0,p}(\mathbb{T})$ and write

$$f(x)=\sum_{n\in\mathbb{Z}} a_n \mathrm{e}^{inx}.$$

Denote by $u_0(t)=\mathrm{e}^{it\Delta}f$ the free Schrödinger evolution of f. Then

$$u_0(t,x)=\mathrm{e}^{it\Delta}f(x)=\sum_{n\in\mathbb{Z}} a_n\mathrm{e}^{-in^2t}\mathrm{e}^{inx}. \tag{4.7}$$

Let $v \in X_1^{s,b}(\mathbb{R}\times\mathbb{T})$, and let $\psi_0 \in \mathcal{C}_0^\infty(\mathbb{R})$ be so that $\psi_0 = 1$ on $[-1,1]$ and $\operatorname{supp}\psi_0 \subset [-2,2]$. Then we consider the following Fourier expansion

$$\psi_0(t)\, v(t,x) = \sum_{n\in\mathbb{Z}} b_n(t)\mathrm{e}^{inx}. \tag{4.8}$$

Thus by Definition 1.2 we have

$$\|v\|^2_{X_1^{s,b}} \leq \|\psi_0(t)\, v\|^2_{X^{s,b}} = \sum_{n\in\mathbb{Z}}\int_{\mathbb{R}} \langle \tau+n^2\rangle^{2b}\langle n\rangle^{2s}|\widehat{b}_n(\tau)|^2\mathrm{d}\tau. \tag{4.9}$$

Now, use the expressions (4.7) and (4.8) to compute

$$\begin{aligned}\psi_0(t)\, u_0\, v(t,x) &= \sum_{(j,k)\in\mathbb{Z}^2} a_j\, b_k(t)\mathrm{e}^{-itj^2}\mathrm{e}^{i(j+k)x}\\ &= \sum_{n\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}} a_{-n-k}\, b_k(t)\mathrm{e}^{-it(n+k)^2}\Big)\mathrm{e}^{-inx},\end{aligned}$$

therefore

$$\psi_0(t)\,\overline{u_0}\,\overline{v}(t,x) = \sum_{n\in\mathbb{Z}} c_n(t)\mathrm{e}^{inx}, \tag{4.10}$$

with

$$c_n(t) = \sum_{k\in\mathbb{Z}} \overline{a_{-n-k}}\,\overline{b_k}(t)\mathrm{e}^{it(n+k)^2}.$$

Now from the properties (2.5) of the time-Fourier transform, we deduce

$$\begin{aligned}\widehat{c_n}(\tau) &= \sum_{k\in\mathbb{Z}} \overline{a_{-n-k}}\;\overline{\widehat{b_k(t)\mathrm{e}^{-it(n+k)^2}}(\tau)} = \sum_{k\in\mathbb{Z}} \overline{a_{-n-k}}\;\overline{\widehat{b_k(t)\mathrm{e}^{-it(n+k)^2}}(-\tau)}\\ &= \sum_{k\in\mathbb{Z}} \overline{a_{-n-k}}\;\overline{\widehat{b_k}}(-\tau+(n+k)^2).\end{aligned}$$

Now write

$$\widehat{c_n}(\tau) = \sum_{k\in\mathbb{Z}} \frac{\overline{a_{-n-k}}}{\langle k\rangle^s\langle -\tau+(n+k)^2+k^2\rangle^b}\;\langle k\rangle^s\langle -\tau+(n+k)^2+k^2\rangle^b\overline{\widehat{b_k}}(-\tau+(n+k)^2),$$

and by the Cauchy-Schwarz inequality we obtain

$$|\widehat{c_n}(\tau)|^2 \leq \Big(\sum_{j\in\mathbb{Z}} A_{j,n}(\tau)\Big)\Big(\sum_{k\in\mathbb{Z}} B_{k,n}(\tau)\Big), \tag{4.11}$$

where

$$A_{j,n}(\tau) = \frac{|a_{-n-j}|^2}{\langle j\rangle^{2s}\langle -\tau+(n+j)^2+j^2\rangle^{2b}}, \tag{4.12}$$

and

$$B_{k,n}(\tau) = \langle k\rangle^{2s}\langle -\tau+(n+k)^2+k^2\rangle^{2b}|\widehat{b_k}|^2(-\tau+(n+k)^2). \tag{4.13}$$

Now by Proposition 1.3, for $\frac{1}{2}<b<1$ and $s\in\mathbb{R}$

$$\|\int_0^t \mathrm{e}^{i(t-t')\Delta}\overline{u_0}\,\overline{v}(t',\cdot)\mathrm{d}t'\|_{X_1^{s,b}} \lesssim \|\overline{u_0 v}\|_{X_1^{s,b-1}} \le \|\psi_0(t)\,\overline{u_0}\,\overline{v}\|_{X^{s,b-1}},$$

where the second inequality is a consequence of Definition 1.2.

Then by (4.10) and (4.11) we obtain

$$\begin{aligned}
\|\psi_0(t)\,\overline{u_0}\,\overline{v}\|^2_{X^{s,b-1}} &= \sum_{n\in\mathbb{Z}}\int_{\mathbb{R}}\langle\tau+n^2\rangle^{2(b-1)}\langle n\rangle^{2s}|\widehat{c_n}(\tau)|^2\mathrm{d}\tau\\
&\le \sum_{n\in\mathbb{Z}}\int_{\mathbb{R}}\frac{\langle n\rangle^{2s}}{\langle\tau+n^2\rangle^{2(1-b)}}\Big(\sum_{j\in\mathbb{Z}}A_{j,n}(\tau)\Big)\Big(\sum_{k\in\mathbb{Z}}B_{k,n}(\tau)\Big)\mathrm{d}\tau\\
&= \sum_{k\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}}\left(\sum_{j\in\mathbb{Z}}\frac{\langle n\rangle^{2s}A_{j,n}(\tau)}{\langle\tau+n^2\rangle^{2(1-b)}}\right)B_{k,n}(\tau)\mathrm{d}\tau.
\end{aligned}$$

Now, thanks to the change of variables $\tau'=-\tau+(n+k)^2$ and (4.13) we deduce

$$\begin{aligned}
&\|\psi_0(t)\,\overline{u_0}\,\overline{v}\|^2_{X^{s,b-1}} \le\\
&\le \sum_{k\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}}\left(\sum_{j\in\mathbb{Z}}\frac{\langle n\rangle^{2s}A_{j,n}(-\tau'+(n+k)^2)}{\langle-\tau'+(n+k)^2+n^2\rangle^{2(1-b)}}\right)B_{k,n}(-\tau'+(n+k)^2)\mathrm{d}\tau'\\
&= \sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}\left(\sum_{(n,j)\in\mathbb{Z}^2}\frac{\langle n\rangle^{2s}A_{j,n}(-\tau'+(n+k)^2)}{\langle-\tau'+(n+k)^2+n^2\rangle^{2(1-b)}}\right)\langle k\rangle^{2s}\langle\tau'+k^2\rangle^{2b}|\widehat{b_k}|^2(\tau')\mathrm{d}\tau'\\
&\le \sup_{(k,\tau)\in\mathbb{Z}\times\mathbb{R}}\left[\sum_{(n,j)\in\mathbb{Z}^2}\frac{\langle n\rangle^{2s}A_{j,n}(-\tau+(n+k)^2)}{\langle-\tau+(n+k)^2+n^2\rangle^{2(1-b)}}\right]\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}\langle k\rangle^{2s}\langle\tau'+k^2\rangle^{2b}|\widehat{b_k}|^2(\tau')\mathrm{d}\tau'\\
&= \|v\|^2_{X_1^{s,b}}\sup_{(k,\tau)\in\mathbb{Z}\times\mathbb{R}}\left[\sum_{(n,j)\in\mathbb{Z}^2}\frac{\langle n\rangle^{2s}A_{j,n}(-\tau+(n+k)^2)}{\langle-\tau+(n+k)^2+n^2\rangle^{2(1-b)}}\right],
\end{aligned}$$

by (4.9).

It remains to estimate the term

$$I(k,\tau) := \sum_{(n,j)\in\mathbb{Z}^2}\frac{\langle n\rangle^{2s}A_{j,n}(-\tau+(n+k)^2)}{\langle-\tau+(n+k)^2+n^2\rangle^{2(1-b)}},$$

uniformly in $(k,\tau)\in\mathbb{Z}\times\mathbb{R}$.

By the definition (4.12) of $A_{j,n}$ and the change of indexes $m=-n-j$, we have

(4.14)

$$
\begin{aligned}
I(k,\tau) &= \\
&= \sum_{(n,j)\in\mathbb{Z}^2} \frac{\langle n\rangle^{2s}|a_{-n-j}|^2}{\langle j\rangle^{2s}\langle -\tau+(n+k)^2+n^2\rangle^{2(1-b)}\langle \tau-(n+k)^2+(n+j)^2+j^2\rangle^{2b}} \\
&= \sum_{(n,m)\in\mathbb{Z}^2} \frac{\langle n\rangle^{2s}|a_m|^2}{\langle n+m\rangle^{2s}\langle -\tau+(n+k)^2+n^2\rangle^{2(1-b)}\langle \tau-(n+k)^2+m^2+(n+m)^2\rangle^{2b}} \\
&:= \sum_{(n,m)\in\mathbb{Z}^2} I_{n,m}(k,\tau).
\end{aligned}
$$

Denote by

$$
R_1 = R_1(\tau,n,k) = -\tau+(n+k)^2+n^2,
$$

$$
R_2 = R_2(\tau,n,k,m) = \tau-(n+k)^2+m^2+(n+m)^2.
$$

Denote by $\sigma=-s>0$ and $\sigma_0=-s_0\geq 0$. Write $b=\frac{1}{2}+\varepsilon$. Then introduce

$$
\beta_1 = 2(1-b) = 1-2\varepsilon < 1 \quad \text{and} \quad \beta_2 = 2b = 1+2\varepsilon > 1.
$$

Therefore, $I_{n,m}$ can be rewritten

$$
I_{n,m}(k,\tau) = \frac{\langle n+m\rangle^{2\sigma}}{\langle n\rangle^{2\sigma}} \frac{|a_m|^2}{\langle R_1\rangle^{\beta_1}\langle R_2\rangle^{\beta_2}}. \tag{4.15}
$$

We treat the cases $m\neq k$ and $m=k$ separately.

• Case $m\neq k$. Observe that

$$
\begin{aligned}
\langle R_1\rangle\langle R_2\rangle &= \max\big(\langle R_1\rangle,\langle R_2\rangle\big)\min\big(\langle R_1\rangle,\langle R_2\rangle\big) \\
&\gtrsim \langle R_1+R_2\rangle \min\big(\langle R_1\rangle,\langle R_2\rangle\big).
\end{aligned}
$$

Thus

$$
\langle R_1\rangle^{\beta_1}\langle R_2\rangle^{\beta_2} \gtrsim \langle n^2+m^2\rangle^{\beta_1}\min\big(\langle R_1\rangle,\langle R_2\rangle\big)^{\beta_2},
$$

and from (4.15) we deduce that

$$
\begin{aligned}
I_{n,m}(k,\tau) &\lesssim \frac{|a_m|^2}{\langle n\rangle^{2\sigma}\langle n^2+m^2\rangle^{1-\sigma-2\varepsilon}} \frac{1}{\min\big(\langle R_1\rangle,\langle R_2\rangle\big)^{\beta_2}} \\
&\lesssim \frac{|a_m|^2}{\langle m\rangle^{2-2\sigma-4\varepsilon}} \frac{1}{\min\big(\langle R_1\rangle,\langle R_2\rangle\big)^{\beta_2}}
\end{aligned}
$$

Now, by Corollary 4.3 (using that $\beta_2>1$),

$$
\sum_{n\in\mathbb{Z}} I_{n,m}(k,\tau) \lesssim \frac{|a_m|^2}{\langle m\rangle^{2-2\sigma-4\varepsilon}} = \frac{|a_m|^2}{\langle m\rangle^{2\sigma_0}}\frac{1}{\langle m\rangle^{\eta}}, \tag{4.16}
$$

with $\eta := 2 - 2\sigma_0 - 2\sigma - 4\varepsilon$.
Now we sum up (4.16) and apply Hölder : For all $p \geq 2$ and $1/q = 1 - 2/p$ so that $q\eta > 1$, we can write

$$\sum_{(n,m)\in\mathbb{Z}^2, m\neq k} I_{n,m}(k,\tau) \lesssim \Big(\sum_{m\in\mathbb{Z}} \frac{|a_m|^p}{\langle m\rangle^{\sigma_0 p}}\Big)^{\frac{2}{p}} \Big(\sum_{j\in\mathbb{Z}} \frac{1}{\langle j\rangle^{q\eta}}\Big)^{\frac{1}{q}}.$$

The condition $q\eta > 1$ is equivalent to

$$2 - 2\sigma_0 - 2\sigma - 4\varepsilon = \eta > \frac{1}{q} = 1 - \frac{2}{p},$$

or

$$\sigma < \frac{1}{2} - \sigma_0 + \frac{1}{p} - 2\varepsilon. \tag{4.17}$$

Assume that (4.1) is satisfied. Then for $0 < \varepsilon \leq \varepsilon_1$ (for ε_1 small enough), the condition (4.17) is also satisfied and we have

$$\sum_{(n,m)\in\mathbb{Z}^2, m\neq k} I_{n,m}(k,\tau) \lesssim \|f\|^2_{\mathcal{H}^{s_0,p}}.$$

• We now consider the case $m = k$.

By (4.14), we have to bound, uniformly in $(k,\tau) \in \mathbb{Z}\times\mathbb{R}$, the term

$$\sum_{n\in\mathbb{Z}} I_{n,k}(k,\tau) = |a_k|^2 \sum_{n\in\mathbb{Z}} \frac{\langle n+k\rangle^{2\sigma}}{\langle n\rangle^{2\sigma}} \frac{1}{\langle -\tau + (n+k)^2 + n^2\rangle^{\beta_1} \langle \tau + k^2\rangle^{\beta_2}}.$$

By the inequality $\langle a+b\rangle \leq \langle a\rangle\langle b\rangle$ we obtain (recall that $\beta_1 = 1 - 2\varepsilon$)

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} I_{n,k}(k,\tau) &\leq |a_k|^2 \sum_{n\in\mathbb{Z}} \frac{\langle n+k\rangle^{2\sigma}}{\langle n\rangle^{2\sigma}} \frac{1}{\langle k^2 + (n+k)^2 + n^2\rangle^{\beta_1}} \\
&\leq |a_k|^2 \sum_{n\in\mathbb{Z}} \frac{\langle n+k\rangle^{2\sigma}}{\langle n\rangle^{2\sigma}\langle k^2+n^2\rangle^{1-2\varepsilon}} \\
&\lesssim |a_k|^2 \sum_{n\in\mathbb{Z}} \frac{1}{\langle n\rangle^{2\sigma}\langle k^2+n^2\rangle^{1-\sigma-2\varepsilon}} \\
&\lesssim |a_k|^2 \sum_{n\in\mathbb{N}} \frac{1}{\langle n\rangle^{2\sigma}\langle k^2+n^2\rangle^{1-\sigma-2\varepsilon}}.
\end{aligned}$$

Now we compare this sums with an integral : Thanks to the change of variables $x=|k|\,y$ we obtain, as $\sigma<\frac{1}{2}$

$$\begin{aligned}\sum_{n\in\mathbb{Z}} I_{n,k}(k,\tau) &\lesssim |a_k|^2\int_0^{+\infty}\frac{\mathrm{d}x}{\langle x\rangle^{2\sigma}\langle k^2+x^2\rangle^{1-\sigma-2\varepsilon}}\\ &\lesssim \frac{|a_k|^2}{\langle k\rangle^{1-4\varepsilon}}\int_0^{+\infty}\frac{\mathrm{d}y}{y^{2\sigma}\langle 1+y^2\rangle^{1-\sigma-2\varepsilon}}\\ &\lesssim \frac{|a_k|^2}{\langle k\rangle^{1-4\varepsilon}}\lesssim \|f\|^2_{\mathcal{H}^{s_0,p}}\ ,\end{aligned}$$

whenever $1-4\varepsilon\geq 2\sigma_0=-2s_0$, i.e. for $0<\varepsilon\leq\varepsilon_2$.
Finally, set $b_2=\frac{1}{2}+\varepsilon$, with $\varepsilon=\min(\varepsilon_1,\varepsilon_2)$. This concludes the proof. □

5. Proof of the main theorem

We now have all the ingredients to prove Theorem 2.2 (observe that Theorem 2.1 is a particular case of the latter).

Proof of Theorem 2.2. — To take profit of the gain of regularity of the first Picard iterate (Proposition 3.2) we write $u=\mathrm{e}^{it\Delta}f+v$ where v lives in a smaller space than u. This idea was used by N. Burq and N. Tzvetkov [**4, 5**] in the context of supercritical wave equations.
We plug this expression in the integral equation

$$u=\mathrm{e}^{it\Delta}f-i\kappa\int_0^t \mathrm{e}^{i(t-t')\Delta}(\overline{u}^2)(t',\cdot)\mathrm{d}t',$$

then we will show that the map K defined by

$$\begin{aligned}K(v) &= -i\kappa\int_0^t \mathrm{e}^{i(t-t')\Delta}(\overline{u_0}^2)(t',\cdot)\mathrm{d}t'-2i\kappa\int_0^t \mathrm{e}^{i(t-t')\Delta}\overline{u_0}\,\overline{v}(t',\cdot)\mathrm{d}t'\\ &\quad -i\kappa\int_0^t \mathrm{e}^{i(t-t')\Delta}(\overline{v}^2)(t',\cdot)\mathrm{d}t',\end{aligned}$$

is a contraction.
Let $2\leq p<4$ and $s_0>-\frac{1}{2}$ satisfy the condition (2.3), i.e. $\dfrac{3}{p}+s_0>\dfrac{1}{2}$, then there exists $s>-\frac{1}{2}$ so that $-\dfrac{1}{2}-s_0-\dfrac{1}{p}<s<-1+\dfrac{2}{p}$, and we can use the estimates (1.3), (3.3) and (4.2) to obtain : There exist $b>\frac{1}{2}$ and $C\geq 1$ such that

$$\|K(v)\|_{X_1^{s,b}}\leq C\big(\|f\|^2_{\mathcal{H}^{s_0,p}}+\|f\|_{\mathcal{H}^{s_0,p}}\|v\|_{X_1^{s,b}}+\|v\|^2_{X_1^{s,b}}\big), \tag{5.1}$$

and

$$\|K(v_1)-K(v_2)\|_{X_1^{s,b}} \leq C\big(\|f\|_{\mathcal{H}^{s_0,p}} + \|v_1+v_2\|_{X_1^{s,b}}\big)\|v_1-v_2\|_{X_1^{s,b}}. \tag{5.2}$$

• The case of small initial data. We assume that $\|f\|_{\mathcal{H}^{s_0,p}} = \mu \ll 1$. Then we show that K is a contraction on the ball of radius $C\mu$ in $X_1^{s,b}$, for μ small enough. For $\|v_1\|_{X_1^{s,b}}, \|v_2\|_{X_1^{s,b}} \leq C\mu$, we deduce from (5.1) and (5.2) that

$$\|K(v)\|_{X_1^{s,b}} \leq C\big(\mu^2 + \mu\|v\|_{X_1^{s,b}} + \|v\|^2_{X_1^{s,b}}\big) \leq 3C^2\mu^2,$$

and

$$\|K(v_1)-K(v_2)\|_{X_1^{s,b}} \leq C\big(\mu + \|v_1+v_2\|_{X_1^{s,b}}\big)\|v_1-v_2\|_{X_1^{s,b}} \leq 3C^2\mu\|v_1-v_2\|_{X_1^{s,b}},$$

and the result follows if we choose μ so that $3C^2\mu < 1$.
The argument to show the uniqueness of the solution in the whole space is similar to the argument given in [**15**], we do not give more details here.

• The general case. Let u be a solution of (1.1), then for all $\lambda > 0$, u_λ defined by $u_\lambda(t,x) = \lambda^2 u(\lambda^2 t, \lambda x)$ is also a solution of the equation, but on a torus of period $2\pi/\lambda$. It is easy to check that the estimates (1.3), (3.3) and (4.2) still hold uniformly w.r.t $\lambda > 0$, if we replace $\mathbb{R}/(2\pi\mathbb{Z})$ with $\mathbb{R}/(\frac{2\pi}{\lambda}\mathbb{Z})$ (see Molinet [**17**] for more details). Now as

$$\|f_\lambda\|_{\mathcal{H}^{s_0,p}} = \|u_\lambda(0,\cdot)\|_{\mathcal{H}^{s_0,p}} \sim \lambda^{1+s_0+\frac{1}{p}},$$

which tends to 0, we can apply the result of the previous case, and find a unique solution $u \in X^{s,b}([-\lambda^2,\lambda^2]\times\mathbb{T})$, for λ small enough.

• The argument showing the regularity of the flow map is exactly the same as in [**15**], hence we omit the proof here. □

Remark 5.1. — We may compute the following Picard iterates of u. Therefore we could look for a solution to (1.1) of the form $u = u_0 + u_1 + \cdots + u_n + v$, where the u_j's are known explicitly and where the unknown v is more regular than u_n. A fixed point argument on v would improve a bit the range (2.3). However we do not pursue this strategy as we do not think this will give an optimal result.

Remark 5.2. — The conclusion of Theorem 2.2 may be improved using estimates in $X^{s,b}_{p,q}$ space, i.e. $X^{s,b}$ spaces based on L^p in the space frequency variable and L^q in the variable τ. See [**13**] for such a strategy for the DNLS equation.

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