A REMARK ON THE SCHRÖDINGER SMOOTHING EFFECT

by

Laurent Thomann

Abstract. — We prove the equivalence between the smoothing effect for a Schrödinger operator and the decay of the associate spectral projectors. We give two applications to the Schrödinger operator in dimension one.

Résumé. — On donne une caractérisation de l'effet régularisant pour un opérateur de Schrödinger par la décroissance de ses projecteurs spectraux. On en déduit deux applications à l'opérateur de Schrödinger en dimension un.

1. Introduction

Let $d \geq 1$, and consider the linear Schrödinger equation

$$\begin{cases} i\partial_t u = Hu, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ u(0,x) = f(x) \in L^2(\mathbb{R}^d), \end{cases} \tag{1.1}$$

where H is a self-adjoint operator on $L^2(\mathbb{R}^d)$.

By the Hille-Yoshida theorem, the equation (1.1) admits a unique solution $u(t) = \mathrm{e}^{-itH} f \in \mathcal{C}\big(\mathbb{R}; L^2(\mathbb{R}^d)\big)$. Under suitable conditions on H, this solution enjoys a local gain of regularity (in the space variable) : For all $T > 0$ there exists $C > 0$ so that

$$\Big(\int_0^T \|\Psi(x) \, \langle H \rangle^{\frac{\gamma}{2}} \mathrm{e}^{-itH} f\|^2_{L^2(\mathbb{R}^d)} \mathrm{d}\, t \Big)^{\frac{1}{2}} \leq C \|f\|_{L^2(\mathbb{R}^d)},$$

for some weight Ψ and exponent $\gamma > 0$.

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This phenomenon has been discovered by T. Kato [**7**] in the context of KdV equations. For the Schrödinger equation in the case $H = -\Delta$, it has been proved by P. Constantin- J.-C. Saut [**2**], P. Sjölin [**11**], L. Vega [**12**] and K. Yajima [**13**]. The variable coefficients case has been obtained by S. Doï [**3, 4, 5, 6**].
The more general results are due to L. Robbiano-C. Zuily [**9, 10**] for equations with obstacles and potentials.

Let H be a self adjoint operator on $L^2(\mathbb{R}^d)$. It can be represented thanks to the spectral measure by

$$H = \int \lambda \mathrm{d}E_\lambda.$$

In the sequel we moreover assume that $H \geq 0$. For $N \geq 0$, we can then define the spectral projector P_N associated to H by

$$P_N = \mathbf{1}_{[N,N+1[}(H) = \int \mathbf{1}_{[N,N+1[}(\lambda)\mathrm{d}E_\lambda. \tag{1.2}$$

Our main result is a characterisation of the smoothing effect by the decay of the spectral projectors. Denote by $\langle H\rangle = (1+H^2)^{\frac{1}{2}}$.

Theorem 1.1 **(Smoothing effect vs. decay)**. —
Let $\gamma > 0$ and $\Psi \in \mathcal{C}(\mathbb{R}^d, \mathbb{R})$. Then the following conditions are equivalent
(i) There exists $C_1 > 0$ so that for all $f \in L^2(\mathbb{R}^d)$

$$\Big(\int_0^{2\pi} \|\Psi(x)\,\langle H\rangle^{\frac{\gamma}{2}}\, e^{-itH} f\|^2_{L^2(\mathbb{R}^d)}\, dt\Big)^{\frac{1}{2}} \leq C_1 \|f\|_{L^2(\mathbb{R}^d)}. \tag{1.3}$$

(ii) There exists $C_2 > 0$ so that for all $N \geq 1$ and $f \in L^2(\mathbb{R}^d)$

$$\|\Psi\, P_N f\|_{L^2(\mathbb{R}^d)} \leq C_2 N^{-\frac{\gamma}{2}} \|P_N f\|_{L^2(\mathbb{R}^d)}. \tag{1.4}$$

The interesting point is that we can take the same function Ψ and exponent $\gamma > 0$ in both statements (1.3) and (1.4).

By the works cited in the introduction, in the case $H = -\Delta$ on $\mathbb{R}^d$, (1.3) is known to hold with $\gamma = \frac{1}{2}$ and $\Psi(x) = \langle x\rangle^{-\frac{1}{2}-\nu}$, for any $\nu > 0$.

There is also a class of operators H on $L^2(\mathbb{R}^d)$ for which (1.3) is well understood. Let $V \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}_+)$, and assume that for $|x|$ large enough $V(x) \geq C\langle x\rangle^k$ and that for any $j \in \mathbb{N}^d$, there exists $C_j > 0$ so that $|\partial_x^j V(x)| \leq C_j \langle x\rangle^{k-|j|}$. Then L. Robbiano and C. Zuily [**9**] show that the smoothing effect (1.3) holds for the operator $H = -\Delta + V(x)$, with $\gamma = \frac{1}{k}$ and $\Psi(x) = \langle x\rangle^{-\frac{1}{2}-\nu}$, for any $\nu > 0$.

We now turn to the case of dimension $d = 1$, and consider the operator $H = -\Delta + V(x)$. We make the following assumption on V

Assumption 1. — *We suppose that $V \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}_+)$, and that there exist $2 < m \leq k$ so that for $|x|$ large enough*

(i) There exists $C > 1$ so that $\frac{1}{C}\langle x\rangle^k \leq V(x) \leq C\langle x\rangle^k$.

(ii) $V''(x) > 0$ and $xV'(x) \geq mV(x) > 0$

(iii) For any $j \in \mathbb{N}$, there exists $C_j > 0$ so that $|\partial_x^j V(x)| \leq C_j \langle x\rangle^{k-|j|}$.

For instance $V(x) = \langle x\rangle^k$ with $k > 2$ satisfies Assumption 1.

It is well known that under Assumption 1, the operator H has a self-adjoint extension on $L^2(\mathbb{R})$ (still denoted by H) and has eigenfunctions $\big(e_n\big)_{n\geq 1}$ which form an Hilbertian basis of $L^2(\mathbb{R})$ and satisfy

$$He_n = \lambda_n^2 e_n, \quad n \geq 1,$$

with $\lambda_n \longrightarrow +\infty$, when $n \longrightarrow +\infty$.

For $N \in \mathbb{N}$ the spectral projector P_N defined in (1.2) can be written in the following way. Let $f = \sum_{n\geq 1} \alpha_n e_n \in L^2(\mathbb{R})$, then

$$P_N f = \sum_{N \leq \lambda_n^2 < N+1} \alpha_n e_n.$$

Observe that we then have $f = \sum_{N\geq 0} P_N f$.

For such a potential, we can remove the spectral projectors in (1.4) and deduce from Theorem 1.1

Corollary 1.2. —
Let $\gamma > 0$ and $\Psi \in \mathcal{C}(\mathbb{R}, \mathbb{R})$. Let $H = \Delta + V(x)$ so that $V(x) = x^2$ or $V(x)$ satisfies Assumption 1. Then the following conditions are equivalent

(i) There exists $C_1 > 0$ so that for all $f \in L^2(\mathbb{R})$

$$\Big(\int_0^{2\pi} \|\Psi(x)\,\langle H\rangle^{\frac{\gamma}{2}}\, e^{-itH} f\|_{L^2(\mathbb{R})}^2\, dt\Big)^{\frac{1}{2}} \leq C_1 \|f\|_{L^2(\mathbb{R})}. \tag{1.5}$$

(ii) There exists $C_2 > 0$ so that for all $n \geq 1$

$$\|\Psi\, e_n\|_{L^2(\mathbb{R})} \leq C_2 \lambda_n^{-\gamma}, \quad \forall\, n \geq 1. \tag{1.6}$$

The statements (1.5) and (1.6) were obtained by K. Yajima & G. Zhang in [**16**] when Ψ is the indicator of a compact $K \subset \mathbb{R}$ and with $\gamma = \frac{1}{k}$.

The statement (1.5) holds for $\Psi(x) = \langle x \rangle^{-\frac{1}{2}-\nu}$, by [**9**], but as far as we know, the bound (1.6) with this Ψ was unknown.

With Theorem 1.1 we are also able to prove the following smoothing effect for the usual Laplacian Δ on $\mathbb{R}$.

Proposition 1.3. — *Let $\Psi \in L^2(\mathbb{R})$. Then there exists $C > 0$ so that for all $f \in L^2(\mathbb{R})$*

$$\Big(\int_0^{2\pi} \|\Psi(x) \, \langle \Delta \rangle^{\frac{1}{4}} e^{-it\Delta} f\|^2_{L^2(\mathbb{R})} \, dt \Big)^{\frac{1}{2}} \leq C \|\Psi\|_{L^2(\mathbb{R})} \|f\|_{L^2(\mathbb{R})}.$$

From the works cited in the introduction, we have

$$\Big(\int_{\mathbb{R}} \|\Psi(x) \, \langle \Delta \rangle^{\frac{1}{4}} \mathrm{e}^{-it\Delta} f\|^2_{L^2(\mathbb{R})} \mathrm{d}\, t \Big)^{\frac{1}{2}} \leq C \|f\|_{L^2(\mathbb{R})},$$

for $\Psi(x) = \langle x \rangle^{-\frac{1}{2}-\nu}$, for any $\nu > 0$. Hence Proposition 1.3 shows that we can extend the class of the weights, but we are only able to prove local integrability in time.

Notation. — *We use the notation $a \lesssim b$ if there exists a universal constant $C > 0$ so that $a \leq Cb$.*

2. Proof of the results

We define the self adjoint operator $A = [H]$ (entire part of H) by

$$A = \int [\lambda] \mathrm{d} E_\lambda.$$

Notice that we immediately have that $A - H$ is bounded on $L^2(\mathbb{R}^d)$.
The first step in the proof of Theorem 1.1 is to show that we can replace e^{-itH} by e^{-itA} in (1.3)

Lemma 2.1. — *Let $\gamma > 0$ and $\Psi \in \mathcal{C}(\mathbb{R}^d, \mathbb{R})$. Then the following conditions are equivalent*

(i) There exists $C_1 > 0$ so that for all $f \in L^2(\mathbb{R}^d)$

$$\Big(\int_0^{2\pi} \|\Psi(x) \, \langle H \rangle^{\frac{\gamma}{2}} e^{-itA} f\|^2_{L^2(\mathbb{R}^d)} \, dt \Big)^{\frac{1}{2}} \leq C_1 \|f\|_{L^2(\mathbb{R}^d)}. \tag{2.1}$$

(ii) There exists $C_2 > 0$ so that for all $f \in L^2(\mathbb{R}^d)$

$$\Big(\int_0^{2\pi} \|\Psi(x) \, \langle H \rangle^{\frac{\gamma}{2}} e^{-itH} f\|^2_{L^2(\mathbb{R}^d)} \, dt \Big)^{\frac{1}{2}} \leq C_2 \|f\|_{L^2(\mathbb{R}^d)}. \tag{2.2}$$

Proof. — We assume (2.1) and we prove (2.2). Let $f \in L^2(\mathbb{R}^d)$ and Define $v = \mathrm{e}^{-itH} f$. This function solves the problem

$$(i\partial_t - A)v = (H - A)v, \quad v(0, x) = f(x).$$

Then by the Duhamel formula

$$\begin{aligned} \mathrm{e}^{-itH} f = v &= \mathrm{e}^{-itA} f - i \int_0^t \mathrm{e}^{-i(t-s)A} (H - A)v \, \mathrm{d}s \\ &= \mathrm{e}^{-itA} f - i \int_0^{2\pi} \mathbf{1}_{\{s<t\}} \mathrm{e}^{-i(t-s)A} (H - A)v \, \mathrm{d}s. \end{aligned}$$

Therefore by (2.1) and Minkowski

$$\begin{aligned} \|\Psi \langle H \rangle^{\frac{\gamma}{2}} \mathrm{e}^{-itH} v\|_{L^2_{2\pi} L^2} &\lesssim \|\Psi \langle H \rangle^{\frac{\gamma}{2}} \mathrm{e}^{-itA} v\|_{L^2_{2\pi} L^2} \\ &\quad + \int_0^{2\pi} \|\Psi \langle H \rangle^{\frac{\gamma}{2}} \mathbf{1}_{\{s<t\}} \mathrm{e}^{-i(t-s)A} (H - A)v\|_{L^2_t L^2_x} \, \mathrm{d}s \\ &\lesssim \|f\|_{L^2} + \int_0^{2\pi} \|(H - A)v\|_{L^2} \, \mathrm{d}s. \end{aligned} \tag{2.3}$$

Now use that the operator $(H - A) : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is bounded, and by (2.3) we obtain

$$\|\Psi \langle H \rangle^{\frac{\gamma}{2}} \mathrm{e}^{-itH} v\|_{L^2_{2\pi} L^2} \lesssim \|f\|_{L^2},$$

which is (2.2).
The proof of the converse implication is similar. □

Proof of Theorem 1.1. — The proof is based on Fourier analysis in time. This idea comes from [**8**] and has also been used in [**16**], but this proof was inspired by [**1**].

(i) $\Longrightarrow$ *(ii)* : To prove this implication, we use the characterisation (2.1). From (1.2) and the definition of A, $\mathrm{e}^{-itA} P_N f = \mathrm{e}^{-itN} P_N f$. Hence it suffices to replace f with $P_N f$ in (1.3) and (1.4) follows.

(ii) $\Longrightarrow$ *(i)* : Again we will use Lemma 2.1. We assume (2.2) and we first prove that

$$\|\Psi \langle A \rangle^{\frac{\gamma}{2}} \mathrm{e}^{-itA} f\|_{L^2(0,2\pi; L^2(\mathbb{R}^d))} \lesssim \|f\|_{L^2(\mathbb{R}^d)}. \tag{2.4}$$

Write $f = \sum_{N \geq 0} P_N f$, then

$$\Psi \langle A \rangle^{\frac{\gamma}{2}} \mathrm{e}^{-itA} f = \sum_{N \geq 0} \mathrm{e}^{-iNt} \langle N \rangle^{\frac{\gamma}{2}} \Psi \; P_N f.$$

Now by Parseval in time

$$\|\Psi\,\langle A\rangle^{\frac{\gamma}{2}}\mathrm{e}^{-itA}f\|^2_{L^2(0,2\pi)}\lesssim\sum_{N\geq 0}\langle N\rangle^{\gamma}|\Psi\ P_Nf|^2,$$

and by integration in the space variable and (1.4)

$$\begin{aligned}\|\Psi\,\langle A\rangle^{\frac{\gamma}{2}}\mathrm{e}^{-itA}f\|^2_{L^2(0,2\pi;L^2(\mathbb{R}^d))} &\lesssim \sum_{N\geq 0}\langle N\rangle^{\gamma}\|\Psi\ P_Nf\|^2_{L^2(\mathbb{R}^d)}\\ &\lesssim \sum_{N\geq 0}\|P_Nf\|^2_{L^2(\mathbb{R}^d)}=\|f\|^2_{L^2(\mathbb{R}^d)},\end{aligned}$$

which yields (2.4).
Now since the operator $\langle A\rangle^{-\gamma/2}\langle H\rangle^{\gamma/2}$ is bounded on L^2 and commutes with e^{-itA}, we have by (2.4)

$$\begin{aligned}&\|\Psi\,\langle H\rangle^{\frac{\gamma}{2}}\mathrm{e}^{-itA}f\|_{L^2(0,2\pi;L^2(\mathbb{R}^d))}=\\ &\qquad=\|\Psi\,\langle A\rangle^{\frac{\gamma}{2}}\mathrm{e}^{-itA}(\langle A\rangle^{-\frac{\gamma}{2}}\langle H\rangle^{\frac{\gamma}{2}}f)\|_{L^2(0,2\pi;L^2(\mathbb{R}^d))}\\ &\qquad\lesssim\|\langle A\rangle^{-\frac{\gamma}{2}}\langle H\rangle^{\frac{\gamma}{2}}f\|_{L^2(\mathbb{R}^d)}\\ &\qquad\lesssim\|f\|_{L^2(\mathbb{R}^d)},\end{aligned}$$

which is (2.1). □

Proof of Corollary 1.2. — Let V satisfy Assumption 1. Then by [**14**, Lemma 3.3] there exists $C>0$ such that

$$|\lambda^2_{n+1}-\lambda^2_n|\geq C\lambda_n^{1-\frac{2}{m}},$$

for n large enough. This implies that $[\lambda_n^2]<[\lambda^2_{n+1}]$ for n large enough, because $m>2$ and $\lambda_n\longrightarrow+\infty$. As a consequence

$$P_Nf=\alpha_ne_n,\ \ \text{with } n \text{ so that }\ N\leq\lambda_n^2<N+1,$$

and this yields the result.
We now consider $V(x)=x^2$. In this case, the eigenvalues are the integers $\lambda_n^2=2n+1$, and the claim follows. □

Remark 2.2. — With this time Fourier analysis, we can prove the following smoothing estimate for H which satisfies Assumption 1

$$\|\langle H\rangle^{\frac{\theta(q,k)}{2}}\mathrm{e}^{-itH}f\|_{L^p(\mathbb{R};L^2(0,T))}\lesssim\|f\|_{L^2(\mathbb{R})},$$

where θ is defined by

$$\theta(q,k) = \begin{cases} \frac{2}{k}(\frac{1}{2}-\frac{1}{q}) & \text{if} \quad 2 \leq q < 4, \\ \frac{1}{2k} - \eta \text{ for any } \eta > 0 & \text{if} \quad q = 4, \\ \frac{1}{2} - \frac{2}{3}(1-\frac{1}{q})(1-\frac{1}{k}) & \text{if} \quad 4 < q < \infty, \\ \frac{4-k}{6k} & \text{if} \quad q = \infty. \end{cases}$$

This was done in [**16**] with a slightly different formulation.

Proof of Proposition 1.3. — By Theorem 1.1, we have to prove that the operator T defined by

$$Tf(x) = N^{\frac{1}{4}}\Psi(x)\mathbf{1}_{[N,N+1[}(-\Delta)f(x),$$

is continuous from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$ with norm independent of $N \geq 1$. By the usual TT^* argument, it is enough to show the result for TT^*.
The kernel of T is $K(x,y) = N^{\frac{1}{4}}\Psi(x)F_N(x-y)$ where

$$F_N(u) = \frac{1}{2\pi}\int e^{iu\xi}\mathbf{1}_{[\sqrt{N},\sqrt{N+1}[}(|\xi|)\mathrm{d}\xi = 4\cos(D_N u)\frac{\sin(C_N u)}{u}, \tag{2.5}$$

with $C_N = (\sqrt{N+1}-\sqrt{N})/2$ and $D_N = (\sqrt{N+1}+\sqrt{N})/2$.
The kernel of TT^* is given by

$$\Lambda(x,z) = \int K(x,y)\overline{K}(z,y)\mathrm{d}y,$$

and by Parseval and (2.5)

$$\begin{aligned} \Lambda(x,z) &= N^{\frac{1}{2}}\Psi(x)\Psi(z)\int F_N(x-y)\overline{F_N}(z-y)\mathrm{d}y \\ &= \frac{1}{4}N^{\frac{1}{2}}\Psi(x)\Psi(z)\int e^{i(x-z)\xi}\mathbf{1}_{[\sqrt{N},\sqrt{N+1}[}(|\xi|)\mathrm{d}\xi \\ &= \pi N^{\frac{1}{2}}\Psi(x)\Psi(z)\cos(D_N(x-z))\frac{\sin(C_N(x-z))}{x-z}. \end{aligned}$$

Now, since $C_N \lesssim 1/\sqrt{N}$ and $|\sin(x)| \leq |x|$, we deduce that $|\Lambda(x,z)| \leq C|\Psi(x)||\Psi(z)|$ (independent of $N \geq 1$), and TT^* is continuous for $\Psi \in L^2(\mathbb{R})$. □

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Laurent Thomann, Université de Nantes, Laboratoire de Mathématiques J. Leray, UMR CNRS 6629, 2, rue de la Houssinière, F-44322 Nantes Cedex 03, France. • *E-mail :* `laurent.thomann@univ-nantes.fr`
Url : `http://www.math.sciences.univ-nantes.fr/~thomann/`