INSTABILITIES FOR SUPERCRITICAL SCHRÖDINGER EQUATIONS IN ANALYTIC MANIFOLDS

by Laurent Thomann

Abstract. — In this paper we consider supercritical nonlinear Schrödinger equations in an analytic Riemannian manifold (M^d, g), where the metric g is analytic. Using an analytic WKB method, we are able to construct an Ansatz for the semiclassical equation for times independent of the small parameter. These approximate solutions will help to show two different types of instabilities. The first is in the energy space, and the second is an immediate loss of regularity in higher Sobolev norms.

1. Introduction

Let (M^d, g) be an analytic Riemannian manifold of dimension $d \geq 3$. In all the paper we assume that the metric g is analytic. Let p an odd integer. We consider the nonlinear Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta_g u = \omega |u|^{p-1} u, & (t,x) \in \mathbb{R} \times M^d, \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

with either $\omega = 1$ (defocusing equation) or $\omega = -1$ (focusing equation). Here $\Delta = \Delta_g$ denotes the Laplace-Beltrami operator defined by $\Delta = \mathrm{div}\nabla$. It is known that the mass

$$\|u(t)\|_{L^2(M^d)} = \|u_0\|_{L^2(M^d)}, \tag{1.2}$$

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and the energy

$$(1.3)\qquad H(u)(t)=\int_{M^d}\Big(\frac{1}{2}|\nabla u|^2+\frac{\omega}{p+1}|u|^{p+1}\Big)\mathrm{d}x=H(u_0,\omega),$$

are conserved by the flow of (1.1), at least formally.
Denote also by

$$(1.4)\qquad H^+(u)=\int_{M^d}\Big(\frac{1}{2}|\nabla u|^2+\frac{1}{p+1}|u|^{p+1}\Big)\mathrm{d}x.$$

In the following we will need the definition of uniform well-posedness :

Definition 1.1. — Let X be a Banach space. We say that the Cauchy problem (1.1) is locally uniformly well-posed in X, if for any bounded subset $\mathcal{B}\subset X$, there exists $T>0$ and a solution $u\in\mathcal{C}\big([-T,T];X\big)$ of (1.1) and such that the flow map

$$u_0\in\mathcal{B}\longmapsto u(t)=\Phi_t(u_0)\in X,$$

is uniformly continuous for any $-T\leq t\leq T$.

1.1. Instability in the energy space. —

By the works of J. Ginibre and G. Velo [10], T. Cazenave and F. B. Weissler [7], we know that (1.1) is locally uniformly well-posed in the energy space $X=H^1(\mathbb{R}^d)\cap L^{p+1}(\mathbb{R}^d)=H^1(\mathbb{R}^d)$ when $p<(d+2)/(d-2)$.
Our first result states that this result does not hold when $p>(d+2)/(d-2)$ is an odd integer.

Theorem 1.2. — *Let $p>(d+2)/(d-2)$ be an odd integer, $\omega\in\{-1,1\}$, and let H^+ be given by (1.4). Let $m\in M^d$. There exist a positive sequence $r_n\longrightarrow 0$, and two sequences $u_0^n,\widetilde{u_0}^n\in\mathcal{C}_0^\infty(M^d)$ of Cauchy data with support in the ball $\big\{|x-m|_g\leq r_n\big\}$, a sequence of times $t_n\longrightarrow 0$, and constants $c,C>0$ such that*

$$(1.5)\qquad H^+(u_0^n)\leq C,\quad H^+(\widetilde{u_0}^n)\leq C,$$

$$(1.6)\qquad H^+(u_0^n-\widetilde{u_0}^n)\longrightarrow 0,\quad \textit{when}\quad n\longrightarrow+\infty,$$

and such that the solutions u^n, $\tilde{u}^n$ of (1.1) satisfy

$$(1.7)\qquad \limsup_{n\to+\infty}\int_{M^d}\big|(u^n-\widetilde{u}^n)(t_n)\big|^{p+1}dx>c.$$

Moreover, the sequences $u_0^n,\widetilde{u_0}^n$ can be chosen such that there exist $\nu_0>0$ and $q_0>p+1$, such that for all $0\leq\nu<\nu_0$ and $p+1\leq q<q_0$,

$$(1.8)\qquad \|u_0^n-\widetilde{u_0}^n\|_{H^{1+\nu}(M^d)}+\|u_0^n-\widetilde{u_0}^n\|_{L^q(M^d)}\longrightarrow 0.$$

For $k \in \mathbb{R}$, the norm $\|\cdot\|_{H^k(M^d)}$ is defined by

$$\|f\|_{H^k(M^d)} = \|(1-\Delta)^{k/2} f\|_{L^2(M^d)}.$$

R. Carles [6] obtains a similar result for the defocusing cubic equation in $\mathbb{R}^d$. An analog of Theorem 1.2 was proved by G. Lebeau [13] for the supercritical wave equation, but for a nonlinearity of the form u^p. After a rescaling of (1.1) to a semiclassical equation, we also have an almost finite speed of propagation principle. This is one reason why such a result was expected for nonlinear supercritical Schrödinger equations.

1.2. Ill-posedness in Sobolev spaces. —

Assume here that (M^d, g) is the euclidian space with the canonical metric $(M^d, g) = (\mathbb{R}^d, \mathrm{can})$. Let $T > 0$ and let $u :]-T, T[\times \mathbb{R}^d \longrightarrow \mathbb{C}$ satisfy (1.1). Then for all $\lambda \in \mathbb{R}$

$$\begin{array}{rcl} u^\lambda :]-\lambda^{-2}T, \lambda^{-2}T[\times\mathbb{R}^d & \longrightarrow & \mathbb{C} \\ (t,x) & \longmapsto & u^\lambda(t,x) = \lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x), \end{array}$$

is also a solution of (1.1).
Define the critical index for Sobolev well-posedness

$$\sigma_{\mathrm{c}} = \frac{d}{2} - \frac{2}{p-1}. \tag{1.9}$$

Then, for all $f \in \dot{H}^{\sigma_{\mathrm{c}}}(\mathbb{R}^d)$ (the homogeneous Sobolev space) and $\lambda \in \mathbb{R}$

$$\lambda^{\frac{2}{p-1}} \|f(\lambda \cdot)\|_{\dot{H}^{\sigma_{\mathrm{c}}}(\mathbb{R}^d)} = \|f\|_{\dot{H}^{\sigma_{\mathrm{c}}}(\mathbb{R}^d)}.$$

This scaling notion is relevant, as we have the following results :

• Let $\sigma > \sigma_{\mathrm{c}}$, then the equation (1.1) is locally uniformly well-posed in $X = H^\sigma(\mathbb{R}^d)$, [10],[7].

• If $0 < \sigma < \sigma_{\mathrm{c}}$, the problem (1.1) is ill-posed in $H^\sigma(\mathbb{R}^d)$, in the sense that there exist a sequence of initial data u_0^n so that

$$\|u_0^n\|_{H^\sigma(\mathbb{R}^d)} \longrightarrow 0,$$

and a sequence of times $t_n \longrightarrow 0$ such that the solution u^n of (1.1) satisfies

$$\|u^n(t_n)\|_{H^\rho(\mathbb{R}^d)} \longrightarrow +\infty,$$

for $\rho = \sigma$ (see Christ-Colliander-Tao [9]), or even for all $\rho \in]\sigma/(\frac{d}{2} - \sigma), \sigma]$ in the particular case $\omega = 1$ and $p = 3$, (see Carles [5] and Alazard-Carles [2]).

Here we prove

THEOREM 1.3. — *Assume that* $(M^d, g) = (\mathbb{R}^d, can)$. *Let* $p \geq 3$ *be an odd integer,* $\omega \in \{-1, 1\}$, *and let* $0 < \sigma < d/2 - 2/(p-1)$. *There exist a sequence* $\check{u}_0^n \in \mathcal{C}^\infty(\mathbb{R}^d)$ *of Cauchy data and a sequence of times* $\tau_n \longrightarrow 0$ *such that*

$$\|\check{u}_0^n\|_{H^\sigma(\mathbb{R}^d)} \longrightarrow 0, \quad \text{when} \quad n \longrightarrow +\infty, \tag{1.10}$$

and such that the solution $\check{u}^n$ *of* (1.1) *satisfies*

$$\|\check{u}^n(\tau_n)\|_{H^\rho(\mathbb{R}^d)} \longrightarrow +\infty, \quad \text{when} \quad n \longrightarrow +\infty, \quad \text{for all} \quad \rho \in \Big] \frac{\sigma}{\frac{p-1}{2}(\frac{d}{2} - \sigma)}, \sigma \Big]. \tag{1.11}$$

In the general case of an analytic manifold (M^d, g) with an analytic metric g, we obtain the weaker result

THEOREM 1.4. — *Let* $p \geq 3$ *be an odd integer,* $\omega \in \{-1, 1\}$, *and let* $0 < \sigma < d/2 - 2/(p-1)$. *Let* $m \in M^d$. *There exist a positive sequence* $r_n \longrightarrow 0$ *and a sequence* $\check{u}_0^n \in \mathcal{C}_0^\infty(M^d)$ *of Cauchy data with support in the ball* $\{|x - m|_g \leq r_n\}$, *a sequence of times* $\tau_n \longrightarrow 0$ *such that*

$$\|\check{u}_0^n\|_{H^\sigma(M^d)} \longrightarrow 0, \quad \text{when} \quad n \longrightarrow +\infty,$$

and such that the solution $\check{u}^n$ *of* (1.1) *satisfies*

$$\|\check{u}^n(\tau_n)\|_{H^\rho(M^d)} \longrightarrow +\infty, \quad \text{when} \quad n \longrightarrow +\infty, \quad \text{for all} \quad \rho \in \Big] I(\sigma), \sigma \Big],$$

where $I(\sigma)$ *is defined by*

$$I(\sigma) = \begin{cases} \frac{\sigma}{2} & \text{for} \quad 0 < \sigma \leq \frac{d}{2} - \frac{4}{p-1}, \\ \frac{\sigma}{\frac{p-1}{2}(\frac{d}{2} - \sigma)} & \text{for} \quad \frac{d}{2} - \frac{4}{p-1} \leq \sigma < \frac{d}{2} - \frac{2}{p-1}. \end{cases}$$

In the case $p = 3$ and $\omega = 1$, Theorem 1.3 was shown by R. Carles [5] using the convergence of the WKB method for $\mathcal{C}^\infty$ data (see [11], [12]). Recently, T. Alazard and R. Carles [1] have obtained a justification for nonlinear geometric optics when $p > 3$ with H^∞ data.
Consider now the semiclassical equation

$$ih\partial_t v + h^2 \Delta v = |v|^{p-1} v. \tag{1.12}$$

In [2], T. Alazard and R. Carles prove that for all non trivial initial condition $v(0, \cdot) \in \mathcal{S}(\mathbb{R}^d)$, the solution v of (1.12) oscillates immediately: There exists $\tau > 0$ so that

$$\liminf_{h \to 0} \||h\nabla|^s v(\tau)\|_{L^2(\mathbb{R}^d)} > 0,$$

for all $s \in]0, 1]$. This yields the result of Theorem 1.3 for the defocusing equation in the euclidien space for any smooth Cauchy condition. Their method does not apply to the focusing case.

Denote by σ_{sob} the Sobolev exponent so that $\dot{H}^{\sigma_{\text{sob}}}(\mathbb{R}^d) \subset L^{p+1}(\mathbb{R}^d)$, i.e.

$$\sigma_{\text{sob}} = \frac{d}{2} - \frac{d}{p+1}. \tag{1.13}$$

Let $p > (d+2)/(d-2)$, then $\sigma_{\text{sob}} < \sigma_{\text{c}}$. As pointed out by G. Lebeau and R. Carles, for $\sigma = \sigma_{\text{sob}}$, Theorem 1.3 yields

$$\|\check{u}_0^n\|_{H^{\sigma_{\text{sob}}}(\mathbb{R}^d)} \longrightarrow 0, \quad \|\check{u}^n(\tau_n)\|_{H^{\rho}(\mathbb{R}^d)} \longrightarrow +\infty,$$

for all $\rho \in]1, \sigma_{\text{sob}}]$. This interval can not be enlarged. Indeed, for all $\rho \leq 1$, the conservation of the quantities (1.2) and (1.3) together with the embedding $\dot{H}^{\sigma_{\text{sob}}}(\mathbb{R}^d) \subset L^{p+1}(\mathbb{R}^d)$ yield for all $\tau > 0$

$$\|\check{u}^n(\tau)\|_{H^{\rho}(\mathbb{R}^d)} \longrightarrow 0.$$

See also [4].

G. Lebeau [14] obtains a stronger result for the wave equation in $(\mathbb{R}^d, \text{can})$, with the same range for ρ in (1.11), but the loss of derivatives is obtained with only one one Cauchy condition, instead of a sequence.

Theorem 1.2 can not be deduced from Theorem 1.3. In fact, the sequences constructed with $\sigma = 1$ such that

$$\|\check{u}_0^n\|_{H^1(M^d)} \longrightarrow 0, \quad \|\check{u}^n(\tau_n)\|_{H^1(M^d)} \longrightarrow +\infty,$$

satisfy $H^+(\check{u}_0^n) \longrightarrow +\infty$, when n tends to infinity.

The instabilities of Theorems 1.2, 1.3 and 1.4 are not geometrical effects, they are only caused by the high exponent of the nonlinearity.
We could also consider more general analytic nonlinearities, for instance $\pm(1+|u|^2)^{\alpha/2}u$ with $\alpha > (d+2)/(d-2)$.
Notice that the focusing case with non analytic Cauchy conditions is more intricate, as other phenomenons are involved, like finite time explosion.

The main ingredient of the proof of our results is the construction of approximate solutions of (1.1), via analytic nonlinear geometric optics, as done by P. Gérard in [11]. This work will be adapted to the case $(M^d, g) = (\mathbb{R}^d, \text{can})$. We will work in weighted spaces, so that these solutions concentrate in a point of $\mathbb{R}^d$, and then the construction in (M^d, g) will follow directly, as we are able to work only in one local chart.

The plan of the paper is the following

1. We first construct a formal solution of (1.1).
   a) In Section 2 we deal with the case $(M^d, g) = (\mathbb{R}^d, \text{can})$: First we reduce (1.1) to a semiclassical equation as done in [13] and [5], then we adapt the analytic WKB method given in [11] to $\mathbb{R}^d$.
   b) In Section 2.2 we consider the general case of an analytic manifold with

an analytic metric.
2. We obtain a family of approximate solutions of (1.1). (Section 3)
3. Using two different rates of concentration of this family, we prove the main results. (Section 4)

NOTATIONS 1.5. — *In this paper* c, C *denote constants the value of which may change from line to line. These constants will always be independent of* h. *We use the notations* $a \sim b$, $a \lesssim b$ *if* $\frac{1}{C}b \leq a \leq Cb$, $a \leq Cb$ *respectively. We write* $a \ll b$ *if* $a \leq Kb$ *for some large constant* K *which is independent of* h.

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2. Nonlinear geometric optics

2.1. The Euclidian case. —

2.1.1. Reduction to a semiclassical equation. — Following [13], [5], we reduce the equation (1.1) to a semiclassical equation, and therefore make the following change of variables and unknown function

$$\begin{cases} t = \hbar^{\alpha} s, \quad x = \hbar z, \quad h = \hbar^{\beta}, \\ u(\hbar^{\alpha} s, \hbar z) = \hbar^{\gamma} v(s, z, h), \end{cases} \tag{2.1}$$

where $h \in]0, 1]$ is a small parameter, and where $\beta > 0$. The value of β will be given in Section 4, in terms of p and d to prove Theorem 1.2, and in terms of p, d and σ to prove Theorem 1.3.
If we choose

$$\alpha = \beta + 2, \quad (p-1)\gamma = -2(\beta + 1), \tag{2.2}$$

we are lead to studying the Cauchy problem

$$\begin{cases} ih\partial_s v(s, z) + h^2 \Delta v(s, z) = \omega |v|^{p-1} v(s, z), \\ v(0, z) = v_0(z). \end{cases} \tag{2.3}$$

Following the ideas of nonlinear geometric optics, we can search a solution of (2.3) for small times (but independent of h) of the form

$$v(s, z, h) = a(s, z, h) \mathrm{e}^{iS(s,z)/h}, \tag{2.4}$$

where formally

$$a(s,z,h)=\sum_{j\geq 0} a_j(s,z)h^j. \tag{2.5}$$

Then v is a formal solution of equation (2.3) if the couple (S,a) satisfies the system

$$\begin{cases} \partial_s S+(\nabla S)^2+\omega|a_0|^{p-1}=0, \\ \partial_s a+2\nabla S\cdot\nabla a+a\Delta S-ih\Delta a+\dfrac{i\omega a}{h}(|a|^{p-1}-|a_0|^{p-1})=0, \\ S(0,z)=S^0(z),\ a(0,z,h)=a^0(z,h), \end{cases} \tag{2.6}$$

where $v(0,z,h)=a^0(z,h)\mathrm{e}^{iS^0(z)/h}$.
In fact to obtain the system (2.6), plug (2.4) in equation (2.3) and identify the coefficients in the expansion in powers of h. The first equation of (2.6) corresponds to the coefficients of h^0, and the second to the others, after division by h. Notice that S will be a real function, if the data $S(0,\cdot)$ is real.
The WKB method consits now in plugging the developement given by (2.5) in (2.6). Annihilating the coefficients of h^j, for $j\geq 0$, yields a cascade of equations. And if we are able to solve them, this gives an approximate solution v_{app} of (2.3)

$$ih\partial_s v_{\text{app}}+h^2\Delta v_{\text{app}}=|v_{\text{app}}|^{p-1}v_{\text{app}}+\mathcal{O}(h^\infty). \tag{2.7}$$

Unfortunately, the obtained system is not closed: the equation which gives a_j depends on a_{j+1}.
Moreover, in general, using (2.7), we can show that v_{app} is close to a solution of (2.3) only for times $s\in[0,Ch\log\frac{1}{h}]$. See [11], Corollaire 1.

To obtain an Ansatz for h-independent times, we work in an analytic frame. Thus in the following we will consider z as a complex variable.

2.1.2. Construction of a formal solution of (2.3). — Here we adapt step by step the proof of P. Gérard [11] given in the case of the torus $\mathbb{T}^d$ to the case $\mathbb{R}^d$.
We need Sjöstrand's definition [15] of an analytic symbol.

Definition 2.1. — We say that the formal series $b(s,z,h)=\sum_{j\geq 0} b_j(s,z)h^j$ is an analytic symbol if there exist positive constants $s_0,l,A,B>0$ such that for all $j\geq 0$

$$(s,z)\mapsto b_j(s,z) \text{ is an holomorphic function on } \{|s|<s_0\}\times\{|\mathrm{Im}\, z|<l\}, \tag{2.8}$$

and

$$|b_j(s,z)|\leq AB^j j! \text{ on } \{|s|<s_0\}\times\{|\mathrm{Im}\, z|<l\}. \tag{2.9}$$

Notice that b has to be analytic in both variables, s and z.

To obtain proper estimates in Sobolev norms later, we want to make sure that the functions are small at infinity in the space variable. Therefore we define the weight

$$W(z) = \mathrm{e}^{(1+z^2)^{1/2}}, \tag{2.10}$$

where $z^2 = z_1^2 + \cdots + z_d^2$ for any $z = (z_1, \cdots, z_d) \in \mathbb{C}^d$. Notice that W is analytic in the band $\{|\mathrm{Im}\, z| < \frac{1}{2}\}$, thus in the following we fix $l < \frac{1}{2}$.

We introduce the space $\mathcal{H}(s_0, l, B)$ composed of the analytic symbols $b = \sum_{j\geq 0} b_j h^j$ satisfying: there exist $A, B > 0$ so that

$$|W(z) b_j(s,z)| \leq AB^j j! \;\text{ on }\; \{|s| < s_0\} \times \{|\mathrm{Im}\, z| < l\},\; \forall j \geq 0. \tag{2.11}$$

$$\mathcal{H}(s_0, l, B) = \left\{ \begin{array}{c} b = \sum_{j\geq 0} b_j h^j \text{ is an analytic symbol on} \\ \big(\{|s| < s_0\} \times \{|\mathrm{Im}\, z| < l\}\big) \text{ s.t. } b_j \text{ satisfies (2.11)} \end{array} \right\}. \tag{2.12}$$

Let $\varepsilon < 1/B$. For $0 \leq \theta \leq 1$, we can endow $\mathcal{H}(s_0, l, B)$ with the norms

$$\|b\|_\theta = \sum_{j\geq 0} \frac{\varepsilon^j}{j!} \sup_{0<\tau<1} \sup_{|s|<s_0(1-\tau)} \sup_{|\mathrm{Im}\, z|<l\tau} |W(z) b_j(s,z)| \Big(1 - \tau - \frac{|s|}{s_0}\Big)^{j+\theta}.$$

Each of these norms makes $\mathcal{H}(s_0, l, B)$ a complete space.
In the following, fix $0 < \varepsilon < 1/B$, and let $0 < h < \varepsilon$. Fix also s_0, $B > 0$ and $l < \frac{1}{2}$. Denote by

$$\mathcal{H} = \mathcal{H}(s_0, l, B),$$

and define

$$\mathcal{H}^0 = \mathcal{H}(0, l, B),$$

the restriction to $s = 0$ of $\mathcal{H}$, endowed with the induced norms. This is the space of the initial conditions.
We will solve the system (2.6) in $(\mathcal{H}, \|\cdot\|_1)$ with a fixed point argument. The choice of the space and norms are inspired by abstract versions of the Cauchy-Kowaleski theorem [3].

We first give some properties of these norms.

Lemma 2.2. — *There exists $C > 0$ such that for all $\theta \in [0,1]$ and $b^1, b^2 \in \mathcal{H}$*

$$\|b^1\, b^2\|_\theta \leq C \|b^1\|_0 \|b^2\|_\theta. \tag{2.13}$$

Proof. — Set

$$\Omega = \big\{(\tau, s, z) \mid 0 < \tau < 1,\; |s| < s_0(1-\tau),\; |\mathrm{Im}\, z| < l\tau\big\},$$

and denote by

$$\sup_{\Omega} = \sup_{0<\tau<1} \sup_{|s|<s_0(1-\tau)} \sup_{|\mathrm{Im}\, z|<l\tau} .$$

Let

$$b^1 = \sum_{j\geq 0} b^1_j \, h^j, \quad \text{and} \quad b^2 = \sum_{j\geq 0} b^2_j \, h^j,$$

be two elements of $\mathcal{H}$, then $b^1 \, b^2$ can be written

$$b^1 \, b^2 = \sum_{j=0}^{\infty} \big(\sum_{k=0}^{j} b^1_k \, b^2_{j-k} \big) h^j. \tag{2.14}$$

It is easy to check that there exists $C > 0$ so that

$$|W(z)| \leq C |W(z)|^2, \tag{2.15}$$

on $|\mathrm{Im}\, z| < \frac{1}{2}$. Therefore by (2.14) and (2.15)

$$\begin{aligned}
\|b^1 \, b^2\|_\theta &= \sum_{j=0}^{\infty} \frac{\varepsilon^j}{j!} \sup_{\Omega} \big| W(z) \sum_{k=0}^{j} b^1_k \, b^2_{j-k}(s,z) \big| \big(1 - \tau - \frac{|s|}{s_0} \big)^{j+\theta} \\
&\leq C \sum_{k=0}^{\infty} \sum_{j=k}^{\infty} \frac{\varepsilon^k}{k!} \sup_{\Omega} |W(z) b^1_k(s,z)| \big(1 - \tau - \frac{|s|}{s_0} \big)^{k} \\
&\qquad \cdot \frac{\varepsilon^{j-k}}{(j-k)!} \sup_{\Omega} |W(z) b^2_{j-k}(s,z)| \big(1 - \tau - \frac{|s|}{s_0} \big)^{j-k+\theta} \\
&= \|b^1\|_0 \|b^2\|_\theta.
\end{aligned}$$

□

For $|s| < s_0$, denote by ∂_s^{-1} the operator defined by

$$\partial_s^{-1} b = \int_0^s b(\sigma) \mathrm{d}\sigma \quad \text{for} \quad b \in \mathcal{H}, \tag{2.16}$$

and $\partial_s^{-2} = \partial_s^{-1} \circ \partial_s^{-1}$. We then have the following

Lemma 2.3. — *i) Let A be one of the operators*

$$b \mapsto \nabla_z b, \ b \mapsto h \Delta_z b, \ b \mapsto \frac{1}{h}(b - b_0),$$

then there exists $C > 0$ such that for all $h \in]0, \varepsilon[$ and $b \in \mathcal{H}$

$$\|\partial_s^{-1} A b\|_1 \leq C s_0 \|b\|_1. \tag{2.17}$$

ii) For all $\theta \in]0,1]$, there exists C_θ such that for all $h \in]0, \varepsilon[$ and $b \in \mathcal{H}$

$$\|\partial_s^{-1} b\|_\theta \leq C_\theta s_0 \|b\|_1. \tag{2.18}$$

iii) There exists $C > 0$ such that for all $h \in]0, \varepsilon[$ and $b \in \mathcal{H}$

$$\|\partial_s^{-2} b\|_0 \leq C s_0 \|b\|_1. \tag{2.19}$$

Proof. — We can assume that $\|b\|_1 = 1$. Then there exists a nonnegative sequence $d = (d_j)_{j\geq 0}$ satisfying $\sum_{j\geq 0} d_j = 1$ so that, for all $j \geq 0$, for all $0 < \tau < 1$, $|s| < s_0(1-\tau)$, $|\text{Im } z| < l\tau$, we have

$$|W(z)\, b_j(s,z)| \leq C\frac{j!}{\varepsilon^j}\frac{d_j}{(1-\tau-\frac{|s|}{s_0})^{j+1}}. \tag{2.20}$$

Denote by $\nabla = \nabla_z$.
Proof of i)
• We prove the inequality $\|\partial_s^{-1}\nabla b\|_1 \leq Cs_0\|b\|_1$. Let $0 < \tau < 1$, $|s| < s_0(1-\tau)$ and $|\text{Im } z| < l\tau$. Let $\tau < \tau' < 1$. By the Cauchy formula we deduce that for all $|s'| \leq |s|$ and $|\text{Im } z| < l\tau$

$$|\nabla b_j(s',z)| \leq \frac{C}{\tau'-\tau}\sup_{|\text{Im } z'|<l\tau'}|b_j(s',z')|.$$

Thus, as $|\nabla W| \leq |W|$, for all $|s'| \leq |s|$ and $|\text{Im } z| < l\tau$

$$|W(z)\nabla b_j(s',z)| \leq \frac{C}{\tau'-\tau}\sup_{|\text{Im } z'|<l\tau'}|W(z')\, b_j(s',z')|. \tag{2.21}$$

Then by (2.20) and (2.21) we obtain

$$\Big|W(z)\int_0^s \nabla b_j(s',z)\text{d}s'\Big| \leq C\frac{j!}{\varepsilon^j}d_j\int_0^{|s|}\frac{1}{\tau'-\tau}\frac{\text{d}|s'|}{(1-\tau'-\frac{|s'|}{s_0})^{j+1}}.$$

We now make the choice

$$\tau'-\tau = 1-\tau'-\frac{|s'|}{s_0}, \quad \text{i.e.} \quad \tau' = \frac{1}{2}(1+\tau-\frac{|s'|}{s_0}), \tag{2.22}$$

then τ' satisfies $\tau < \tau' < 1$ because $0 < \tau < 1$ and $|s'| < (1-\tau)s_0$.
Moreover, (2.22) yields

$$1-\tau'-\frac{|s'|}{s_0} = \frac{1}{2}(1-\tau-\frac{|s'|}{s_0}),$$

therefore

$$\begin{aligned}\Big|W(z)\int_0^s \nabla b_j(s',z)\text{d}s'\Big| &\leq C\frac{j!}{\varepsilon^j}d_j\int_0^{|s|}\frac{\text{d}|s'|}{(1-\tau-\frac{|s'|}{s_0})^{j+2}}\\ &\leq Cs_0\frac{j!}{\varepsilon^j}d_j\big((1-\tau-\frac{|s|}{s_0})^{-j-1}-(1-\tau)^{-j-1}\big).\end{aligned}$$

And thus, as $|s| < s_0(1-\tau)$

$$\begin{aligned}\frac{\varepsilon^j}{j!}\Big|W(z)\int_0^s \nabla b_j(s',z)\text{d}s'\Big|\Big(1-\tau-\frac{|s|}{s_0}\Big)^{j+1} &\leq Cs_0\Big((1-(1-\frac{|s|}{s_0(1-\tau)})^{j+1}\Big)d_j\\ &\leq Cs_0d_j.\end{aligned}$$

Finally, by the previous inequality

$$\begin{aligned}\|\partial_s^{-1}\nabla b\|_1 &= \sum_{j\geq 0}\frac{\varepsilon^j}{j!}\sup_{0<\tau<1}\ \sup_{|s|<s_0(1-\tau)}\ \sup_{|\mathrm{Im}\, z|<l\tau}\Big|W(z)\int_0^s \nabla b_j(s',z)\mathrm{d}s'\Big|\big(1-\tau-\frac{|s|}{s_0}\big)^{j+1}\\ &\leq Cs_0\sum_{j\geq 0} d_j \leq Cs_0,\end{aligned}$$

which was the claim.

• The inequality $h\|\partial_s^{-1}\Delta b\|_1 \leq Cs_0\|b\|_1$ can be shown by the same manner, using that $h<\varepsilon$ compensates the loss of one more derivative.

• Denote by $b'=(b-b_0)/h$, then for all $j\geq 0$, $b'_j=b_{j+1}$. By (2.20)

$$\Big|W(z)\int_0^s b_{j+1}(s',z)\mathrm{d}s'\Big| \leq \frac{s_0}{j+1}\frac{(j+1)!}{\varepsilon^{j+1}}d_{j+1}\big((1-\tau-\frac{|s|}{s_0})^{-j-1}-(1-\tau)^{-j-1}\big),$$

and therefore, for all $0<\tau<1$, $|s|<s_0(1-\tau)$, $|\mathrm{Im}\, z|<l\tau$ and $j\geq 0$

$$\begin{aligned}\frac{\varepsilon^j}{j!}\Big|W(z)\int_0^s b_{j+1}(s',z)\mathrm{d}s'\Big|(1-\tau-\frac{|s|}{s_0})^{j+1} &\leq \frac{s_0}{\varepsilon}\big(1-(1-\frac{|s|}{s_0(1-\tau)})^{j+1}\big)d_{j+1}\\ &\leq Cs_0 d_{j+1}.\end{aligned}$$

This yields $h^{-1}\|\partial_s^{-1}(b-b_0)\|_1 \leq Cs_0\|b\|_1$ for fixed $\varepsilon>h$.

Proof of *ii)*

By integration of inequality (2.20), we obtain for all $j\geq 1$

$$\begin{aligned}\Big|W(z)\int_0^s b_j(s',z)\mathrm{d}s'\Big| &\leq Cs_0\frac{j!}{j\varepsilon^j}d_j\big((1-\tau-\frac{|s|}{s_0})^{-j}-(1-\tau)^{-j}\big).\\ &\leq Cs_0\frac{j!}{\varepsilon^j}d_j(1-\tau-\frac{|s|}{s_0})^{-j},\end{aligned}$$

hence

$$\frac{\varepsilon^j}{j!}\Big|W(z)\int_0^s b_j(s',z)\mathrm{d}s'\Big|\Big(1-\tau-\frac{|s|}{s_0}\Big)^{j+\theta} \leq Cs_0 d_j. \tag{2.23}$$

For $j=0$ we obtain

$$\Big|W(z)\int_0^s b_0(s',z)\mathrm{d}s'\Big| \leq Cs_0\Big(\log(1-\tau)-\log(1-\tau-\frac{|s|}{s_0})\Big),$$

then

$$\Big|W(z)\int_0^s b_0(s',z)\mathrm{d}s'\Big|\Big(1-\tau-\frac{|s|}{s_0}\Big)^{\theta} \leq Cs_0 d_0. \tag{2.24}$$

By the definition of $\|\cdot\|_\theta$, inequalities (2.23) and (2.24) give the result.

The proof of *iii)* is similar, and is left here. □

Lemma 2.4. — *There exists $C>0$ such that for all $h\in]0,\varepsilon[$ and $b^1,b^2\in\mathcal{H}$*

$$\|(\partial_s^{-1}b^1)\,(\partial_s^{-1}b^2)\|_1 \leq Cs_0^2\|b^1\|_1\|b^2\|_1. \tag{2.25}$$

Proof. — Write

$$\begin{aligned}
\|(\partial_s^{-1}b^1)\,(\partial_s^{-1}b^2)\|_1 &= \sum_{j=0}^{\infty}\frac{\varepsilon^j}{j!}\sup_{\Omega}\big|W(z)\sum_{k=0}^{j}\partial_s^{-1}b^1_k\,\partial_s^{-1}b^2_{j-k}(s,z)\big|\big(1-\tau-\frac{|s|}{s_0}\big)^{j+1}\\
&\leq C\sum_{k=0}^{\infty}\sum_{j=k}^{\infty}\frac{\varepsilon^k}{k!}\sup_{\Omega}|W(z)\partial_s^{-1}b^1_k(s,z)|\big(1-\tau-\frac{|s|}{s_0}\big)^{k+\frac12}\\
&\qquad\cdot\frac{\varepsilon^{j-k}}{(j-k)!}\sup_{\Omega}|W(z)\partial_s^{-1}b^2_{j-k}(s,z)|\big(1-\tau-\frac{|s|}{s_0}\big)^{j-k+\frac12}\\
&= C\,\|\partial_s^{-1}b^1\|_{\frac12}\|\partial_s^{-1}b^2\|_{\frac12},
\end{aligned}$$

and then by Lemma 2.3 *ii)* with $\theta=1/2$, we deduce

$$\|(\partial_s^{-1}b^1)\,(\partial_s^{-1}b^2)\|_1 \leq Cs_0^2\|b^1\|_1\|b^2\|_1.$$

□

Proposition 2.5. — *Let $S^0\in\mathcal{H}^0(l,B)$ be a real analytic function, and let $a^0\in\mathcal{H}^0(l,B)$ be an analytic symbol. Then there exist $s_0>0$, a real analytic function $S\in\mathcal{H}(s_0,l,B)$, and an analytic symbol $a\in\mathcal{H}(s_0,l,B)$, such that $v=ae^{iS/h}$ is a formal solution of equation (2.3) with Cauchy data $v_0=a^0e^{iS^0/h}$.*

Remark 2.6. — By the Cauchy formula, the function $v=ae^{iS/h}$ satisfies for all $k\in\mathbb{N}$

$$\sup_{|s|<s_0}\ \sup_{|\mathrm{Im}\ z|<l/2}\ \big|\big(1-h^2\Delta\big)^{k/2}v\big|\lesssim \mathrm{e}^{-|z|}, \tag{2.26}$$

and

$$\sup_{|s|<s_0}\ \sup_{|\mathrm{Im}\ z|<l/2}\ \big|\big(1-h^2\Delta\big)^{k/2}|v|^{p-1}v\big|\lesssim \mathrm{e}^{-p|z|}. \tag{2.27}$$

This will be usefull in the sequel.

Proof. — The proof is based on a fixed point argument in $(\mathcal{H},\|\cdot\|_1)$.
Set $\varphi=\nabla S$ and differentiate the first equation of (2.6) with respect to the space variable, then we obtain

$$\begin{cases}\partial_s\varphi=-2\varphi\cdot\nabla\varphi-\omega\nabla f(a_0)\\[2mm] \partial_s a=-2\varphi\cdot\nabla a-a\,\mathrm{div}\varphi+ih\Delta a-\dfrac{i\omega a}{h}\big(f(a)-f(a_0)\big),\end{cases} \tag{2.28}$$

where $\overline{a}(s,z)=\overline{a(\overline{s},\overline{z})}$ and $f(b)=(b\,\overline{b})^{\frac{p-1}{2}}$
Differentiate the system (2.28) with respect to s and obtain

$$\left\{\begin{aligned}\partial_s^2\varphi=&-2\partial_s\varphi\cdot\nabla\varphi-2\varphi\cdot\nabla\partial_s\varphi-\omega\partial_s\nabla f(a_0)\\\partial_s^2 a=&-2\partial_s\varphi\cdot\nabla-2\varphi\cdot\nabla\partial_s a-a\,\mathrm{div}\partial_s\varphi-\partial_s a\,\mathrm{div}\varphi+ih\Delta\partial_s a\\&-\frac{i\omega\partial_s a}{h}\big(f(a)-f(a_0)\big)-\frac{i\omega a}{h}\partial_s\big(f(a)-f(a_0)\big).\end{aligned}\right.\tag{2.29}$$

Write

$$\left\{\begin{aligned}\partial_s\varphi&=\partial_s^{-1}(\partial_s^2\varphi)+\partial_s\varphi(0,\cdot),\\\partial_s a&=\partial_s^{-1}(\partial_s^2 a)+\partial_s a(0,\cdot),\end{aligned}\right.\tag{2.30}$$

and

$$\left\{\begin{aligned}\varphi&=\partial_s^{-2}(\partial_s^2\varphi)+s\partial_s\varphi(0,\cdot)+\varphi(0,\cdot),\\a&=\partial_s^{-2}(\partial_s^2 a)+s\partial_s a(0,\cdot)+a(0,\cdot).\end{aligned}\right.\tag{2.31}$$

Now introduce the new unknown function $u=(u_1,u_2)=(\partial_s^2\varphi,\partial_s^2 a)$. Hence we are lead to solving a system of the form

$$u=F(s,u).\tag{2.32}$$

We will show that for $0<s_0<1$ small enough F is a contraction in a ball in $(\mathcal{H},\|\cdot\|_1)$. Let $R>0$ be such that

$$\|\varphi(0,\cdot)\|_0,\|\partial_s\varphi(0,\cdot)\|_0,\|\nabla\varphi(0,\cdot)\|_0,\|\nabla\partial_s\varphi(0,\cdot)\|_0,\|\Delta\partial_s\varphi(0,\cdot)\|_0\leq R,$$

and

$$\begin{gathered}\|a(0,\cdot)\|_0,\|\partial_s a(0,\cdot)\|_0,\|\nabla a(0,\cdot)\|_0,\|\nabla\partial_s a(0,\cdot)\|_0,\|\Delta\partial_s a(0,\cdot)\|_0,\\\|(a-a_0)(0,\cdot)/h\|_0,\|\partial_s(a-a_0)(0,\cdot)/h\|_0\leq R.\end{gathered}$$

- Write

$$\begin{aligned}\partial_s\varphi\nabla\varphi=&\big(\partial_s^{-1}\partial_s^2\varphi+\partial_s\varphi(0,\cdot)\big)\cdot\\&\big(\partial_s^{-1}(\partial_s^{-1}\nabla)(\partial_s^2\varphi)+s\nabla\partial_s\varphi(0,\cdot)+\nabla\varphi(0,\cdot)\big)\end{aligned}$$

Then by (2.25) and (2.13)

$$\begin{aligned}\|\partial_s\varphi\nabla\varphi\|_1\lesssim&\, s_0^2\|\partial_s^2\varphi\|_1\|\partial_s^{-1}\nabla(\partial_s^2\varphi)\|_1+R\|\partial_s^{-1}\partial_s^2\varphi\|_1\\&+R\|\partial_s^{-1}(\partial_s^{-1}\nabla)(\partial_s^2\varphi)\|_1+R^2,\end{aligned}$$

and by (2.17) and (2.18)

$$\begin{aligned}\|\partial_s\varphi\nabla\varphi\|_1&\lesssim s_0^3\|\partial_s^2\varphi\|_1^2+s_0R\|\partial_s^2\varphi\|_1+R^2\\&\lesssim s_0^2\|\partial_s^2\varphi\|_1^2+R^2.\end{aligned}\tag{2.33}$$

- Similarly we obtain

$$\|\varphi\nabla\partial_s\varphi\|_1\lesssim s_0^2\|\partial_s^2\varphi\|_1^2+R^2,\tag{2.34}$$

and
(2.35)
$$\|\partial_s\varphi\nabla a\|_1, \|\varphi\nabla\partial_s a\|_1, \|a\operatorname{div}\partial_s\varphi\|_1, \|\partial_s a\operatorname{div}\varphi\|_1 \lesssim s_0^2\|\partial_s^2\varphi\|_1\|\partial_s^2 a\|_1 + R^2.$$

• We have
$$\nabla\partial_s f(a_0) = \nabla\partial_s^{-1}\partial_s^2 f(a_0) + \nabla\partial_s f(a_0)(0,\cdot). \tag{2.36}$$
By the Leibniz rule and (2.13)
$$\|\partial_s^2 f(a_0)\|_1 \lesssim \|\partial_s^2 a_0\|_1\|a_0\|_0^{p-2} + \|(\partial_s a_0)^2\|_1\|a_0\|_0^{p-3}.$$
From (2.31) and (2.19)
$$\|a_0\|_0 \lesssim \|\partial_s^{-2}\partial_s^2 a_0\|_0 + R \lesssim s_0\|\partial_s^2 a_0\|_1 + R. \tag{2.37}$$
Now use (2.30), (2.25) and (2.18)
$$\|(\partial_s a_0)^2\|_1 \lesssim s_0^2\|\partial_s^2 a_0\|_1^2 + R^2. \tag{2.38}$$
Finally, from (2.36), (2.37) and (2.38) we deduce
$$\|\nabla\partial_s f(a_0)\|_1 \lesssim s_0^{\frac{p-1}{2}}\|\partial_s^2 a_0\|_1^{\frac{p-1}{2}} + R^{\frac{p-1}{2}} \lesssim s_0^{\frac{p-1}{2}}\|\partial_s^2 a\|_1^{\frac{p-1}{2}} + R^{\frac{p-1}{2}}. \tag{2.39}$$
• Write
$$h\Delta\partial_s a = h\partial_s^{-1}\Delta\partial_s^2 a + h\Delta\partial_s a(0,\cdot),$$
therefore by (2.17)
$$\|h\Delta\partial_s a\|_1 \lesssim s_0\|\partial_s^2 a\|_1 + R. \tag{2.40}$$
• We now estimate the term $\frac{\partial_s a}{h}\big(f(a) - f(a_0)\big)$. Observe that
$$\begin{aligned} f(a) - f(a_0) &= (a\overline{a})^{\frac{p-1}{2}} - (a_0\overline{a_0})^{\frac{p-1}{2}} \\ &= (a\overline{a} - a_0\overline{a_0})\big((a\overline{a})^{\frac{p-3}{2}} + \cdots + (a_0\overline{a_0})^{\frac{p-3}{2}}\big), \end{aligned}$$
and
$$a\overline{a} - a_0\overline{a_0} = (a - a_0)\overline{a} + (\overline{a} - \overline{a_0})a_0.$$
Then by (2.13)
$$\|\frac{\partial_s a}{h}\big(f(a) - f(a_0)\big)\|_1 \lesssim \|\partial_s a\,\frac{a - a_0}{h}\|_1\|a\|_0^{p-2}. \tag{2.41}$$
Use (2.30), (2.31) to write
$$\begin{aligned} \partial_s a\,\frac{a - a_0}{h} = &\Big(\partial_s^{-1}(\partial_s^2 a) + \partial_s a(0,\cdot)\Big) \\ &\Big(\partial_s^{-1}\partial_s^{-1}\frac{\partial_s^2(a - a_0)}{h} + s\frac{\partial_s(a - a_0)(0,\cdot)}{h} + \frac{(a - a_0)(0,\cdot)}{h}\Big), \end{aligned}$$
then by (2.25), (2.13) and (2.18)
$$\|\partial_s a\,\frac{a - a_0}{h}\|_1 \lesssim \big(s_0\|\partial_s^2 a\|_1 + Rs_0\big)\big(\|\partial_s^{-1}\frac{\partial_s^2(a - a_0)}{h}\|_1 + R\big). \tag{2.42}$$

Moreover from (2.17) we have

$$\|\partial_s^{-1}\frac{\partial_s^2(a-a_0)}{h}\|_1 \lesssim s_0\|\partial_s^2 a\|_1. \tag{2.43}$$

Therefore inequalities (2.41), (2.42) and (2.43) yield

$$\|\frac{\partial_s a}{h}\big(f(a)-f(a_0)\big)\|_1 \lesssim s_0^p\|\partial_s^2 a\|_1^p + R^p. \tag{2.44}$$

• Similar arguments are used to show that

$$\|\frac{a}{h}\partial_s\big(f(a)-f(a_0)\big)\|_1 \lesssim s_0^p\|\partial_s^2 a\|_1^p + R^p. \tag{2.45}$$

Inequalities (2.33), (2.34), (2.35), (2.39), (2.40), (2.44) and (2.45) show that, if $s_0 > 0$ is small enough, there exists $R_1 > R$ such that F maps the ball of radius R_1 $\big(\text{in } (\mathcal{H}, \|\cdot\|_1)\big)$ into itself.

With analogous arguments, we can show that F is a contraction in $(\mathcal{H}, \|\cdot\|_1)$. Hence by the fixed point theorem, there exists a unique $u = (\partial_s^2\varphi, \partial_s^2 a) \in \mathcal{H}\times\mathcal{H}$ which satisfies (2.32).
Let $(\varphi^0, a^0) \in \mathcal{H}\times\mathcal{H}$, and consider the couple $(\partial_s\varphi(0,\cdot), \partial_s a(0,\cdot)) \in \mathcal{H}\times\mathcal{H}$ which solves the system (2.28) at $s=0$. Let u be the solution of (2.32) with these initial conditions. Then with the formula (2.31) we recover the couple (φ, a) which is a solution of (2.28). Moreover, (2.31) shows that $(\varphi, a) \in \mathcal{H}\times\mathcal{H}$.

Let $S^0 \in \mathcal{H}^0$ and take $\varphi^0 = \nabla S^0$. The function φ (with Cauchy condition $\varphi(0,\cdot) = \varphi^0$) is irrotational, as it satisfies the equation

$$\partial_s\varphi = -2\varphi\cdot\nabla\varphi - \omega\nabla f(a_0).$$

Therefore there exists S so that $\varphi = \nabla S$ and which is solution of

$$\nabla\big(\partial_s S + (\nabla S)^2 + \omega f(a_0)\big) = 0.$$

Moreover, it is possible to choose S such that

$$\partial_s S + (\nabla S)^2 + \omega f(a_0) = 0.$$

Now the formula

$$S(s,z) = \int_0^s \partial_s S(\sigma,z)\mathrm{d}\sigma + S^0(z) = -\int_0^s \big(\varphi\cdot\varphi + \omega f(a_0)\big)(\sigma,z)\mathrm{d}\sigma + S^0(z), \tag{2.46}$$

shows that $S \in \mathcal{H}$.

Finally, we have shown the existence of a solution $(S,a) \in \mathcal{H}\times\mathcal{H}$ of the system

$$\begin{cases} \partial_s S + (\nabla S)^2 + \omega|a_0|^{p-1} = 0, \\ \partial_s a + 2\nabla S\cdot\nabla a + a\Delta S - ih\Delta a + \dfrac{i\omega a}{h}(|a|^{p-1} - |a_0|^{p-1}) = 0, \\ S(0,z) = S^0(z) \in \mathcal{H}^0,\ a(0,z,h) = a^0(z,h) \in \mathcal{H}^0. \end{cases}$$

With a Gronwall inequality, it is straightforward to check that S is real analytic. □

Remark 2.7. — The inequality $\|\partial_s^{-1}b\|_0 \leq Cs_0\|b\|_1$ fails, and that is the reason why we have to differentiate the system (2.28) with respect to the time variable, before applying the contraction method.

2.2. The general case of an analytic manifold (M^d, g). —
Let (M^d, g) an analytic Riemannian manifold of dimension d. We assume moreover that g is analytic. Let $m \in M^d$. Then there exist a neighbourhood $\mathcal{U} \subset M^d$ of m, a neighbourhood $\mathcal{V} \subset \mathbb{R}^d$ of 0, and an homeomorphism

$$\kappa\colon \mathcal{U} \longrightarrow \mathcal{V}. \tag{2.47}$$

In the chart $(\mathcal{U}, \kappa)$ the metric g can be written

$$g = \sum_{1\leq j,k\leq d} g_{jk}(x)\mathrm{d}x_j\mathrm{d}x_k,$$

where $G = (g_{jk})$ is a positive symmetric matrix and analytic in $\mathcal{V}$.
In these coordinates, we have the explicit formula for the Laplace-Beltrami operator

$$\begin{aligned}\Delta_g = \Delta_g(x) &= \frac{1}{\sqrt{\det G}}\mathrm{div}\big(\sqrt{\det G}\; G^{-1}\nabla\,\cdot\,\big)\\ &= \frac{1}{\sqrt{\det G}}\sum_{1\leq j,k\leq d}\frac{\partial}{\partial x_j}\big(\sqrt{\det G}\; g^{jk}\frac{\partial}{\partial x_k}\big),\end{aligned}$$

where $(g^{jk}) = G^{-1}$. Every function involved in the former expression is analytic.
We now make the rescaling (2.1). The function

$$v(s,z,h) = \hbar^{-\gamma}u(\hbar^{\alpha}s, \hbar z),$$

satisfies

$$ih\partial_t v(s,z) + h^2\Delta(\hbar z)v(s,z) = \omega|v|^{p-1}v(s,z), \quad (s,z)\in\mathbb{R}\times\hbar^{-1}\mathcal{V}. \tag{2.48}$$

We now adapt the analysis of Section 2 to the equation (2.48), in $\hbar^{-1}\mathcal{V}$ instead of $\mathbb{R}^d$.
Let $r > 0$ such that

$$\mathcal{B}(0,2r) \subset \mathcal{V}. \tag{2.49}$$

Notice that on the set $\big\{(|\hbar z| < r) \cap (|\mathrm{Im}\ z| < l)\big\}$, the coefficients of Δ_g are uniformly bounded with respect to $\hbar$, as well as their derivatives.
Here again, we want to find a formal solution of (2.48) of the form

$$v(s,z,h) = a(s,z,h)\mathrm{e}^{iS(s,z)/h} = \big(\sum_{j\geq 0} a_j(s,z)h^j\big)\mathrm{e}^{iS(s,z)/h}.$$

Therefore (S,a) has to satisfy the system

$$\begin{cases} \partial_s S + (\nabla_g S)^2 + \omega |a_0|^{p-1} = 0, \\ \partial_s a + 2\nabla_g S \cdot \nabla_g a + a\,\Delta_g S - ih\Delta_g a + \dfrac{i\omega a}{h}(|a|^{p-1} - |a_0|^{p-1}) = 0, \\ S(0,z) = S^0(z),\ a(0,z,h) = a^0(z,h), \end{cases} \tag{2.50}$$

with $\nabla_g = \nabla_g(\hbar z)$, $\Delta_g = \Delta_g(\hbar z)$ and where $v(0,z,h) = a^0(z,h)\mathrm{e}^{iS^0(z)/h}$. For $\hbar > 0$ small enough, denote by

$$\mathcal{D}_\hbar = \{|s| < s_0\} \times \big\{(|\hbar z| < r) \cap (|\mathrm{Im}\ z| < l)\big\},$$

and by $\mathcal{H}_\hbar = \mathcal{H}_\hbar(s_0,l,r,B)$ the space of the analytic symbols $b(s,z,h) = \sum_{j\geq 0} b_j(s,z)h^j$ (see Definition 2.1) satisfying

$$|W(z)\, b_j(s,z)| \leq AB^j j! \ \text{ on } \ \mathcal{D}_\hbar,\ \forall j \geq 0.$$

Define also $\mathcal{H}^0_\hbar = \mathcal{H}_\hbar(0,l,r,B)$ the space of the initial conditions. Let $\varepsilon < 1/B$. For $0 \leq \theta \leq 1$, we endow $\mathcal{H}_\hbar(s_0,l,r,B)$ with the norms

$$\|b\|_{\theta,\hbar} = \sum_{j\geq 0} \frac{\varepsilon^j}{j!} \sup_{0<\tau<1} \sup_{|s|<s_0(1-\tau)} \sup_{\Gamma_\tau} |W(z)\, b_j(s,z)| \Big(1 - \tau - \frac{|s|}{s_0}\Big)^{j+\theta},$$

where $\Gamma_\tau = \big\{(|\hbar z| < r\tau) \cap (|\mathrm{Im}\ z| < l\tau)\big\}$.

Now it is straightforward to check that the results of Lemma 2.2, Lemma 2.3 and Lemma 2.4 hold when $\|\cdot\|_\theta$ is replaced with $\|\cdot\|_{\theta,\hbar}$ and that the constants involved in the estimates do not depend on $\hbar$. Notice that the boundedness of $\mathcal{D}_\hbar$ with respect to the variable $|\hbar z|$ is dealt with in exaclty the same way as was done with the boundedness with respect to $|\mathrm{Im}\ z|$. This yields the following analog of Proposition 2.5

Proposition 2.8. — *Let $S^0 \in \mathcal{H}^0_\hbar(l,r,B)$ be a real analytic function, and let $a^0 \in \mathcal{H}^0_\hbar(l,r,B)$ be an analytic symbol. Then there exist $s_1 > 0$ independent of $\hbar$, a real analytic function $S \in \mathcal{H}_\hbar(s_1,l,r,B)$, and an analytic symbol $a \in \mathcal{H}_\hbar(s_1,l,r,B)$, such that $v = a e^{iS/h}$ is a formal solution of equation* (2.3) *with Cauchy data $v_0 = a^0 e^{iS^0/h}$.*

Remark 2.9. — Proposition 2.5 is contained in Proposition 2.8: In the case $(M^d,g) = (\mathbb{R}^d, \mathrm{can})$, $r = +\infty$ and $\mathcal{H}_\hbar(s_1,l,r,B) = \mathcal{H}(s_1,l,B)$.

We are now able to construct an approximate solution of the problem (2.48).

Let c_0 such that $c_0/h =: n \in \mathbb{N}$. Define

$$a^{(n)}(s,z,h) = \sum_{j\leq n} a_j(s,z)h^j,$$

and

$$v_{\mathrm{app}}(s,z,h)=a^{(n)}(s,z,h)\mathrm{e}^{iS(s,z)/h}. \tag{2.51}$$

where the a_j's and S are given by Proposition 2.8. The choice of the initial condition $v_{\mathrm{app}}(0,z,h)$ will be made in Section 4.

We now show that if c_0 is small enough, v_{app} is a good approximation to the problem (2.3).

Proposition 2.10. — *Let $s_1>0$ be given by Proposition 2.8. If $c_0\ll 1$, there exists $\delta_1>0$ such that the function v_{app} defined by (2.51) satisfies*

$$ih\partial_s v_{app}+h^2\Delta v_{app}=\omega(v_{app}\overline{v_{app}})^{\frac{p-1}{2}}v_{app}+e^{-\delta_1/h}g, \tag{2.52}$$

with $\overline{v_{app}}=\overline{v_{app}(s,\overline{z})}$ and where g is an analytic function on $\{|s|<s_1\}\times\big\{(|\hbar z|<r)\cap(|Im\ z|<l)\big\}$ such that for all $k\in\mathbb{N}$, there exists $C_k>0$ independent of h so that

$$\sup_{|s|<s_1}\ \sup_{|Im\ z|<l/2}\|(1-h^2\Delta)^{k/2}g(s,\cdot+iIm\ z)\|_{L^2(\mathcal{B}(0,r/\hbar))}\leq C_k. \tag{2.53}$$

Here we have used the convention that $\mathcal{B}(0,r/\hbar)=\mathbb{R}^d$ if $r=+\infty$.

Proof. — Denote by $f(b)=(b\,\overline{b})^{\frac{p-1}{2}}$, with $\overline{b}=\overline{b(s,\overline{z})}$. The function v_{app} satisfies the equation

$$\begin{aligned}&ih\partial_s v_{\mathrm{app}}+h^2\Delta v_{\mathrm{app}}-\omega f(v_{\mathrm{app}})v_{\mathrm{app}}\\&=-a^{(n)}\big(\partial_s S+(\nabla S)^2+\omega f(a_0)\big)\mathrm{e}^{iS/h}\\&\qquad+ih\Big(\partial_s a^{(n)}+2\nabla S\cdot\nabla a^{(n)}+a^{(n)}\Delta S-ih\Delta a^{(n)}\\&\qquad\qquad+\frac{i\omega a^{(n)}}{h}\big(f(a^{(n)})-f(a_0)\big)\Big)\mathrm{e}^{iS/h}.\end{aligned} \tag{2.54}$$

For $m=n,n+1$ write the expansion in h

$$\frac{i\omega a^{(m)}}{h}\big(f(a^{(m)})-f(a_0)\big):=\sum_{j=0}^{pm-1}b_{j,m}h^j. \tag{2.55}$$

By construction the following system is satisfied

$$\begin{cases}\partial_s S+(\nabla S)^2+\omega(a_0\overline{a_0})^{\frac{p-1}{2}}=0,\\[2mm] \partial_s a^{(n)}+2\nabla S\cdot\nabla a^{(n)}+a^{(n)}\Delta S-ih\Delta a^{(n-1)}+\displaystyle\sum_{j=0}^{n}b_{j,n+1}h^j=0.\end{cases} \tag{2.56}$$

Notice that

$$b_{j,n}=b_{j,n+1}\quad\text{for all}\quad j\leq n-1. \tag{2.57}$$

Therefore by (2.57) and (2.56), (2.54) rewrites

$$\begin{aligned} &ih\partial_s v_{\text{app}} + h^2\Delta v_{\text{app}} - \omega(v_{\text{app}}\overline{v_{\text{app}}})^{\frac{p-1}{2}} v_{\text{app}} \\ &= ih\big(- ih^{n+1}\Delta a_n - \sum_{j=0}^{n} b_{j,n+1}h^j + \sum_{j=0}^{pn-1} b_{j,n}h^j\big)\mathrm{e}^{iS/h} \\ &= \big(h^{n+2}\Delta a_n - ih^{n+1}b_{n,n+1} + ih\sum_{j=n}^{pn-1} b_{j,n}h^j\big)\mathrm{e}^{iS/h}. \end{aligned} \tag{2.58}$$

We now estimate each term of the r.h.s. of (2.58). By (2.55) we have

$$hb_{j,n} = i\omega\Big(\sum_{i_1+\cdots+i_p=j} \widetilde{a_{i_1}}\cdots\widetilde{a_{i_p}} - (a_0\overline{a_0})^{\frac{p-1}{2}}a_j\Big),$$

with $\widetilde{a_{i_k}} = a_{i_k}$ or $\widetilde{a_{i_k}} = \overline{a_{i_k}}$.
Now by (2.11), $|a_{i_k}| \lesssim B^{i_k}(i_k)!\,\mathrm{e}^{-|z|}$, thus

$$h|b_{j,n}| \lesssim B^j\Big(\sum_{i_1+\cdots+i_p=j}(i_1)!\,\cdots(i_p)!\,+j!\,\Big)\mathrm{e}^{-p|z|} \lesssim B^j j!\,\mathrm{e}^{-p|z|}, \tag{2.59}$$

and by the Stirling formula,

$$(pn)!\,\lesssim n^{1/2}\big(\frac{pn}{\mathrm{e}}\big)^{pn},$$

we deduce from (2.59)

$$\begin{aligned} |h\sum_{j=n}^{pn-1} b_{j,n}h^j| &\lesssim \big(\sum_{j=n}^{pn-1} B^j j!\,h^j\big)\mathrm{e}^{-p|z|} \\ &\leq (p-1)n(Bh)^{pn}(pn)!\,\mathrm{e}^{-p|z|} \\ &\lesssim h^{-\frac{3}{2}}\big(\frac{Bc_0p}{\mathrm{e}}\big)^{\frac{c_0p}{h}}\mathrm{e}^{-p|z|}, \end{aligned}$$

as we have $n = c_0/h$. Now choose $c_0 < e/(Bp)$, then there exists $\delta > 0$ such that

$$|h\sum_{j=n}^{pn-1} b_{j,n}h^j| \lesssim \sum_{j=n}^{pn-1} h|b_{j,n}h^j| \lesssim \mathrm{e}^{-\delta/h}\mathrm{e}^{-p|z|}.$$

Similarly, for some $\delta > 0$

$$|h^{n+1}b_{n,n+1}| \lesssim \mathrm{e}^{-\delta/h}\mathrm{e}^{-|z|},$$
$$|h^{n+2}\Delta a_n| \lesssim \mathrm{e}^{-\delta/h}\mathrm{e}^{-|z|}.$$

Finally use that the function $\phi : (\operatorname{Re} z, \operatorname{Im} z) \longmapsto \mathrm{e}^{-|\operatorname{Re} z + i\operatorname{Im} z|}$ satisfies

$$\sup_{|\operatorname{Im} z|<l} \|\phi(\cdot, \operatorname{Im} z)\|_{L^2(\mathcal{B}(0,r/\hbar))} \lesssim 1.$$

We have therefore proved the estimate (2.53) for $k = 0$.
To treat the case $k \geq 0$, use the Cauchy formula to obtain

$$\sup_{|s|<s_1} \sup_{|\mathrm{Im}\ z|<l/2} \left|(1 - h^2\Delta)^{k/2} a_j\right| \lesssim \sup_{|s|<s_1} \sup_{|\mathrm{Im}\ z|<l} |a_j| \lesssim B^j j!\, \mathrm{e}^{-|z|},$$

and

$$\sup_{|s|<s_1} \sup_{|\mathrm{Im}\ z|<l/2} \left|(1 - h^2\Delta)^{k/2} \mathrm{e}^{iS/h}\right| \lesssim 1,$$

and we can easily adapt the previous computations. □

3. Validity of the Ansatz

Proposition 3.1. — *Let v_{app} be the function defined by* (2.51). *Let v be the solution of*

$$\begin{cases} ih\partial_s v + h^2\Delta v = \omega|v|^{p-1}v, & (s,z) \in \mathbb{R}^{1+d}, \\ v(0,z) = v_{app}(0,z). \end{cases} \tag{3.1}$$

Then there exist $s_2 > 0$ and $\delta_2 > 0$ such that for all $k \in \mathbb{N}$

$$\sup_{0<s<s_2} \|(1 - h^2\Delta)^{k/2}(v - v_{app})(s)\|_{L^2(\mathcal{B}(0,r/\hbar))} \leq C_k\, e^{-\delta_2/h},$$

with $C_k > 0$.

Proof. — It is given in [11], but we reproduce it in the appendix. □

We are now able to define the Ansatz to the equation (1.1).

In the case $(M^d, g) = (\mathbb{R}^d, \mathrm{can})$, we consider the function u_{app} given by (2.51) and define

$$u_{\mathrm{app}}(t,x) = \hbar^{\gamma} v_{\mathrm{app}}(\hbar^{-\alpha}t, \hbar^{-1}x), \tag{3.2}$$

where γ and α satisfy the relations (2.2) and $h = \hbar^{\beta}$. The initial condition will be given in the next section.
From Proposition 3.1 we deduce

Corollary 3.2. — *(The case $(M^d, g) = (\mathbb{R}^d, can)$) Let s_2 be given by Proposition 3.1, let u_{app} be given by* (3.2), *and let u be the solution of*

$$\begin{cases} i\partial_t u + \Delta u = \omega|u|^{p-1}u, & (t,x) \in \mathbb{R}^{1+d}, \\ u(0,x) = u_{app}(0,x). \end{cases}$$

Then for all $k \in \mathbb{N}$

$$\|u - u_{app}\|_{L^{\infty}([0,\hbar^{\alpha}s_2];H^k(\mathbb{R}^d))} \longrightarrow 0,$$

when $h \longrightarrow 0$.

In the general case of an analytic manifold (M^d, g), we have to construct an approximate solution supported in $\mathcal{B}(0,r) \subset \mathcal{U}$.
Let $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$, $\chi \geq 0$, such that

$$\chi(\xi) = \begin{cases} 1 & \text{for } |\xi| \leq r/2, \\ 0 & \text{for } |\xi| \geq r. \end{cases} \tag{3.3}$$

Let $0 < \eta < 1$, let v_{app} be given by (2.51) and consider

$$\underline{u_{\text{app}}}(t,x) = \hbar^\gamma \chi(\hbar^{-\eta}|x|) v_{\text{app}}(\hbar^{-\alpha} t, \hbar^{-1} x), \tag{3.4}$$

where γ and α are given by the relations (2.2), and $h = \hbar^\beta$.
We have

$$\text{supp } \underline{u_{\text{app}}} \subset \{(t,x) \in \mathbb{R}^{1+d} \;:\; |x| \leq r\hbar^\eta\},$$

which concentrates in $x = 0$.
Hence if $\hbar$ is small enough, u_{app} is supported in $\mathcal{V}$, and we can transport this function to $\mathcal{U}$ by the chart κ (see (2.47)).We therefore define the approximate solution u^M_{app} of (1.1) by

$$u^M_{\text{app}} = \underline{u_{\text{app}}} \circ \kappa. \tag{3.5}$$

In the following we write $u^M_{\text{app}} = u_{\text{app}}$.
Then, as u_{app} is compactly supported, it can not be analytic. We now consider all the functions only with real variables.
Up to now, we did not use the rate of decrease of the weight W^{-1} introduced in (2.11), but it is needed now because of the truncation. However, because of the error $\mathrm{e}^{-c/\hbar}$ induced from this cutoff, we obtain the following weaker result

Corollary 3.3. — *(The general case) Let s_2 be given by Proposition 3.1, let u_{app} be given by* (3.4)*, and let u be the solution of*

$$\begin{cases} i\partial_t u + \Delta u = \omega |u|^{p-1} u, & (t,x) \in \mathbb{R} \times M^d, \\ u(0,x) = u_{app}(0,x). \end{cases} \tag{3.6}$$

Let $\kappa \geq 0$ such that $\beta + \eta - \kappa < 1$. Then for all $k \in \mathbb{N}$

$$\|u - u_{app}\|_{L^\infty([0,\hbar^{\alpha+\kappa} s_2]; H^k(M^d))} \longrightarrow 0,$$

when $h \longrightarrow 0$.

Proof. — Let $k > d/2$ an integer, and set

$$\|f\|_{H^k_\hbar} = \|\big(1 - \hbar^{2(\beta+1)}\Delta\big)^{k/2} f\|_{L^2(M^d)}.$$

With the Leibniz rule and interpolation we check that for all $f \in H^k(M^d)$ and $g \in W^{k,\infty}(M^d)$

$$\|f\,g\|_{H^k_\hbar} \lesssim \|f\|_{H^k_\hbar} \|g\|_{L^\infty(M^d)} + \|f\|_{L^2(M^d)} \|\big(1 - \hbar^{2(\beta+1)}\Delta\big)^{k/2} g\|_{L^\infty(M^d)}. \tag{3.7}$$

Moreover, as $k > d/2$, for all $f_1, f_2 \in H^k(M^d)$

$$\|f_1\, f_2\|_{H^k_\hbar} \lesssim \hbar^{-(\beta+1)k} \|f_1\|_{H^k_\hbar} \|f_2\|_{H^k_\hbar} \tag{3.8}$$

The function u_{app} satisfies

$$i\partial_t u_{\text{app}} + \Delta u_{\text{app}} = \omega |u_{\text{app}}|^{p-1} u_{\text{app}} + \mathrm{e}^{-c/\hbar^{1-\eta}} q,$$

with

$$\|q\|_{H^k_\hbar} \lesssim 1.$$

Let u be the solution of (3.6) and define $w = u - u_{\text{app}}$. Then w satisfies

$$\begin{cases} i\partial_t w + \Delta w = \omega\big(|w + u_{\text{app}}|^{p-1}(w + u_{\text{app}}) - |u_{\text{app}}|^{p-1} u_{\text{app}}\big) + \mathrm{e}^{-c/\hbar^{1-\eta}} q \\ w(0,x) = 0. \end{cases} \tag{3.9}$$

We expand the r.h.s. of (3.9), apply the operator $\big(1 - \hbar^{2(\beta+1)}\Delta\big)^{k/2}$ to the equation, and take the L^2- scalar product with $\big(1 - \hbar^{2(\beta+1)}\Delta\big)^{k/2} w$. Then we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \|w\|_{H^k_\hbar} \lesssim \sum_{j=1}^{p} \|w^j\, u_{\text{app}}^{p-j}\|_{H^k_\hbar} + \mathrm{e}^{-c/\hbar^{1-\eta}}. \tag{3.10}$$

We now have to estimate the terms $\|w^j\, u_{\text{app}}^{p-j}\|_{H^k_\hbar}$, for $1 \le j \le p$. From (3.7) we deduce

$$\begin{aligned} \|w^j\, u_{\text{app}}^{p-j}\|_{H^k_\hbar} &\lesssim \|w^j\|_{H^k_\hbar} \|u_{\text{app}}^{p-j}\|_{L^\infty(M^d)} \\ &\quad + \|w^j\|_{L^2(M^d)} \|\big(1 - \hbar^{2(\beta+1)}\Delta\big)^{k/2} u_{\text{app}}^{p-j}\|_{L^\infty(M^d)}. \end{aligned} \tag{3.11}$$

By (3.8), and as we have

$$\|u_{\text{app}}^{p-j}\|_{L^\infty(M^d)} \lesssim \hbar^{\gamma(p-j)}, \quad \|\big(1 - \hbar^{2(\beta+1)}\Delta\big)^{k/2} u_{\text{app}}^{p-j}\|_{L^\infty(M^d)} \lesssim \hbar^{\gamma(p-j)}, \tag{3.12}$$

thus inequality (3.11) yields

$$\|w^j\, u_{\text{app}}^{p-j}\|_{H^k_\hbar} \lesssim \hbar^{\gamma(p-j)} \hbar^{-(\beta+1)(j-1)k} \|w\|^j_{H^k_\hbar}.$$

Therefore, from (3.10) we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \|w\|_{H^k_\hbar} \lesssim \hbar^{\gamma(p-1)} \|w\|_{H^k_\hbar} + \hbar^{-(\beta+1)(p-1)k} \|w\|^p_{H^k_\hbar} + \mathrm{e}^{-c/\hbar^{1-\eta}}.$$

Observe that $\|w(0)\|_{H^k_\hbar} = 0$. Now, for times t so that

$$\|w\|_{H^k_\hbar} \lesssim \hbar^{\gamma + (\beta+1)k}, \tag{3.13}$$

we can remove the nonlinear term in (3.11), and by the Gronwall Lemma,

$$\|w\|_{H^k_\hbar} \lesssim \mathrm{e}^{-c/\hbar^{1-\eta}} \mathrm{e}^{C\hbar^{\gamma(p-1)} t}. \tag{3.14}$$

By (2.2), $\alpha = 2 + \beta$ and $\gamma(p-1) = -2(\beta+1)$, thus for all $0 \leq t \leq s_2 \hbar^{\alpha+\kappa}$,

$$\hbar^{\gamma(p-1)} t \leq s_2 \hbar^{-\beta+\kappa},$$

and if $\beta + \eta - \kappa < 1$, the r.h.s. in (3.14) tends to 0. Then the inequality (3.13) is satisfied for all $0 \leq t \leq s_2 \hbar^{\alpha+\kappa}$, and with a continuity argument, we infer that (3.14) holds for $0 \leq t \leq s_2 \hbar^{\alpha+\kappa}$.
Finally,

$$\|w(t)\|_{H^k(M^d)} \lesssim \hbar^{-(\beta+1)k} \|w(t)\|_{H^k_\hbar} \longrightarrow 0,$$

for $0 \leq t \leq s_2 \hbar^{\alpha+\kappa}$, when $\hbar \longrightarrow 0$, what we wanted to prove.

□

4. The instability argument

We have now the tools to show our main results.
We consider Cauchy conditions $v^0 = a^0 \mathrm{e}^{iS^0/h}$ of (2.3) which do not oscillate, i.e. such that $S^0 = 0$. We have seen in the previous section, that for some analytic amplitudes a^0, the solution writes $v = a\mathrm{e}^{iS/h}$ and therefore oscillates immediately with magnitude $\sim \frac{1}{h}$.

Let χ be given by (3.3) and $a^0 \in \mathcal{H}^0(l, B)$ nontrivial (for instance $a^0(y) = \mathrm{e}^{-y^2}$). Now set

$$u_0^h(x) = \hbar^\gamma \chi(\hbar^{-\eta}|\kappa(x)|) a^0(\hbar^{-1}\kappa(x)), \tag{4.1}$$

as initial data for (1.1).
Then we have the Ansatz (2.51), (3.4), (3.5)

$$u_{\mathrm{app}}(t,x) = \hbar^\gamma \chi(\hbar^{-\eta}|\kappa(x)|) a(\hbar^{-\alpha}t, \hbar^{-1}\kappa(x)) \mathrm{e}^{iS(\hbar^{-\alpha}t, \hbar^{-1}\kappa(x))/h}, \tag{4.2}$$

with $u_{\mathrm{app}}(0,\cdot) = u_0^h$.
For $0 < c_0 \ll 1$ satisfying Proposition 2.10, set

$$h = \hbar^\beta = \frac{c_0}{n}$$

with $n \in \mathbb{N}$, and this induces the sequences in the statements of our main results. In particular

$$\mathrm{supp}\ u_0^h \subset \{(t,x) \in \mathbb{R} \times M^d \ : \ |\kappa(x)| \leq r\hbar^\eta\},$$

and hence we can choose

$$r_n = \max_{|x| \leq r\hbar^\eta} |\kappa^{-1}(x)|_g \longrightarrow 0,$$

in Theorems 1.2 and 1.3. Here we have assumed $m = 0$, reduction which is always possible.

4.1. Proof of Theorem 1.2. —

Let $0 < \varepsilon < 1$ and define

$$\delta_h = \hbar^{\varepsilon\beta} \log \frac{1}{\hbar} = \frac{1}{\beta} h^{\varepsilon} \log \frac{1}{h}, \tag{4.3}$$

which tends to 0 with h. This choice will become clear later. Consider

$$\widetilde{u_0}^h = (1+\delta_h) u_0^h, \tag{4.4}$$

and the associate function $\widetilde{u_{\text{app}}}$.

In all this subsection we take

$$\gamma = -\frac{d}{p+1}.$$

This is the right parameter γ so that u_{app} and $\widetilde{u_{\text{app}}}$ are normalized in $L^{p+1}(M^d)$ uniformly for $h \in]0, \varepsilon[$.

Lemma 4.1. — *Let $p \geq (d+2)/(d-2)$ be an odd integer, and let $u_0^h, \widetilde{u_0}^h$ be defined by* (4.1), (4.4). *Then*

$$H^+(u_0^h) \lesssim 1, \quad H^+(\widetilde{u_0}^h) \lesssim 1.$$

There exist $\nu_0 > 0$ and $q_0 > p+1$, such that for all $0 < \nu < \nu_0$ and $p+1 \leq q < q_0$

$$\|u_0^h - \widetilde{u_0}^h\|_{H^{1+\nu}(M^d)}, \quad \|u_0^h - \widetilde{u_0}^h\|_{L^q(M^d)} \longrightarrow 0, \tag{4.5}$$

when $h \longrightarrow 0$.

Proof. — We make the change of variables $y = \hbar^{-1}\kappa(x)$, then

$$\begin{aligned} \|\nabla u_0^h\|^2_{L^2(M^d)} &\sim \hbar^{2\gamma+d-2} \int_{|y| \leq r\hbar^{-1+\eta}} \big|\nabla\big(\chi(\hbar^{1-\eta}y) a^0(y)\big)\big|^2 \mathrm{d}y \\ &\sim \hbar^{2\gamma+d-2} \int \big|\nabla a^0(y)\big|^2 \mathrm{d}y, \end{aligned}$$

as $0 < \eta < 1$.

As $2\gamma + d - 2 = -2d/(p+1) + d - 2 > 0$ when $p > 2d/(d-2) - 1$, it follows that

$$\|\nabla u_0^h\|_{L^2(M^d)} \longrightarrow 0 \quad \text{for} \quad h \longrightarrow 0.$$

Compute

$$\|u_0^h\|^{p+1}_{L^{p+1}(M^d)} \sim \hbar^{(p+1)\gamma+d} \int |a^0(y)|^{p+1} \mathrm{d}y \sim \hbar^{(p+1)\gamma+d}.$$

By definition $(p+1)\gamma + d = 0$, hence $\|u_0^h\|_{L^{p+1}(M^d)}$ remains bounded when h tends to 0, as well as $H^+(u_0^h)$.

Similarly, $H^+(\widetilde{u_0}^h) \lesssim 1$.

By the definition (4.3) of δ_h we also have for all $\sigma \geq 0$

$$\|u_0^h - \widetilde{u_0}^h\|^2_{H^\sigma(M^d)} \sim \hbar^{2\gamma+d-2\sigma} \delta_h^2 \sim \hbar^{2\gamma+d-2\sigma+2\varepsilon\beta} \big(\log \frac{1}{\hbar}\big)^2. \tag{4.6}$$

The terms in (4.6) tend to 0 if

$$\sigma < \gamma + \frac{d}{2} + \varepsilon\beta.$$

But, by (2.2) and as $p > (d+2)/(d-2)$,

$$\gamma + \frac{d}{2} > 1,$$

hence we can choose $\nu_0 = \varepsilon\beta$ in the statement.
The proof of the other part is similar. □

Proof of Theorem 1.2. — The statements (1.5), (1.6) and (1.8) have already been proved in Lemma 4.1.
Let $0 < \varepsilon < 1$ which appears in (4.3), and set $s_h = h^{1-\varepsilon} = \hbar^{\beta(1-\varepsilon)}$ and $t_h = \hbar^{\alpha} s_h = \hbar^{\alpha+\beta(1-\varepsilon)}$. Denote by $S = S(\hbar^{-\alpha}t_h, \hbar^{-1}\kappa(x))$ and by $b = \chi(\hbar^{-\eta}|\kappa(x)|)a(\hbar^{-\alpha}t_h, \hbar^{-1}\kappa(x))$. Then we have

$$\begin{aligned} \|(u_{\mathrm{app}} - \widetilde{u_{\mathrm{app}}})(t_h)\|_{L^{p+1}(M^d)} &= \hbar^{\gamma}\|b\,\mathrm{e}^{iS/h} - \widetilde{b}\,\mathrm{e}^{i\widetilde{S}/h}\|_{L^{p+1}(\mathcal{U})} \\ &\geq \hbar^{\gamma}\|b\,(\mathrm{e}^{i(\widetilde{S}-S)/h} - 1)\|_{L^{p+1}(\mathcal{U})} - \hbar^{\gamma}\|b - \widetilde{b}\|_{L^{p+1}(\mathcal{U})}. \end{aligned} \tag{4.7}$$

We now estimate the l.h.s. terms of (4.7).
First compute

$$\hbar^{\gamma}\|(b-\widetilde{b})(\hbar^{-\alpha}t_h, \hbar^{-1}\kappa(\cdot))\|_{L^{p+1}(\mathcal{U})} \sim \hbar^{\gamma+d/(p+1)}\|(b-\widetilde{b})(s_h,\cdot)\|_{L^{p+1}(\mathcal{V})}.$$

From the well-posedness of (2.6), we deduce

$$\|(b-\widetilde{b})(s_h,\cdot)\|_{L^{p+1}(\hbar^{-1}\mathcal{V})} \longrightarrow 0,$$

where s_2 is given by Proposition 3.1.
Hence

$$\hbar^{\gamma}\|(b-\widetilde{b})(\hbar^{-\alpha}t_h, \hbar^{-1}\kappa(\cdot))\|_{L^{p+1}(\mathcal{U})} \longrightarrow 0, \quad h \longrightarrow 0. \tag{4.8}$$

Secondly, a Taylor expansion near $s = 0$ shows that

$$(\widetilde{S} - S)(s_h, y) \sim -\omega(p-1)\delta_h s_h \big(\chi(\hbar^{-1-\eta}|y|)a^0(y)\big)^{p-1}. \tag{4.9}$$

Now observe that

$$\frac{\delta_h s_h}{h} \sim \log\frac{1}{h} \longrightarrow +\infty.$$

We then deduce from (4.9) that for all $|y| \leq 1$,

$$\limsup_{h\to 0}\big|\mathrm{e}^{i(\widetilde{S}-S)/h)} - 1\big| = 2,$$

and as $\hbar^{\gamma}\|b(s_h, \hbar^{-1}\kappa(x))\|_{L^{p+1}(\mathcal{U})} \sim 1$, we obtain

$$\limsup_{h\to 0}\hbar^{\gamma}\|b\,(\mathrm{e}^{i(\widetilde{S}-S)/h} - 1)\|_{L^{p+1}(\mathcal{U})} \geq c. \tag{4.10}$$

Thus, according to (4.8) and (4.10)

$$\limsup_{h\to 0} \|(u_{\text{app}} - \widetilde{u_{\text{app}}})(t_h)\|_{L^{p+1}(M^d)} \geq c. \tag{4.11}$$

Finally, if we denote by $L^{p+1} = L^{p+1}(M^d)$,

$$\begin{aligned} \|(u-\widetilde{u})(t_h)\|_{L^{p+1}} \geq \|(u_{\text{app}} - \widetilde{u_{\text{app}}})(t_h)\|_{L^{p+1}} - \|(u - u_{\text{app}})(t_h)\|_{L^{p+1}} \\ -\|(\widetilde{u} - \widetilde{u_{\text{app}}})(t_h)\|_{L^{p+1}}. \end{aligned} \tag{4.12}$$

If $\varepsilon > 0$ is chosen small enough, we can apply Corollary 3.3, with $\kappa = (1-\varepsilon)\beta$, with yields $\|(u - u_{\text{app}})(t_h)\|_{L^{p+1}}, \|(\widetilde{u} - \widetilde{u_{\text{app}}})(t_h)\|_{L^{p+1}} \longrightarrow 0$ with h, and thus from (4.10) and (4.12)

$$\limsup_{h\to 0} \|(u-\widetilde{u})(t_h)\|_{L^{p+1}} \geq \limsup_{h\to 0} \|(u_{\text{app}} - \widetilde{u_{\text{app}}})(t_h)\|_{L^{p+1}} > c,$$

which concludes the proof. □

4.2. Proof of Theorem 1.3. —

Here we deal with the case $(M^d, g) = (\mathbb{R}^d, \text{can})$.
Let $\beta > 0$ and $\gamma(p-1) = -2(\beta+1)$ as prescribed by (2.2). Let $0 < \sigma < d/2 - 2/(p-1)$ and

$$\frac{\sigma}{\frac{p-1}{2}(\frac{d}{2} - \sigma)} < \rho \leq \sigma.$$

Consider u_{app} defined by (4.2) and let $s_2 > 0$ be given by Proposition 3.1.
Then, according to Corollary 3.2, the solution u of (1.1) with initial condition $u(0) = u_{\text{app}}(0)$ satisfies for all $k \in \mathbb{R}$

$$\|(u - u_{\text{app}})(t_h)\|_{H^k(\mathbb{R}^d)} \longrightarrow 0, \quad h \longrightarrow 0,$$

with $t_h = \hbar^\alpha s_2$. To prove that u satisfies (1.10) and (1.11) we only have to check that

$$\|u_{\text{app}}(0)\|_{H^\sigma(\mathbb{R}^d)} \to 0 \quad \text{and} \quad \|u_{\text{app}}(t_h)\|_{H^\rho(\mathbb{R}^d)} \to +\infty.$$

To begin with,

$$\|u_{\text{app}}(0)\|_{H^\sigma(\mathbb{R}^d)} \sim \hbar^{\gamma - \sigma + d/2}. \tag{4.13}$$

Then, use the equations (2.6) to observe that $a\nabla S(s_2, \cdot) \not\equiv 0$. Hence

$$\|u_{\text{app}}(t_h)\|_{H^\sigma(\mathbb{R}^d)} \sim \hbar^{\gamma - (\beta+1)\rho + d/2}. \tag{4.14}$$

By (4.13) and (4.14), we only have to show that we can choose $\beta > 0$ so that

$$\gamma - \sigma + d/2 > 0, \tag{4.15}$$

$$\gamma - (\beta+1)\rho + d/2 < 0. \tag{4.16}$$

Let $\varepsilon > 0$ such that

$$\sigma < d/2 - 2/(p-1) - \varepsilon, \tag{4.17}$$

$$\rho > \frac{\sigma + \varepsilon}{\frac{p-1}{2}(\frac{d}{2} - \sigma - \varepsilon)}, \tag{4.18}$$

and take

$$\gamma = \sigma - d/2 + \varepsilon. \tag{4.19}$$

Therefore by (4.17) and (4.19) we obtain

$$\beta = -\frac{p-1}{2}\gamma - 1 = -\frac{p-1}{2}(\sigma - d/2 + \varepsilon) - 1 > 0. \tag{4.20}$$

Moreover, with the choice (4.19), inequality (4.15) is satisfied.
Finally, using the relations (4.19) and (4.20), we deduce that (4.16) is equivalent to

$$\rho > \frac{1}{\beta + 1}(\gamma + d/2) = \frac{\sigma + \varepsilon}{\frac{p-1}{2}(\frac{d}{2} - \sigma - \varepsilon)},$$

which is satisfied by (4.18).

4.3. Proof of Theorem 1.4. —

Assume here that (M^d, g) is an analytic riemannian manifold with an analytic metric g.
Consider the function u_{app} defined by (4.2) and let $s_2 > 0$ be given by Proposition 3.1.
Let $\kappa \geq 0$ such that $\beta + \eta - \kappa < 1$. Denote by $t_h = \hbar^{\alpha+\kappa} s_2$, then by Corollary 3.3, the solution u of (1.1) with initial condition $u(0) = u_{\text{app}}(0)$ satisfies for all $k \in \mathbb{R}$

$$\|(u - u_{\text{app}})(t_h)\|_{H^k(M^d)} \longrightarrow 0, \quad h \longrightarrow 0. \tag{4.21}$$

• Let

$$\frac{d}{2} - \frac{4}{p-1} < \sigma < \frac{d}{2} - \frac{2}{p-1}. \tag{4.22}$$

Choose $\varepsilon > 0$ so that

$$\sigma < \frac{d}{2} - \frac{2}{p-1} - \varepsilon, \tag{4.23}$$

and define

$$\gamma = \sigma - d/2 + \varepsilon,$$

thus

$$\beta = -\frac{p-1}{2}\gamma - 1 = -\frac{p-1}{2}(\sigma - d/2 + \varepsilon) - 1.$$

Then by (4.22) and (4.23), $0 < \beta < 1$. Choose now $\eta > 0$ so small that $0 < \beta + \eta < 1$. The convergence (4.21) then follows with $t_h = \hbar^\alpha s_2$.
Finally

$$\|u_{\mathrm{app}}(t_h)\|_{H^\sigma(\mathbb{R}^d)} \sim \hbar^{\gamma-(\beta+1)\rho+d/2} \longrightarrow +\infty,$$

for

$$\rho > \frac{1}{\beta+1}(\gamma + d/2) = \frac{\sigma+\varepsilon}{\frac{p-1}{2}(\frac{d}{2}-\sigma-\varepsilon)},$$

which was to prove.

• Assume here that $0 < \sigma < \frac{d}{2} - \frac{4}{p-1}$. For $\beta > 0$ and $\varepsilon > 0$, take $\kappa = \beta - 1 + 2\varepsilon$ and $\eta = \varepsilon$, so that $\beta + \eta - \kappa = 1 - \varepsilon < 1$. Then (4.21) holds and
(4.24)

$$\|u_{\mathrm{app}}(0)\|_{H^\sigma(\mathbb{R}^d)} \sim \hbar^{\gamma-\sigma+d/2} \quad \text{and} \quad \|u_{\mathrm{app}}(t_h)\|_{H^\sigma(\mathbb{R}^d)} \sim \hbar^{\gamma-(\beta+1-\kappa)\rho+d/2}.$$

Define $\gamma = \sigma - d/2 + \varepsilon$, then

$$\beta = -\frac{p-1}{2}\gamma - 1 = -\frac{p-1}{2}(\sigma - d/2 + \varepsilon) - 1 > 0,$$

and

$$\|u_{\mathrm{app}}(0)\|_{H^\sigma(\mathbb{R}^d)} \longrightarrow 0 \quad \text{when} \quad h \longrightarrow 0.$$

The second term in (4.24) tends to $+\infty$ when

$$\rho > -\frac{\gamma + d/2}{\beta + 1 - \kappa} = \frac{\sigma+\varepsilon}{2(1-\varepsilon)},$$

which concludes the proof, as $\varepsilon > 0$ is arbitrary.

A
Appendix

Here we reproduce a part of the work of P. Gérard [11].

PROPOSITION A.1. — *Let $r > 0$ be given by (2.49). Let v_{app} be the function defined by (2.51), and let v be the solution of*

$$\begin{cases} ih\partial_s v + h^2\Delta v = \omega|v|^{p-1}v, & (s,z) \in \mathbb{R}^{1+d}, \\ v(0,z) = v_{app}(0,z). \end{cases}$$

Then there exist $s_2 > 0$, $\lambda > 0$ and $\delta_2 > 0$ such that v can be extended to a continuous function on $[0, s_2]$, holomorphic-valued on $\big\{(|\hbar z| < r) \cap (|Im\ z| < \lambda)\big\}$, and so that for all $k \in \mathbb{N}$
(A.1)

$$\sup_{|s|<s_2} \sup_{|Im\, z|<\lambda} \|(1-h^2\Delta)^{k/2}(v - v_{app})(s, \cdot + iIm\, z)\|_{L^2(\mathcal{B}(0,r/\hbar))} \le C_k e^{-\delta_2/h},$$

with $C_k > 0$.

Proof. — Let

$$r_h = -ih\partial_s v_{\rm app} - h^2\Delta v_{\rm app} + \omega(v_{\rm app}\overline{v_{\rm app}})^{\frac{p-1}{2}} v_{\rm app}.$$

Then $f_1 = v - v_{\rm app}$ satisfies

$$ih\partial_s f_1 + h^2\Delta f_1 = F(s,z,f_1,h) + r_h, \tag{A.2}$$

where F stands for

$$F(s,z,f_2,h) = \omega\big((v_{\rm app}+f_2)^{\frac{p+1}{2}}(\overline{v_{\rm app}}+\overline{f_2})^{\frac{p-1}{2}} - (v_{\rm app}\overline{v_{\rm app}})^{\frac{p-1}{2}} v_{\rm app}\big),$$

with $\overline{f}(s,z) = \overline{f(\overline{s},\overline{z})}$.

We now show, that for $s_2 > 0$ and $\lambda > 0$ small enough, there exists a solution f_1 of (A.2) such that $f(s,z)$ is continuous on $[0,s_2]$, holomorphic-valued on $|\mathrm{Im}\, z| < \lambda$, and exponentially decreasing in h: There exists $\delta > 0$ so that for all $k \in \mathbb{N}$

$$\sup_{0\le s\le s_2}\ \sup_{|\mathrm{Im}\, z|<\lambda} \|(1-h^2\Delta)^{k/2} f_1(s,\cdot + i\mathrm{Im}\, z)\|_{L^2(\mathcal{B}(0,r/\hbar))} \le C_k \mathrm{e}^{-\delta/h}.$$

This will be done thanks to a fixed point argument.

For $0 < \lambda < l$ and $k > d/2$, set

$$\|f\|_h = \sup_{|\mathrm{Im}\, z|<\lambda} \|(1-h^2\Delta)^{k/2} f(\cdot + i\mathrm{Im}\, z)\|_{L^2(\mathcal{B}(0,r/\hbar))}.$$

Let $s_1 > 0$ be given by Proposition 2.8. Let also $\delta > 0$ and $s_2 \in [0,s_1]$. If $f = f(s,z)$ is continuous on $[0,s_1]$ and analytic on $|\mathrm{Im}\, z| < \lambda$ we set

$$N_h(f,s_2) = \sup_{0\le s\le s_2} \mathrm{e}^{\delta(1-\frac{s}{2s_2})/h}\|f(s)\|_h.$$

By the Sobolev embeddings, we have

$$\sup_{s\le s_2}\ \sup_{|\mathrm{Im}\ z|<\lambda} |f(s,z)| \lesssim N_h(f,s_2)\mathrm{e}^{-\frac{\delta}{4h}}.$$

By Proposition 2.10 we can choose $\delta > 0$, $\lambda > 0$ and $K > 0$ so that

$$\|f_1(0)\|_h \le K\mathrm{e}^{-\frac{\delta}{h}}, \quad N_h(r_h,s_1) \le K, \quad \text{and} \quad \sup_{s\le s_1}\ \sup_{|\mathrm{Im}\ z|<\lambda} |v_{\rm app}(s,z)| \le K. \tag{A.3}$$

Now use that

$$\|f\,g\|_h \lesssim h^{-k}\|f\|_h\|g\|_h, \quad \sup_{s\le s_2}\|f(s)\|_h \lesssim \mathrm{e}^{-\frac{\delta}{2h}} N_h(f,s_2), \tag{A.4}$$

to deduce that for all $L > 0$, there exists $C_L > 0$ and $h_L > 0$ so that if $h < h_L$ and $N_h(f,s_2) \le L$, the following estimates hold

$$\|F(s,\cdot,f(s),h)\|_h \le C_L\|f(s)\|_h, \tag{A.5}$$

and

$$\|F(s,\cdot,f_1(s),h) - F(s,\cdot,f_2(s),h)\|_h \le C_L\|f_1(s)-f_2(s)\|_h. \tag{A.6}$$

Let $L > 0$ to be chosen and $h < h_L$. For f such that $N_h(f, s_2) \leq L$, let w be the solution of

$$\begin{cases} ih\partial_t w + h^2\Delta w = F(s, z, f, h) + r_h, \\ w(0) = f_1(0). \end{cases} \tag{A.7}$$

The usual L^2-estimates for the Schrödinger equation and (A.4) yield

$$\|w(s)\|_h \leq \|f_1(0)\|_h + \frac{1}{h}\int_0^s \big(C_L\|f(\tau)\|_h + \|r_h(\tau)\|_h\big)\mathrm{d}\tau. \tag{A.8}$$

Now use the exponential decrease of the norms $\|\cdot\|_h$ to obtain

$$\frac{1}{h}\int_0^s \|f(\tau)\|_h \mathrm{d}\tau \leq \frac{2s_2}{\varepsilon} N_h(f, s_2)\mathrm{e}^{\delta(1-\frac{s}{2s_2})/h}.$$

Then, with (A.3), we deduce from (A.8)

$$N_h(w, s_2) \leq K(1 + \frac{2s_2}{\varepsilon}) + \frac{2s_2}{\varepsilon} C_L N_h(f, s_2). \tag{A.9}$$

Now fix $L \geq 4K$ and $s_2 \leq \min(s_1, \varepsilon/2, \varepsilon/(4C_L))$, by (A.9) we obtain $N_h(w, s_2) \leq L$.

We have proved that the application $f \mapsto w$, induced by the equation (A.7), maps the ball $\{f,\ N_h(f, s_2) \leq L\}$ into itself. Finally use (A.6) to show that this application is a contraction. Hence (A.2) admits a unique solution f_1, which satisfies the estimate (A.1). □

BIBLIOGRAPHY

[1] T. Alazard et R. Carles. Supercritical geometric optics for nonlinear Schrödinger equations. *arXiv:0704.2488*.

[2] T. Alazard and R. Carles. Sequential loss of regularity for super-critical nonlinear Schrödinger equations. *Preprint*.

[3] M. S. Baouendi and C. Goulaouic. Remarks on the abstract form of nonlinear Cauchy-Kovalevsky theorems. *Comm. Partial Differential Equations* 2 (1977), no. 11, 1151–1162.

[4] N. Burq, P. Gérard, and S. Ibrahim. Ill-posedness for supercritical nonlinear Schrödinger and wave equations. *Preprint*.

[5] R. Carles. Geometric optics and instability for semi-classical Schrödinger equations. To appear in *Arch. Ration. Mech. Anal.* 183, no. 3, 525–553, 2007.

[6] R. Carles. On instability for the cubic nonlinear Schrödinger equation. *C. R. Math. Acad. Sci. Paris*, 344, no. 8, 483-486, 2007.

[7] T. Cazenave and F. B. Weissler. The Cauchy problem for the critical nonlinear Schrödinger equation in H^s. *Nonlinear Anal.* 14 (1990), no. 10, 807–836.

[8] M. Christ, J. Colliander, and T. Tao. Asymptotics, frequency modulation, and low regularity ill-posedness for canonical defocusing equations. *Amer. J. Math.*, 125(6):1235–1293, 2003.

[9] M. Christ, J. Colliander, and T. Tao. Ill-posedness for non linear Schrödinger and wave equation. To appear in *Ann. Inst. H. Poincaré Anal. Non Linéaire*.

[10] J. Ginibre and G. Velo. On a class of nonlinear Schrödinger equations. *J. Funct. Anal.* 32 (1979), no. 1, 1–71.

[11] P. Gérard. Remarques sur l'analyse semi-classique de l'équation de Schrödinger non linéaire. *Séminaire sur les Équations aux Dérivées Partielles, 1992–1993*, Exp. No. XIII, 13 pp., École Polytech., Palaiseau, 1993.

[12] E. Grenier. Semiclassical limit of the nonlinear Schrödinger equation in small time. *Proc. Amer. Math. Soc.* 126 (1998), no. 2, 523–530.

[13] G. Lebeau. Non linear optic and supercritical wave equation. *Bull. Soc. Roy. Sci. Liège*, 70 (2001), no. 4-6, 267–306.

[14] G. Lebeau. Perte de régularité pour les équations d'ondes sur-critiques. *Bull. Soc. Math. France* 133 (2005), no. 1, 145–157.

[15] J. Sjöstrand. Singularités analytiques microlocales. *Astérisque, 95, 1–166, Soc. Math. France, Paris, 1982.*