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EXISTENCE OF STRONG SOLUTIONS TO A FLUID-STRUCTURE SYSTEM*

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Abstract. We study a coupled fluid-structure system. The structure corresponds to a part of the boundary of a domain containing an incompressible viscous fluid. The structure displacement is modeled by a damped beam equation. We prove the existence of strong solutions to our system for small data and the existence of local strong solutions for any initial data.

Key words. fluid-structure interaction, Navier–Stokes equations, beam equation, coupled system

AMS subject classifications. 35Q30, 74F10, 76D05, 76D03

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1. Introduction.

We study a fluid-structure system coupling the Navier–Stokes equations in a two-dimensional domain with a damped beam equation located on the boundary of a domain occupied by a fluid flow. For similar systems, the existence of weak solutions has been established in [4, 8] for two-dimensional domains and in [4, 6] for three-dimensional domains.

Here we are interested in the existence of local-in-time strong solutions. In [3], Beirão da Veiga proves the existence of local strong solutions for small data under the assumption $\alpha \geq 0$ (see the beam equation (1.3)). In this paper, we improve this type of result, with $\alpha > 0$, by showing the existence of local strong solutions without any smallness condition (Theorem 3.2), and we also prove the existence of global strong solutions in a given time interval $[0, T]$ for small data (Theorem 3.1).

To the best of the author's knowledge, this problem was introduced in [10] by Quarteroni, Tuveri, and Veneziani to model cardiovascular systems like blood flow in large vessels—arteries, for instance.

Let $L > 0$ and $T > 0$ be, respectively, a length and a time. Let η be a function from $(0, T) \times (0, L)$ to $(-1, +\infty)$. Let $t \in (0, T)$; we can define a domain $\Omega_{\eta(t)}$ depending on time by

$$\Omega_{\eta(t)} = \Big\{(x, y)\ ;\ 0 \leq x \leq L \text{ and } 0 \leq y \leq 1 + \eta(t, x)\Big\}.$$

Here $\eta(t)$ is the displacement of the beam. We note by $\Gamma_s = (0, L) \times \{1\}$ the reference configuration of the beam. The displacement η has to satisfy the assumption

$$\exists \delta_0 > 0 \text{ such that } \forall t \geq 0\ \forall x \in (0, L) \quad 1 + \eta(t, x) \geq \delta_0 > 0 \tag{1.1}$$

to ensure that, for every time t, $\Omega_{\eta(t)}$ is a connected domain. Let us set $\Omega = (0, L) \times (0, 1)$ and $\Gamma = \partial\Omega$, that is,

$$\Gamma = \{0\} \times (0, 1) \bigcup \{L\} \times (0, 1) \bigcup (0, L) \times \{0\} \bigcup (0, L) \times \{1\}.$$

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We also set $\Gamma^0 = \Gamma \setminus \Gamma_s$, the fixed boundary part

$$\Gamma^0 = \{0\} \times (0,1) \bigcup \{L\} \times (0,1) \bigcup (0,L) \times \{0\}$$

and

$$\Gamma_{\eta(t)} = \Big\{(x,y)\ ;\ 0 \leq x \leq L \text{ and } y = 1 + \eta(t,x)\Big\}.$$

Thus $\partial\Omega_{\eta(t)} = \Gamma^0 \cup \Gamma_{\eta(t)}$. We will use other notation:

$$\begin{aligned} &\Sigma_T^0 = (0,T) \times \Gamma^0, && \Sigma_T^s = (0,T) \times \Gamma_s, \\ &Q_T = (0,T) \times \Omega, && \mathcal{Q}_T = \bigcup_{t \in (0,T)} \{t\} \times \Omega_{\eta(t)}, \\ &\Sigma_T = (0,T) \times \Gamma, && \mathcal{S}_T = \bigcup_{t \in (0,T)} \{t\} \times \Gamma_{\eta(t)}. \end{aligned}$$

The velocity $\mathbf{u}$ and the pressure p of the fluid in the domain $\mathcal{Q}_T$ are described by the Navier–Stokes equations

$$\begin{aligned} \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} - \operatorname{div} \sigma(\mathbf{u},p) &= 0 && (\mathcal{Q}_T), \\ \operatorname{div} \mathbf{u} &= 0 && (\mathcal{Q}_T), \\ \mathbf{u} &= \eta_t \mathbf{e}_2 && (\mathcal{S}_T), \\ \mathbf{u} &= \mathbf{0} && (\Sigma_T^0), \\ \mathbf{u}(0) &= \mathbf{u}^0 && (\Omega_{\eta^0}). \end{aligned} \tag{1.2}$$

The displacement η satisfies the following beam equation:

$$\begin{aligned} \eta_{tt} - \beta\eta_{xx} - \gamma\eta_{txx} + \alpha\eta_{xxxx} &= \phi[\mathbf{u},p,\eta] && (\Sigma_T^s), \\ \eta(0) &= \eta^0 && (\Gamma_s), \\ \eta_t(0) &= \eta^1 && (\Gamma_s). \end{aligned} \tag{1.3}$$

In these equations, σ and ϕ are defined by

$$\begin{aligned} \sigma(\mathbf{u},p) &= -pI + \nu(\nabla\mathbf{u} + (\nabla\mathbf{u})^T), \\ \phi[\mathbf{u},p,\eta] &= -\sigma(\mathbf{u},p)(-\eta_x\mathbf{e}_1 + \mathbf{e}_2) \cdot \mathbf{e}_2, \end{aligned}$$

where $\mathbf{e}_1 = (1,0)$ and $\mathbf{e}_2 = (0,1)$ and $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2$, $\nu > 0$ is the viscosity of the fluid; $\alpha > 0$, $\beta \geq 0$, $\gamma > 0$ are constants relative to the structure (see [3] for more details).

2. Functional settings. We have to define the function spaces for the solutions $(\mathbf{u},p,\eta)$ of (1.2)–(1.3).

In the fixed domain Ω, we define the classical Hilbert space in two dimensions $\mathbf{L}^2(\Omega) = L^2(\Omega;\mathbb{R}^2)$ and in the same way the Sobolev spaces $\mathbf{H}^s(\Omega) = H^s(\Omega;\mathbb{R}^2)$. We introduce

$$\mathbf{V}^\sigma(\Omega) = \Big\{\mathbf{u} \in \mathbf{H}^\sigma(\Omega)\ ;\ \operatorname{div} \mathbf{u} = 0 \text{ in } \Omega\Big\},$$

$$\mathbf{H}^{\sigma,\tau}(Q_T) = L^2(0,T;\mathbf{H}^\sigma(\Omega)) \cap H^\tau(0,T;\mathbf{L}^2(\Omega)),$$

$$\mathbf{V}^{\sigma,\tau}(Q_T) = L^2(0,T;\mathbf{V}^\sigma(\Omega)) \cap H^\tau(0,T;\mathbf{V}^0(\Omega)).$$

We need a definition of Sobolev spaces in the time-dependent domain $\Omega_{\eta(t)}$.

DEFINITION 2.1. *We say that* $\mathbf{u}$ *belongs to* $H^\tau(\bigcup_{t\in(0,T)} \{t\} \times \mathbf{H}^\sigma(\Omega_{\eta(t)}))$ *(resp., to* $H^\tau(\bigcup_{t\in(0,T)} \{t\} \times \mathbf{V}^\sigma(\Omega_{\eta(t)}))$*) if*

- *for almost every t in $(0,T)$, $\mathbf{u}(t)$ belongs to $\mathbf{H}^\sigma(\Omega_{\eta(t)})$ (resp., in $\mathbf{V}^\sigma(\Omega_{\eta(t)})$),*
- *$t \mapsto \|\mathbf{u}(t)\|_{\mathbf{H}^\sigma(\Omega_{\eta(t)})}$ (resp., $t \mapsto \|\mathbf{u}(t)\|_{\mathbf{V}^\sigma(\Omega_{\eta(t)})}$) is in $H^\tau(0,T;\mathbb{R})$.*

We finally define

$$\mathbf{H}^{\sigma,\tau}(\mathcal{Q}_T) = L^2\left(\bigcup_{t\in(0,T)} \{t\}\times \mathbf{H}^\sigma(\Omega_{\eta(t)})\right) \bigcap H^\tau\left(\bigcup_{t\in(0,T)} \{t\}\times \mathbf{L}^2(\Omega_{\eta(t)})\right),$$

$$\mathbf{V}^{\sigma,\tau}(\mathcal{Q}_T) = L^2\left(\bigcup_{t\in(0,T)} \{t\}\times \mathbf{V}^\sigma(\Omega_{\eta(t)})\right) \bigcap H^\tau\left(\bigcup_{t\in(0,T)} \{t\}\times \mathbf{V}^0(\Omega_{\eta(t)})\right).$$

Solutions $(\mathbf{u}, p, \eta)$ of (1.2)–(1.3) satisfy

$$\begin{aligned} 0 &= \int_{\Omega_{\eta(t)}} \operatorname{div}\, \mathbf{u}(t) = \int_{\partial\Omega_{\eta(t)}} \mathbf{u}(t)\cdot\mathbf{n}(t) = \int_{\Gamma_{\eta(t)}} \mathbf{u}(t)\cdot\mathbf{n}(t) + \int_{\Gamma^0} \mathbf{u}(t)\cdot\mathbf{n}_0 \\ &= \int_{\Gamma_s} \eta_t(t) + 0 = \int_0^L \eta_t(t,x)\mathrm{d}x, \end{aligned}$$

where $\mathbf{n}(t) = \frac{1}{\sqrt{1+\eta_x^2(t)}}\big(-\eta_x(t)\mathbf{e}_1 + \mathbf{e}_2\big)$ is the unit normal to $\Gamma_{\eta(t)}$ outward $\Omega_{\eta(t)}$ and $\mathbf{n}_0$ is the unit normal to each part of Γ^0 outward $\Omega_{\eta(t)}$, that is,

$$\mathbf{n}_0 = \mathbf{e}_1 \text{ on } \{L\}\times(0,1), \qquad \mathbf{n}_0 = -\mathbf{e}_1 \text{ on } \{0\}\times(0,1), \quad \text{or} \quad \mathbf{n}_0 = -\mathbf{e}_2 \text{ on } (0,L)\times\{0\}.$$

Thus we must choose η^1 in $L^2_0(\Gamma_s) = \big\{\eta \in L^2(\Gamma_s)\ ;\ \int_{\Gamma_s}\eta = 0\big\}$. Furthermore, we can choose $\eta^0 \in L^2_0(\Gamma_s)$, and then we shall have

$$\int_{\Gamma_s} \eta(t) = 0 \quad \text{and} \quad \int_{\Gamma_s} \eta_t(t) = 0 \quad \forall t \geq 0. \tag{2.1}$$

We have to choose boundary conditions for η too. Here, we decide to fix η and η_x on $(0,T)\times\{0,L\}$ as follows:

$$\eta(t,0) = \eta(t,L) = 0 \quad \text{and} \quad \eta_x(t,0) = \eta_x(t,L) = 0 \quad \forall t \in (0,T). \tag{2.2}$$

We could have chosen periodic boundary conditions as in [3]. The result obtained in the following may be directly translated to this situation.

With (2.1) and (2.2), we get

$$\int_{\Gamma_s} \eta_{tt} = 0\ , \int_{\Gamma_s} \eta_{xx} = 0, \ \text{ and } \int_{\Gamma_s} \eta_{txx} = 0 \quad \forall\ t \geq 0.$$

We use M_s the orthogonal projection from $L^2(\Gamma_s)$ onto $L^2_0(\Gamma_s)$ to rewrite (1.3). We will use a special trace function γ_s defined by

$$\gamma_s p = M_s(p_{|\Gamma_s}) = p_{|\Gamma_s} - \frac{1}{|\Gamma_s|}\int_{\Gamma_s} p_{|\Gamma_s} \quad \forall p \in H^\sigma(\Omega) \text{ with } \sigma > \frac{1}{2}.$$

Equation (1.3) becomes

$$\eta_{tt} - \beta\eta_{xx} - \gamma\eta_{txx} + \alpha M_s \eta_{xxxx} = \gamma_s p + \overline{\phi}[\mathbf{u},\eta], \tag{2.3}$$

with $\overline{\phi}[\mathbf{u},\eta] = -\nu\gamma_s\big(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\big)(-\eta_x\mathbf{e}_1 + \mathbf{e}_2)\cdot\mathbf{e}_2$.

Let us introduce the spaces

(2.4)
$$H^\sigma_{(0)}(\Gamma_s) = \begin{cases} \Big\{\mu \in H^\sigma(\Gamma_s) \cap L^2_0(\Gamma_s) \text{ s.t. } \mu = \mu_x = 0 \text{ at } x = 0, L\Big\} & \text{for } \frac{3}{2} < \sigma, \\ \Big\{\mu \in H^\sigma(\Gamma_s) \cap L^2_0(\Gamma_s) \text{ s.t. } \mu = 0 \text{ at } x = 0, L\Big\} & \text{for } \frac{1}{2} < \sigma \leq \frac{3}{2}, \\ H^\sigma(\Gamma_s) \cap L^2_0(\Gamma_s) & \text{for } 0 \leq \sigma \leq \frac{1}{2}. \end{cases}$$

Due to (2.1) and (2.2), we look for η in the spaces

$$H^{\sigma,\tau}_{(0)}(\Sigma^s_T) = L^2(0,T; H^\sigma_{(0)}(\Gamma_s)) \cap H^\tau(0,T; L^2_0(\Gamma_s)).$$

The pressure term p is defined in the Navier–Stokes equations up to an additive constant. Then we define the space $\mathcal{H}^\sigma(\Omega)$ by

$$\mathcal{H}^\sigma(\Omega) = \Big\{p \in H^\sigma(\Omega) \ \text{ s.t. } \int_\Omega p = 0\Big\}.$$

We will look for p in $L^2(\bigcup_{t\in(0,T)}\{t\} \times \mathcal{H}^1(\Omega_{\eta(t)}))$ (see Definition 2.1).

3. Main results. We can now state the two main theorems of this paper. First, we consider global strong solutions of system (1.2)–(1.3) with a condition on the size of the initial data only. Second, we prove the existence of a local strong solution for the same system.

THEOREM 3.1. *Let* $(\mathbf{u}^0, \eta^0, \eta^1) \in \mathbf{V}^1(\Omega_{\eta^0}) \times H^3_{(0)}(\Gamma_s) \times H^1_{(0)}(\Gamma_s)$. *There exists* $R > 0$ *such that for any initial data satisfying* $\|\mathbf{u}^0\|^2_{\mathbf{V}^1(\Omega_{\eta^0})} + \|\eta^0\|^2_{H^3_{(0)}(\Gamma_s)} + \|\eta^1\|^2_{H^1_{(0)}(\Gamma_s)} \leq R^2$ *and the compatibility condition*

$$\begin{aligned} \mathbf{u}^0 &= \mathbf{0} && (\Gamma^0), \\ \mathbf{u}^0 &= \eta^1 \mathbf{e}_2 && (\Gamma_{\eta^0}), \end{aligned} \tag{3.1}$$

system (1.2)–(1.3) *has a unique global strong solution* $(\mathbf{u}, p, \eta)$ *in*

$$\mathbf{V}^{2,1}(\mathcal{Q}_T) \times L^2\left(\bigcup_{t\in(0,T)} \{t\} \times \mathcal{H}^1(\Omega_{\eta(t)})\right) \times H^{4,2}_{(0)}(\Sigma^s_T).$$

THEOREM 3.2. *Let* $(\mathbf{u}^0, \eta^0, \eta^1) \in \mathbf{V}^1(\Omega_{\eta^0}) \times H^3_{(0)}(\Gamma_s) \times H^1_{(0)}(\Gamma_s)$*, satisfying the compatibility condition* (3.1). *There exists a time* $T_0 > 0$ *such that system* (1.2)–(1.3) *has a unique strong solution* $(\mathbf{u}, p, \eta) \in \mathbf{V}^{2,1}(\mathcal{Q}_{T_0}) \times L^2(\bigcup_{t\in(0,T_0)} \{t\} \times \mathcal{H}^1(\Omega_{\eta(t)})) \times H^{4,2}_{(0)}(\Sigma^s_{T_0})$.

The core of the paper consists in the proof of these theorems. First of all, thanks to a suitable change of variables, we introduce an equivalent problem (4.5) in a cylindrical domain Q_T. Due to the change of variables, new nonlinear terms appear in the equations. The proof of existence of solutions for system (4.5) is split into different steps:

(i) We study the nonhomogeneous linearized system (5.1), where the nonlinearities in (4.5) are now considered as right-hand sides. The proof of existence for this system uses a fixed point method for another equivalent system (5.10) introduced in section 5.2 thanks to the splitting method due to Raymond [11]. Indeed, we see in section 5.1 that we cannot apply a fixed point method directly to system (5.1).

(ii) From the linearized system, we prove the existence of strong solutions for system (4.5) thanks to another fixed point method in section 6.

In section 7, we complete the proof by checking that the change of variables defined in section 4 is suitable in the sense of Definition 4.1.

4. An equivalent problem in the fixed domain Ω. We want to use a change of variables to rewrite system (1.2)–(1.3) in the domain $Q_T = (0,T)\times\Omega$. This change of variables introduces nonlinear terms in the variables $(\mathbf{u}, p, \eta)$ that we will treat as right-hand sides in section 5. As in [3], for a fixed $t \in (0,T)$, we introduce the following change of variables:

$$\begin{array}{ccc} \Omega_{\eta(t)} & \longrightarrow & \Omega, \\ (x,y) & \longmapsto & (x,z) = \left(x, \dfrac{y}{1+\eta(t,x)}\right) \end{array} . \tag{4.1}$$

Setting $\hat{f}(x,z) = f(x,y)$, we have the formulas

$$\begin{cases} \hat{f}(x,z) = f(x,(1+\eta(t,x))z), \\ f(x,y) = \hat{f}\left(x, \dfrac{y}{1+\eta(t,x)}\right). \end{cases}$$

Then we can calculate the derivatives of $f(x,y)$ using the derivatives of $\hat{f}(x,z)$:

$$\begin{cases} f_t = \hat{f}_t - z\dfrac{\eta_t}{1+\eta}\hat{f}_z, \\ f_x = \hat{f}_x - z\dfrac{\eta_x}{1+\eta}\hat{f}_z, \\ f_y = \dfrac{1}{1+\eta}\hat{f}_z, \\ f_{xx} = \hat{f}_{xx} - 2z\dfrac{\eta_x}{1+\eta}\hat{f}_{xz} + \left(z\dfrac{\eta_x}{1+\eta}\right)^2\hat{f}_{zz} - z\dfrac{(1+\eta)\eta_{xx} - \eta_x^2}{(1+\eta)^2}\hat{f}_z, \\ f_{yy} = \dfrac{1}{(1+\eta)^2}\hat{f}_{zz}. \end{cases}$$

Now, we state the system satisfied by $\hat{\mathbf{u}}(x,z) = \mathbf{u}(x,y)$ and $\hat{p}(x,z) = p(x,y)$:

$$\begin{array}{rll} \hat{\mathbf{u}}_t - \operatorname{div}\,\sigma(\hat{\mathbf{u}},\hat{p}) = \hat{\mathbf{F}}\,[\hat{\mathbf{u}},\hat{p},\eta] & (Q_T), \\ \operatorname{div}\,\hat{\mathbf{u}} = \operatorname{div}\,\hat{\mathbf{w}}\,[\hat{\mathbf{u}},\eta] & (Q_T), \\ \hat{\mathbf{u}}(0) = \hat{\mathbf{u}}^0 & (\Omega), \\ \hat{\mathbf{u}} = \eta_t(t,x)\mathbf{e}_2 & (\Sigma_T^s), \\ \hat{\mathbf{u}} = \mathbf{0} & (\Sigma_T^0), \end{array} \tag{4.2}$$

with

$$\begin{aligned} \hat{\mathbf{F}}(t,x,z) &= \hat{\mathbf{F}}\,[\hat{\mathbf{u}},\hat{p},\eta] \\ &= -\eta\hat{\mathbf{u}}_t + \left[z\eta_t + \nu z\left(\frac{\eta_x^2}{1+\eta} - \eta_{xx}\right)\right]\hat{\mathbf{u}}_z \\ &\quad + \nu\left\{-2z\eta_x\hat{\mathbf{u}}_{xz} + \eta\hat{\mathbf{u}}_{xx} + \frac{z^2\eta_x^2 - \eta}{1+\eta}\hat{\mathbf{u}}_{zz}\right\} \\ &\quad + z(\eta_x\hat{p}_z - \eta\hat{p}_x)\mathbf{e}_1 - (1+\eta)\hat{u}_1\hat{\mathbf{u}}_x + (z\eta_x\hat{u}_1 - \hat{u}_2)\hat{\mathbf{u}}_z, \end{aligned} \tag{4.3}$$

$$\begin{aligned} \hat{\mathbf{w}}(t,x) &= \hat{\mathbf{w}}\,[\hat{\mathbf{u}},\eta] \\ &= -\eta\hat{u}_1\mathbf{e}_1 + z\eta_x\hat{u}_1\mathbf{e}_2. \end{aligned}$$

For instance, to calculate the divergence term, we write $u_{1,x}+u_{2,z}$ in terms of $\hat{\mathbf{u}}$, and taking $1+\eta$ as a multiplier, we get

$$0=(1+\eta)\hat{u}_{1,x}-z\eta_x\hat{u}_{1,z}+\hat{u}_{2,z}.$$

Then we see that

$$\hat{u}_{1,x}+\hat{u}_{2,z}=\text{div }\hat{\mathbf{u}}=-\eta\hat{u}_{1,x}+z\eta_x\hat{u}_{1,z}=\text{div }\hat{\mathbf{w}}.$$

The beam equation (2.3) becomes

$$\eta_{tt}-\beta\eta_{xx}-\gamma\eta_{txx}+\alpha M_s\eta_{xxxx}=\gamma_s\hat{p}-2\nu\gamma_s\hat{u}_{2,z}+\gamma_s\hat{H}[\hat{\mathbf{u}},\eta],$$

with

$$\hat{H}[\hat{\mathbf{u}},\eta]=\nu\left(\frac{\eta_x}{1+\eta}\hat{u}_{1,z}+\eta_x\hat{u}_{2,x}-\frac{\eta_x^2-2\eta}{1+\eta}\hat{u}_{2,z}\right). \tag{4.4}$$

To simplify the notation, we drop out the symbol $\hat{}$ and obtain the system

$$\begin{array}{rll} \mathbf{u}_t-\text{div }\sigma(\mathbf{u},p)=\mathbf{F}[\mathbf{u},p,\eta] & & (Q_T),\\ \text{div }\mathbf{u}=\text{div }\mathbf{w}[\mathbf{u},\eta] & & (Q_T),\\ \mathbf{u}=\mathbf{0} & & (\Sigma_T^0),\\ \mathbf{u}=\eta_t\mathbf{e}_2 & & (\Sigma_T^s),\\ \mathbf{u}(0)=\mathbf{u}^0 & & (\Omega),\\ \eta_{tt}-\gamma\eta_{txx}-\beta\eta_{xx}+\alpha M_s\eta_{xxxx}=\gamma_s p-2\nu\gamma_s u_{2,z}+\gamma_s H[\mathbf{u},\eta] & & (\Sigma_T^s),\\ \eta(0)=\eta^0 & & (\Gamma_s),\\ \eta_t(0)=\eta^1 & & (\Gamma_s). \end{array} \tag{4.5}$$

The previous system is equivalent to system (1.2)–(1.3). More precisely, we state the following definition.

DEFINITION 4.1. $(\mathbf{u},p,\eta)$ *in* $\mathbf{H}^{2,1}(\mathcal{Q}_T)\times L^2(\bigcup_{t\in(0,T)}\{t\}\times\mathcal{H}^1(\Omega_{\eta(t)}))\times H^{4,2}_{(0)}(\Sigma_T^s)$ *is a solution of* (1.2)–(1.3) *when the following conditions are satisfied:*

(i) $(\hat{\mathbf{u}},\hat{p},\eta)$ *obtained for the change of variables* $\hat{\mathbf{u}}(x,z)=\mathbf{u}(x,y)$, $\hat{p}(x,z)=p(x,y)$ *with* $z=\frac{y}{1+\eta(t,x)}$ *is a solution of* (4.5)*;*

(ii) *for any time* t *in* $(0,T)$*, the previous change of variables is a* $\mathcal{C}^1$*-diffeomorphism from* $\Omega_{\eta(t)}$ *into* Ω*;*

(iii) η *satisfies condition* (1.1).

If we set $\mathbf{u}=\mathbf{v}+\mathbf{w}[\mathbf{u},\eta]$, we notice that div $\mathbf{v}=0$ and the system satisfied by $(\mathbf{v},p,\eta)$ is

$$\begin{array}{rll} \mathbf{v}_t+\text{div }\sigma(\mathbf{v},p)=\mathbf{f}[\mathbf{u},p,\eta] & & (Q_T),\\ \text{div }\mathbf{v}=0 & & (Q_T),\\ \mathbf{v}=-\mathbf{w}[\mathbf{u},\eta] & & (\Sigma_T^0),\\ \mathbf{v}=\eta_t\mathbf{e}_2-\mathbf{w}[\mathbf{u},\eta] & & (\Sigma_T^s),\\ \mathbf{v}(0)=\mathbf{u}^0-\mathbf{w}[\mathbf{u},\eta](0) & & (\Omega),\\ \eta_{tt}-\gamma\eta_{txx}-\beta\eta_{xx}+\alpha M_s\eta_{xxxx}=\gamma_s p-2\nu\gamma_s v_{2,z}-2\nu\gamma_s w_{2,z}[\mathbf{u},\eta]+\gamma_s H[\mathbf{u},\eta] & & (\Sigma_T^s),\\ \eta(0)=\eta^0 & & (\Gamma_s),\\ \eta_t(0)=\eta^1 & & (\Gamma_s), \end{array} \tag{4.6}$$

with

$$\mathbf{f}[\mathbf{u},p,\eta]=\mathbf{F}[\mathbf{u},p,\eta]+\nu\Delta\mathbf{w}[\mathbf{u},\eta]-\partial_t\mathbf{w}[\mathbf{u},\eta], \tag{4.7}$$

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 \quad \text{and} \quad \mathbf{w}[\mathbf{u},\eta] = w_1[\mathbf{u},\eta]\mathbf{e}_1 + w_2[\mathbf{u},\eta]\mathbf{e}_2.$$

The explicit expression of $\mathbf{w}[\mathbf{u},\eta] = -\eta u_1\mathbf{e}_1 + z\eta_x u_1\mathbf{e}_2$ depends only on u_1 and η. Thus, the boundary conditions on Σ_T^0 and Σ_T^s are

$$\begin{aligned} \mathbf{v} &= \mathbf{0} && (\Sigma_T^0),\\ \mathbf{v} &= \eta_t\mathbf{e}_2 && (\Sigma_T^s). \end{aligned}$$

Moreover, the term $-2\nu\gamma_s v_{2,z}$ in $(4.6)_6$ vanishes. Indeed, $v_{1,x} + v_{2,z} = 0$ in Q_T and $v_1 = 0$ on Σ_T^s. Furthermore, if $\mathbf{v}$ is in $\mathbf{H}^{2,1}(Q_T)$, then $v_{1,x|\Sigma_T^s} = 0$ and $v_{2,z|\Sigma_T^s} = 0$. That is why we are considering the following system:

$$(4.8)\qquad \begin{aligned} \mathbf{v}_t + \operatorname{div}\sigma(\mathbf{v},p) &= \mathbf{f}[\mathbf{u},p,\eta] && (Q_T),\\ \operatorname{div}\mathbf{v} &= 0 && (Q_T),\\ \mathbf{v} &= \mathbf{0} && (\Sigma_T^0),\\ \mathbf{v} &= \eta_t\mathbf{e}_2 && (\Sigma_T^s),\\ \mathbf{v}(0) &= \mathbf{v}^0 && (\Omega),\\ \eta_{tt} - \gamma\eta_{txx} - \beta\eta_{xx} + \alpha M_s\eta_{xxxx} &= \gamma_s p + h[\mathbf{u},\eta] && (\Sigma_T^s),\\ \eta(0) &= \eta^0 && (\Gamma_s),\\ \eta_t(0) &= \eta^1 && (\Gamma_s), \end{aligned}$$

where

$$(4.9)\qquad h[\mathbf{u},\eta] = -2\nu\gamma_s w_{2,z}[\mathbf{u},\eta] + \gamma_s H[\mathbf{u},\eta]$$

and

$$(4.10)\qquad \mathbf{v}^0 = \mathbf{u}^0 - \mathbf{w}[\mathbf{u},\eta](0) = \mathbf{u}^0 + \eta^0 u_1^0\mathbf{e}_1 - z\eta_x^0 u_1^0\mathbf{e}_2.$$

On the other hand, to have continuity on [0,T), the previous conditions on $\mathbf{v}$ must be checked at time $t = 0$. Thus, we have to add a compatibility condition at time $t = 0$:

$$(4.11)\qquad \begin{aligned} \operatorname{div}\mathbf{v}^0 &= 0 && (\Omega),\\ \mathbf{v}^0 &= \mathbf{0} && (\Gamma^0),\\ \mathbf{v}^0 &= \eta^1\mathbf{e}_2 && (\Gamma_s), \end{aligned}$$

which is written in terms of $(\mathbf{u}^0, \eta^0, \eta^1)$ as follows:

$$(4.12)\qquad \begin{aligned} \operatorname{div}(\mathbf{u}^0 + \eta^0 u_1^0\mathbf{e}_1 - z\eta_x^0 u_1^0\mathbf{e}_2) &= 0 && (\Omega),\\ \mathbf{u}^0 &= \mathbf{0} && (\Gamma^0),\\ \mathbf{u}^0 &= \eta^1\mathbf{e}_2 && (\Gamma_s). \end{aligned}$$

5. Study of an auxiliary linear system. In this section, we prove existence and uniqueness of solutions to the system

$$(5.1)\qquad \begin{aligned} \mathbf{v}_t + \operatorname{div}\sigma(\mathbf{v},p) &= \mathbf{f} && (Q_T),\\ \operatorname{div}\mathbf{v} &= 0 && (Q_T),\\ \mathbf{v} &= \mathbf{0} && (\Sigma_T^0),\\ \mathbf{v} &= \eta_t\mathbf{e}_2 && (\Sigma_T^s),\\ \mathbf{v}(0) &= \mathbf{v}^0 && (\Omega),\\ \eta_{tt} - \gamma\eta_{txx} - \beta\eta_{xx} + \alpha M_s\eta_{xxxx} &= \gamma_s p + h && (\Sigma_T^s),\\ \eta(0) &= \eta^0 && (\Gamma_s),\\ \eta_t(0) &= \eta^1 && (\Gamma_s) \end{aligned}$$

for a right-hand side

$$(\mathbf{f}, h) \in Z_T = \mathbf{L}^2(Q_T) \times L^2(0,T; H^{1/2}_{(0)}(\Gamma_s)), \tag{5.2}$$

and initial data $(\mathbf{v}^0, \eta^0, \eta^1)$ in $X^0_{\mathbf{cc}}$, where

$$X^0 = \mathbf{H}^1(\Omega) \times H^3_{(0)}(\Gamma_s) \times H^1_{(0)}(\Gamma_s)$$

and

$$X^0_{\mathbf{cc}} = \Big\{ (\mathbf{z}^0, \mu^0, \mu^1) \in X^0 \text{ s.t. } (\mathbf{z}^0, \mu^0, \mu^1) \text{ satisfies (4.11)} \Big\}.$$

The space X^0 will be equipped with the norm

$$\|(\mathbf{z}^0, k^0, k^1)\|_{X^0} = \Big(\|\mathbf{z}^0\|^2_{\mathbf{H}^1(\Omega)} + \|\eta^0\|^2_{H^3(\Gamma_s)} + \|\eta^1\|^2_{H^1(\Gamma_s)} \Big)^{1/2}.$$

The main result of this section is the following theorem.

THEOREM 5.1. *Let $(\mathbf{v}^0, \eta^0, \eta^1)$ be in $X^0_{\mathbf{cc}}$ and $(\mathbf{f}, h)$ be in Z_T. Then system* (5.1) *admits one and only one solution $(\mathbf{v}, p, \eta)$ in*

$$\begin{aligned} X_T = \Big\{ &(\mathbf{z}, q, \mu) \in \mathbf{H}^{2,1}(Q_T) \times L^2(0,T; \mathcal{H}^1(\Omega)) \times H^{4,2}_{(0)}(\Sigma^s_T) \\ &\text{s.t. } \mathbf{z} = \mathbf{0} \text{ on } \Sigma^0_T \text{ and } \mathbf{z} = \mu_t \mathbf{e}_2 \text{ on } \Sigma^s_T \Big\}. \end{aligned} \tag{5.3}$$

Moreover, we get the estimate

$$\|(\mathbf{v}, p, \eta)\|_{X_T} \le C(\|(\mathbf{v}^0, \eta^0, \eta^1)\|_{X^0} + \|(\mathbf{f}, h)\|_{Z_T}). \tag{5.4}$$

5.1. Why a fixed point method on the pressure term p does not work. A way to find solutions of the coupled system (5.1) is to use a fixed point method. For a given $\overline{p}$ in $L^2(0,T; \mathcal{H}^1(\Omega))$, we consider the following system:

$$\begin{aligned} \mathbf{v}_t + \operatorname{div} \sigma(\mathbf{v}, p) &= \mathbf{f} && (Q_T), \\ \operatorname{div} \mathbf{v} &= 0 && (Q_T), \\ \mathbf{v} &= \mathbf{0} && (\Sigma^0_T), \\ \mathbf{v} &= \eta_t \mathbf{e}_2 && (\Sigma^s_T), \\ \mathbf{v}(0) &= \mathbf{v}^0 && (\Omega), \\ \eta_{tt} - \gamma \eta_{txx} - \beta \eta_{xx} + \alpha M_s \eta_{xxxx} &= \gamma_s \overline{p} + h && (\Sigma^s_T), \\ \eta(0) &= \eta^0 && (\Gamma_s), \\ \eta_t(0) &= \eta^1 && (\Gamma_s). \end{aligned} \tag{5.5}$$

Given fixed (η^0, η^1) and h, we can solve the beam equation. Next, knowing η, we can find solutions to the Stokes system with right-hand side $\mathbf{f}$, initial data $\mathbf{v}^0$, and a boundary condition depending on η_t.

This idea cannot be applied directly with isomorphism theorems for the two equations separately. Indeed, we first get the following proposition.

PROPOSITION 5.2. *Let (η^0, η^1) be in $H_s = H^3_{(0)}(\Gamma_s) \times H^1_{(0)}(\Gamma_s)$. For $\gamma_s \overline{p}$, h in $L^2(0,T; L^2_0(\Gamma_s))$, equation*

$$\begin{aligned} \eta_{tt} - \beta \eta_{xx} - \gamma \eta_{txx} + \alpha M_s \eta_{xxxx} &= \gamma_s \overline{p} + h && (\Sigma^s_T), \\ \eta(0) &= \eta^0 && (\Gamma_s), \\ \eta_t(0) &= \eta^1 && (\Gamma_s) \end{aligned}$$

admits a solution η *in* $H^{4,2}_{(0)}(\Sigma^s_T)$ *satisfying the estimate*

$$\|\eta\|_{H^{4,2}_{(0)}(\Sigma^s_T)} \le C\Big(\|(\eta^0,\eta^1)\|_{H^s} + \|h\|_{L^2(\Sigma^s_T)} + \|\gamma_s\overline{p}\|_{L^2(\Sigma^s_T)}\Big).$$

Then the result for the Stokes equations is the following.

PROPOSITION 5.3. *Let* $\mathbf{v}^0$ *be in* $\mathbf{V}^1(\Omega)$. *For* $\mathbf{f}$ *and* g, *respectively, in* $\mathbf{L}^2(Q_T)$ *and* $H^{2,1}_{(0)}(\Sigma^s_T)$ *with the compatibility condition* $\mathbf{v}^0 = g(0)\mathbf{e}_2$ *on* Γ_s *and* $\mathbf{v}^0 = \mathbf{0}$ *on* Γ^0, *the system*

$$\begin{aligned} \mathbf{v}_t - \operatorname{div}\,\sigma(\mathbf{v},p) &= \mathbf{f} && (Q_T),\\ \operatorname{div}\,\mathbf{v} &= 0 && (Q_T),\\ \mathbf{v} &= g\mathbf{e}_2 && (\Sigma^s_T),\\ \mathbf{v} &= \mathbf{0} && (\Sigma^0_T),\\ \mathbf{v}(0) &= \mathbf{v}^0 && (\Omega) \end{aligned} \tag{5.6}$$

admits a unique solution $(\mathbf{v},p)$ *in* $\mathbf{V}^{2,1}(Q_T)\times L^2(0,T;\mathcal{H}^1(\Omega))$ *and furthermore*

$$\|(\mathbf{v},p)\|_{\mathbf{V}^{2,1}(Q_T)\times L^2(0,T;\mathcal{H}^1(\Omega))} \le C\Big(\|\mathbf{v}^0\|_{\mathbf{V}^1(\Omega)} + \|g\|_{H^{2,1}_{(0)}(\Sigma^s_T)} + \|\mathbf{f}\|_{\mathbf{L}^2(Q_T)}\Big).$$

The first proposition comes from regularity results for the beam equation proved in Proposition 5.9. The second proposition is a result from [11] in the case when g belongs to $H^{2,1}_{(0)}(\Sigma^s_T)$.

To conclude, the solution $(\mathbf{v},p,\eta)$ of system (5.5) obeys

$$\|(\mathbf{v},p,\eta)\|_{X_T} \le C\Big(\|(\mathbf{v}^0,\eta^0,\eta^1)\|_{X^0} + \|(\mathbf{f},h)\|_{Z_T} + \|\overline{p}\|_{L^2(0,T;\mathcal{H}^1(\Omega))}\Big).$$

Thus this method gives directly the solution of system (5.5) in the expected spaces (thanks to the isomorphism theorems), but we cannot act on the constant C to get a contraction. That is why we have to consider a new equivalent system.

5.2. New equivalent system. Let us define the so-called Leray projection P from $\mathbf{L}^2(\Omega)$ in $\mathbf{V}^0_{\mathbf{n}}(\Omega)$, where

$$\mathbf{V}^0_{\mathbf{n}}(\Omega) = \Big\{\mathbf{u}\in\mathbf{L}^2(\Omega) \text{ s.t. } \mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \Gamma \text{ and } \operatorname{div}\,\mathbf{u} = 0 \text{ in } \Omega\Big\}.$$

We want to split system (5.1) into two parts in order to construct a contraction mapping acting on a part of the pressure term only. More precisely, following the idea of [11, 12], the Stokes system can be expressed in terms of $\mathbf{v}_e = P\mathbf{v}$, $\mathbf{v}_s = (I-P)\mathbf{v}$ and their associated pressures p_e, p_s; then we will construct a contraction mapping acting on p_e to obtain the expected result.

To express simply the Stokes system in the variables $(\mathbf{v}_e,\mathbf{v}_s,p_e,p_s)$, we have to introduce some operators. Let us denote by N the operator defined from $H^\sigma(\Gamma)$ to $H^{\sigma+3/2}(\Omega)$ (for $\sigma\ge -1/2$) by $q = N(g)$ (for g in $H^\sigma(\Gamma)$) if

$$\begin{cases} \Delta q(t) = 0 & (\Omega),\\ \dfrac{\partial q(t)}{\partial \mathbf{n}} = g & (\Gamma). \end{cases} \tag{5.7}$$

Then, for $\mathbf{z}$ in $\mathbf{L}^2(\Omega)$, the solution π of

$$\begin{cases} \Delta\pi = \operatorname{div}\,\mathbf{z} & (\Omega),\\ \dfrac{\partial\pi}{\partial\mathbf{n}} = \mathbf{z}\cdot\mathbf{n} & (\Gamma) \end{cases}$$

is a sum of two terms π_1 and π_2 in $H^1(\Omega)$ satisfying

$$\begin{cases} \pi_1 \in H^1_0(\Omega), \\ \Delta \pi_1 = \operatorname{div} \mathbf{z} \quad (\Omega) \end{cases} \qquad \text{and} \qquad \pi_2 = N\Big((\mathbf{z} - \nabla \pi_1)\cdot \mathbf{n}\Big).$$

Setting $\pi_1 = -(-\Delta_D)^{-1}(\operatorname{div} \mathbf{z})$, we get $\pi_2 = N((\mathbf{z} + \nabla(-\Delta_D)^{-1}(\operatorname{div} \mathbf{z}))\cdot \mathbf{n})$. Thus, we can define the operator π from $\mathbf{L}^2(\Omega)$ into $H^1(\Omega)$ by

$$\pi(\mathbf{z}) = -(-\Delta_D)^{-1}(\operatorname{div} \mathbf{z}) + N((\mathbf{z} + \nabla(-\Delta_D)^{-1}(\operatorname{div} \mathbf{z}))\cdot \mathbf{n}) \quad \text{for } \mathbf{z} \in \mathbf{L}^2(\Omega). \tag{5.8}$$

Finally we denote by N_s the restriction on $H^\sigma(\Gamma_s)$ of N, that is, $N_s(g) = N(g\chi_s)$ for any g in $H^\sigma(\Gamma_s)$ $(\sigma \geq -1/2)$.

With these notations, system $(5.1)_{1-5}$ is equivalent to

$$\begin{aligned} \mathbf{v}_{e,t} - \nu \Delta \mathbf{v}_e + \nabla p_e &= P\mathbf{f} && (Q_T), \\ \mathbf{v}_e &= -\gamma_\tau \mathbf{v}_s && (\Sigma_T), \\ \mathbf{v}_e(0) &= P\mathbf{v}^0 && (\Omega), \\ \mathbf{v}_s &= \nabla N_s(\eta_t) && (Q_T), \\ p_s &= \pi(\mathbf{f}) - N_s(\eta_{tt}) && (Q_T), \\ p &= p_e + p_s && (Q_T). \end{aligned} \tag{5.9}$$

The explications to obtain this system are detailed in [11].

The pressure term on the right-hand side of the beam equation is

$$\gamma_s p = \gamma_s p_e + \gamma_s \pi(\mathbf{f}) - \gamma_s N_s(\eta_{tt}).$$

System (5.1) is equivalent to the following system in terms of $(\mathbf{v}_e, p_e, \mathbf{v}_s, p_s, \eta)$:

$$\begin{aligned} \mathbf{v}_{e,t} - \nu \Delta \mathbf{v}_e + \nabla p_e &= P\mathbf{f} && (Q_T), \\ \mathbf{v}_e &= -\gamma_\tau \mathbf{v}_s && (\Sigma_T), \\ \mathbf{v}_e(0) &= P\mathbf{v}^0 && (\Omega), \\ \mathbf{v}_s &= \nabla N_s(\eta_t) && (Q_T), \\ (I + \gamma_s N_s)\eta_{tt} - \beta \eta_{xx} - \gamma \eta_{txx} + \alpha M_s \eta_{xxxx} &= \gamma_s p_e + \tilde{h} && (\Sigma_T^s), \\ \eta(0) &= \eta^0 && (\Gamma_s), \\ \eta_t(0) &= \eta^1 && (\Gamma_s), \\ p &= p_e + p_s && (Q_T), \\ \mathbf{v} &= \mathbf{v}_e + \mathbf{v}_s && (Q_T), \\ p_s &= \pi(\mathbf{f}) - N_s(\eta_{tt}) && (Q_T), \end{aligned} \tag{5.10}$$

with

$$\tilde{h} = h + \gamma_s \pi(\mathbf{f}). \tag{5.11}$$

We want to find solutions to system (5.10). With $(\mathbf{f}, h)$ and $(\mathbf{v}^0, \eta^0, \eta^1)$ fixed, our method is to set the pressure term $\overline{p}_e \in L^2(0,T;\mathcal{H}^1(\Omega))$ only on the right-hand side of the beam equation. Then, considering $\overline{p}_e$ only in $L^{2-\varepsilon}(0,T,\mathcal{H}^1(\Omega))$ (for a small parameter $\varepsilon > 0$), we find a solution η of the modified beam equation in a space E_T^ε. The next step is, with η in E_T^ε, to get $\mathbf{v}_e$, $\mathbf{v}_s$, and p_e, respectively, in $\mathbf{V}^{2,1}(Q_T)$, $L^2(0,T;\mathbf{H}^2(\Omega)) \cap H^{3/4}(0,T;\mathbf{H}^{1/2}(\Omega))$, and $L^2(0,T;\mathcal{H}^1(\Omega))$. All of these results will allow us to define a contraction mapping from a ball of the space of pressure term $L^2(0,T;\mathcal{H}^1(\Omega))$ into itself for a small time T_0 in $(0,T)$. Then, because of the linearity of system (5.10), we will have the existence and uniqueness of a strong solution in $(0,T)$ corresponding with fixed initial data $(\mathbf{v}^0, \eta^0, \eta^1)$ in $X^0_{\mathbf{cc}}$ and right-hand members $(\mathbf{f}, h)$ in Z_T.

5.3. Existence of solutions for each part of (5.1) and estimates. We begin this loop by fixing a pressure term $\overline{p}_e$ in the beam equation. We will suppose that $\overline{p}_e$ is in $L^2(0,T;\mathcal{H}^1(\Omega))$. By a classic embedding theorem, we get $\gamma_s\overline{p}_e \in L^{2-\varepsilon}(0,T;H^{1/2}_{(0)}(\Gamma_s))$ for any $0<\varepsilon<1$. Then we have the estimate

$$\|\gamma_s\overline{p}_e\|_{L^{2-\varepsilon}(0,T;H^{1/2}_{(0)}(\Gamma_s))} \le CT^\theta\|\gamma_s\overline{p}_e\|_{L^2(0,T;H^{1/2}_{(0)}(\Gamma_s))} \quad \text{for } \theta = \frac{1}{2-\varepsilon}-\frac{1}{2}. \tag{5.12}$$

Thus, we can prove the following proposition.

PROPOSITION 5.4. *Let $0<\varepsilon<1$, let (η^0,η^1) be in H_s, and let $(\mathbf{f},h)$ be in Z_T, as defined in Proposition 5.2 and in (5.2). Then first $\tilde{h}$ defined by (5.11) is in $L^2(0,T;H^{1/2}_{(0)}(\Gamma_s))$ and, second, with $\overline{p}_e$ in $L^{2-\varepsilon}(0,T;\mathcal{H}^1(\Omega))$, the equation*

$$\begin{aligned} (I+\gamma_sN_s)\eta_{tt}-\beta\eta_{xx}-\gamma\eta_{txx}+\alpha M_s\eta_{xxxx} &= \gamma_s\overline{p}_e+\tilde{h} && (\Sigma^s_T),\\ \eta(0)&=\eta^0 && (\Gamma_s),\\ \eta_t(0)&=\eta^1 && (\Gamma_s) \end{aligned} \tag{5.13}$$

admits a unique solution η in

$$E^\varepsilon_T = L^{2-\varepsilon}(0,T;H^{4+\varepsilon/2}_{(0)}(\Gamma_s))\cap W^{2,2-\varepsilon}(0,T;H^{\varepsilon/2}_{(0)}(\Gamma_s)), \tag{5.14}$$

satisfying

$$\|\eta\|_{E^\varepsilon_T} \le C\Big(\|(\eta^0,\eta^1)\|_{H_s}+\|\overline{p}_e\|_{L^{2-\varepsilon}(0,T;\mathcal{H}^1(\Omega))}+\|\tilde{h}\|_{L^{2-\varepsilon}(0,T;H^{1/2}_{(0)}(\Gamma_s))}\Big). \tag{5.15}$$

Furthermore, η_t belongs to $H^{3/2,3/4}_{(0)}(\Sigma^s_T)$.

Proof. First, $\tilde{h}$ is in $L^2(0,T;H^{1/2}_{(0)}(\Gamma_s))$ thanks to the regularity of $\mathbf{f}$ and h via formula (5.11).

We want to rewrite (5.13) as a first order system. For that, we set

$$Y(t) = \begin{pmatrix} \eta(t) \\ \eta_t(t) \end{pmatrix}, \quad Y^0 = \begin{pmatrix} \eta^0 \\ \eta^1 \end{pmatrix},$$

and

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ (I+\gamma_sN_s)^{-1}(-\alpha M_s\Delta^2+\beta\Delta) & \gamma(I+\gamma_sN_s)^{-1}\Delta \end{pmatrix}. \tag{5.16}$$

Then $D(-\mathcal{A}) = H^4_{(0)}(\Gamma_s)\times H^2_{(0)}(\Gamma_s)$. We denote $\mathbb{H} = H^2_{(0)}(\Gamma_s)\times L^2_0(\Gamma_s)$. Y is the solution of the equation

$$\begin{aligned} Y'(t) &= \mathcal{A}Y(t)+\begin{pmatrix} 0 \\ (I+\gamma_sN_s)^{-1}(\gamma_s\overline{p}_e+\tilde{h}) \end{pmatrix} && (\Sigma^s_T),\\ Y(0)&=Y_0 && (\Gamma_s). \end{aligned} \tag{5.17}$$

We use the well-known Duhamel's formula, with $\mathcal{B} = \begin{pmatrix} 0 \\ (I+\gamma_sN_s)^{-1}(\gamma_s\overline{p}_e+\tilde{h}) \end{pmatrix}$,

$$Y(t) = \mathrm{e}^{t\mathcal{A}}Y_0+\int_0^t \mathrm{e}^{(t-\tau)\mathcal{A}}\mathcal{B}(\tau)\mathrm{d}\tau.$$

For $\kappa > 0$, we have formally

$$(-\mathcal{A})^\kappa Y(t) = (-\mathcal{A})^\kappa \mathrm{e}^{t\mathcal{A}} Y_0 + \int_0^t (-\mathcal{A})^\kappa \mathrm{e}^{(t-\tau)\mathcal{A}} \mathcal{B}(\tau)\mathrm{d}\tau,$$

and because Y_0 is in $[D(-\mathcal{A}), \mathbb{H}]_{1/2}$ and $\mathcal{B}(\tau)$ is in $[D(-\mathcal{A}), \mathbb{H}]_{3/4}$, we get

$$(-\mathcal{A})^\kappa Y(t) = (-\mathcal{A})^{\kappa-1/2} \mathrm{e}^{t\mathcal{A}} (-\mathcal{A})^{1/2} Y_0 + \int_0^t (-\mathcal{A})^{\kappa-1/4} \mathrm{e}^{(t-\tau)\mathcal{A}} (-\mathcal{A})^{1/4} \mathcal{B}(\tau)\mathrm{d}\tau.$$

Now, using triangular inequality in $\mathbb{H}$, we have for $r > 1$

$$\begin{aligned} \|(-\mathcal{A})^\kappa Y(t)\|_{\mathbb{H}}^r \leq c\Bigg(& \left\|(-\mathcal{A})^{\kappa-1/2} \mathrm{e}^{t\mathcal{A}}\right\|_{\mathcal{L}(\mathbb{H})}^r \left\|(-\mathcal{A})^{1/2} Y_0\right\|_{\mathbb{H}}^r \\ & + \left\|\int_0^t (-\mathcal{A})^{\kappa-1/4} \mathrm{e}^{(t-\tau)\mathcal{A}} (-\mathcal{A})^{1/4} \mathcal{B}(\tau)\mathrm{d}\tau\right\|_{\mathbb{H}}^r \Bigg). \end{aligned}$$

Because $(-\mathcal{A})$ is a generator of an analytic semigroup (see the proof in [12], which relies on a result in [5]), we get the estimates (see [9])

$$\left\|(-\mathcal{A})^\kappa \mathrm{e}^{t\mathcal{A}}\right\|_{\mathcal{L}(\mathbb{H})} \leq \frac{c}{t^\kappa} \quad \text{for } \kappa > 0.$$

With Young's formula, we have

$$\begin{aligned} & \int_0^T \left\|(-\mathcal{A})^\kappa Y(t)\right\|_{\mathbb{H}}^r \mathrm{d}t \\ & \leq \int_0^T \left\|(-\mathcal{A})^{\kappa-1/2} \mathrm{e}^{t\mathcal{A}}\right\|_{\mathcal{L}(\mathbb{H})}^r \mathrm{d}t \left\|(-\mathcal{A})^{1/2} Y_0\right\|_{\mathbb{H}}^r \\ & \qquad + \left(\int_0^T \left\|(-\mathcal{A})^{\kappa-1/4} \mathrm{e}^{(\cdot)\mathcal{A}}\right\|_{\mathcal{L}(\mathbb{H})}^p\right)^{r/p} \left(\int_0^T \left\|(-\mathcal{A})^{1/4} \mathcal{B}(\cdot)\right\|_{\mathbb{H}}^q\right)^{r/q} \\ & \leq c\int_0^T \frac{\mathrm{d}t}{t^{(\kappa-\frac{1}{2})r}} \left\|(-\mathcal{A})^{1/2} Y_0\right\|_{\mathbb{H}}^r + c\left\|t \mapsto \frac{1}{t^{(\kappa-\frac{1}{4})p}}\right\|_{L^p(0,T)}^r \left\|(-\mathcal{A})^{1/4} \mathcal{B}(\cdot)\right\|_{L^q(0,T;\mathbb{H})}^r, \end{aligned}$$

with $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, and κ satisfying

$$\begin{cases} (\kappa - \frac{1}{2})r < 1, \\ (\kappa - \frac{1}{4})p < 1. \end{cases}$$

Then the triplet $(p, q, r) = (1, 2-\varepsilon, 2-\varepsilon)$ is suitable. For this choice, κ has only to obey $\kappa < 1 + \frac{\varepsilon}{4-2\varepsilon}$, and thus $\kappa = 1 + \varepsilon/4$ is convenient. This gives us first

$$Y \in L^{2-\varepsilon}(0, T; [D((-\mathcal{A})^2), D(-\mathcal{A})]_{1-\varepsilon/4}) = L^{2-\varepsilon}(0, T; H_{(0)}^{4+\varepsilon/2}(\Gamma_s) \times H_{(0)}^{2+\varepsilon/2}(\Gamma_s))$$

and second

$$Y' \in L^{2-\varepsilon}(0, T; [D(-\mathcal{A}), \mathbb{H}]_{1-\varepsilon/4}) = L^{2-\varepsilon}(0, T; H_{(0)}^{2+\varepsilon/2}(\Gamma_s) \times H_{(0)}^{\varepsilon/2}(\Gamma_s)).$$

Thus, the solution η of (5.13) belongs to $L^{2-\varepsilon}(0, T; H_{(0)}^{4+\varepsilon/2}(\Gamma_s)) \cap W^{2,2-\varepsilon}(0, T; H_{(0)}^{\varepsilon/2}(\Gamma_s))$. The estimate comes from Duhamel's formula and the different calculations above.

The last part is to prove that η_t is in $H^{3/2,3/4}_{(0)}(\Sigma^s_T)$. We use different interpolation formulas: η belongs to E^ε_T, and thus η_t is in $L^{2-\varepsilon}(0,T;H^{2+\varepsilon/2}_{(0)}(\Gamma_s))\cap W^{1,2-\varepsilon}(0,T;H^{\varepsilon/2}_{(0)}(\Gamma_s))$, which can be embedded continuously in $W^{\lambda,2-\varepsilon}(0,T;[H^{2+\varepsilon/2}_{(0)}(\Gamma_s),H^{\varepsilon/2}_{(0)}(\Gamma_s)]_\lambda)$ for $0<\lambda<1$. A quick calculation gives us

$$[H^{2+\varepsilon/2}_{(0)}(\Gamma_s),H^{\varepsilon/2}_{(0)}(\Gamma_s)]_\lambda = H^{2+\varepsilon/2-2\lambda}_{(0)}(\Gamma_s).$$

An injection formula in Sobolev spaces of fractional order (see [1]) gives

$$W^{\lambda,2-\varepsilon}(0,T)\hookrightarrow W^{0,2}(0,T) \quad \text{when } \lambda = \frac{1}{2-\varepsilon}-\frac{1}{2}.$$

So $W^{\lambda,2-\varepsilon}(0,T;[H^{2+\varepsilon/2}_{(0)}(\Gamma_s),H^{\varepsilon/2}_{(0)}(\Gamma_s)]_\lambda)\hookrightarrow L^2(0,T;H^{3/2}_{(0)}(\Gamma_s))$. In the same way we can prove that $W^{\lambda,2-\varepsilon}(0,T;[H^{2+\varepsilon/2}_{(0)}(\Gamma_s),H^{\varepsilon/2}_{(0)}(\Gamma_s)]_\lambda)\hookrightarrow H^{3/4}(0,T;L^2_0(\Gamma_s))$. ∎

We use a new definition of solutions for the Stokes system (5.6). Indeed, we look for a solution $(\mathbf{v}_e,\mathbf{v}_s,p_e)$ of the equivalent system (see section 5.2)

$$\begin{aligned}
\mathbf{v}_{e,t}-\nu\Delta\mathbf{v}_e+\nabla p_e &= P\mathbf{f} && (Q_T),\\
\mathbf{v}_e &= -\gamma_\tau\mathbf{v}_s && (\Sigma_T),\\
\mathbf{v}_e(0) &= P\mathbf{v}^0 && (\Omega),\\
\mathbf{v}_s &= \nabla N_s(g) && (Q_T),\\
\mathbf{v} &= \mathbf{v}_e+\mathbf{v}_s && (Q_T),\\
p &= p_s+p_e && (Q_T),\\
p_s &= \pi(\mathbf{f})-N_s(g_t) && (Q_T),
\end{aligned} \tag{5.18}$$

where $\pi(\mathbf{f})$ is given in (5.8). We now can state the following result on solutions of the Stokes equivalent system (5.18).

PROPOSITION 5.5. *Let g be in $H^{3/2,3/4}_{(0)}(\Sigma^s_T)$, $\mathbf{f}$ in $\mathbf{L}^2(Q_T)$, and $\mathbf{v}^0$ in $\mathbf{V}^1(\Omega)$ with the compatibility condition $\mathbf{v}^0=\mathbf{0}$ on Γ^0 and $\mathbf{v}^0=g(0)\mathbf{e}_2$ on Γ_s. Then (5.18) admits a unique solution $(\mathbf{v}_e,\mathbf{v}_s,p_e)$ in $X^{e,s}_T=\mathbf{V}^{2,1}(Q_T)\times L^2(0,T;\mathbf{H}^2(\Omega))\cap H^{3/4}(0,T;\mathbf{H}^{1/2}(\Omega))\times L^2(0,T;\mathcal{H}^1(\Omega))$. We have the estimate*

$$\|(\mathbf{v}_e,\mathbf{v}_s,p_e)\|_{X^{e,s}_T}\le c\Big(\|\mathbf{v}^0\|_{\mathbf{V}^1(\Omega)}+\|g\|_{H^{3/2,3/4}_{(0)}(\Sigma^s_T)}+\|\mathbf{f}\|_{\mathbf{L}^2(Q_T)}\Big).$$

Proof. In [11], system (5.18) is solved with $\mathbf{f}=0$. Thus, we have to look for solution $(\mathbf{v}_e,\mathbf{v}_s,p_e)$ of (5.18) as a sum of two terms, one $(\mathbf{v}^1_e,\mathbf{v}^1_s,p^1_e)$ solution of system

$$\begin{aligned}
\mathbf{v}^1_{e,t}-\nu\Delta\mathbf{v}^1_e+\nabla p^1_e &= \mathbf{0} && (Q_T),\\
\mathbf{v}^1_e &= -\gamma_\tau\mathbf{v}^1_s && (\Sigma_T),\\
\mathbf{v}^1_e(0) &= P\mathbf{v}^0 && (\Omega),\\
\mathbf{v}^1_s &= \nabla N_s(g) && (Q_T),\\
\mathbf{v}^1 &= \mathbf{v}^1_e+\mathbf{v}^1_s && (Q_T),\\
p^1 &= p^1_s+p^1_e && (Q_T),\\
p^1_s &= -N_s(g_t) && (Q_T)
\end{aligned} \tag{5.19}$$

and the other $(\mathbf{v}^2_e,\mathbf{v}^2_s,p^2_e)$ solution of system

$$\begin{aligned}
\mathbf{v}^2_{e,t}-\nu\Delta\mathbf{v}^2_e+\nabla p^2_e &= P\mathbf{f} && (Q_T),\\
\mathbf{v}^2_e &= \mathbf{0} && (\Sigma_T),\\
\mathbf{v}^2_e(0) &= \mathbf{0} && (\Omega),\\
\mathbf{v}^2_s &= \mathbf{0} && (Q_T),\\
\mathbf{v}^2 &= \mathbf{v}^2_e+\mathbf{v}^2_s && (Q_T),\\
p^2 &= p^2_s+p^2_e && (Q_T),\\
p^2_s &= \pi(\mathbf{f}) && (Q_T).
\end{aligned} \tag{5.20}$$

Thanks to the compatibility condition, g and $\mathbf{v}^0$ satisfy the hypothesis of Theorem 2.7 in [11], and then there exists a solution $(\mathbf{v}_e^1, \mathbf{v}_s^1, p_e^1)$ in $X_T^{e,s}$ of (5.19) satisfying the estimate

$$\|(\mathbf{v}_e^1, \mathbf{v}_s^1, p_e^1)\|_{X_T^{e,s}} \leq C\Big(\|P\mathbf{v}^0\|_{\mathbf{V}^1(\Omega)} + \|g\|_{H_{(0)}^{3/2,3/4}(\Sigma_T^s)}\Big).$$

Thanks to classical results on the Stokes system, there exists a solution $(\mathbf{v}_e^2, \mathbf{v}_s^2, p_e^2)$ of (5.20) with $(\mathbf{v}_e^2, p_e^2)$ in $\mathbf{V}^{2,1}(Q_T) \times L^2(0,T;\mathcal{H}^1(\Omega))$ for $\mathbf{f}$ in $\mathbf{L}^2(Q_T)$ and $\mathbf{v}_s^2 = \mathbf{0}$. Furthermore $(\mathbf{v}_e^2, \mathbf{v}_s^2, p_e^2)$ obeys

$$\|(\mathbf{v}_e^2, \mathbf{v}_s^2, p^2)\|_{X_T^{e,s}} \leq C\|P\mathbf{f}\|_{\mathbf{L}^2(Q_T)} \leq C\|\mathbf{f}\|_{\mathbf{L}^2(Q_T)}.$$

Finally, $(\mathbf{v}_e, \mathbf{v}_s, p)$, defined by $\mathbf{v}_e = \mathbf{v}_e^1 + \mathbf{v}_e^2$, $\mathbf{v}_s = \mathbf{v}_s^1 + \mathbf{v}_s^2$, and $p_e = p_e^1 + p_e^2$, is a solution of (5.18), belongs to $X_T^{e,s}$, and satisfies the estimate

$$\|(\mathbf{v}_e, \mathbf{v}_s, p_e)\|_{X_T^{e,s}} \leq C\Big(\|P\mathbf{v}^0\|_{\mathbf{V}^1(\Omega)} + \|g\|_{H_{(0)}^{3/2,3/4}(\Sigma_T)} + \|\mathbf{f}\|_{\mathbf{L}^2(Q_T)}\Big). \qquad \square$$

5.4. Construction of a solution of system (5.1). In order to prove the existence of solutions for system (5.1), we have to construct a contraction mapping for the equivalent system (5.10). Initial data $(\mathbf{v}^0, \eta^0, \eta^1)$ in $X_{\mathbf{cc}}^0$ and right-hand sides $(\mathbf{f}, h)$ in Z_T are fixed in this section. For $\overline{p}_e$, we consider the mapping $\mathcal{G}$ defined by

$$\begin{aligned} \mathcal{G}: \quad L^2(0,T;\mathcal{H}^1(\Omega)) \quad &\longrightarrow \quad X_T^{\varepsilon} = \Big\{(\mathbf{v}_e, \mathbf{v}_s, p_e, \eta) \in X_T^{e,s} \times E_T^{\varepsilon}\Big\}, \\ \overline{p}_e \quad &\longmapsto \quad (\mathbf{v}_e, \mathbf{v}_s, p_e, \eta) \text{ the solution of system (5.21)}, \end{aligned}$$

$$\begin{aligned} \mathbf{v}_{e,t} - \nu\Delta\mathbf{v}_e + \nabla p_e &= P\mathbf{f} && (Q_T), \\ \mathbf{v}_e &= -\gamma_\tau \mathbf{v}_s && (\Sigma_T), \\ \mathbf{v}_e(0) &= P\mathbf{v}^0 && (\Omega), \\ \mathbf{v}_s &= \nabla N_s(\eta_t) && (Q_T), \\ (I + \gamma_s N_s)\eta_{tt} - \beta\eta_{xx} - \gamma\eta_{txx} + \alpha M_s \eta_{xxxx} &= \gamma_s \overline{p}_e + \tilde{h} && (\Sigma_T^s), \\ \eta(0) &= \eta^0 && (\Gamma_s), \\ \eta_t(0) &= \eta^1 && (\Gamma_s), \\ p &= p_e + p_s && (Q_T), \\ \mathbf{v} &= \mathbf{v}_e + \mathbf{v}_s && (Q_T), \\ p_s &= \pi(\mathbf{f}) - N_s(\eta_{tt}) && (Q_T), \end{aligned} \tag{5.21}$$

where $\tilde{h}$ is defined from $\mathbf{f}$ and h in (5.11), and E_T^{ε} and $X_T^{e,s}$ are defined, respectively, in (5.14) and in Proposition 5.5.

PROPOSITION 5.6. *The mapping $\mathcal{G}$ is well-defined from $L^2(0,T;\mathcal{H}^1(\Omega))$ into X_T^{ε}. We have, moreover, the following estimate for $\theta > 0$ defined in* (5.12)*:*

$$\|\mathcal{G}(\overline{p}_e)\| \leq C\Big(\|(\mathbf{v}^0, \eta^0, \eta^1)\|_{X^0} + \|(\mathbf{f}, h)\|_{Z_T} + T^{\theta}\|\overline{p}_e\|_{L^2(0,T;\mathcal{H}^1(\Omega))}\Big). \tag{5.22}$$

Furthermore, for two pressures $\overline{p}_{e,1}$ and $\overline{p}_{e,2}$ in $L^2(0,T;\mathcal{H}^1(\Omega))$, we have the solution $\mathcal{G}(\overline{p}_{e,1}) - \mathcal{G}(\overline{p}_{e,2}) = (\mathbf{v}_{e,1} - \mathbf{v}_{e,2}, \mathbf{v}_{s,1} - \mathbf{v}_{s,2}, p_{e,1} - p_{e,2}, \eta_1 - \eta_2)$ corresponding with $\mathcal{G}(\overline{p}_{e,1} - \overline{p}_{e,2})$ in (5.21)*, with zero for initial data and right-hand sides. Moreover, $\mathcal{G}(\overline{p}_{e,1}) - \mathcal{G}(\overline{p}_{e,2})$ satisfies the estimate*

$$\|\mathcal{G}(\overline{p}_{e,1}) - \mathcal{G}(\overline{p}_{e,2})\|_{X_T^{\varepsilon}} \leq cT^{\theta}\|\overline{p}_{e,1} - \overline{p}_{e,2}\|_{L^2(0,T;\mathcal{H}^1(\Omega))}.$$

Proof. Thanks to Proposition 5.4 in section 5.3, we get η in E_T^ε and η_t in $H^{3/2,3/4}_{(0)}(\Sigma_T^s)$; together with Proposition 5.5 (for $g = \eta_t$), it follows that $(\mathbf{v}_e, \mathbf{v}_s, p_e, \eta)$ belongs to X_T^ε and satisfies estimate (5.22).

The proof of the second part of this proposition relies on the linearity of the system and the same propositions. ∎

We now are able to construct a contraction mapping from a ball of $L^2(0,T;\mathcal{H}^1(\Omega))$ into itself. Let us consider the linear operator $\mathcal{F}$ from $L^2(0,T;\mathcal{H}^1(\Omega))$ into itself defined by

$$\mathcal{F} = \mathcal{P} \circ \mathcal{G},$$

where $\mathcal{P}$ is the projection from X_T^ε into $L^2(0,T;\mathcal{H}^1(\Omega))$ defined obviously by

$$\mathcal{P}(\mathbf{v}_e, \mathbf{v}_s, p_e, \eta) = p_e.$$

We detail some properties on $\mathcal{F}$ in the following proposition.

PROPOSITION 5.7. *$\mathcal{F}$ is well-defined from $L^2(0,T;\mathcal{H}^1(\Omega))$ into itself, and, for any $R > 0$, there exists a time $T_0 > 0$ such that $\mathcal{F}$ is a contraction in*

$$\mathcal{B}_{L^2(0,T_0;\mathcal{H}^1(\Omega))}(R) = \Big\{ q_e \in L^2(0,T_0;\mathcal{H}^1(\Omega)) \ \textit{s.t.}\ \|q_e\|_{L^2(0,T_0;\mathcal{H}^1(\Omega))} \le R \Big\}.$$

Proof.

Step 1. The well-posedness of $\mathcal{F}$ comes from Proposition 5.6.

Step 2. Furthermore, from estimates

$$\|(\mathbf{v}_e, \mathbf{v}_s, p_e)\|_{X_T^{\varepsilon,s}} \le c\Big(\|\mathbf{v}^0\|_{\mathbf{V}^1(\Omega)} + \|\eta\|_{E_T^\varepsilon} + \|\mathbf{f}\|_{\mathbf{L}^2(Q_T)} \Big)$$

and

$$\|\eta\|_{E_T^\varepsilon} \le C\Big(\|(\eta^0, \eta^1)\|_{H^s} + \|\gamma_s \overline{p}_e\|_{L^{2-\varepsilon}(0,T;H^{1/2}_{(0)}(\Gamma_s))} \Big)$$

we get

$$\|(\mathbf{v}_e, p_e, \eta)\|_{X_T^\varepsilon} \le C\Big(\|(\mathbf{v}^0, \eta^0, \eta^1)\|_{X^0} + \|(\mathbf{f}, h)\|_{Z_T} + T^\theta \|\overline{p}_e\|_{L^2(0,T;\mathcal{H}^1(\Omega))} \Big),$$

and thanks to

$$\|p_e\|_{L^2(0,T;\mathcal{H}^1(\Omega))} \le C\|(\mathbf{v}_e, \mathbf{v}_s, p_e, \eta)\|_{X_T^\varepsilon},$$

we have finally

$$\begin{aligned} &\|p_e\|_{L^2(0,T;\mathcal{H}^1(\Omega))} \\ &\quad \le C\Big(\|(\mathbf{v}^0, \eta^0, \eta^1)\|_{X^0} + \|(\mathbf{f}, h)\|_{Z_T} + T^\theta \|\overline{p}_e\|_{L^2(0,T;\mathcal{H}^1(\Omega))} \Big). \end{aligned} \tag{5.23}$$

Thus, we now introduce $R > 0$ such that $C(\|(\mathbf{v}^0, \eta^0, \eta^1)\|_{X^0} + \|(\mathbf{f}, h)\|_{Z_T}) \le R/2$. If we take $\overline{p}_e$ in $\mathcal{B}_{L^2(0,T;\mathcal{H}^1(\Omega))}(R)$, then, for any time T_0, we get

$$\|p_e\|_{L^2(0,T_0;\mathcal{H}^1(\Omega))} \le R/2 + CT_0^\theta R,$$

which gives

$$\|p_e\|_{L^2(0,T_0;\mathcal{H}^1(\Omega))} < R$$

for T_0 such that $CT_0^\theta < 1/2$, for instance.

Step 3. The contraction is obtained for two pressure terms $\overline{p}_{e,1}, \overline{p}_{e,2}$ thanks to Proposition 5.6. Indeed, we have for two pressure terms $\overline{p}_{e,1}$ and $\overline{p}_{e,2}$ in $L^2(0,T;\mathcal{H}^1(\Omega))$ the estimate

$$\|\mathcal{F}(\overline{p}_{e,1}) - \mathcal{F}(\overline{p}_{e,2})\|_{L^2(0,T;\mathcal{H}^1(\Omega))} \leq cT^\theta \|\overline{p}_{e,1} - \overline{p}_{e,2}\|_{L^2(0,T;\mathcal{H}^1(\Omega))}.$$

Thus, for T_0 such that $cT_0^\theta < 1/2$, we get the contraction. ◻

We have now all the arguments to prove Theorem 5.1.

Proof of Theorem 5.1. By the Banach fixed point theorem, Proposition 5.7 is equivalent to the existence of a unique solution $(\mathbf{v}_e, \mathbf{v}_s, p_e, \eta)$ in $X^\varepsilon_{T_0}$ of system (5.21) on $(0,T_0)$. To get the existence of solutions on $(0,T)$, we use the same idea as that in Proposition 5.7 but initializing with $\overline{p}_e$ on $(0,2T_0)$ defined by $\overline{p}_e = p_e$ on $(0,T_0)$ (with p_e coming from the solution $(\mathbf{v}_e, \mathbf{v}_s, p_e, \eta)$ obtained above) and $\overline{p}_e = 0$ on $(T_0, 2T_0)$. By linearity of the system, the same estimates occur, and we have the existence and uniqueness on $(0,2T_0)$ in $X^\varepsilon_{2T_0}$. Step by step, we get the existence of a solution $(\mathbf{v}_e, \mathbf{v}_s, p_e, \eta)$ of (5.10) in X^ε_T.

To conclude the proof of Theorem 5.1, we need to prove the regularity of the solution $(\mathbf{v}, p, \eta)$ of system (5.1) with $\mathbf{v} = \mathbf{v}_e + \mathbf{v}_s$ and $p = p_e + p_s$. We already have $(\mathbf{v}_e, \mathbf{v}_s, p_e, \eta) \in X^\varepsilon_T = \mathbf{V}^{2,1}(Q_T) \times L^2(0,T;\mathbf{H}^2(\Omega)) \cap H^{3/4}(0,T;\mathbf{H}^{1/2}(\Omega)) \times L^2(0,T;\mathcal{H}^1(\Omega)) \times E^\varepsilon_T$. Now, we use the following theorem (see [2]).

THEOREM 5.8. *Assume that $\mathcal{A}$ is the generator of a analytic semigroup, $\mathcal{B} \in L^2(0,T;\mathbb{H})$, and $Y^0 \in [D(-\mathcal{A}), \mathbb{H}]_{1/2}$. Then the problem*

$$\begin{aligned} Y'(t) &= \mathcal{A}Y(t) + \mathcal{B}(t), \\ Y(0) &= Y^0 \end{aligned}$$

has a unique solution in $H^1(0,T;\mathbb{H}) \cap L^2(0,T;D(-\mathcal{A}))$.

In our case, remember that $D(-\mathcal{A}) = H^4_{(0)}(\Gamma_s) \times H^2_{(0)}(\Gamma_s)$, where $\mathcal{A}$ is defined in (5.16), $\mathbb{H} = H^2_{(0)}(\Gamma_s) \times L^2_0(\Gamma_s)$, and $\mathcal{B} = (0, (I + \gamma_s N_s)^{-1}(\gamma_s p_e + \tilde{h}))^T$. Then we have $[D(-\mathcal{A}), \mathbb{H}]_{1/2} = H_s$ and the following proposition.

PROPOSITION 5.9. *Let (η^0, η^1) be in H_s. For p_e in $L^2(0,T;\mathcal{H}^1(\Omega))$ and $\tilde{h}$ in $L^2(0,T;L^2_0(\Gamma_s))$, the equation*

$$\begin{aligned} (I + \gamma_s N_s)\eta_{tt} - \beta\eta_{xx} - \gamma\eta_{txx} + \alpha M_s \eta_{xxxx} &= \gamma_s p_e + \tilde{h} && (\Sigma^s_T), \\ \eta(0) &= \eta^0 && (\Gamma_s), \\ \eta_t(0) &= \eta^1 && (\Gamma_s) \end{aligned}$$

admits a solution η in $H^{4,2}_{(0)}(\Sigma^s_T)$ satisfying the estimate

$$\|\eta\|_{H^{4,2}_{(0)}(\Sigma^s_T)} \leq C\Big(\|(\eta^0, \eta^1)\|_{H^s} + \|\tilde{h}\|_{L^2(0,T;L^2_0(\Gamma_s))} + \|p_e\|_{L^2(0,T;\mathcal{H}^1(\Omega))}\Big).$$

The regularity of η gives η_t in $H^1(0,T;L^2_0(\Gamma_s))$ and then $\mathbf{v}_s$ in $H^1(0,T;\mathbf{H}^{1/2}(\Omega))$. Consequently, we have $(\mathbf{v}_s, p_s)$ in $\mathbf{H}^{2,1}(Q_T) \times L^2(0,T;\mathcal{H}^1(\Omega))$.

Thus, the solution $(\mathbf{v}, p, \eta)$ of (5.1) belongs to X_T. The estimate of $(\mathbf{v}, p, \eta)$ in X_T comes from all of the previous ones. ◻

6. Proof of Theorems 3.1 and 3.2 in the fixed domain Q_T. In this section, we want to prove Theorems 3.1 and 3.2 in the fixed domain in the sense of Definition 4.1. That is, we will find a solution $(\mathbf{u}, p, \eta)$ of system (4.5). We will use a fixed point method from a space of solutions of system (5.1) into itself. We begin the proof by an estimate on $(\mathbf{F}, \mathbf{w}, h)$, where $(\mathbf{F}, \mathbf{w})$ are defined in (4.3) and $h = \gamma_s H$ with H defined in (4.4).

PROPOSITION 6.1. *For* $(\mathbf{u}, p, \eta)$ *in* X_T, $(\mathbf{F}[\mathbf{u}, p, \eta], \mathbf{w}[\mathbf{u}, \eta], h[\mathbf{u}, \eta])$ *belongs to*

$$W_T = \Big\{(\mathbf{G}, \mathbf{z}, K) \in \mathbf{L}^2(Q_T) \times \mathbf{H}^{2,1}(Q_T) \times L^2(0, T; H^{1/2}_{(0)}(\Gamma_s)) \quad s.t.\ \mathbf{z} = 0 \ on\ \Gamma\Big\}, \tag{6.1}$$

and there exists $\delta > 0$ *such that*

$$\|(\mathbf{F}[\mathbf{u}, p, \eta], \mathbf{w}[\mathbf{u}, \eta], h[\mathbf{u}, \eta])\|_{W_T} \le CT^\delta (1 + \|(\mathbf{u}, p, \eta)\|_{X_T})\|(\mathbf{u}, p, \eta)\|^2_{X_T}. \tag{6.2}$$

Let $(\mathbf{u}_1, p_1, \eta_1)$ *and* $(\mathbf{u}_2, p_2, \eta_2)$ *be two triplets in* X_T *such that for* $i = 1, 2$

$$\|(\mathbf{u}_i, p_i, \eta_i)\|_{X_T} \le R$$

for some $R > 0$*. Thus we get*

$$\|(\mathbf{F}_1, \mathbf{w}_1, h_1) - (\mathbf{F}_2, \mathbf{w}_2, h_2)\|_{W_T} \le C(1 + R)RT^\delta \|(\mathbf{u}_1, p_1, \eta_1) - (\mathbf{u}_2, p_2, \eta_2)\|_{X_T} \tag{6.3}$$

with the notation $(\mathbf{F}_i, \mathbf{w}_i, h_i) = (\mathbf{F}[\mathbf{u}_i, p_i, \eta_i], \mathbf{w}[\mathbf{u}_i, \eta_i], h[\mathbf{u}_i, \eta_i])$.

To prove Proposition 6.1, we use two lemmas.

LEMMA 6.2. *For* $0 < \varepsilon' < \varepsilon$, *we get* $H^{1/2+\varepsilon}(0, T) \hookrightarrow H^{1/2+\varepsilon'}(0, T)$, *and if* a *belongs to* $H^{1/2+\varepsilon}(0, T)$, *then*

$$\|a\|_{H^{1/2+\varepsilon'}(0,T)} \le cT^{(1-\theta)/2}\|a\|_{H^{1/2+\varepsilon}(0,T)}, \qquad where\ \theta = \frac{1/2 + \varepsilon'}{1/2 + \varepsilon}.$$

LEMMA 6.3. *Let* $\mathbf{b}$ *and* a *be, respectively, in* $\mathbf{H}^{2,1}(Q_T)$ *and* $H^{1,1/2}(Q_T)$*; then* $a\mathbf{b}$ *belongs to* $\mathbf{L}^2(Q_T)$ *and there exists* $\delta > 0$ *such that*

$$\|a\mathbf{b}\|_{\mathbf{L}^2(Q_T)} \le CT^\delta \|a\|_{H^{1,1/2}(Q_T)}\|\mathbf{b}\|_{\mathbf{H}^{2,1}(Q_T)}.$$

Proof of Lemma 6.2. By interpolation,

$$H^{1/2+\varepsilon'}(0, T) = [H^{1/2+\varepsilon}(0, T), L^2(0, T)]_{1-\theta}, \qquad \text{where } \theta = \frac{1/2 + \varepsilon'}{1/2 + \varepsilon} \quad (0 < \theta < 1).$$

Then if a is in $H^{1/2+\varepsilon}(0, T)$, then a is in the interpolated space $H^{1/2+\varepsilon'}(0, T)$ with the estimate

$$\|a\|_{H^{1/2+\varepsilon'}(0,T)} \le C\|a\|^\theta_{H^{1/2+\varepsilon}(0,T)}\|a\|^{1-\theta}_{L^2(0,T)}.$$

On the other hand, the embedding $L^\infty(0, T) \hookrightarrow L^2(0, T)$ and a Hölder inequality in $(0, T)$ of finite mass gives $\|a\|_{L^2(0,T)} \le CT^{1/2}\|a\|_{L^\infty(0,T)}$. The embedding $H^{1/2+\varepsilon}(0, T) \hookrightarrow L^\infty(0, T)$ concludes the proof. ∎

Proof of Lemma 6.3. By Theorem B.3 in [7], for $\mathbf{b} \in \mathbf{H}^{2,1}(Q_T)$ and $a \in H^{1,1/2}(Q_T)$, $a\mathbf{b}$ belongs to $\mathbf{H}^{1-2\kappa,1/2-\kappa}(Q_T)$ for $0 \le \kappa < 1/2$. We now use the following two classical embeddings:

- $H^{1/2-\kappa}(0, T; \mathbb{R}) \hookrightarrow L^{1/\kappa}(0, T; \mathbb{R})$ (see [1]);
- $L^{1/\kappa}(0, T; \mathbb{R}) \hookrightarrow L^2(0, T; \mathbb{R})$ (because $2 < 1/\kappa \le +\infty$) with the estimate

$$\|c\|_{L^2(0,T;\mathbb{R})} \le T^{1/2-\kappa}\|c\|_{L^{1/\kappa}(0,T;\mathbb{R})} \qquad \text{for } c \in L^{1/\kappa}(0, T; \mathbb{R}).$$

Together, these two estimates give that $\|a\mathbf{b}\|_{\mathbf{L}^2(\Omega)}$, which is in $H^{1/2-\kappa}(0,T;\mathbb{R})$, belongs to $L^2(0,T;\mathbb{R})$ with the estimate (for $1/2-\kappa>0$)

$$\begin{aligned}\|a\mathbf{b}\|_{\mathbf{L}^2(Q_T)} &\leq CT^{1/2-\kappa}\|a\mathbf{b}\|_{H^{1/2-\kappa}(0,T;\mathbf{L}^2(\Omega))}\\ &\leq C'T^{1/2-\kappa}\|a\mathbf{b}\|_{\mathbf{H}^{1-2\kappa,1/2-\kappa}(Q_T)}\\ &\leq C''T^{1/2-\kappa}\|a\|_{H^{1,1/2}(Q_T)}\|\mathbf{b}\|_{\mathbf{H}^{2,1}(Q_T)}. \qquad \square\end{aligned}$$

We can now prove Proposition 6.1.

Proof of Proposition 6.1. Thanks to Lemmas 6.2 and 6.3, we can estimate the norms of the right-hand sides. We use the strong regularity of η and $\mathbf{u}$. Indeed, η in $H^{4,2}_{(0)}(\Sigma^s_T)$ gives

$$\eta \in H^{2\kappa}(0,T;H^{4(1-\kappa)}_{(0)}(\Gamma_s)) \quad \text{for} \quad 0<\kappa<1.$$

This gives us directly that

$$\begin{aligned}&\eta \in H^{7/4-\varepsilon/2}(0,T;H^{1/2+\varepsilon}_{(0)}(\Gamma_s)),\\ &\eta \in H^{5/4-\varepsilon/2}(0,T;H^{3/2+\varepsilon}_{(0)}(\Gamma_s)),\\ &\eta \in H^{3/4-\varepsilon/2}(0,T;H^{5/2+\varepsilon}_{(0)}(\Gamma_s)),\\ &\eta \in H^{1/4-\varepsilon/2}(0,T;H^{7/2+\varepsilon}_{(0)}(\Gamma_s)).\end{aligned} \tag{6.4}$$

The first three equations of (6.4) gives, respectively, η, η_x, and η_{xx} in $L^\infty(\Sigma^s_T)$ with the following estimates:

$$\|\eta\|_{\mathbf{L}^\infty(\Sigma^s_T)}+\|\eta_x\|_{\mathbf{L}^\infty(\Sigma^s_T)}+\|\eta_{xx}\|_{\mathbf{L}^\infty(\Sigma^s_T)} \leq cT^\chi\|\eta\|_{H^{4,2}_{(0)}(\Sigma^s_T)} \quad \text{for} \quad \chi>0.$$

From the last equation in (6.4), we get only $\eta_{xxx}\in L^2(0,T;L^\infty(\Gamma_s))$.

Let us check some terms of $\mathbf{F}[\mathbf{u},p,\eta]$, $\mathbf{w}[\mathbf{u},\eta]$, or $h[\mathbf{u},\eta]$.

• For $\mathbf{F}[\mathbf{u},p,\eta]$, we only need to check that all the terms are in $\mathbf{L}^2(Q_T)$. The first term in $\mathbf{F}[\mathbf{u},p,\eta]$ is $-\eta\mathbf{u}_t$:

$$\|-\eta\mathbf{u}_t\|_{\mathbf{L}^2(Q_T)} \leq \|\eta\|_{L^\infty(\Sigma^s_T)}\|\mathbf{u}_t\|_{\mathbf{L}^2(Q_T)}.$$

Then, via the embeddings $H^{\frac{1}{2}+\varepsilon}(0,T)\hookrightarrow H^{\frac{1}{2}+\varepsilon'}(0,T)\hookrightarrow\mathcal{C}(0,T)$ and the smoothness of η, we get

$$\|\eta\|_{L^\infty(\Sigma^s_T)} \leq cT^{\frac{1}{2}(1-\theta)}\|\eta\|_{H^{1/2+\varepsilon'}(0,T;H^{3-2\varepsilon'}(\Gamma_s))} \quad \text{for } \varepsilon'<\varepsilon \text{ and } \varepsilon \text{ s.t. } 0\leq\varepsilon<1.$$

Thus $\|\eta\|_{L^\infty(\Sigma^s_T)} \leq cT^{\frac{1}{2}(1-\theta)}\|\eta\|_{H^{4,2}_{(0)}(\Sigma^s_T)}$ and finally

$$\|-\eta\mathbf{u}_t\|_{L^2(Q_t)} \leq cT^{\frac{1}{2}(1-\theta)}\|\eta\|_{H^{4,2}_{(0)}(\Sigma^s_T)}\|\mathbf{u}\|_{\mathbf{H}^{2,1}(Q_t)}.$$

Another term is $\frac{\eta_x^2}{1+\eta}\mathbf{u}_z$, which satisfies

$$\left\|\frac{\eta_x^2}{1+\eta}\mathbf{u}_z\right\|_{\mathbf{L}^2(Q_T)} \leq \left\|\frac{1}{1+\eta}\right\|_{L^\infty(\Sigma^s_T)}\|\eta_x\|^2_{L^\infty(\Sigma^s_T)}\|\mathbf{u}_z\|_{\mathbf{L}^2(Q_T)}$$

and becomes, thanks to Lemma 6.2,

$$\left\|\frac{\eta_x^2}{1+\eta}\mathbf{u}_z\right\|_{\mathbf{L}^2(Q_T)} \leq cT^{1-\theta}\|\eta\|^2_{H^{4,2}_{(0)}(\Sigma^s_T)}\|\mathbf{u}\|_{\mathbf{H}^{2,1}(Q_T)}.$$

Terms with a product of $\mathbf{u}$ and a derivative of $\mathbf{u}$ like $(1+\|\eta\|)u_1\mathbf{u}_x$, $\eta_x u_1 \mathbf{u}_z$ or $u_2\mathbf{u}_z$ must be carefully studied. Thanks to Lemma 6.3, because $\mathbf{u}$ belongs to $\mathbf{H}^{2,1}(Q_T)$ and then $\mathbf{u}_x$ and $\mathbf{u}_z$ are in $\mathbf{H}^{1,1/2}(Q_T)$, we get that $u_1\mathbf{u}_x$, $u_2\mathbf{u}_z$, and $u_2\mathbf{u}_z$ belong to $\mathbf{L}^2(Q_T)$ with, for $0 \le \kappa < 1/2$,

$$\begin{aligned}
&\|-(1+\eta)u_1\mathbf{u}_x + (z\eta_x u_1 - u_2)\mathbf{u}_z\|_{\mathbf{L}^2(Q_T)}\\
&\le CT^{1/2-\kappa}\Big(1+\|\eta\|_{L^\infty(\Sigma_T^s)} + \|\eta_x\|_{L^\infty(\Sigma_T^s)}\Big)\|\mathbf{u}\|_{\mathbf{H}^{2,1}(Q_T)}\|\nabla\mathbf{u}\|_{\mathbf{H}^{1,1/2}(Q_T)}\\
&\le CT^{1/2-\kappa}\Big(1+\|\eta\|_{H^{4,2}_{(0)}(\Sigma_T^s)}\Big)\|\mathbf{u}\|^2_{\mathbf{H}^{2,1}(Q_T)}.
\end{aligned}$$

• For $\mathbf{w}[\mathbf{u},\eta]$, we have to prove that all the terms belong to $\mathbf{H}^{2,1}(Q_T)$ with the expected estimate. First of all, the calculations of the derivatives of $\mathbf{w}[\mathbf{u},\eta]$ are

(6.5)
$$\begin{aligned}
\mathbf{w}_x &= -\eta_x u_1\mathbf{e}_1 - \eta u_{1,x}\mathbf{e}_1 + z\eta_{xx}u_1\mathbf{e}_2 + z\eta_x u_{1,x}\mathbf{e}_2,\\
\mathbf{w}_z &= -\eta u_{1,z}\mathbf{e}_1 + \eta_x u_1\mathbf{e}_2 + z\eta_x u_{1,z}\mathbf{e}_2,\\
\mathbf{w}_{xx} &= -\eta_{xx}u_1\mathbf{e}_1 - 2\eta_x u_{1,x}\mathbf{e}_1 - \eta u_{1,xx}\mathbf{e}_1 + z\eta_{xxx}u_1\mathbf{e}_2 + 2z\eta_{xx}u_{1,x}\mathbf{e}_2 + z\eta_x u_{1,xx}\mathbf{e}_2,\\
\mathbf{w}_{zz} &= -\eta u_{1,zz}\mathbf{e}_1 + 2\eta_x u_{1,z}\mathbf{e}_2 + \eta_x u_{1,zz}\mathbf{e}_2,\\
\mathbf{w}_t &= -\eta_t u_1\mathbf{e}_1 - \eta u_{1,t}\mathbf{e}_1 + z\eta_{x,t}u_1\mathbf{e}_2 + z\eta_x u_{1,t}\mathbf{e}_2.
\end{aligned}$$

Then almost all of the estimates of the derivatives in $\mathbf{L}^2(Q_T)$ are obtained as for $\mathbf{F}[\mathbf{u},p,\eta]$. Others terms like $\eta_{xxx}\mathbf{u_1}\mathbf{e}_1$ are estimated as follows:

$$\|\eta_{xxx}u_1\|_{L^2(Q_T)} \le CT^\theta\|\eta_{xxx}\|_{L^2(0,T;L^\infty(\Gamma_s))}\|\mathbf{u}\|_{L^\infty(0,T;\mathbf{L}^2(\Omega))}. \tag{6.6}$$

• For $h[\mathbf{u},\eta]$. We can remark that h defined in Proposition 6.1 is the trace of function H on Γ_s, and we can prove that the lifting H of h belongs to $L^2(0,T;H^1(\Omega))$. We have to calculate the different terms of H and their derivatives:

$$\begin{aligned}
H &= \nu\left[\frac{\eta_x}{1+\eta}u_{1,z} + \eta_x u_{2,x} - \frac{\eta_x^2-2\eta}{1+\eta}u_{2,z}\right],\\
H_x &= \nu\left[\left(\frac{\eta_{xx}(1+\eta)-\eta_x^2}{(1+\eta)^2}\right)u_{1,z} + \frac{\eta_x}{1+\eta}u_{1,xz} + \eta_{xx}u_{2,x} + \eta_x u_{2,xx}\right.\\
&\qquad\left. - \left(2\eta_x\frac{\eta_{xx}-1}{1+\eta} - \eta_x\frac{\eta_x^2-2\eta}{(1+\eta)^2}\right)u_{2,z} - \frac{\eta_x^2-2\eta}{1+\eta}u_{2,xz}\right],\\
H_z &= \nu\left[\frac{\eta_x}{1+\eta}u_{1,zz} + \eta_x u_{2,xz} - \frac{\eta_x^2-2\eta}{1+\eta}u_{2,zz}\right].
\end{aligned} \tag{6.7}$$

Because of the regularity of η, we always get the expected estimates.

The second point comes from the at least quadratic nonlinearity of the right-hand sides with respect to $(\mathbf{u},p,\eta)$. Some calculations give estimates (6.3). ∎

PROPOSITION 6.4. *For a given triplet* $(\overline{\mathbf{u}},\overline{p},\overline{\eta})$ *in* X_T*, system* (4.5) *with right-hand sides* $(\overline{\mathbf{F}},\overline{\mathbf{w}},\overline{H}) = (\mathbf{F}[\overline{\mathbf{u}},\overline{p},\overline{\eta}],\mathbf{w}[\overline{\mathbf{u}},\overline{\eta}],H[\underline{u},\overline{\eta}])$ *and initial data* $(\mathbf{u}^0,\eta^0,\eta^1)$ *in* X^0 *satisfying* (4.12) *admits a unique solution* $(\mathbf{u},p,\eta)$ *in* X_T *with the estimate*

$$\|(\mathbf{u},p,\eta)\|_{X_T} \le c_1(\|(\mathbf{u}^0,\eta^0,\eta^1)\|_{X^0} + c_2T^\delta(1+\|(\overline{\mathbf{u}},\overline{p},\overline{\eta})\|_{X_T})\|(\overline{\mathbf{u}},\overline{p},\overline{\eta})\|^2_{X_T}), \tag{6.8}$$

where $\delta>0$ *is defined in Proposition* 6.1*. In other terms, we can construct a mapping*

(6.9)
$$\begin{aligned}
\mathcal{X}:\quad X_T &\longrightarrow X_T\\
(\overline{\mathbf{u}},\overline{p},\overline{\eta}) &\longmapsto \mathcal{X}(\overline{\mathbf{u}},\overline{p},\overline{\eta}) = (\mathbf{u},p,\eta) \textit{ is a solution of the system } (4.5)\\
&\qquad \textit{with } (\mathbf{F}[\overline{\mathbf{u}},\overline{p},\overline{\eta}],\mathbf{w}[\overline{\mathbf{u}},\overline{\eta}],H[\overline{\mathbf{u}},\overline{\eta}]) \textit{ for right-hand sides,}
\end{aligned}$$

which satisfies

$$\|\mathcal{X}(\overline{\mathbf{u}},\overline{p},\overline{\eta})\|_{X_T} \leq c_1\Big(\|(\mathbf{u}^0,\eta^0,\eta^1)\|_{X^0} + c_2T^\delta(1+\|(\overline{\mathbf{u}},\overline{p},\overline{\eta})\|_{X_T})\|(\overline{\mathbf{u}},\overline{p},\overline{\eta})\|^2_{X_T}\Big). \tag{6.10}$$

Proof. Let us notice that $(\mathbf{u},p,\eta)$ is solution of (4.5) with right-hand sides $(\mathbf{F}[\overline{\mathbf{u}},\overline{p},\overline{\eta}],\mathbf{w}[\overline{\mathbf{u}},\overline{\eta}],H[\overline{\mathbf{u}},\overline{\eta}])$ if and only if $(\mathbf{v},p,\eta)=(\mathbf{u}-\mathbf{w}[\overline{\mathbf{u}},\overline{\eta}],p,\eta)$ is a solution of (4.8) with $(\mathbf{f}[\overline{\mathbf{u}},\overline{p},\overline{\eta}],h[\overline{\mathbf{u}},\overline{\eta}])$ as right-hand sides and $(\mathbf{v}^0,\eta^0,\eta^1)$ for initial data (see (4.7), (4.9), and (4.10) for the definitions of $\mathbf{f}$, h, and $\mathbf{v}^0$). Then this proposition relies first on the result of existence of solutions for system (5.1) in Theorem 5.1 and second on Proposition 6.1 for the estimate. ∎

We can conclude this section showing existence of solutions in the fixed domain.

PROPOSITION 6.5. *Let* $(\mathbf{u}^0,\eta^0,\eta^1)$ *be in* X^0 *satisfying* (4.12).

(i) *There exists a time* $T_0>0$ *such that system* (4.5) *admits a unique local strong solution* $(\mathbf{u},p,\eta)$ *in* X_{T_0}.

(ii) *There exists* r *small enough such that, under condition* $\|(\mathbf{u}^0,\eta^0,\eta^1)\|_{X^0}\leq r$, *system* (4.5) *admits a unique global strong solution* $(\mathbf{u},p,\eta)$ *in* X_T.

Proof. Let $(\mathbf{u}^0,\eta^0,\eta^1)$ be in X^0 satisfying (4.12). We denote $r=\|(\mathbf{u}^0,\eta^0,\eta^1)\|_{X^0}$ and set $R=2c_1r$ (where c_1 is the constant in (6.10)).

(i) Let us define $T_0=\big(\frac{1}{2c_1c_2R(R+1)}\big)^\delta$ and

$$B_{X_{T_0}}(R)=\Big\{(\mathbf{u},p,\eta)\in X_{T_0} \text{ with } \|(\mathbf{u},p,\eta)\|_{X_{T_0}}\leq R\Big\}.$$

Then $\mathcal{X}$ is a contraction mapping in $B_{X_{T_0}}(R)$. Indeed, let $(\mathbf{u}_1,p_1,\eta_1)$ and $(\mathbf{u}_2,p_2,\eta_2)$ be two triplets in $B_{X_{T_0}}(R)$. With the previous notation, we get solutions $\mathcal{X}(\mathbf{u}_i,p_i,\eta_i)$ $(i=1,2)$ of system (4.5) corresponding with right-hand sides $(\mathbf{F}[\mathbf{u}_i,p_i,\eta_i],\mathbf{w}[\mathbf{u}_i,\eta_i],H[\mathbf{u}_i,\eta_i])$ $(i=1,2)$ and initial data $(\mathbf{u}^0,\eta^0,\eta^1)$. First, each solution obeys the estimate (6.10) thanks to Proposition 6.4, which gives, for R and T_0 as above,

$$\|\mathcal{X}(\mathbf{u}_i,p_i,\eta_i)\|_{X_{T_0}}\leq\frac{R}{2}+\frac{R}{2}=R.$$

Second, the difference satisfies

$$\|\mathcal{X}(\mathbf{u}_1,p_1,\eta_1)-\mathcal{X}(\mathbf{u}_2,p_2,\eta_2)\|_{X_{T_0}} \leq c_1c_2T_0^\delta(1+R)R\|(\mathbf{u}_1,p_1,\eta_1)-(\mathbf{u}_2,p_2,\eta_2)\|_{X_{T_0}} \tag{6.11}$$

thanks to (6.3), that is,

$$\|\mathcal{X}(\mathbf{u}_1,p_1,\eta_1)-\mathcal{X}(\mathbf{u}_2,p_2,\eta_2)\|_{X_T}\leq\frac{1}{2}\|(\mathbf{u}_1,p_1,\eta_1)-(\mathbf{u}_2,p_2,\eta_2)\|_{X_T}.$$

(ii) We choose r such that $c_2T^\delta r(1+2c_1r)=1$, that is,

$$r=\frac{1}{c_1^2c_2T^\delta(1+\sqrt{1+\frac{2}{c_1c_2T^\delta}})}.$$

Then $\mathcal{X}$ is a contraction mapping in $B_{X_T}(R)$ (see (i) for details).

7. Back to the moving domain. Thanks to Definition 4.1, the proof of Theorems 3.1 and 3.2 in the moving domain consists in proving that the change of variables

$$\phi_t : \begin{array}{ccc} \Omega & \longrightarrow & \Omega_{\eta(t)} \\ (x,z) & \longmapsto & (x,y) \end{array}$$

is well-defined as a $\mathcal{C}^1$-diffeomorphism from Ω into $\Omega_{\eta(t)}$ for every $t \in [0,T)$ and that condition (1.1) is checked for the solution $(\mathbf{u}, p, \eta)$ of (4.5). Then we will have the solution $(\tilde{\mathbf{u}}, \tilde{p}, \eta) = (\phi_t(\mathbf{u}), \phi_t(p), \eta)$ of (1.2)–(1.3) in $\mathbf{V}^{2,1}(\mathcal{Q}_T) \times L^2(\bigcup_{t\in(0,T)} \{t\} \times \mathcal{H}^1(\Omega_{\eta(t)})) \times H^{4,2}_{(0)}(\Sigma^s_T)$. Furthermore, by the change of variables, we will be able to check which compatibility condition corresponds in $\mathcal{Q}_T$ to (4.12).

We have to show that condition (1.1) is checked. In the case of the existence of solutions for small data, because we then have

$$\|(\mathbf{u}, p, \eta)\|_{X_T} \le r,$$

we easily get from $\|\eta\|_{L^\infty(\Sigma^s_T)} \le \|\eta\|_{H^{4,2}_{(0)}(\Sigma^s_T)}$ (obtained thanks to the continuous embedding $H^{4,2}_{(0)}(\Sigma^s_T) \hookrightarrow L^\infty(\Sigma^s_T)$) that

$$\|\eta\|_{L^\infty(\Sigma^s_T)} \le r \le 1 - \delta_0 \quad \text{for } r \text{ small enough.}$$

Condition (1.1) is checked for local solutions too thanks to the continuity of the embeddings, for $0 < \varepsilon < 1$ (see the proof of Proposition 6.1),

$$H^{4,2}_{(0)}(\Sigma^s_{T_0}) \quad \hookrightarrow \quad H^{1/2+\varepsilon}(0, T_0; H^{3-2\varepsilon}_{(0)}(\Gamma_s)) \quad \hookrightarrow \quad L^\infty(\Sigma^s_{T_0}),$$

which gives $\|\eta\|_{L^\infty(\Sigma^s_{T_0})} \le cT_0^\theta \|\eta\|_{H^{4,2}_{(0)}(\Sigma^s_{T_0})}$ (for $\theta > 0$) and then $\|\eta\|_{L^\infty(\Sigma^s_{T_0})} \le cT_0^\theta R \le 1 - \delta_0$ for T_0 small enough.

The embedding $H^{4,2}_{(0)}(\Sigma^s_T) \hookrightarrow \mathcal{C}([0,T); \mathcal{C}^1(\Gamma_s))$ together with the condition $1+\eta \ge \delta_0 > 0$ show that ϕ_t is a $\mathcal{C}^1$-diffeomorphism from Ω into $\Omega_{\eta(t)}$.

All the derivatives of the solutions written in the variable (x,y) are combinations of those in the variable (x,z) multiplied at most by η or one of its derivatives which are smooth enough to get $(\tilde{\mathbf{u}}, \tilde{p})$ in $\mathbf{H}^{4,2}(\mathcal{Q}_T) \times L^2(\bigcup_{t\in(0,T)} \{t\} \times \mathcal{H}^1(\Omega_{\eta(t)}))$ (the calculations are exactly the ones proving that $\mathbf{F}[\mathbf{u}, p, \eta]$ belongs to $\mathbf{L}^2(Q_T)$ for $(\mathbf{u}, p, \eta)$ in X_T).

The compatibility conditions became, after the change of variables,

$$\begin{aligned} \operatorname{div} \mathbf{u}^0 &= 0 && (\Omega_{\eta^0}), \\ \mathbf{u}^0 &= \eta^1 \mathbf{e}_2 && (\Gamma_{\eta^0}), \\ \mathbf{u}^0 &= \mathbf{0} && (\Gamma^0). \end{aligned}$$

REFERENCES

[1] R. A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[2] C. Bardos, *A regularity theorem for parabolic equations*, J. Funct. Anal., 7 (1971), pp. 311–322.

[3] H. Beirão da Veiga, *On the existence of strong solutions to a coupled fluid-structure evolution problem*, J. Math. Fluid Mech., 6 (2004), pp. 21–52.

[4] A. Chambolle, B. Desjardins, M. J. Esteban, and C. Grandmont, *Existence of weak solutions for the unsteady interaction of a viscous fluid with an elastic plate*, J. Math. Fluid Mech., 7 (2005), pp. 368–404.

[5] S. Chen and R. Triggiani, *Proof of extensions of two conjectures on structural damping for elastic systems*, Pacific J. Math., 1 (1989), pp. 15–55.

[6] C. Grandmont, *Existence of weak solutions for the unsteady interaction of a viscous fluid with an elastic plate*, SIAM J. Math. Anal., 40 (2008), pp. 716–737.

[7] G. Grubb and V. A. Solonnikov, *Boundary value problems for the nonstationary Navier-Stokes equations treated by pseudo-differential methods*, Math. Scand., 69 (1991), (1992) pp. 217–290.

[8] M. Guidorzi, M. Padula, and P. I. Plotnikov, *Hopf solutions to a fluid-elastic interaction model*, Math. Models Methods Appl. Sci., 18 (2008), pp. 215–269.

[9] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

[10] A. Quarteroni, M. Tuveri, and A. Veneziani, *Computational vascular fluid dynamics: Problems, models and methods*, Comput. Vis. Sci., 2 (2000), pp. 163–197.

[11] J.-P. Raymond, *Stokes and Navier-Stokes equations with nonhomogeneous boundary conditions*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 24 (2007), pp. 125–169.

[12] J.-P. Raymond, *Feedback stabilization of a fluid-structure model*, SIAM J. Control Optim., 48 (2010), pp. 5398–5443.