Existence of strong solutions for a system coupling the Navier-Stokes equations and a damped wave equation.

Julien Lequeurre.

February 17, 2012

AMS subject classifications.: 35Q30, 74F10, 76D05, 76D03

Keywords: fluid-structure interaction, Navier-Stokes equations, damped wave equation, coupled system

Abstract: We consider a fluid - structure interaction problem coupling the Navier-Stokes equations with a damped wave equation which describes the displacement of a part of the boundary of the fluid domain. The system is considered first in the two dimensional setting and in a second part it is adapted to the three dimensional setting.

1 Introduction.

This paper is devoted to the study of a coupled fluid-structure system modeling the blood flow in a large vessel which was introduced in [10]. The system couples the Navier-Stokes equations which models an incompressible newtonian viscous fluid and a beam equation which models the displacement of the moving boundary. Thus, the fluid domain depends on the displacement of the beam.

Let us introduce the notations. Let $L > 0$ be a length and let η be a function *a priori* from $(0,T) \times \mathbb{R}/L\mathbb{Z}$ into $\mathbb{R}$ satisfying the assumption:

$$\exists \delta_0 > 0 \text{ such that } \forall t \geq 0\ \forall x \in (0,L) \quad 1 + \eta(t,x) \geq \delta_0 > 0. \tag{1.1}$$

The function η models the displacement of the beam in the upper part of the boundary of the domain. Assumption (1.1) ensures that the domain $\Omega_{\eta(t)}$ (see Figure 1) defined by

$$\Omega_{\eta(t)} = \left\{ (x,y) \in \mathbb{R}^2 \text{ s.t. } x \in (0,L) \text{ and } 0 < y < 1 + \eta(t,x) \right\}$$

is a connected domain for any time $t \geq 0$. We introduce the moving boundary $\Gamma^s_{\eta(t)}$ defined by

$$\Gamma^s_{\eta(t)} = \left\{ (x,y) \in \mathbb{R}^2 \text{ s.t. } x \in (0,L) \text{ and } y = 1 + \eta(t,x) \right\}.$$

The other part of the boundary is denoted by Γ, that is $\Gamma = (0,L) \times \{0\}$. Finally, we introduce $\Omega_0 = (0,L) \times (0,1)$ and $\Gamma^s_0 = (0,L) \times \{1\}$ respectively the reference domain and reference state of the beam corresponding with the case $\eta = 0$, that is when the beam is «at rest».

We define also $\Gamma_0 = \partial\Omega_0 = \Gamma \cup \Gamma^s_0$ and give some other notations:

$$Q_T = \bigcup_{t\in(0,T)} \{t\} \times \Omega_{\eta(t)}, \quad \Sigma^s_T = \bigcup_{t\in(0,T)} \{t\} \times \Gamma^s_{\eta(t)}, \quad \Sigma_T = (0,T) \times \Gamma,$$
$$Q^0_T = (0,T) \times \Omega_0, \quad \Sigma^{s,0}_T = (0,T) \times \Gamma^s_0, \quad \Sigma^0_T = (0,T) \times \Gamma_0.$$

We introduce here the two partial differential equations of our system. First the Navier-Stokes equations in the variables $(\mathbf{u},p)$ (respectively the velocity and the pressure of the fluid)

$$\begin{array}{rcll} \mathbf{u}_t - \operatorname{div} \sigma(\mathbf{u},p) + (\mathbf{u} \cdot \nabla)\mathbf{u} & = & \mathbf{0} & \text{in } Q_T, \\ \operatorname{div} \mathbf{u} & = & 0 & \text{in } Q_T, \\ \mathbf{u} & = & \eta_t \mathbf{e}_2 & \text{on } \Sigma^s_T, \\ \mathbf{u} & = & \mathbf{0} & \text{on } \Sigma_T, \\ \mathbf{u}(0) & = & \mathbf{u}^0 & \text{in } \Omega_{\eta_{1,0}} \end{array} \tag{1.2}$$

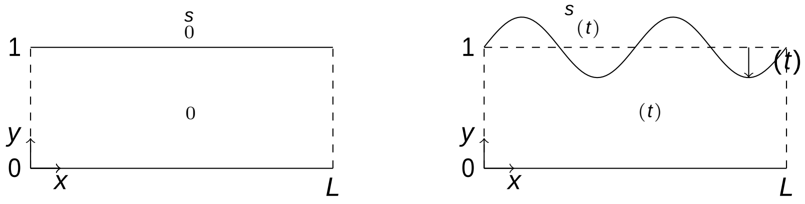

Figure 1: The domains $_0$ (left) and $_{(t)}$ (right).

and second, the damped beam equation:

$$\begin{array}{rcll} {}_{tt} + \quad {}_{xxxx} \quad {}_{xx} \quad {}_{txx} & = & (\mathbf{u}; p)(\quad {}_x \mathbf{e}_1 + \mathbf{e}_2) \quad \mathbf{e}_2 & \text{on } {}^{s;0}_{T}, \\ ((0); \; {}_t(0)) & = & (\; {}^{1;0}, \; {}^{2;0}) & \text{in } {}^{s}_{0}. \end{array} \tag{1.3}$$

In equations (1.2) and (1.3), $(\mathbf{u}; p)$ is the Cauchy stress tensor defined by $(\mathbf{u}; p) = \quad r\,\mathbf{u} + (r\,\mathbf{u})^{\mathbf{tr}}$ $p\mathbf{I}$ where $\mathbf{I}$ is the identity $2 \quad 2$ matrix. The vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ are defined by $\mathbf{e}_1 = (1; 0)^{\mathbf{tr}}$ and $\mathbf{e}_2 = (0; 1)^{\mathbf{tr}}$. The coefficient $\quad > 0$ is the viscosity of the fluid and $\quad 0$, $\quad 0$, $\quad > 0$ are constants relative to the structure, namely the rigidity, the stretching and the friction of the beam.

The question of existence of solutions for this kind of systems has already been studied. For instance, existence of weak solutions has been proved in [4, 5] in three dimensions.

Here we are interested in existence of strong solutions for this system. It has already been studied in [3] and [7]. In [3], the author considers in two dimensions the Navier-Stokes equations coupled with a damped beam equation. In both cases $\quad > 0$ and $\quad = 0$ in (1.3), he proves the local existence of strong solutions for small initial data and for a small parameter (namely the ratio between the volumic masses of the fluid and the beam). In [7], we consider the same setting in the case $\quad > 0$. We prove the local existence of strong solutions for arbitrary large data and the global existence on $[0; T]$ for any $T > 0$ of strong solutions for small initial data. These results have been extended to the three dimensional setting in [8] following the exact same proof as in [7].

In this paper, we study both the two and the three dimensional settings in the case $\quad = 0$. In fact, the main results and their proofs in both cases are quite similar. Thus, in the first part of the paper, namely section 2, we detail the two dimensional case and in the second part, namely in section 3, we state the results in the three dimensional case and we especially stress on the differences with the two dimensional case.

The plan of this paper follows the one in [7] where a similar system is studied. Nevertheless, we cannot apply the exact same strategy as in [7] due to the less regular displacement of the beam (see Proposition 2.11 in section 2.4.3 for the two dimensional system and Theorem 3.3 in section 3.3 for the three dimensional case to be compared to Proposition 5.4 in [7]). In particular, we use another lifting of the nonzero divergence term (see section 2.3) and we have to take more regular initial data for the displacement in the three dimensional case in order to make the fixed point method work (see section 3.4 for details).

2 The two dimensional system.

As already mentioned, we consider in this section system (1.2)–(1.3) in the case $\quad = 0$, that is the system coupling the Navier-Stokes equations

$$\begin{array}{rcll} \mathbf{u}_t \quad \operatorname{div} \; (\mathbf{u}; p) + (\mathbf{u} \quad r)\mathbf{u} & = & \mathbf{0} & \text{in } Q_T; \\ \operatorname{div} \mathbf{u} & = & 0 & \text{in } Q_T; \\ \mathbf{u} & = & {}_t \mathbf{e}_2 & \text{on } {}^{s;}_{T}; \\ \mathbf{u} & = & \mathbf{0} & \text{on } {}_T; \\ \mathbf{u}(0) & = & \mathbf{u}^0 & \text{in } {}_{1;0} \end{array} \tag{2.1}$$

and the damped wave equation

$$\begin{array}{rcll} {}_{tt} \quad {}_{xx} \quad {}_{txx} & = & (\mathbf{u}; p)(\quad {}_{x}\mathbf{e}_1 + \mathbf{e}_2) \quad \mathbf{e}_2 & \text{on } {}^{s;0}_{T}, \\ ((0); \ {}_{t}(0)) & = & (\ {}^{1;0}, \ {}^{2;0}) & \text{in } {}^{s}_{0}. \end{array} \tag{2.2}$$

We prove the local existence of strong solutions for any initial data or the global existence of strong solutions for small initial data. After introducing the different function spaces in section 2.1, we set the results for this system in section 2.2. Then, we give their proofs which follow the different steps of [7]. Namely, thanks to a change of variables, we set the system in the fixed cylinder Q_T^0 in section 2.3. Then, in section 2.4, we prove the existence of strong solution for the linearized system with nonhomogeneous right-hand sides. Finally, we use a fixed point method in section 2.5 to conclude the proof.

2.1 Functional settings.

We have to give a definition for functions in time dependent domains. Furthermore, the different functions are periodic in the first space variable.

We introduce the classic Hilbert space $L^2_\#(\ _0)$ as the space of $L^2_{\mathrm{loc}}(\mathbb{R} \quad (0;1))$ which are L periodic. In the same way, we set $\mathbf{L}^2_\#(\ _0) = L^2_\#(\ _0;\mathbb{R}^2)$ and $\mathbf{H}_\#(\ _0) = H_\#(\ _0)$. We introduce the Stokes space

$$\mathbf{V}^0_\#(\ _0) = \quad \mathbf{z} \ 2 \ \mathbf{L}^2_\#(\ _0) \text{ s.t. div } \mathbf{u} = 0 \text{ in } \quad _0$$

and

$$\begin{array}{rcl} \mathbf{H}_\#^{;} \ (Q_T^0) & = & L^2(0;T;\mathbf{H}_\#(\ _0)) \setminus H \ (0;T;\mathbf{L}^2_\#(\ _0)); \\ \mathbf{V}_\#^{;} \ (Q_T^0) & = & L^2(0;T;\mathbf{V}_\#(\ _0)) \setminus H \ (0;T;\mathbf{V}^0_\#(\ _0)). \end{array}$$

We define functions in the time dependent cylinder Q_T as follows.

Definition 2.1. *We say that* $\mathbf{u}$ *belongs to* $H \ (\bigcup_{t\in(0;T)} ftg \quad \mathbf{H}_\#(\quad _{(t)}))$ *(respectively to* $H \ (\bigcup_{t\in(0;T)} ftg$ $\mathbf{V}_\#(\quad _{(t)})))$ *if*

- *for a.e.* t *in* $(0;T)$, $\mathbf{u}(t)$ *belongs to* $\mathbf{H}_\#(\quad _{(t)})$ *(resp. to* $\mathbf{V}_\#(\quad _{(t)})$*),*
- $t \ 7! \ k\mathbf{u}(t)k_{\mathbf{H}_\#(\Omega \ _{(t)})}$ *(resp.* $t \ 7! \ k\mathbf{u}(t)k_{\mathbf{V}_\#(\Omega \ _{(t)})}$*) is in* $H \ (0;T;\mathbb{R})$.

Because of the divergence free condition in (2.1), the solutions $(\mathbf{u}; p; \ )$ of (2.1)–(2.2) have to satisfy

$$0 = \int_{\Omega \ _{(t)}} \text{div } \mathbf{u}(t) = \int_{\Gamma^s \ _{(t)}} \mathbf{u}(t) \quad \mathbf{n}(t) \qquad \int_{\Gamma} \mathbf{u}(t) \quad \mathbf{e}_2 = \int_{\Gamma_0^s} \ _t(t).$$

Thus, $\ _t$ has to satisfy $\int_{\Gamma_0^s} \ _t = 0$. In the same way, we will consider displacements satisfying $\int_{\Gamma_0^s} \ = 0$. Thus, we must take the initial data for the beam $\ ^{1;0}$ and $\ ^{2;0}$ in $L^2_{\#;0}(\ ^s_0)$ the space of x periodic function in $L^2_{\mathrm{loc}}(\mathbb{R})$ of period L and of zero mean value in $\ ^s_0$:

$$L^2_{\#;0}(\ ^s_0) = \left\{ \quad 2 \ L^2_\#(\ ^s_0) \text{ s.t. } \int_{\Gamma_0^s} \ = 0 \right\}.$$

We introduce the projection $M^s_\#$ from $L^2_\#(\ ^s_0)$ onto $L^2_{\#;0}(\ ^s_0)$ defined by

$$M^s_\#(\ ) = \qquad \frac{1}{j \ ^s_0 j} \int_{\Gamma_0^s} \qquad \text{for all} \quad 2 \ L^2_\#(\ ^s_0).$$

Then, we define a new trace function $\ ^s_\#$ to set the right-hand side of the beam equation on the space $L^2_{\#;0}(\ ^s_0)$ as follows

$$\ ^s_\#(q) = M^s_\#(q_{|\Gamma_0^s}) = q_{|\Gamma_0^s} \quad \frac{1}{j \ ^s_0 j} \int_{\Gamma_0^s} q_{|\Gamma_0^s} \ \text{ for all } q \ 2 \ H_\#(\ ^s_0) \text{ (with } \quad > 1=2).$$

The beam equation (2.2) becomes

$$
\begin{array}{rcll}
{}_{tt} \quad {}_{xx} \quad {}_{txx} & = & {}^{s}_{\#}[\ (\mathbf{u};p)(\quad {}_{x}\mathbf{e}_1 + \mathbf{e}_2)\ \ \mathbf{e}_2)] & \text{on } {}^{s;0}_{T}; \\
(\ (0);\ {}_{t}(0)) & = & (\ {}^{1;0};\ {}^{2;0}) & \text{in } {}^{s}_{0}.
\end{array}
\tag{2.3}
$$

Then, we define the Sobolev spaces for the displacement as $H_{\#}(\ {}^{s}_{0}) = H\ \ (\ {}^{s}_{0}) \setminus L^2_{\#;0}(\ {}^{s}_{0})$ and the spaces on ${}^{s;0}_{T}$ as follows

$$H_{\#}^{\ ;}\ (\ {}^{s;0}_{T}) = L^2(0;T;H_{\#}(\ {}^{s}_{0})) \setminus H\ \ (0;T;L^2_{\#;0}(\ {}^{s}_{0})).$$

The pressure is defined in equations (2.1) and (2.3) up to an additive constant. Thus, to obtain the uniqueness of the pressure, we look for the pressure in Sobolev spaces with zero mean value on ${}_0$. That is, we introduce the spaces

$$H_{\#}(\ {}_0) = \quad q \in H_{\#}(\ {}_0) \text{ s.t.} \int_{\Omega_0} q = 0 \qquad \text{for} \qquad 0$$

and we look for pressure p with $(\mathbf{u};p;\ \ )$ solution of (2.1)–(2.3) in the space $L^2 \quad \bigcup_{t\in(0;T)} ftg \quad H^1_{\#} \quad (t) \quad$.

2.2 Main results.

We can now state the main results of this section. They correspond to the global existence of strong solutions for small initial data in Theorem 2.2 or local existence of strong solutions for any initial data in Theorem 2.3.

We want to prove the following results:

Theorem 2.2. *Let* $T > 0$. *Let* $(\mathbf{u}^0;\ {}^{1;0};\ {}^{2;0})$ *be in* $\mathbf{V}^1_{\#}(\quad {}_{1;0}) \quad H^2_{\#}(\ {}^{s}_{0}) \quad H^1_{\#}(\ {}^{s}_{0})$. *There exists* $R > 0$ *such that for any initial data satisfying*

$$k\mathbf{u}^0k^2_{\mathbf{V}^1_{\#}(\Omega\ \ {}_{1;0})} + k\ \ {}^{1;0}k^2_{H^2_{\#}(\Gamma^s_0)} + k\ \ {}^{2;0}k^2_{H^1_{\#}(\Gamma^s_0)} \quad R^2$$

and the compatibility condition

$$\mathbf{u}^0 = \mathbf{0} \quad \textit{on} \qquad \textit{and} \quad \mathbf{u}^0 = \ {}^{2;0}\mathbf{e}_2 \quad \textit{on} \ \ {}^{s}\,{}_{1;0}; \tag{2.4}$$

system (2.1)–(2.3) *has a unique global strong solution* $(\mathbf{u};p;\ \ )$ *in*

$$\mathbf{V}^{2;1}_{\#}(Q_T) \quad L^2 \left(\bigcup_{t\in(0;T)} ftg \quad H^1_{\#} \quad (t) \right) \quad E^0_T.$$

The space E^0_T *is defined by*

$$E^0_T = H^1(0;T;H^2_{\#}(\ {}^{s}_{0})) \setminus H^2(0;T;L^2_{\#;0}(\ {}^{s}_{0})).$$

Theorem 2.3. *Let* $(\mathbf{u}^0;\ {}^{1;0};\ {}^{2;0})$ *be in* $\mathbf{V}^1_{\#}(\quad {}_{1;0}) \quad H^2_{\#}(\ {}^{s}_{0}) \quad H^1_{\#}(\ {}^{s}_{0})$ *satisfying the compatibility condition* (2.4). *There exists a time* $T_0 > 0$ *such that system* (2.1)–(2.3) *has a unique strong solution* $(\mathbf{u};p;\ \ ) \in \mathbf{V}^{2;1}_{\#}(Q_{T_0}) \quad L^2(\bigcup_{t\in(0;T_0)} ftg \quad H^1_{\#}(\quad (t))) \quad E^0_{T_0}$.

The different steps of the proof are quite classical. First, thanks to a change of variables in section 2.3, we set the problem in the fixed cylinder $Q^0_T = (0;T) \quad {}_0$. Then, in section 2.4, we study the linearized system with nonhomogeneous right-hand sides. Finally, by a fixed point procedure set in section 2.5, we are able to prove existence for the nonlinear system set in the fixed cylinder. We conclude the proof thanks to the regularity of the change of variables.

2.3 Change of variables.

We introduce the change of variables

$$
\begin{array}{c}
{}_{(t)} : \quad {}_{(t)} \quad ! \quad {}_0 \\
(x, y) \quad 7! \quad (x, z) = \quad x, \dfrac{y}{1 + \quad (t, x)} \quad :
\end{array}
$$

Following the calculations in [3, 12, 7], system (2.1)–(2.3) becomes

$$
\begin{array}{rcll}
\mathbf{u}_t \quad \mathrm{div} \quad (\mathbf{u}, p) & = & \mathbf{F}[\mathbf{u}, p, \ ] & \text{in } Q_T^0, \\
\mathrm{div}\ \mathbf{u} & = & \mathrm{div}\ \mathbf{w}[\mathbf{u}, \ ] & \text{in } Q_T^0, \\
\mathbf{u} & = & {}_t\mathbf{e}_2 & \text{on } {}_T^{s,0}, \\
\mathbf{u} & = & \mathbf{0} & \text{on } {}_T, \\
{}_{tt} \quad {}_{xx} \quad {}_{txx} & = & {}^s_{\#}(p \quad 2 \quad u_{2,z}) + H[\mathbf{u}, \ ] & \text{on } {}_T^{s,0}, \\
(\mathbf{u}(0), \ (0), \ {}_t(0)) & = & (\mathbf{u}^0, \ {}^{1,0}, \ {}^{2,0}) &
\end{array}
\tag{2.5}
$$

where

$$
\begin{array}{rcl}
\mathbf{F}[\mathbf{u}, p, \ ] & = & \mathbf{u}_t + \ z \ {}_t + \ z \ \dfrac{{}^2_x}{1 + } \quad {}_{xx} \quad \mathbf{u}_z \\
& & + \quad 2z \ {}_x\mathbf{u}_{xz} + \ \mathbf{u}_{xx} + \dfrac{z^2 \ {}^2_x}{1 + }\mathbf{u}_{zz} \\
& & + z(\ {}_x p_z \quad p_x)\mathbf{e}_1 \quad (1 + \ )u_1\mathbf{u}_x + (z \ {}_x u_1 \quad u_2)\mathbf{u}_z, \\
\mathbf{w}[\mathbf{u}, \ ] & = & u_1\mathbf{e}_1 + z \ {}_x u_1\mathbf{e}_2, \\
H[\mathbf{u}, \ ] & = & {}^s_{\#} \quad \dfrac{{}_x}{1 + } u_{1,z} + \ {}_x u_{2,x} \quad \dfrac{{}^2_x \quad 2}{1 + } u_{2,z} \quad :
\end{array}
\tag{2.6}
$$

System (2.5) is equivalent to system (2.1)–(2.3) in the sens of

Definition 2.4. $(\mathbf{u}, p, \ )$ *in* $\mathbf{H}^{2,1}_{\#}(Q_T) \quad L^2(\bigcup_{t\in(0,T)} \{t\} \quad H^1_{\#}(\ {}_{(t)})) \quad E^0_T$ *is solution of* (2.1)–(2.3) *when the following conditions are satisfied:*

(i) $(\hat{\mathbf{u}}, \hat{p}, \ )$ *obtained for the change of variables* $\hat{\mathbf{u}}(x, z) = \mathbf{u}(x, y)$ *and* $\hat{p}(x, z) = p(x, y)$ *with* $z = \frac{y}{1+ \ (t,x)}$ *is a solution of* (2.5),

(ii) for any time t *in* $(0, T)$, *the previous change of variables is a* C^1*-diffeomorphism from* ${}_{(t)}$ *into* ${}_0$,

(iii) *satisfies condition* (1.1).

In [13], the author considers a lifting of both the divergence condition and the nonhomogeneous Dirichlet condition. We introduce here his notation (with some modifications due to the periodic boundary conditions). For $1/2 \quad {}_1 \quad 2$ and ${}_2 \quad 0$, we define

$$
\mathbf{H}^{1, 2}_{\Gamma_0,\Omega_0} = \Big\{ (\mathbf{g}, h) \in \mathbf{H}^{1}_{\#}(\ {}_0) \quad H^{2}_{\#}(\ {}_0) \quad \text{s.t. } \langle \mathbf{g} \quad \mathbf{n}, 1\rangle_{\mathbf{H}^{1}_{\#}(\Gamma_0),\mathbf{H}^{-1}_{\#}(\Gamma_0)} = \int_{\Omega_0} h
$$

and for $1/2 \quad {}_1 \quad 2$ and $1 \quad {}_2 \quad 0$

$$
\mathbf{H}^{1, 2}_{\Gamma_0,\Omega_0} = \quad (\mathbf{g}, h) \in \mathbf{H}^{1}_{\#}(\ {}_0) \quad H^{-2}_{\#}(\ {}_0)' \quad \text{s.t. } \langle \mathbf{g} \quad \mathbf{n}, 1\rangle_{\mathbf{H}^{1}_{\#}(\Gamma_0),\mathbf{H}^{-1}_{\#}(\Gamma_0)} = \langle h, 1\rangle_{(H^{-2}_{\#}(\Omega_0))', H^{-2}_{\#}(\Omega_0)}
$$

Then, for $(\mathbf{g}, h)$ in $\mathbf{H}^{3/2,1}_{\Gamma_0,\Omega_0}$, there exists a unique solution $(\mathbf{z}, \ ) = (L(\mathbf{g}, h), L_p(\mathbf{g}, h))$ in $\mathbf{H}^2_{\#}(\ {}_0) \quad H^1_{\#}(\ {}_0)$ of the following equation:

$$
\mathbf{z} + r \quad = \mathbf{0} \quad \text{and} \quad \mathrm{div}\ \mathbf{z} = h \quad \text{in } {}_0 \quad \text{and} \quad \mathbf{z} = \mathbf{g} \quad \text{on } {}_0.
$$

The liftings L and L_p define two linear operators with more general regularity:

Proposition 2.5 (Corollary 8.4 in [13]). *The operator L is linear and continuous from $\mathbf{H}^{s+1/2,s}_{\Gamma_0,\Omega_0}$ into $\mathbf{H}^{s+1}_{\#}(\Omega_0)$ for all $-1 \le s \le 1$ and the operator L_p is linear and continuous from $\mathbf{H}^{s+1/2,s}_{\Gamma_0,\Omega_0}$ into $H^{s}_{\#}(\Omega_0)$ for all $-1 \le s \le 1$.*

This result will be used in particular for $(\mathbf{g}, h) = (\mathbf{0}, \operatorname{div} \mathbf{w}[\mathbf{u}, \ ])$ with $\mathbf{w}[\mathbf{u}, \ ]$ defined in (2.6). We will see in section 2.5 that, for $(\mathbf{u}, p, \ )$ smooth enough, $\mathbf{w}[\mathbf{u}, \ ]$ belongs to

$$
\begin{aligned}
G_T \quad = \quad \Big\{ & \mathbf{k} \in \mathbf{L}^2_{\#}(Q^0_T) \text{ s.t. } \operatorname{div} \mathbf{k} \in L^2(0, T; \mathbf{H}^1_{\#}(\Omega_0)), \\
& \mathbf{k}_t \in \mathbf{L}^2_{\#}(Q^0_T) \text{ and } \mathbf{k} = 0 \text{ on } \ ^0_T \Big\} .
\end{aligned}
$$

Thus, with $\mathbf{w}[\mathbf{u}, \ ]$ in G_T, we get $\operatorname{div} \mathbf{w}_t[\mathbf{u}, \ ]$ in $L^2(0, T; H^{-1}_{\#}(\Omega_0))$ thanks to first $\mathbf{w}[\mathbf{u}, \ ] = \mathbf{0}$ on $\partial\Omega_0$ and second the property

$$
\begin{aligned}
& \langle \operatorname{div} \mathbf{w}_t[\mathbf{u}, \ ], h \rangle_{H^{-1}_{\#}(\Omega_0), H^1_{\#}(\Omega_0)} \\
= \quad & -\langle \mathbf{w}_t[\mathbf{u}, \ ], \nabla h \rangle_{\mathbf{L}^2_{\#}(\Omega_0), \mathbf{L}^2_{\#}(\Omega_0)} + \langle \mathbf{w}_t[\mathbf{u}, \ ] \cdot \mathbf{n}, h \rangle_{H^{-1/2}_{\#}(\Gamma_0), H^{1/2}_{\#}(\Gamma_0)}
\end{aligned}
$$

for any h in $H^1_{\#}(\Omega_0)$. Then, for $\mathbf{w}[\mathbf{u}, \ ]$ in G_T, we get that $(\mathbf{0}, \operatorname{div} \mathbf{w}[\mathbf{u}, \ ])$ belongs to $L^2\left(0, T; \mathbf{H}^{3/2,1}_{\Gamma_0,\Omega_0}\right) \cap$

H^1 0w[u;] ,Tf]4.926 T-0.997 Td [(0)]TJ/F10 68]TJ9626 Tf 11.505 3.005 Td [(( ) .9738 Tf 3.971 0 Td [(;)]TJ/F7 6.9738

Note that the term $-2\nu v_{2,z}$ vanishes in the right-hand side of the damped wave equation, because $\operatorname{div} \mathbf{v} = v_{1,x} + v_{2,z} = 0$ in Q_T^0 and $v_1 = 0$ on $\Sigma_T^{s,0}$ and for $\mathbf{v}$ in $\mathbf{H}^{2,1}_{\#}(Q_T^0)$, $v_{1,x|\Sigma_T^{s,0}} = 0$. Thus $v_{2,z|\Sigma_T^{s,0}} = 0$.

The compatibility conditions in terms of $(\mathbf{v}^0, \eta^{1,0}, \eta^{2,0})$ are

$$\operatorname{div} \mathbf{v}^0 = 0 \text{ in } \Omega_0, \quad \mathbf{v}^0 = \eta^{2,0}\mathbf{e}_2 \text{ on } \Gamma_0^s \quad \text{and} \quad \mathbf{v}^0 = \mathbf{0} \text{ on } \Gamma. \tag{2.12}$$

That is, in terms of $(\mathbf{u}^0, \eta^{1,0}, \eta^{2,0})$:

$$\begin{aligned} &\operatorname{div} \left(\mathbf{u}^0 - \mathbf{z}[\mathbf{u}^0, \eta^{1,0}]\right) = 0 \quad \text{in } \Omega_0, \\ &\mathbf{u}^0 = \eta^{2,0}\mathbf{e}_2 \quad \text{on } \Gamma_0^s \quad \text{and} \quad \mathbf{u}^0 = \mathbf{0} \quad \text{on } \Gamma. \end{aligned} \tag{2.13}$$

From now on, we can follow the different steps in [7]. We will have to adapt the functional space for η (from $H^{4,2}_{(0)}(\Sigma_T^{s,0})$ when $\beta > 0$ to E_T^0 here) and the proof of existence of solution for the damped wave equation.

Let us begin by proving existence and uniqueness of strong solutions for the linearized system.

2.4 Study of an auxiliary linear system.

In this section, we prove existence and uniqueness of solutions to the system

$$\begin{array}{rcll} \mathbf{v}_t - \operatorname{div} \sigma(\mathbf{v}, q) & = & \mathbf{f} & \text{in } Q_T^0, \\ \operatorname{div} \mathbf{v} & = & 0 & \text{in } Q_T^0, \\ \mathbf{v} & = & \eta_t \mathbf{e}_2 & \text{on } \Sigma_T^{s,0}, \\ \mathbf{v} & = & \mathbf{0} & \text{on } \Sigma_T, \\ \eta_{tt} - \beta\eta_{xx} - \gamma\eta_{txx} & = & \gamma^s_{\#} q + h & \text{on } \Sigma_T^{s,0}, \\ (\mathbf{v}(0), \eta(0), \eta_t(0)) & = & (\mathbf{v}^0, \eta^{1,0}, \eta^{2,0}). & \end{array} \tag{2.14}$$

In system (2.14), the initial data $(\mathbf{v}^0, \eta^{1,0}, \eta^{2,0})$ belongs to $X^0_{\mathbf{cc}}$ where

$$X^0 = \mathbf{H}^1_{\#}(\Omega_0) \times H^2_{\#}(\Gamma_0^s) \times H^1_{\#}(\Gamma_0^s)$$

and

$$X^0_{\mathbf{cc}} = \left\{ (\mathbf{z}^0, \eta^{1,0}, \eta^{2,0}) \in X^0 \text{ s.t. } (\mathbf{z}^0, \eta^{1,0}, \eta^{2,0}) \text{ satisfies (2.12)} \right\}. \tag{2.15}$$

The right-hand side $(\mathbf{f}, h)$ in system (2.14) belongs to

$$Z_T = \mathbf{L}^2_{\#}(Q_T^0) \times L^2(0,T; H^{1/2}_{\#}(\Gamma_0^s)).$$

The main result of this section is the following.

Theorem 2.6. *Let* $(\mathbf{v}^0, \eta^{1,0}, \eta^{2,0})$ *be in* $X^0_{\mathbf{cc}}$ *and* $(\mathbf{f}, h)$ *be in* Z_T*.Then, system* (2.14) *admits a unique solution* $(\mathbf{v}, q, \eta)$ *in*

$$\begin{aligned} X_T = \Big\{ &(\mathbf{z}, r, \eta) \in \mathbf{H}^{2,1}_{\#}(Q_T^0) \times L^2(0,T; H^1_{\#}(\Omega_0)) \times E_T^0 \\ &\text{s.t. } \mathbf{z} = \mathbf{0} \text{ on } \Sigma_T \text{ and } \mathbf{z} = \eta_t\mathbf{e}_2 \text{ on } \Sigma_T^{s,0} \Big\}. \end{aligned} \tag{2.16}$$

We have the estimate

$$\|(\mathbf{v}, q, \eta)\|_{X_T} \leq C\left(\|(\mathbf{v}^0, \eta^{1,0}, \eta^{2,0})\|_{X^0} + \|(\mathbf{f}, h)\|_{Z_T}\right). \tag{2.17}$$

To prove Theorem 2.6, we rewrite system (2.14) using the Leray projection from $\mathbf{L}^2_{\#}(\Omega_0)$ onto

$$\mathbf{V}^0_{\#,\mathbf{n}}(\Omega_0) = \left\{\mathbf{z} \in \mathbf{L}^2_{\#}(\Omega_0) \text{ s.t. } \operatorname{div} \mathbf{z} = 0 \text{ in } \Omega_0 \text{ and } \mathbf{z}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega_0\right\}.$$

More precisely, we split the velocity $\mathbf{v}$ into two parts, namely $\mathbf{v}_e = P\mathbf{v}$ and $\mathbf{v}_s = (I - P)\mathbf{v}$. The velocity $\mathbf{v}_e$ is solution of an evolutionary partial differential equation associated with a pressure term q_e and the velocity $\mathbf{v}_s$ is solution of a stationary partial differential equation assicated to another pressure term q_s.

Thanks to the splitting of system (2.6), we are able to prove the existence of a unique solution to the equivalent system. Then, using the equivalence between the two systems, we can conclude the proof (see section 2.4.3 for details).

2.4.1 Equivalent system.

The Leray projection maps $\mathbf{L}^2_\#(\Omega_0)$ onto $\mathbf{V}^0_{\#,\mathbf{n}}(\Omega_0)$ along $\nabla H^1_\#(\Omega_0)$, that is for every $\mathbf{z}$ in $\mathbf{L}^2_\#(\Omega_0)$, there exists a function $\pi(\mathbf{z})$ in $H^1_\#(\Omega_0)$ such that $(I-P)\mathbf{z} = \nabla\pi(\mathbf{z})$. Furthermore, we can calculate $\pi(\mathbf{z})$ from $\mathbf{z}$. Indeed, taking the divergence and the normal trace in the identity $(I-P)\mathbf{z} = \nabla\pi(\mathbf{z})$, we get

$$\begin{aligned} \operatorname{div}((I-P)\mathbf{z}) &= \operatorname{div}\mathbf{z} = \operatorname{div}(\nabla\pi(\mathbf{z})) = \Delta\pi(\mathbf{z}) && \text{in } \Omega_0, \\ ((I-P)\mathbf{z})\cdot\mathbf{n} &= \mathbf{z}\cdot\mathbf{n} = \nabla\pi(\mathbf{z})\cdot\mathbf{n} = \frac{\partial\pi(\mathbf{z})}{\partial\mathbf{n}} && \text{on } \Gamma_0. \end{aligned}$$

The previous system in $\pi(\mathbf{z})$ is ill-posed because, for $\mathbf{z}$ in $\mathbf{L}^2_\#(\Omega_0)$, the normal trace of $\mathbf{z}$ is not necessary defined. But, we can decompose $\pi(\mathbf{z})$ into $\pi_1(\mathbf{z})$ and $\pi_2(\mathbf{z})$ defined by

$$\begin{aligned} &\Delta\pi_1(\mathbf{z}) = \operatorname{div}\mathbf{z} \text{ in } \Omega_0, \\ &\pi_1(\mathbf{z}) \in H^1_{\#,0}(\Omega_0) = \{r \in H^1_\#(\Omega_0) \text{ s.t. } r = 0 \text{ on } \Gamma_0\} \end{aligned}$$

and

$$\Delta\pi_2(\mathbf{z}) = 0 \text{ in } \Omega_0 \quad \text{and} \quad \frac{\partial\pi_2(\mathbf{z})}{\partial\mathbf{n}} = (\mathbf{z} - \nabla\pi_1(\mathbf{z}))\cdot\mathbf{n} \text{ on } \Gamma_0.$$

We denote by N the operator from $H^\sigma_\#(\Gamma_0)$ into $H^{\sigma+3/2}_\#(\Omega_0)$ (for $\sigma \geq -1/2$) defined for g in $H^\sigma_\#(\Gamma_0)$ (with $\sigma \geq -1/2$) by $Ng = r$ if and only if

$$\Delta r = 0 \text{ in } \Omega_0 \quad \text{and} \quad \frac{\partial r}{\partial\mathbf{n}} = g \text{ on } \Gamma_0.$$

Then, with $\pi_1(\mathbf{z}) = (\Delta_D)^{-1}(\operatorname{div}\mathbf{z})$ and $\pi_2(\mathbf{z}) = N((\mathbf{z} - \nabla\pi_1(\mathbf{z}))\cdot\mathbf{n})$, we get first that

$$\pi_2(\mathbf{z}) = N\left(\left(\mathbf{z} + \nabla(\Delta_D)^{-1}(\operatorname{div}\mathbf{z})\right)\cdot\mathbf{n}\right)$$

and second that

$$\pi(\mathbf{z}) = (\Delta_D)^{-1}(\operatorname{div}\mathbf{z}) + N\left(\left(\mathbf{z} + \nabla(\Delta_D)^{-1}(\operatorname{div}\mathbf{z})\right)\cdot\mathbf{n}\right). \tag{2.18}$$

In the case of $\mathbf{v}$, with $(\mathbf{v}, p, \eta)$ solution of system (2.14), we know that $\operatorname{div}\mathbf{v} = 0$, thus if q is defined by $\nabla q = (I-P)\mathbf{v}$, then $q = N(\mathbf{v}\cdot\mathbf{n}) = N(\eta_t\chi_{\Gamma^s_0})$. We define by N_s the restriction of N to Γ^s_0, that is $N_s = N(\cdot\,\chi_{\Gamma^s_0})$ defined from $H^\sigma_\#(\Gamma^s_0)$ into $H^{\sigma+3/2}_\#(\Omega_0)$ (for $\sigma \geq -1/2$).

Finally, we get that system (2.14) is equivalent to the following one:

$$\begin{array}{rcll} \mathbf{v}_{e,t} - \operatorname{div}\sigma(\mathbf{v}_e, q_e) &=& P\mathbf{f} & \text{in } Q^0_T, \\ \mathbf{v}_e &=& -\nabla N_s(\eta_t) & \text{on } \Sigma^0_T, \\ \mathbf{v}_e(0) &=& P\mathbf{v}^0 & \text{in } \Omega_0, \\ \mathbf{v}_s &=& \nabla N_s(\eta_t) & \text{in } Q^0_T, \\ (I + \rho\gamma^s_\# N_s)\eta_{tt} - \beta\eta_{xx} - \gamma\eta_{txx} &=& \gamma^s_\# q_e + \hat{h} & \text{on } \Sigma^{s,0}_T, \\ (\eta(0), \eta_t(0)) &=& (\eta^{1,0}, \eta^{2,0}) & \\ q &=& q_e + q_s & \text{in } Q^0_T, \\ \mathbf{v} &=& \mathbf{v}_e + \mathbf{v}_s & \text{in } Q^0_T, \\ q_s &=& \pi(\mathbf{f}) - N_s(\eta_{tt}) & \text{in } Q^0_T \end{array} \tag{2.19}$$

where

$$\hat{h} = h + \gamma^s_\#\pi(\mathbf{f})$$

with the operator π from $\mathbf{L}^2_\#(\Omega_0)$ into $H^1_\#(\Omega_0)$ is defined in (2.18). The whole decomposition of system (2.14) into system (2.19) can be found in [7] or in [12] and the decomposition for the Stokes system in [11].

2.4.2 Existence of solution and regularity of each equation separately.

The next proposition gives existence of solution of the damped wave equation with a pressure term $\overline{q}_e$ in the right-hand side in $L^{2-}(0;T;H^{1=2}_{\#}(\ {}^{s}_{0}))$, for $0 < \ < 1$:

Proposition 2.7. *Let* $0 < \ < 1$. *Let* $(\ ^{1;0};\ ^{2;0})$ *be in* $H^2_{\#}(\ ^{s}_{0})\quad H^1_{\#}(\ ^{s}_{0})$ *and* $(\mathbf{f};h)$ *be in* Z_T. *Then, first* $\hat{h}$ *belongs to* $L^2(0;T;H^{1=2}_{\#}(\ ^{s}_{0}))$ *and second, with* $\overline{q}_e$ *in* $L^{2-}(0;T;H^1_{\#}(\ {}_{0}))$, *equation*

$$\begin{array}{rcll} (I + \ ^{s}_{\#}N_s)\ _{tt} \quad _{xx} \quad _{txx} & = & ^{s}_{\#}\overline{q}_e + \hat{h} & \textit{on} \ ^{s;0}_{T}; \\ (\ (0);\ _t(0)) & = & (\ ^{1;0};\ ^{2;0}) & \textit{in} \ ^{s}_{0} \end{array} \tag{2.20}$$

admits a unique solution $\ $ *in*

$$E_T = W^{1;2-}(0;T;H^2_{\#}(\ ^{s}_{0})) \setminus W^{2;2-}(0;T;H^2_{\#}(\ ^{s}_{0})):$$

Furthermore, $\ _t$ *belongs to* $H^{3=2;3=4}_{\#}(\ ^{s;0}_{T})$.

Proof. The first point is obvious by definition of $\hat{h}$. To prove the second point, we write equation (2.20) as a first order system. We define $H_s = H^2_{\#}(\ ^{s}_{0})\quad L^2_{\#;0}(\ ^{s}_{0})$ endowed with the norm

$$k(\ _1;\ _2)k^2_{H_s} = k(\quad _s)\ _1k^2_{L^2_{\#;0}(\Gamma^s_0)} + k\ _2k^2_{L^2_{\#;0}(\Gamma^s_0)} \qquad 8(\ _1;\ _2)\ 2\ H_s:$$

The operator $\ _s$ is a operator with domain $H^2_{\#}(\ ^{s}_{0})$ on $L^2_{\#;0}(\ ^{s}_{0})$ defined by

$$\ _s\ =\ \ _{xx} \text{ for all } \ 2\ D(\ _s) = H^2_{\#}(\ ^{s}_{0}):$$

Setting

$$Y = \quad _t \quad ; \qquad Y^0 = \begin{array}{c} ^{1;0} \\ ^{2;0} \end{array}$$

and defining the operator $A\ _{;}$ with domain $D(A\ _{;}\ ) = H^2_{\#}(\ ^{s}_{0})\quad H^2_{\#}(\ ^{s}_{0})$ on H_s by

$$A\ _{;}\ = \begin{array}{cc} I & 0 \\ 0 & (I + \ ^{s}_{\#}N_s)^{-1} \end{array} \qquad \begin{array}{cc} 0 & I \\ _s & _s \end{array} \ ;$$

equation (2.20) becomes

$$\begin{array}{rcl} Y' & = & A\ _{;}\ Y + \begin{array}{c} 0 \\ (I + \ ^{s}_{\#}N_s)^{-1} \left[^{s}_{\#}\overline{q}_e + \hat{h} \right] \end{array} \\ Y(0) & = & Y^0: \end{array} \tag{2.21}$$

First, using Proposition 2.2 in [14] and Lemma 3.1 in [12], we get that $A\ _{;}$ is a generator of an analytic semigroup on H_s. Then, following exactly the proof of Proposition $5{:}4$ in [7], we get that the solution of system (2.21) can be written with the Duhamel formula:

$$Y(t) = \mathrm{e}^{t\mathcal{A}\ _{;}}\ Y^0 + \int_0^t \mathrm{e}^{(t-\ )\mathcal{A}\ _{;}}\ B(\ )\mathrm{d}$$

with $B = \begin{array}{c} 0 \\ (I + \ ^{s}_{\#}N_s)^{-1} \left[^{s}_{\#}\overline{q}_e + \hat{h} \right] \end{array}$ in $L^{2-}(0;T;D(A^{1=4}_{\ ;}))$ and that Y belongs to

$$L^{2-}(0;T;[D(A^2_{\ ;}\ );D(A\ _{;}\ )]_{1-}\ ) \setminus W^{1;2-}(0;T;[D(A\ _{;}\ );H_s]_{1-}\ ):$$

A simple calculation gives the interpolated spaces (see [9]):

$$\begin{array}{rcl} D(A^2_{\ ;}\ );D(A\ _{;}\ )\ _{1-} & = & (\ _1;\ _2)\ 2\ H^2_{\#}(\ ^{s}_{0})\quad H^2_{\#}(\ ^{s}_{0}) \\ & & \text{s.t.} \quad _1 + \quad _2\ 2\ H^2_{\#}(\ ^{s}_{0})\ ; \\ {[D(A\ _{;}\ );H_s]}_{1-} & = & H^2_{\#}(\ ^{s}_{0})\quad H^2_{\#}(\ ^{s}_{0}): \end{array}$$

That is, η belongs to
$$W^{1,2-}(0,T;H^2_\#(\Sigma^s_0))\cap W^{2,2-}(0,T;H^{2}_\#(\Sigma^s_0))$$
and then η_t belongs to
$$L^{2-}(0,T;H^2_\#(\Sigma^s_0))\cap W^{1,2-}(0,T;H^{2}_\#(\Sigma^s_0)).$$
Two interpolation formula give that η_t belongs to $H^{3/2,3/4}_\#(\Sigma^{s,0}_T)$. Furthermore, we get the expected estimates from the Duhamel formula. $\square$

The regularity of solution of the Stokes problem stays the same as in Proposition 5.5 in [7]. Namely, we look for solution of the Stokes equivalent system:

$$\begin{array}{rcll}
\mathbf{v}_{e,t} - \Delta\mathbf{v}_e + \nabla q_e &=& P\mathbf{f} & \text{in } Q^0_T,\\
\mathbf{v}_e &=& -\mathbf{v}_s & \text{on } \Sigma^0_T,\\
\mathbf{v}_e(0) &=& P\mathbf{v}^0 & \text{in } \Omega_0,\\
\mathbf{v}_s &=& \nabla N_s(g) & \text{in } Q^0_T,\\
\mathbf{v} &=& \mathbf{v}_e+\mathbf{v}_s & \text{in } Q^0_T,\\
q &=& q_s+q_e & \text{in } Q^0_T,\\
q_s &=& (\mathbf{f}) - N_s(g_t) & \text{in } Q^0_T.
\end{array} \tag{2.22}$$

We have the following result

Proposition 2.8. *Let g be in $H^{3/2,3/4}_\#(\Sigma^{s,0}_T)$, $\mathbf{f}$ in $\mathbf{L}^2_\#(Q^0_T)$ and $\mathbf{v}^0$ in $\mathbf{V}^1_\#(\Omega_0)$ with the compatibility condition $\mathbf{v}^0=\mathbf{0}$ on Γ and $\mathbf{v}^0=g(0)\mathbf{e}_2$ on Σ^s_0. Then, (2.22) admits a unique solution $(\mathbf{v}_e,\mathbf{v}_s,q_e)$ in $X^{e,s}_T=\mathbf{V}^{2,1}_\#(Q^0_T)\times L^2(0,T;\mathbf{H}^2_\#(\Omega_0))\cap H^{3/4}(0,T;\mathbf{H}^{1/2}_\#(\Omega_0))\times L^2(0,T;H^1_\#(\Omega_0))$. We have the estimate*

$$\|(\mathbf{v}_e,\mathbf{v}_s,q_e)\|_{X^{e,s}_T}\le c\left(\|\mathbf{v}^0\|_{\mathbf{V}^1_\#(\Omega)}+\|g\|_{H^{3/2,3/4}_\#(\Sigma^{s,0}_T)}+\|\mathbf{f}\|_{\mathbf{L}^2_\#(Q^0_T)}\right).$$

Proof. We refer to Proposition 5.5 in [7]. $\square$

We can now construct the contraction mapping to prove Theorem 2.6.

2.4.3 Contruction of a solution of system (2.14).

In this section, the initial data $(\mathbf{v}^0,\eta^{1,0},\eta^{2,0})$ and the right-hand side $(\mathbf{f},h)$ are fixed respectively in $X^0_{\mathbf{cc}}$ and Z_T.

We consider the mapping G defined by

$$\begin{array}{rccl}
G: & L^2(0,T;H^1_\#(\Omega_0)) & \to & X^{e,s,\eta}_T=X^{e,s}_T\times E_T\\
& \bar q_e & \mapsto & (\mathbf{v}_e,\mathbf{v}_s,q_e,\eta) \text{ solution of system (2.23)}
\end{array}$$

$$\begin{array}{rcll}
\mathbf{v}_{e,t} - \operatorname{div}\sigma(\mathbf{v}_e,q_e) &=& P\mathbf{f} & \text{in } Q^0_T,\\
\mathbf{v}_e &=& -\nabla N_s(\eta_t) & \text{on } \Sigma^0_T,\\
\mathbf{v}_e(0) &=& P\mathbf{v}^0 & \text{in } \Omega_0,\\
\mathbf{v}_s &=& \nabla N_s(\eta_t) & \text{in } Q^0_T,\\
(I+\gamma^s_\# N_s)\eta_{tt} - \eta_{xx} - \eta_{txx} &=& \gamma^s_\#\bar q_e + h & \text{on } \Sigma^{s,0}_T,\\
(\eta(0),\eta_t(0)) &=& (\eta^{1,0},\eta^{2,0}) & \\
q &=& q_e+q_s & \text{in } Q^0_T,\\
\mathbf{v} &=& \mathbf{v}_e+\mathbf{v}_s & \text{in } Q^0_T,\\
q_s &=& (\mathbf{f}) - N_s(\eta_{tt}) & \text{in } Q^0_T.
\end{array} \tag{2.23}$$

We have the following result.

Proposition 2.9. *The mapping* G *is well-defined from* $L^2(0;T;H^1_\#(\ _0))$ *into* $X_T^{e;s;}$. *Moreover, we have the estimate, for* $\ = \frac{1}{2-} \quad \frac{1}{2} > 0$:

$$kG(\overline{q}_e)k_{X_T^{e;s;}} \qquad C \ \ k(\mathbf{v}^0; \ ^{1;0}; \ ^{2;0})k_{X^{0;"}} + k(\mathbf{f};h)k_{Z_T} + T \ \ k\overline{q}_e k_{L^2(0;T;\mathcal{H}^1_\#(\Omega_0))} \ : \tag{2.24}$$

Furthermore, for two pressures $\overline{q}_{e;1}$ *and* $\overline{q}_{e;2}$ *in* $L^2(0;T;H^1_\#(\ _0))$, *the term* $G(\overline{p}_{e;1}) \quad G(\overline{p}_{e;2}) = (\mathbf{v}_{e;1} \quad \mathbf{v}_{e;2};\mathbf{v}_{s;1} \quad \mathbf{v}_{s;2};q_{e;1} \quad q_{e;2}; \ _1 \quad \ _2)$ *is the solution of the system corresponding with* $G(\overline{q}_{e;1} \quad \overline{q}_{e;2})$ *in* $(2{:}23)$ *with zero for initial data and right-hand sides. Moreover,* $G(\overline{q}_{e;1}) \quad G(\overline{q}_{e;2})$ *satisfies the estimate*

$$kG(\overline{q}_{e;1}) \quad G(\overline{q}_{e;2})k_{X_T^{e;s;}} \qquad cT \ \ k\overline{q}_{e;1} \quad \overline{q}_{e;2}k_{L^2(0;T;\mathcal{H}^1_\#(\Omega_0))}:$$

From the mapping G, we define another mapping F from $L^2(0;T;H^1(\ ))$ into itself defined by $F = P \quad G$ where P is the projection from $X_T^{e;s;}$ into $L^2(0;T;H^1_\#(\ _0))$ definen from $\qquad X^{e;s;}$;

Thus, if $Y^0 = (\ ^{1;0};\ ^{2;0})^{\mathbf{tr}}$ belongs to $[D(A\ _{;}\ ); H_s]_{1=2}$, f belongs to $L^2(\ ^{s;0}_T)$, we know that equation (2.21) admits a unique solution $Y = (\ ;\ _t)^{\mathbf{tr}}$ in the space

$$L^2(0;T;D(A\ _{;}\ )) \setminus H^1(0;T;H_s) \setminus C([0;T];[D(A\ _{;}\ );H_s]^{1=2}):$$

Thanks to the calculations above, $D(A\ _{;}\ ) = H^2_{\#}(\ ^s_0)\quad H^2_{\#}(\ ^s_0)$ on $H_s = H^2_{\#}(\ ^s_0)\quad L^2_{\#;0}(\ ^s_0)$ and the solution $Y = (\ ;\ _t)^{\mathbf{tr}}$ of (2.21) belongs to

$$\begin{aligned} &L^2(0;T;H^2_{\#}(\ ^s_0)\quad H^2_{\#}(\ ^s_0)) \setminus H^1(0;T;H^2_{\#}(\ ^s_0)\quad L^2_{\#;0}(\ ^s_0)) \\ \setminus\quad &C^0([0;T];H^2_{\#}(\ ^s_0)\quad H^1_{\#}(\ ^s_0)); \end{aligned}$$

that is, equation (2.20) admits a solution in the space

$$H^1(0;T;H^2_{\#}(\ ^s_0)) \setminus H^2(0;T;L^2_{\#;0}(\ ^s_0)):$$

$\square$

2.5 Proof of Theorems 2.2 and 2.3.

We now have all the tools to prove the main results of this section. We first use a fixed point procedure to prove existence ans uniqueness of system (2.5) in the fixed cylinder Q^0_T. Then, using Definition 2.4, we prove the existence and uniqueness of system (2.1)–(2.3).

2.5.1 In the cylindrical domain $Q^0_T = (0;T)\quad _0$.

We use a second fixed point procedure. First, we have to estimate $(\mathbf{F}[\mathbf{u};p;\ ];\mathbf{w}[\mathbf{u};\ ];h[\mathbf{u};\ ])$ (defined in (2.6)) in terms of $(\mathbf{u};p;\ )$ in X_T. Namely, we have the following result:

Proposition 2.13. *Let* $(\mathbf{u};p;\ )$ *be in* X_T*, defined in* (2.16)*, then the triplet* $(\mathbf{F}[\mathbf{u};p;\ ];\mathbf{w}[\mathbf{u};\ ];H[\mathbf{u};\ ])$*, obtained from* $(\mathbf{u};p;\ )$ *in* (2.6)*, belongs to*

$$W_T = \mathbf{L}^2_{\#}(Q^0_T)\quad G_T\quad L^2(0;T;H^{1=2}_{\#}(\ ^s_0)):$$

Furthermore, there exists $\ > 0$ *such that*

$$k(\mathbf{F}[\mathbf{u};p;\ ];\mathbf{w}[\mathbf{u};\ ];H[\mathbf{u};\ ])k_{W_T}\quad c_2T\ \ (1 + k(\mathbf{u};p;\ )k_{X_T})k(\mathbf{u};p;\ )k^2_{X_T}: \tag{2.25}$$

Let $(\mathbf{u}_1;p_1;\ _1)$ *and* $(\mathbf{u}_2;p_2;\ _2)$ *be two triplets in* X_T *such that for* $i = 1;2$, $k(\mathbf{u}_i;p_i;\ _i)k_{X_T}\quad R$ *for some* $R > 0$*, we get*

$$k(\mathbf{F}_1;\mathbf{w}_1;H_1)\quad (\mathbf{F}_2;\mathbf{w}_2;H_2)k_{W_T}\quad C(1+R)RT\ \ k(\mathbf{u}_1;p_1;\ _1)\quad (\mathbf{u}_2;p_2;\ _2)k_{X_T}$$

with the notations $(\mathbf{F}_i;\mathbf{w}_i;H_i) = (\mathbf{F}[\mathbf{u}_i;p_i;\ _i];\mathbf{w}[\mathbf{u}_i;\ _i];H[\mathbf{u}_i;\ _i])$.

Proof. The proof relies on the Lemmas $6{:}2$ and $6{:}3$ in [7]. More precisely, the smoothness of gives the good estimates of the different products. Indeed, in E^0_T gives by interpolation

$$2\ H^{1+}\quad 0;T;H^{2(1-\ )}_{\#}(\ ^s_0)\ \ ;\quad 0 < \ < 1:$$

Then,

$$\begin{aligned} 2\ C([0;T];H^2_{\#}(\ ^s_0)); &\qquad _x\ 2\ L^{\infty}(\ ^{s;0}_T); \\ _{xx}\ 2\ H^1(0;T;L^2_{\#;0}(\ ^s_0)) &\qquad _{tx}\ 2\ L^{\infty}(0;T;L^2_{\#;0}(\ ^s_0)): \end{aligned}$$

As already mentionned, we have to prove that $\mathbf{w}[\mathbf{u};\ ]$ satisfies

$$\text{div}\ \mathbf{w}[\mathbf{u};\ ]\ 2\ L^2(0;T;H^1_{\#}(\ _0)); \qquad \mathbf{w}_t[\mathbf{u};\ ]\ 2\ \mathbf{L}^2_{\#}(Q^0_T):$$

The worst term to estimate in all the different calculations is $_{xx}u_{1;z}$. Using that (respectively $u_{1;z}$) belongs to $H^1(0;T;L^2_{\#;0}(\ ^s_0))$ (respectively to $H^1_{\#}(\ _0) = L^2_{\#}(0;L;H^1(0;1)) \setminus H^1_{\#}(0;L;L^2(0;1))$), we get that

$$k\ \ _{xx}u_{1;z}k_{L^2_{\#}(\Omega_0)}\quad Ck\ \ _{xx}k_{L^2_{\#}(\Gamma^s_0)}ku_{1;z}k_{H^1_{\#}(0;L;L^2(0;1))}:$$

Then, because η_{xx} belongs to $H^1(0,T;L^2_{\#,0}(\Gamma^s_0))$ and $H^1(0,T) \hookrightarrow L^\infty(0,T)$, we have

$$\|\eta_{xx}\|_{L^\infty(0,T;L^2_{\#,0}(\Gamma^s_0))} \leq CT^\theta\|\eta\|_{H^1(0,T;L^2_{\#,0}(\Gamma^s_0))}, \qquad \text{for } \theta > 0.$$

This gives

$$\begin{aligned}\|\eta_{xx}u_{1,z}\|_{L^2_\#(Q^0_T)} &\leq CT^\theta\|\eta\|_{H^1(0,T;L^2_{\#,0}(\Gamma^s_0))}\|u_{1,z}\|_{L^2(0,T;H^1_\#(\Omega_0))}\\ &\leq CT^\theta\|(\mathbf{u},p,\eta)\|^2_{X_T}.\end{aligned}$$

The other terms can be estimates using the classic Sobolev embeddings. □

With this proposition, we follow exactly the proof of the fixed point procedure in [7]. Namely, we now state

Proposition 2.14. *For $(\bar{\mathbf{u}},\bar p,\bar\eta)$ in X_T, system (2.5) with right-hand sides $(\mathbf{F}[\bar{\mathbf{u}},\bar p,\bar\eta],\mathbf{w}[\bar{\mathbf{u}},\bar\eta],H[\bar{\mathbf{u}},\bar\eta])$ and initial data $(\mathbf{u}^0,\eta^{1,0},\eta^{2,0})$ in X^0 satisfying (2.13) admits a unique solution $(\mathbf{u},p,\eta)$ in X_T with the estimate*

$$\|(\mathbf{u},p,\eta)\|_{X_T} \leq c_1(\|(\mathbf{u}^0,\eta^{1,0},\eta^{2,0})\|_{X^0} + c_2T^\theta(1+\|(\bar{\mathbf{u}},\bar p,\bar\eta)\|_{X_T})\|(\bar{\mathbf{u}},\bar p,\bar\eta)\|^2_{X_T})$$

where $\theta > 0$ is defined in Proposition 2.13. In other terms, we can construct a mapping $\mathcal{X}_T$:

$$\begin{array}{ccl} X_T & \longrightarrow & X_T \\ (\bar{\mathbf{u}},\bar p,\bar\eta) & \longmapsto & \begin{array}{l}\mathcal{X}_T(\bar{\mathbf{u}},\bar p,\bar\eta) = (\mathbf{u},p,\eta) \text{ is a solution of the system (2.5)}\\ \text{with } (\mathbf{F}[\bar{\mathbf{u}},\bar p,\bar\eta],\mathbf{w}[\bar{\mathbf{u}},\bar\eta],H[\bar{\mathbf{u}},\bar\eta]) \text{ for right-hand sides.}\end{array}\end{array}$$

which satisfies

$$\begin{aligned}&\|\mathcal{X}_T(\bar{\mathbf{u}},\bar p,\bar\eta)\|_{X_T}\\ &\leq c_1\left(\|(\mathbf{u}^0,\eta^{1,0},\eta^{2,0})\|_{X^0} + c_2T^\theta(1+\|(\bar{\mathbf{u}},\bar p,\bar\eta)\|_{X_T})\|(\bar{\mathbf{u}},\bar p,\bar\eta)\|^2_{X_T}\right).\end{aligned} \tag{2.26}$$

Proof. The proof relies directly on Theorem 2.6 and Proposition 2.13. Indeed, from Proposition 2.13, we know that for a triplet $(\bar{\mathbf{u}},\bar p,\bar\eta)$ in X_T, the triplet $(\mathbf{F}[\bar{\mathbf{u}},\bar p,\bar\eta],\mathbf{w}[\bar{\mathbf{u}},\bar\eta],H[\bar{\mathbf{u}},\bar\eta])$ belongs to W_T and in particular that $\mathbf{w}[\bar{\mathbf{u}},\bar\eta]$ belongs to G_T. Thus first, the lifting $(\mathbf{z}[\bar{\mathbf{u}},\bar\eta],\pi[\bar{\mathbf{u}},\bar\eta])$ belongs to $\mathbf{H}^{2,1}_\#(Q^0_T) \times L^2(0,T;H^1_\#(\Omega_0))$ and that $(\mathbf{z}[\mathbf{u}^0,\eta^{1,0}]) = \mathbf{z}[\mathbf{u},\eta](0)$ is well-defined thanks to the embedding $\mathbf{H}^{2,1}_\#(Q^0_T) \hookrightarrow C([0,T];\mathbf{H}^1_\#(\Omega_0))$ and satisfies the estimates

$$\|\mathbf{z}[\bar{\mathbf{u}},\bar\eta]\|_{\mathbf{H}^{2,1}_\#(Q^0_T)} + \|\pi[\bar{\mathbf{u}},\bar\eta]\|_{L^2(0,T;\mathcal{H}^1_\#(\Omega_0))} \leq C\|\mathbf{w}[\bar{\mathbf{u}},\bar\eta]\|_{G_T}.$$

and

$$\|\mathbf{z}[\mathbf{u}^0,\eta^{1,0}]\|_{\mathbf{H}^1_\#(\Omega_0)} \leq C\|(\mathbf{u}^0,\eta^{1,0},\eta^{2,0})\|_{X^0}.$$

Second, the couple $(\mathbf{f}[\bar{\mathbf{u}},\bar\eta],h[\bar{\mathbf{u}},\bar\eta])$ defined in (2.9) and (2.10) belongs to Z_T and with the initial data $(\mathbf{u}^0,\eta^{1,0},\eta^{2,0})$ in X^0 satisfying (2.13), we get that $(\mathbf{v}^0,\eta^{1,0},\eta^{2,0})$, with $\mathbf{v}^0$ defined in (2.11), belongs to $X^0_{\mathbf{cc}}$. We now apply Theorem 2.6 to obtain a unique solution $(\mathbf{v},q,\eta)$ of system (2.8) with $(\mathbf{f}[\bar{\mathbf{u}},\bar\eta],h[\bar{\mathbf{u}},\bar\eta])$ for right-hand side and $(\mathbf{v}^0,\eta^{1,0},\eta^{2,0})$ for initial data. Then, the correspondance between systems (2.8) and (2.5) gives that system (2.5) admits a unique solution $(\mathbf{u},p,\eta)$ in X_T satsifying the estimate

$$\|(\mathbf{u},p,\eta)\|_{X_T} \leq C_1\left(\|(\mathbf{u}^0,\eta^{1,0},\eta^{2,0})\|_{X^0} + \|(\mathbf{F}[\bar{\mathbf{u}},\bar p,\bar\eta],\mathbf{w}[\bar{\mathbf{u}},\bar\eta],H[\bar{\mathbf{u}},\bar\eta])\|_{W_T}\right).$$

Now, using the estimate in Proposition 2.13, we get the estimate of the proposition. □

Then, we can write Theorems 2.2 and 2.3 in the fixed domain, acting in (2.26) either on the time of existence or on the smallness of the initial data.

Proposition 2.15. *Let $(\mathbf{u}^0,\eta^{1,0},\eta^{2,0})$ be in X^0 satisifying (2.13).*

(i) There exists a time $T_0 > 0$ such that system (2.5) admits a unique local strong solution $(\mathbf{u},p,\eta)$ in X_{T_0}.

(ii) There exists r small enough such that, under condition

$$\|(\mathbf{u}^0,\eta^{1,0},\eta^{2,0})\|_{X^0} \leq r,$$

system (2.5) admits a unique global strong solution $(\mathbf{u},p,\eta)$ in X_T.

2.5.2 In the domain Q_T.

The regularity of the solution $(\mathbf{u}, p, \eta)$ of system (2.5) in both cases of Proposition 2.15 gives that the change of variables

$$\Phi_{(t)}^{-1} : \begin{array}{ccc} \Omega_0 & \to & \Omega_{(t)} \\ (x, z) & \mapsto & (x, y) = (x, (1 + \eta(t, x))z) \end{array}$$

is a C^1 diffeomorphism from Ω_0 into $\Omega_{(t)}$ because η is smooth and satisfies condition (1.1). Indeed, by a Sobolev embedding, we have

$$\|\eta\|_{L^\infty(\Sigma_T^{s,0})} \leq c\|\eta\|_{\mathcal{E}_T^0}$$

and up to a change of T_0 in the first case of Proposition 2.15 or a change to r in the second case, we can always prescribe $\|\eta\|_{\mathcal{E}_T^0} \leq \frac{1-\delta_0}{c}$ and thus

$$\|\eta\|_{L^\infty(\Sigma_T^{s,0})} < 1 - \delta_0,$$

that is η satisfying assumption (1.1).

All these results set us in the case of Definition 2.4.

3 The three dimensional case.

In this section, we consider the corresponding three dimensional system of system (2.1)–(2.3). That is, from now on, for L_1, $L_2 > 0$, we define $\omega_0 = \mathbb{R}/L_1\mathbb{Z} \times \mathbb{R}/L_2\mathbb{Z}$. Then, we define the fixed domain $\Omega_0 = \omega_0 \times (0, 1)$. The boundary Γ_0 of the domain Ω_0 is split into two parts, the reference state of the membrane $\Gamma_0^s = \omega_0 \times \{1\}$ and the bottom part $\Gamma = \omega_0 \times \{0\}$, see Figure 2. For a displacement η, we define in the same way as in section 2 the moving part of the boundary:

$$\Gamma^s_{(t)} = \left\{ (x, y, z) \in \mathbb{R}^3 \text{ s.t. } (x, y) \in \omega_0 \quad \text{and} \quad z = 1 + \eta(t, x, y) \right\}.$$

Then, the domain occupied by the fluid at time $t \geq 0$ is

$$\Omega_{(t)} = \left\{ (x, y, z) \in \mathbb{R}^3 \text{ s.t. } (x, y) \in \omega_0 \quad \text{and} \quad 0 < z < 1 + \eta(t, x, y) \right\}.$$

The displacement η has to satisfy the condition (1.1) too. With this change of notations, we can keep the definitions, for $T > 0$,

$$Q_T = \bigcup_{t\in(0,T)} \{t\} \times \Omega_{(t)}, \quad \Sigma_T^{s} = \bigcup_{t\in(0,T)} \{t\} \times \Gamma^s_{(t)}, \quad \Sigma_T^0 = (0, T) \times \Gamma_0,$$
$$Q_T^0 = (0, T) \times \Omega_0, \quad \Sigma_T^{s,0} = (0, T) \times \Gamma_0^s, \quad \Sigma_T = (0, T) \times \Gamma.$$

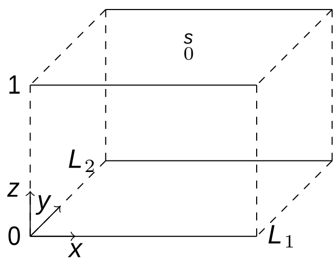

Figure 2: The domain Ω_0 in three dimensions.

Then, the system in this setting becomes

$$
\begin{aligned}
\mathbf{u}_t - \operatorname{div} \sigma(\mathbf{u}, p) + (\mathbf{u} \cdot \nabla)\mathbf{u} &= \mathbf{0} && \text{in } Q_T, \\
\operatorname{div} \mathbf{u} &= 0 && \text{in } Q_T, \\
\mathbf{u} &= \eta_t \mathbf{e}_3 && \text{on } \Sigma^{s,\eta}_T, \\
\mathbf{u} &= \mathbf{0} && \text{on } \Sigma_T, \\
\eta_{tt} - \beta \Delta_s \eta - \gamma \Delta_s \eta_t &= \mathcal{F}^s_{\#}\left[\sigma(\mathbf{u}, p)\hat{\mathbf{n}} \cdot \mathbf{e}_3 \right] && \text{on } \Sigma^{s,0}_T, \\
(\mathbf{u}(0), \eta(0), \eta_t(0)) &= (\mathbf{u}^0, \eta^{1,0}, \eta^{2,0})
\end{aligned}
\tag{3.1}
$$

Here, $\mathcal{F}^s_{\#}$ is the trace function from $H^{\sigma}_{\#}(\Omega_0)$ into $H^{-1/2}_{\#}(\Gamma^s_0)$ for $\sigma > 1/2$ (see section 2 for details). The operator Δ_s is the Laplace operator on Γ^s_0 defined by $D(\Delta_s) = H^2_{\#}(\Gamma^s_0)$ on $L^2_{\#,0}(\Gamma^s_0)$ by $\Delta_s \eta = \frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2}$ for all η in $D(\Delta_s)$. The vector $\hat{\mathbf{n}}$ is defined from the unit normal $\mathbf{n} = \frac{1}{\sqrt{1+\eta_x^2+\eta_y^2}}(-\eta_x \mathbf{e}_1 - \eta_y \mathbf{e}_2 + \mathbf{e}_3)$ to $\Gamma(t)$ outward $\Omega(t)$ as follows

$$
\hat{\mathbf{n}} = \sqrt{1 + \eta_x^2 + \eta_y^2}\,\mathbf{n} = -\eta_x \mathbf{e}_1 - \eta_y \mathbf{e}_2 + \mathbf{e}_3.
$$

Note that, the spaces $L^2_{\#,0}(\Gamma^s_0)$ and $H^{\sigma}_{\#}(\Gamma^s_0)$ have been defined in section 2 in the one dimensional periodic case. Here they are the generalization to the two dimensions periodic space (in the variables x and y). Thus, $L^2_{\#,0}(\Gamma^s_0)$ is the space of all periodic (in the variables x and y) functions in $L^2(\Gamma^s_0)$ of zero mean value on Γ^s_0 and the spaces $H^{\sigma}_{\#}(\Gamma^s_0)$ are defined as $H^{\sigma}(\Gamma^s_0) \cap L^2_{\#,0}(\Gamma^s_0)$.

3.1 Main results.

The aim of this section is to prove the same alternative for system (3.1) than for system (2.1)–(2.3), that is either existence and uniqueness of local in time strong solution for any initial data or existence and uniqueness of global in time on $[0, T]$ ($T > 0$ fixed) strong solution for small initial data. We can now state these results.

Theorem 3.1. *Let* $(\mathbf{u}^0, \eta^{1,0}, \eta^{2,0}) \in \mathbf{V}^1_{\#}(\Omega_{\eta^{1,0}}) \times H^{5/2}_{\#}(\Gamma^s_0) \times H^{3/2}_{\#}(\Gamma^s_0)$*. There exists* $R > 0$ *such that for any initial data satisfying*

$$
\|\mathbf{u}^0\|^2_{\mathbf{V}^1_{\#}(\Omega_{\eta^{1,0}})} + \|\eta^{1,0}\|^2_{H^{5/2}_{\#}(\Gamma^s_0)} + \|\eta^{2,0}\|^2_{H^{3/2}_{\#}(\Gamma^s_0)} \leq R^2
$$

and the compatibility condition

$$
\mathbf{u}^0 = \mathbf{0} \quad \text{on } \Gamma \quad \text{and} \quad \mathbf{u}^0 = \eta^{2,0}\mathbf{e}_3 \quad \text{in } \Gamma^s_{\eta^{1,0}}, \tag{3.2}
$$

system (3.1) *has a unique global strong solution* $(\mathbf{u}, p, \eta)$ *in*

$$
\mathbf{V}^{2,1}_{\#}(Q_T) \times L^2\left(\bigcup_{t \in (0,T)} \{t\} \times H^1_{\#}(\Omega(t)) \right) \times E_T
$$

where

$$
E_T = H^1\left(0, T; H^{5/2}_{\#}(\Gamma^s_0)\right) \cap H^2\left(0, T; H^{1/2}_{\#}(\Gamma^s_0)\right). \tag{3.3}
$$

Theorem 3.2. *Let* $(\mathbf{u}^0, \eta^{1,0}, \eta^{2,0}) \in \mathbf{V}^1_{\#}(\Omega_{\eta^{1,0}}) \times H^{5/2}_{\#}(\Gamma^s_0) \times H^{3/2}_{\#}(\Gamma^s_0)$ *satisfying the compatibility condition* (3.2)*. There exists a time* $T_0 > 0$ *such that system* (3.1) *has a unique strong solution* $(\mathbf{u}, p, \eta)$ *in*

$$
\mathbf{V}^{2,1}_{\#}(Q_{T_0}) \times L^2\left(\bigcup_{t \in (0,T_0)} \{t\} \times H^1_{\#}(\Omega(t)) \right) \times E_{T_0}.
$$

The proof of these results is quite the same as in the previous section where Theorems 2.2 and 2.3 are the equivalent results in the two dimensional case. Namely, we use a change of variables to set the problem in the fixed cylinder Q^0_T. Then, we prove existence and uniqueness of solution for the linearized system. Next, using a fixed point procedure, we prove the existence and uniqueness of the nonlinear system in the fixed domain. Finally, the regularity of the change of variables give the solution in the moving domain (in the sense of Definition 2.4).

3.2 Change of variables.

As introduced in [3], the change of variables is very simple due to the special form of the domain. Namely, we have

$$\overset{(t)}{} : \quad \overset{(t)}{} \quad \overset{!}{} \quad \overset{0}{}$$
$$(x; y; z) \quad 7\,! \quad (x; y; z_0) \quad \text{where} \quad z_0 = \frac{z}{1 + \quad (t; x; y)}:$$

Then we can calculate the derivatives of $f(x; y; z)$ using the derivatives of $\hat{f}(x; y; z_0)$:

$$f_t = \hat{f}_t \quad z_0 \frac{t}{1 +} \hat{f}_{z_0}; \qquad f_x = \hat{f}_x \quad z_0 \frac{x}{1 +} \hat{f}_{z_0}; \qquad f_y = \hat{f}_y \quad z_0 \frac{y}{1 +} \hat{f}_{z_0}$$

$$f_z = \frac{1}{1 +} \hat{f}_{z_0}; \qquad f_{zz} = \frac{1}{(1 + \quad)^2} \hat{f}_{z_0 z_0};$$

$$f_{xx} = \hat{f}_{xx} \quad 2z_0 \frac{x}{1 +} \hat{f}_{x z_0} + \quad z_0 \frac{x}{1 +} \quad ^2 \hat{f}_{z_0 z_0} \quad z_0 \frac{(1 + \quad) \quad _{xx} \quad \, ^2_x}{(1 + \quad)^2} \hat{f}_{z_0};$$

$$f_{yy} = \hat{f}_{yy} \quad 2z_0 \frac{y}{1 +} \hat{f}_{y z_0} + \quad z_0 \frac{y}{1 +} \quad ^2 \hat{f}_{z_0 z_0} \quad z_0 \frac{(1 + \quad) \quad _{yy} \quad \, ^2_y}{(1 + \quad)^2} \hat{f}_{z_0}:$$

With this formulas, we can now state the Navier-Stokes equations in the cylindrical domain Q^0_T. The method is to multiply the equation by $1 +$ and to put all the nonlinear terms in the right-hand side. Let us write the different terms first:

$$\begin{aligned}
\mathbf{u}_t &= \hat{\mathbf{u}}_t \quad \frac{z_0 \quad _t}{1 +} \hat{\mathbf{u}}_{z_0}; \\
\mathbf{u} &= \hat{\mathbf{u}}_{xx} + \hat{\mathbf{u}}_{yy} + \frac{1}{(1 + \quad)^2} \hat{\mathbf{u}}_{z_0 z_0} \quad \frac{2z_0}{1 +} [\, _x \hat{\mathbf{u}}_{x z_0} + \quad _y \hat{\mathbf{u}}_{y z_0}] \\
&\quad + \frac{z_0^2}{(1 + \quad)^2} \quad ^2_x + \, ^2_y \quad \hat{\mathbf{u}}_{z_0 z_0} \\
&\quad \frac{z_0}{1 +} \quad (1 + \quad)(\, _{xx} + \quad _{yy}) \quad (\, ^2_x + \, ^2_y) \quad \hat{\mathbf{u}}_{z_0}; \\
(\mathbf{u} \quad r)\mathbf{u} &= \hat{u}_1 \quad \hat{\mathbf{u}}_x \quad \frac{z_0 \quad _x}{1 +} \hat{\mathbf{u}}_{z_0} \quad + \hat{u}_2 \quad \hat{\mathbf{u}}_y \quad \frac{z_0 \quad _y}{1 +} \hat{\mathbf{u}}_{z_0} \quad + \hat{u}_3 \quad \frac{1}{1 +} \hat{\mathbf{u}}_{z_0} \\
r p &= \hat{p}_x \quad \frac{z_0 \quad _x}{1 +} \hat{p}_{z_0} \quad \mathbf{e}_1 + \quad \hat{p}_y \quad \frac{z_0 \quad _y}{1 +} \hat{p}_{z_0} \quad \mathbf{e}_2 + \quad \frac{1}{1 +} \hat{p}_{z_0} \quad \mathbf{e}_3:
\end{aligned}$$

Then, equation $\mathbf{u}_t \qquad \mathbf{u} + (\mathbf{u} \quad r)\mathbf{u} + \; r p = \mathbf{0}$ becomes

$$\begin{aligned}
&\hat{\mathbf{u}}_t \qquad \hat{} \, \hat{\mathbf{u}} + \hat{r} \hat{p} \\
= &\qquad \hat{\mathbf{u}}_t + \quad (\hat{\mathbf{u}}_{xx} + \hat{\mathbf{u}}_{yy}) \quad \frac{}{1 +} \hat{\mathbf{u}}_{z_0 z_0} + \frac{z_0^2}{1 +} (\, ^2_x + \, ^2_y) \hat{\mathbf{u}}_{z_0 z_0} \\
&\quad 2 \; z_0 (\, _x \hat{\mathbf{u}}_{x z_0} + \quad _y \hat{\mathbf{u}}_{y z_0}) \quad (1 + \quad) \hat{u}_1 \hat{\mathbf{u}}_x \quad (1 + \quad) \hat{u}_2 \hat{\mathbf{u}}_y \quad \hat{u}_3 \hat{\mathbf{u}}_{z_0} \\
&\quad + \; z_0 (\, _x \hat{u}_1 + \, _y \hat{u}_2) \qquad z_0 (1 + \quad)(\, _{xx} + \quad _{yy}) + \quad z_0 (\, ^2_x + \, ^2_y) \quad \hat{\mathbf{u}}_{z_0} \\
&\quad + z_0 \; _t \hat{\mathbf{u}}_{z_0} + \quad z_0 \; _x \hat{p}_{z_0} \qquad \hat{p}_x \; \mathbf{e}_1 + \quad z_0 \; _y \hat{p}_{z_0} \qquad \hat{p}_y \; \mathbf{e}_2:
\end{aligned}$$

In the same way, equation $\operatorname{div} \mathbf{u} = 0$ becomes

$$\begin{aligned}
\widehat{\operatorname{div}} \; \hat{\mathbf{u}} &= \qquad (\hat{u}_{1;x} + \hat{u}_{2;y}) + z_0 (\, _x \hat{u}_{1;z_0} + \quad _y \hat{u}_{2;z_0}) \\
&= \widehat{\operatorname{div}} \; \hat{\mathbf{w}}[\hat{\mathbf{u}}; \quad];
\end{aligned}$$

with

$$\hat{\mathbf{w}}[\hat{\mathbf{u}}; \quad] = \qquad \hat{u}_1 \mathbf{e}_1 \qquad \hat{u}_2 \mathbf{e}_2 + z_0 (\, _x \hat{u}_1 + \quad _y \hat{u}_2) \mathbf{e}_3:$$

Finally, the right-hand side of the plate equation becomes

$$
\begin{aligned}
& (\mathbf{u}; p)\ \sqrt{1 + \ {}^2_x + \ {}^2_y}\,\mathbf{n}\ \ \mathbf{e}_3 \\
= \ & p \quad 2\ u_{3;z} + \quad {}_x\ \Big[u_{1;z} + u_{3;x}\Big] + \quad {}_y\ \Big[u_{2;z} + u_{3;y}\Big] \\
= \ & \hat{p} \quad 2\ \hat{u}_{3;z} + \frac{}{1 + \ } \Big[2 \quad (\ {}^2_x + \ {}^2_y)\ \hat{u}_{3;z_0} + \frac{x}{1 + \ }\hat{u}_{1;z_0} + \frac{y}{1 + \ }\hat{u}_{2;z_0} \\
& + \quad {}_x\hat{u}_{3;x} + \quad {}_y\hat{u}_{3;y} \Big] :
\end{aligned}
$$

Let us drop out the notation ^. The Navier-Stokes equations in the cylindrical domain Q^0_T are

$$
\begin{array}{rcll}
\mathbf{u}_t \quad \text{div} \quad (\mathbf{u}; p) & = & \mathbf{F}[\mathbf{u}; p; \ ] & \text{in } Q^0_T; \\
\text{div } \mathbf{u} & = & \text{div } \mathbf{w}[\mathbf{u}; \ ] & \text{in } Q^0_T; \\
\mathbf{u} & = & \ {}_t\mathbf{e}_3 & \text{on } \ {}^{s;0}_T; \\
\mathbf{u} & = & \mathbf{0} & \text{on } \ {}_T; \\
\mathbf{u}(0) & = & \mathbf{u}^0 & \text{in } \ {}_0
\end{array}
\tag{3.4}
$$

where the right-hand sides $\mathbf{F}[\mathbf{u}; p; \ ]$ and $\mathbf{w}[\mathbf{u}; \ ]$ are

$$
\begin{aligned}
& \mathbf{F}[\mathbf{u}; p; \ ] \\
= \ & \quad \mathbf{u}_t + \quad (\mathbf{u}_{xx} + \mathbf{u}_{yy}) \quad \frac{}{1 + \ }\mathbf{u}_{z_0z_0} \quad 2\ z_0(\ {}_x\mathbf{u}_{xz_0} + \quad {}_y\mathbf{u}_{yz_0}) \\
& + \frac{z_0^2}{1 + \ }(\ {}^2_x + \ {}^2_y)\mathbf{u}_{z_0z_0} \quad z_0\ \Big[(1 + \ )(\ {}_{xx} + \ {}_{yy}) \quad (\ {}^2_x + \ {}^2_y)\Big]\ \mathbf{u}_{z_0} \\
& \Big[(1 + \ )u_1\mathbf{u}_x \quad (1 + \ )u_2\mathbf{u}_y + \Big[z_0(\ {}_xu_1 + \ {}_yu_2)\Big] \quad u_3\Big]\ \mathbf{u}_{z_0} \\
& + z_0 \ {}_t\mathbf{u}_{z_0} + \Big[z_0 \ {}_xp_{z_0} \quad p_x\Big]\ \mathbf{e}_1 + \Big[z_0 \ {}_yp_{z_0} \quad p_y\Big]\ \mathbf{e}_2 :
\end{aligned}
\tag{3.5}
$$

and

$$
\mathbf{w}[\mathbf{u}; \ ] = \quad u_1\mathbf{e}_1 \quad u_2\mathbf{e}_2 + z_0(\ {}_xu_1 + \ {}_yu_2)\mathbf{e}_3 : \tag{3.6}
$$

The damped wave equation becomes

$$
\begin{array}{rcll}
{}_{tt} \quad {}_s \quad {}_s \ {}_t & = & {}^s_{\#}p \quad 2\ {}^s_{\#}u_{3;z} + \ {}^s_{\#}H[\mathbf{u}; \ ] & \text{on } \ {}^{s;0}_T; \\
(\ (0); \ {}_t(0)) & = & (\ {}^{1;0}; \ {}^{2;0}) & \text{in } \ {}^s_0
\end{array}
\tag{3.7}
$$

with

$$
\begin{aligned}
H[\mathbf{u}; \ ] \ = \ & \frac{}{1 + \ }\Big[2 \quad (\ {}^2_x + \ {}^2_y)\ \hat{u}_{3;z_0} + \frac{x}{1 + \ }\hat{u}_{1;z_0} \\
& + \frac{y}{1 + \ }\hat{u}_{2;z_0} + \quad {}_x\hat{u}_{3;x} + \quad {}_y\hat{u}_{3;y} \Big] :
\end{aligned}
\tag{3.8}
$$

Then, we consider $\mathbf{z}[\mathbf{u}; \ ] = \hat{E}\mathbf{w}[\mathbf{u}; \ ]$ and $[\mathbf{u}; \ ] = \hat{E}_p\mathbf{w}[\mathbf{u}; \ ]$ defined by

$$
\begin{array}{rcll}
\mathbf{z}[\mathbf{u}; \ ] + r \ [\mathbf{u}; \ ] & = & \mathbf{0} & \text{in } \ {}_0 \\
\text{div } \mathbf{z}[\mathbf{u}; \ ] & = & \text{div } \mathbf{w}[\mathbf{u}; \ ] & \text{in } \ {}_0 \\
\mathbf{z}[\mathbf{u}; \ ] & = & \mathbf{g} & \text{on } \ {}_0 :
\end{array}
\tag{3.9}
$$

Thanks to Proposition 2.5 (which states a result in [13]) in the previous section, we know that for $\mathbf{w}[\mathbf{u}; \ ]$ in $G^{\#}_T$ defined by

$$
\begin{aligned}
G^{\#}_T \ = \ & \Big\{ \mathbf{k} \in \mathbf{L}^2_{\#}(Q^0_T) \text{ s.t. } \text{div } \mathbf{k} \in L^2(0; T; \mathbf{H}^1_{0\#}(\ {}_0)); \\
& \mathbf{k}_t \in \mathbf{L}^2_{\#}(Q^0_T) \text{ and } \mathbf{k} = 0 \text{ on } \ {}^0_T \ ;
\end{aligned}
$$

we get that equation (3.9) has a unique solution $(\mathbf{z}[\mathbf{u}; \ ]; \ [\mathbf{u}; \ ])$ in $\mathbf{H}^{2;1}_{\#}(Q^0_T) \cap L^2(0; T; H^1_{\#}(\ {}_0))$ satisfying the estimate

$$
\|(\mathbf{z}[\mathbf{u}; \ ]; \ [\mathbf{u}; \ ])\|_{\mathbf{H}^{2;1}_{\#}(Q^0_T) \cap L^2(0;T;\mathcal{H}^1_{\#}(\Omega_0))} \quad C\|\mathbf{w}[\mathbf{u}; \ ]\|_{G^{\#}_T};
$$

where the norm on $G_T^{\#}$ is

$$\|\mathbf{k}\|_{G_T^{\#}} = \left(\|\mathbf{k}\|^2_{\mathbf{L}^2_{\#}(Q_T^0)} + \|\mathbf{k}_t\|^2_{\mathbf{L}^2_{\#}(Q_T^0)} \right)^{1=2} \qquad \text{for all } \mathbf{k} \in G_T^{\#}.$$

Thus, we look for solution $(\mathbf{u}, p, \ )$ of system (3.4)–(3.7) under the form $\mathbf{u} = \mathbf{v} + \mathbf{z}[\mathbf{u}, \ ]$, $p = q + \ [\mathbf{u}, \ ]$ where $\mathbf{z}[\mathbf{u}, \ ]$ and $\ [\mathbf{u}, \ ]$ is the solution of (3.9). Then, $(\mathbf{v}, q, \ )$ is solution of the following system

$$\begin{array}{rcll}
\mathbf{v}_t - \operatorname{div} \ (\mathbf{v}, q) &=& \mathbf{f}[\mathbf{u}, p, \ ] & \text{in } Q_T^0, \\
\operatorname{div} \mathbf{v} &=& 0 & \text{in } Q_T^0, \\
\mathbf{v} &=& \ _t \mathbf{e}_3 & \text{on } \ _T^{s,0}, \\
\mathbf{v} &=& \mathbf{0} & \text{on } \ _T, \\
\mathbf{v}(0) &=& \mathbf{v}^0 & \text{in } \ _0, \\
\ _{tt} - \ _s \ - \ _s \ _t &=& \ ^s_{\#} q + h[\mathbf{u}, \ ] & \text{on } \ _T^{s,0}, \\
(\ (0), \ _t(0)) &=& (\ ^{1,0}, \ ^{2,0}) & \text{in } \ _0^s
\end{array} \tag{3.10}$$

where

$$\mathbf{f}[\mathbf{u}, p, \ ] = \mathbf{F}[\mathbf{u}, p, \ ] - \mathbf{z}_t[\mathbf{u}, \ ],$$

$$h[\mathbf{u}, p, \ ] = \ _s H[\mathbf{u}, p, \ ] - 2 \ (\mathbf{z}[\mathbf{u}, \ ])_{2,z_0} + \ _s \ [\mathbf{u}, \ ].$$

and

$$\mathbf{v}^0 = \mathbf{u}^0 - \mathbf{z}[\mathbf{u}^0, \ ^{1,0}].$$

The compatiblity condition at $t = 0$ becomes in the variables $(\mathbf{v}^0, \ ^{1,0}, \ ^{2,0})$:

$$\operatorname{div} \mathbf{v}^0 = 0 \quad \text{in } \ _0, \quad \mathbf{v}^0 = \mathbf{0} \text{ on } \quad \text{and} \quad \mathbf{v}^0 = \ ^{2,0}\mathbf{e}_3 \text{ on } \ _0^s \tag{3.11}$$

and in the variables $(\mathbf{u}^0, \ ^{1,0}, \ ^{2,0})$:

$$\operatorname{div} (\mathbf{u}^0 - \mathbf{z}[\mathbf{u}^0, \ ^{1,0}]) = 0 \text{ in } \ _0, \quad \mathbf{u}^0 = \mathbf{0} \text{ on } \quad \text{and} \quad \mathbf{u}^0 = \ ^{2,0}\mathbf{e}_3 \text{ on } \ _0^s. \tag{3.12}$$

3.3 Study of a linear auxiliary system.

In this section, we consider the following linear system:

$$\begin{array}{rcll}
\mathbf{v}_t - \operatorname{div} \ (\mathbf{v}, q) &=& \mathbf{f} & \text{in } Q_T^0, \\
\operatorname{div} \mathbf{v} &=& 0 & \text{in } Q_T^0, \\
\mathbf{v} &=& \ _t \mathbf{e}_3 & \text{on } \ _T^{s,0}, \\
\mathbf{v} &=& \mathbf{0} & \text{on } \ _T, \\
\ _{tt} - \ M_s \ _s \ - \ M_s \ _s \ _t &=& \ ^s_{\#} q + h & \text{on } \ _T^{s,0}, \\
(\mathbf{v}(0), \ (0), \ _t(0)) &=& (\mathbf{v}^0, \ ^{1,0}, \ ^{2,0}).
\end{array} \tag{3.13}$$

The initial data $(\mathbf{v}^0, \ ^{1,0}, \ ^{2,0})$ belongs to $X_{\mathbf{cc}}^{0,\#}$ defined from

$$X^{0,\#} = \mathbf{V}^1_{\#}(\ _0) \times H^{5=2}_{\#}(\ _0^s) \times H^{3=2}_{\#}(\ _0^s)$$

by

$$X_{\mathbf{cc}}^{0,\#} = \left\{ (\mathbf{z}^0, \ ^{1,0}, \ ^{2,0}) \in X^{0,\#} \text{ s.t. } (\mathbf{z}^0, \ ^{1,0}, \ ^{2,0}) \text{ satisfies (3.11)} \right\}.$$

The right-hand side $(\mathbf{f}, h)$ belongs to $Z_T^{\#} = \mathbf{L}^2_{\#}(Q_T^0) \times L^2(0, T; H^{1=2}_{\#}(\ _0^s))$. We have the following result:

Theorem 3.3. *Let $(\mathbf{v}^0, \ ^{1,0}, \ ^{2,0})$ be in $X_{\mathbf{cc}}^{0,\#}$ and $(\mathbf{f}, h)$ be in $Z_T^{\#}$. Then, system (3.13) admits a unique solution $(\mathbf{v}, q, \ )$ in*

$$X_T^{\#} = \mathbf{V}^{2,1}_{\#}(Q_T^0) \times L^2(0, T; H^1_{\#}(\ _0)) \times E_T$$

(the space E_T is defined in (3.3)). Furthermore, we have the estimate

$$\|(\mathbf{v}, q, \ )\|_{X_T^{\#}} \leq C_1 \left(\|(\mathbf{v}^0, \ ^{1,0}, \ ^{2,0})\|_{X^{0,\#}} + \|(\mathbf{f}, h)\|_{Z_T^{\#}} \right).$$

Proof. The proof is exactly the same as in the two dimensional case (see section 2). First, the fixed point procedure in the space $X_T^{e,s,\eta}$ (for $0 < \eta < 1$) is the same. Then, to obtain a better regularity of η, we use Theorem 2.12. This gives η in $H^2(0,T;L^2_{\#,0}(\Gamma_0^s)) \cap H^1(0,T;H^2_\#(\Gamma_0^s))$. To obtain the regularity of E_T, we use an interpolation method. We consider $\eta^1 = \eta_x$ and $\eta^2 = \eta_y$. Then, η^1 and η^2 are solution of

$$\begin{aligned} \eta^1_{tt} - \beta\Delta_s\eta^1 - \gamma\Delta_s\eta^1_t &= \left(\gamma^s_\# q + h\right)_x \quad \text{on } \Sigma_T^{s,0}, \\ (\eta^1(0),\eta^1_t(0)) &= (\eta^{1,0}_x,\eta^{2,0}_x) \quad \text{in } \Gamma^s_0. \end{aligned}$$

and

$$\begin{aligned} \eta^2_{tt} - \beta\Delta_s\eta^2 - \gamma\Delta_s\eta^2_t &= \left(\gamma^s_\# q + h\right)_y \quad \text{on } \Sigma_T^{s,0}, \\ (\eta^2(0),\eta^2_t(0)) &= (\eta^{1,0}_y,\eta^{2,0}_y) \quad \text{in } \Gamma^s_0. \end{aligned}$$

Thus, for $\left[\gamma^s_\# q + h\right]$ in $L^2(0,T;H^1_\#(\Gamma^s_0))$ and $(\eta^{1,0},\eta^{2,0})$ in $H^3_\#(\Gamma^s_0) \times H^2_\#(\Gamma^s_0)$, η^1 and η^2 belong to $H^1(0,T;H^2_\#(\Gamma^s_0)) \cap H^2(0,T;L^2_{\#,0}(\Gamma^s_0))$, that is η belongs to

$$H^1(0,T;H^3_\#(\Gamma^s_0)) \cap H^2(0,T;H^1_\#(\Gamma^s_0)).$$

By interpolation, for $0 < \varepsilon < 1$, with a right-hand side in $\left[\gamma^s_\# q + h\right]$ in $L^2(0,T;H^\varepsilon_\#(\Gamma_0))$ and an initial data $(\eta^{1,0},\eta^{2,0})$ in

$$\left[H^3_\#(\Gamma^s_0) \times H^2_\#(\Gamma^s_0), H^2_\#(\Gamma^s_0) \times H^1_\#(\Gamma^s_0)\right]_{1-\varepsilon} = H^{2+\varepsilon}_\#(\Gamma^s_0) \times H^{1+\varepsilon}_\#(\Gamma^s_0),$$

we obtain a unique solution η in

$$H^1\left(0,T;\left[H^3_\#(\Gamma^s_0), H^2_\#(\Gamma^s_0)\right]_{1-\varepsilon}\right) \cap H^2\left(0,T;\left[H^1_\#(\Gamma^s_0), L^2_{0,\#}(\Gamma^s_0)\right]_{1-\varepsilon}\right),$$

that is in $H^1\left(0,T;H^{2+\varepsilon}_\#(\Gamma^s_0)\right) \cap H^2\left(0,T;H^\varepsilon_\#(\Gamma^s_0)\right)$.

The case $\varepsilon = 1/2$ which is the limit case (because both h and $\gamma^s_\# q$ only belong to $L^2(0,T;H^{1/2}_\#(\Gamma^s_0))$) gives that for initial data $(\eta^{1,0},\eta^{2,0})$ in $H^{5/2}_\#(\Gamma^s_0) \times H^{3/2}_\#(\Gamma^s_0)$, the solution η to equation

$$\begin{aligned} \eta_{tt} - \beta\Delta_s\eta - \gamma\Delta_s\eta_t &= \gamma^s_\# q + h \quad \text{on } \Sigma_T^{s,0}, \\ (\eta(0),\eta_t(0)) &= (\eta^{1,0},\eta^{2,0}) \quad \text{in } \Gamma^s_0 \end{aligned}$$

belongs to E_T. □

It is important here to stress that it is the point of the proof where we need the periodic setting. Indeed, thanks to this setting, we can obtain by interpolation a better regularity for η from the fact that h and $\gamma^s_\# q$ belong to $L^2(0,T;H^{1/2}_\#(\Gamma^s_0))$ (and not only $L^2(\Sigma^{s,0}_T)$) and from the corresponding regularity of the initial data. We cannot use the same idea in the case of the Dirichlet homogeneous condition and therefore, we cannot obtain Theorems 3.1 and 3.2 in that case.

3.4 Proof of Theorems 3.1 and 3.2.

We begin by proving this results in the fixed domain. It relies on the same fixed point procedure as in section 2.5. We introduce the space

$$W_T^\# = \mathbf{L}^2_\#(Q^0_T) \times G^\#_T \times L^2(0,T;H^1_\#(\Gamma_0))$$

endowed with the norm

$$\|(\mathbf{G},\mathbf{r},G)\|_{W_T^\#} = \left(\|\mathbf{G}\|^2_{\mathbf{L}^2_\#(Q^0_T)} + \|\mathbf{r}\|^2_{G^\#_T} + \|G\|^2_{L^2(0,T;H^1_\#(\Omega_0))}\right)^{1/2}.$$

Now, we can estimate the nonlinearities in system (3.4)–(3.7) in terms of $(\mathbf{u},p,\eta)$ in $X_T^\#$.

Proposition 3.4. *Let* $(\mathbf{u},p,\eta)$ *be in* $X_T^{\#}$*, then* $(\mathbf{F}[\mathbf{u},p,\eta],\mathbf{w}[\mathbf{u},\eta],H[\mathbf{u},\eta])$*, obtained from* $(\mathbf{u},p,\eta)$ *in (3.5), (3.6) and (3.8), belongs to* $W_T^{\#}$*. Furthermore, there exists* $\gamma > 0$ *such that*

$$\|(\mathbf{F}[\mathbf{u},p,\eta],\mathbf{w}[\mathbf{u},\eta],H[\mathbf{u},\eta])\|_{W_T^{\#}} \leq c_2 T^{\gamma}(1+\|(\mathbf{u},p,\eta)\|_{X_T^{\#}})\|(\mathbf{u},p,\eta)\|^2_{X_T^{\#}}.$$

Let $(\mathbf{u}_1,p_1,\eta_1)$ *and* $(\mathbf{u}_2,p_2,\eta_2)$ *be two triplets in* $X_T^{\#}$ *such that for* $i=1,2$, $\|(\mathbf{u}_i,p_i,\eta_i)\|_{X_T^{\#}} \leq R$ *for some* $R>0$*, we get*

$$\begin{aligned}&\|(\mathbf{F}_1,\mathbf{w}_1,H_1)-(\mathbf{F}_2,\mathbf{w}_2,H_2)\|_{W_T^{\#}}\\ &\leq C(1+R)RT^{\gamma}\|(\mathbf{u}_1,p_1,\eta_1)-(\mathbf{u}_2,p_2,\eta_2)\|_{X_T^{\#}}\end{aligned}$$

with the notations $(\mathbf{F}_i,\mathbf{w}_i,H_i)=(\mathbf{F}[\mathbf{u}_i,p_i,\eta_i],\mathbf{w}[\mathbf{u}_i,\eta_i],H[\mathbf{u}_i,\eta_i])$*.*

Proof. This proposition can be proved using the Sobolev embeddings and the nonlinear estimates in the Appendix B in [6], especially Proposition B.1 and Theorem B.3. From η in E_T, we only get

$$\eta_{xx}\in H^1(0,T;H^{1/2}_{\#}(\Gamma^s_0))\cap H^{5/4}(0,T;L^2_{\#,0}(\Gamma^s_0))$$

and η_{tx} in $H^{3/2,3/4}_{\#}(\Sigma^{s,0}_T)$. But the worst terms to estimate come from the divergence term. Indeed, $\mathbf{w}[\mathbf{u},\eta]=-u_1\mathbf{e}_1-u_2\mathbf{e}_2+z_0(\eta_x u_1+\eta_y u_2)\mathbf{e}_3$ gives the terms $z_0\eta_{tx}u_1\mathbf{e}_3$ and $z_0\eta_{ty}u_2\mathbf{e}_3$ to estimate in $L^2(Q^0_T)$ ($\mathbf{w}_t$ has to be in $\mathbf{L}^2(Q^0_T)$). Because $H^{3/2,3/4}_{\#}(\Sigma^{s,0}_T)\hookrightarrow H^{1,1/2}_{\#}(\Sigma^{s,0}_T)$, we can use the same tedious method as in the proof Proposition 6.1 in [7] or Proposition 2.13 in section 2.

The second «worst» estimate to get is $\|\eta_{xx}u_{1,z_0}\|_{L^2(Q^0_T)}$. Indeed, η_{xx} is only in $H^1(0,T;H^{1/2}_{\#}(\Gamma^s_0))$. But Proposition B.1 in [6], gives for any time t, that $\eta_{xx}(t)u_{1,z_0}(t)$ belongs to $L^2(\Omega_0)$ thanks to the calculation (with the notation of [6])

$$\alpha=0,\quad \beta=\frac{1}{2},\quad \omega=1,\ n=3\quad\text{and}\quad \frac{3}{2}=\alpha+\beta+\omega\geq\frac{n}{2}=\frac{3}{2}.$$

This is the limit case which is possible here because both $\beta>0$ and $\omega>0$. Thus,

$$\|\eta_{xx}(t)u_{1,z_0}(t)\|_{L^2(\Omega_0)}\leq C\|\eta_{xx}(t)\|_{H^{1/2}_{\#}(\Gamma^s_0)}\|u_{1,z_0}\|_{H^1_{\#}(\Omega_0)}. \tag{3.14}$$

We introduce the embeddings $H^1(0,T)\hookrightarrow H^{1/2+\varepsilon}(0,T)\hookrightarrow C([0,T])$ (for $0<\varepsilon<\frac{1}{2}$) with the estimate

$$\|f\|_{L^\infty(0,T)}\leq C\|f\|_{H^{1/2+\varepsilon}(0,T)}\leq CT^{\frac14-\frac{\varepsilon}{2}}\|f\|_{H^1(0,T)}\ \text{for all}\ f\in H^1(0,T).$$

Thus, taking the $L^2(0,T)$ norm in (3.14) and using the $L^2(0,T)$ norm for the velocity u_{1,z_0} and the $L^\infty(0,T)$ norm for the displacement on the right-hand side, we get

$$\begin{aligned}\|\eta_{tx}u_1\|_{L^2(Q^0_T)}&\leq C\|\eta_{tx}\|_{L^\infty(0,T;H^{1/2}_{\#}(\Gamma^s_0))}\|u_1(t)\|_{L^2(0,T;H^1_{\#}(\Omega_0))}\\ &\leq CT^{\frac14-\frac{\varepsilon}{2}}\|\eta\|_{\mathcal{E}_T}\|\mathbf{u}\|_{\mathbf{H}^{2,1}_{\#}(Q^0_T)}.\end{aligned}$$

The other terms containing a derivative of η are estimated in the same way.

We now consider the terms $u_1\mathbf{u}_x$, $u_2\mathbf{u}_y$ and $u_3\mathbf{u}_{z_0}$ in $\mathbf{F}[\mathbf{u},\eta]$. We follow Lemma 6.3 in [7]. From Theorem B.3, for u_1 in $H^{2,1}(Q^0_T)$ and $\mathbf{u}_x$ in $\mathbf{H}^{1,1/2}(Q^0_T)$, we get that $u_1\mathbf{u}_x$ belongs to $\mathbf{H}^{1/2-\varepsilon,1/4-\varepsilon/4}(Q^0_T)$ for $0<\varepsilon<1/2$. Then, by a classic embedding formula, we know $H^{1/4-\varepsilon/2}(0,T)\hookrightarrow L^p(0,T)$ where $\frac{1}{p}=\frac{1}{2}-\frac{1}{4}+\frac{\varepsilon}{2}$ (see [1]), that is $p=\frac{4}{1+2\varepsilon}$. Furthermore, we have $L^p(0,T)\hookrightarrow L^2(0,T)$ (because $2<p$) with the estimate

$$\|f\|_{L^2(0,T)}\leq CT^{\frac12-\frac1p}\|f\|_{L^p(0,T)}\ \text{for all}\ f\in L^p(0,T).$$

But $\frac{1}{2}-\frac{1}{p}=\frac{1}{2}-\frac{1}{2}+\frac{1}{4}-\frac{\varepsilon}{2}=\frac{1}{4}-\frac{\varepsilon}{2}$ which is nonnegative thanks to $0<\varepsilon<1/2$ in the application of Proposition B.1. Finally, going back to the estimate of $u_1\mathbf{u}_x$, we get

$$\begin{aligned}\|u_1\mathbf{u}_x\|_{\mathbf{L}^2(Q^0_T)}&\leq CT^{\frac12-\frac{\varepsilon}{2}}\|u_1\mathbf{u}_x\|_{H^{\frac12-\frac{\varepsilon}{2}}(0,T;\mathbf{L}^2(\Omega_0))}\\ &\leq cT^{\frac12-\frac{\varepsilon}{2}}\|u_1\|_{H^{2,1}(Q^0_T)}\|\mathbf{u}_x\|_{\mathbf{H}^{1,1/2}(Q^0_T)}\\ &\leq CT^{\frac12-\frac{\varepsilon}{2}}\|\mathbf{u}\|^2_{\mathbf{H}^{2,1}(Q^0_T)}.\end{aligned}$$

The second part of the proposition comes directly from the first one and the fact that the functions $\mathbf{F}[\mathbf{u}, p, \ ]$, $\mathbf{w}[\mathbf{u}, \ ]$ and $H[\mathbf{u}, \ ]$ are, by construction, at least quadratic in the variables $(\mathbf{u}, p, \ )$.

□

Proposition 3.5. *Let* $(\mathbf{u}, p, \ )$ *be in* $X_T^{\#}$*, then system* (3.4)–(3.7) *with initial data* $(\mathbf{u}^0, \ ^{1,0}, \ ^{2,0})$ *in* $X^{0,\#}$ *satisfying* (3.12) *and* $(\mathbf{F}[\mathbf{u}, p, \ ], \mathbf{w}[\mathbf{u}, \ ], H[\mathbf{u}, \ ])$ *as right-hand side admits a unique solution* $(\overline{\mathbf{u}}, \overline{p}, \overline{\ })$ *in* $X_T^{\#}$ *with the estimate*

$$\begin{aligned} &\|(\overline{\mathbf{u}}, \overline{p}, \overline{\ })\|_{X_T^{\#}} \\ &\quad C_1 \ \|(\mathbf{u}^0, \ ^{1,0}, \ ^{2,0})\|_{X^{0,\#}} + C_2 T \ (1 + \|(\mathbf{u}, p, \ )\|_{X_T^{\#}})\|(\mathbf{u}, p, \ )\|^2_{X_T^{\#}} \end{aligned}$$

where *is a stricly positive constant. That is, we can construct a mapping* $X_T^{\#}$*:*

$$\begin{array}{rcl} X_T^{\#} & \to & X_T^{\#} \\ (\mathbf{u}, p, \ ) & \mapsto & \begin{array}{l} (\overline{\mathbf{u}}, \overline{p}, \overline{\ }) \text{ is the solution of system (3.4) (3.7)} \\ \text{with } (\mathbf{F}[\mathbf{u}, p, \ ], \mathbf{w}[\mathbf{u}, \ ], H[\mathbf{u}, \ ]) \text{ for right-hand side} \end{array} \end{array}$$

which satisfies the estimate

$$\begin{aligned} &\|X_T(\mathbf{u}, p, \ )\|_{X_T^{\#}} \\ &\quad C_1 \ \|(\mathbf{u}^0, \ ^{1,0}, \ ^{2,0})\|_{X^{0,\#}} + C_2 T \ (1 + \|(\mathbf{u}, p, \ )\|_{X_T^{\#}})\|(\mathbf{u}, p, \ )\|^2_{X_T^{\#}} \ . \end{aligned}$$

The proof of this proposition relies on Theorem 3.3 and Proposition 3.4. One can find the idea in the proof of Proposition 2.14.

We can now show existence of solutions in the fixed domain:

Proposition 3.6. *Let* $(\mathbf{u}^0, \ ^{1,0}, \ ^{2,0})$ *be in* $X^{0,\#}$ *satisfying* (3.12)*. Then,*

(i) there exists a time $T_0 > 0$ *such that system* (3.4)–(3.7) *admits a unique local strong solution* $(\mathbf{u}, p, \ )$ *in* $X_{T_0}^{\#}$.

(ii) there exists r *small enough such that, under the condition*

$$\|(\mathbf{u}^0, \ ^{1,0}, \ ^{2,0})\|_{X^{0,\#}} \quad r,$$

system (3.4)–(3.7) *admits a unique global strong solution* $(\mathbf{u}, p, \ )$ *in* $X_T^{\#}$.

Proof. The proof is clear thanks to the previous proposition. We have to act on the size of the initial data or on the size of the time interval to get the alternative. □

To conclude, thanks to in E_T or E_{T_0}, we can prove that either for r or for T_0 small enough (depending on the case we consider in Proposition 3.6), satisfies condition (1.1). Together with

$$\begin{array}{rcl} ^{-1}_{(t)} : \quad \ ^{0} & \to & \ _{(t)} \\ (x, y, z_0) & \mapsto & (x, y, z) \end{array}$$

is a C^1 diffeomorphism for any time t, that finishes the proof.

Acknowledgment

The author would like to thank C. Grandmont for helpful discussions during the preparation of the paper.

References

[1] R. A. Adams. *Sobolev Spaces.* Academic Press, 1975.

[2] C. Bardos. A Regularity Theorem for Parabolic Equations. *J. Funct. Anal.*, 7:311–322, 1971.

[3] H. Beirão da Veiga. On the existence of strong solutions to a coupled fluid-structure evolution problem. *J. Math. Fluid Mech.*, 6(1):21–52, 2004.

[4] A. Chambolle, B. Desjardins, M. J. Esteban, and C. Grandmont. Existence of weak solutions for the unsteady interaction of a viscous fluid with an elastic plate. *J. Math. Fluid Mech.*, 7(3):368–404, 2005.

[5] C. Grandmont. Existence of weak solutions for the unsteady interaction of a viscous fluid with an elastic plate. *SIAM J. Math. Anal.*, 40(2):716–737, 2008.

[6] G. Grubb and V. A. Solonnikov. Boundary value problems for the nonstationary Navier-Stokes equations treated by pseudo-differential methods. *Math. Scand.*, 69(2):217–290 (1992), 1991.

[7] J. Lequeurre. Existence of strong solutions to a fluid-structure system. *SIAM J. Math. Anal.*, 43(1):389–410, 2011.

[8] J. Lequeurre. *Quelques résultats d'existence, de contrôlabilité et de stabilisation pour des systèmes couplés fluide-structure.* PhD thesis, Institut de Mathématiques de Toulouse, 2011.

[9] J.-L. Lions and E. Magenes. *Problèmes aux limites non homogènes et applications. Vol. 1.* Travaux et Recherches Mathématiques, No. 17. Dunod, Paris, 1968.

[10] A. Quarteroni, M. Tuveri, and A. Veneziani. Computational vascular fluid dynamics: problems, models and methods. *Comput. Visual. Sci.*, 2:163–197, 2000.

[11] J.-P. Raymond. Stokes and Navier-Stokes equations with nonhomogeneous boundary conditions. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 24:125–169, 2007.

[12] J.-P. Raymond. Feedback stabilization of a fluid - structure model. *SIAM J. Control Optim.*, 48(8):5398–5443, 2010.

[13] J.-P. Raymond. Stokes and Navier-Stokes equations with a nonhomogeneous divergence condition. *Discrete Contin. Dyn. Syst. Ser. B*, 14(4):1537–1564, 2010.

[14] G. F. Webb. Existence and asymptotic behavior for a strongly damped nonlinear wave equation. *Canad. J. Math.*, 32(3):631–643, 1980.