Null Controllability of a Fluid - Structure System.

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Abstract : We study a coupled fluid-structure system. The structure corresponds to a part of the boundary of a domain containing an incompressible viscous fluid. The structure displacement is modeled by an ordinary differential equation. We prove the local null controllability of the system when the control acts on a fixed subset of the fluid domain.

1 Introduction.

Controllability for fluid-structure systems has been studied recently. In a series of papers, J.P. Raymond and M. Vanninathan prove null controllability for different kinds of linear coupled systems modeling, with an increasing difficulty, fluid-structure interaction in 2D. The fluid is modeled respectively by the Helmholtz equation [11], the Heat equation [13, 12] and the Stokes equation [14].

In [3], A. Doubova and E. Fernandez-Cara consider a 1D interaction problem of a particle in a fluid modeled by the Burgers equation. They prove null controllability for the linearized model and then local null controllability for the nonlinear system.

Very recently, M. Boulakia, A. Oxel in [2] and O. Imanuvilov, T. Takahashi in [5] prove independently local exact controllability for a system modeling a rigid body moving in a viscous incompressible fluid described by the Navier-Stokes equations in 2D with a control acting in a fixed subset of the fluid domain.

In this paper, we are interested in the null controllability of a system coupling the Navier-Stokes equations and an ordinary differential equation (see equations (1.7)–(1.6)). More precisely, we prove that for any time $T > 0$ and any initial data small enough, we can find a control acting in a subdomain of the fluid part such that the solution of our system vanishes at time T (see Theorem 1.3).

The systems in [3, 2, 5] deal with nonlinear fluid equations. The strategy of the different proofs is quite the same. First, a change of variables sets the problem in a fixed domain. Then, the different authors prove that the obtained linearized system is null controllable with some control. Finally, a fixed point procedure gives the local null controllability.

The way used to prove the controllability of the linear [11, 13, 12, 14] or the linearized [3, 2, 5] systems is based on the duality between the controllability of a system and the existence of an observability inequality for the adjoint system. Such an observability inequality relies in fact on a Carleman estimate. The proofs of Carleman estimates are really tricky and not straightforward.

1.1 The system.

We consider a viscous incompressible fluid in a two dimensional domain. The boundary of the domain is split into two parts. One part is fixed, the other one is a moving beam. At rest, the beam is in its reference state $\Gamma_0^s = (0, L) \times \{1\}$, where $L > 0$ is the characteristic length of the beam. The domain of the fluid at rest is denoted Ω_0. Then its boundary Γ_0 is the union of two curves Γ_0^s and Γ. We suppose that the boundary Γ_0 is smooth, that is at least $\mathcal{C}^4$.

The displacement of the beam is given by a function η depending on the time t and on the position x in the reference state Γ_0^s. Then, *a priori*, the function η is from $(0, +\infty) \times (0, L)$ in $\mathbb{R}$. For any $t \geq 0$, the moving boundary given by the displacement η is

$$\Gamma_{\eta(t)}^s = \Big\{(x, y) \in \mathbb{R}^2 \text{ s.t. } x \in (0, L) \text{ and } y = 1 + \eta(t, x)\Big\}.$$

Then, the fluid at time t occupies a domain noted $\Omega_{\eta(t)}$ which has for boundary $\partial\Omega_{\eta(t)} = \Gamma \bigcup \Gamma^s_{\eta(t)}$.

We have the following assumption on the displacement

$$\exists \varepsilon > 0 \text{ such that } \forall t \in [0,T] \quad \forall x \in (0,L) \quad 1 + \eta(t,x) \geq \varepsilon > 0 \tag{1.1}$$

to ensure that, for every time t, $\Omega_{\eta(t)}$ is a connected domain.

Let us introduce some notations. We fix a time $T > 0$, then

$$\begin{aligned} Q^0_T &= (0,T) \times \Omega_0, & Q^\eta_T &= \bigcup_{t\in(0,T)} \{t\} \times \Omega_{\eta(t)}, & \Sigma_T &= (0,T) \times \Gamma, \\ \Sigma^{s,0}_T &= (0,T) \times \Gamma^s_0, & \Sigma^{s,\eta}_T &= \bigcup_{t\in(0,T)} \{t\} \times \Gamma^s_{\eta(t)}, & \Sigma^0_T &= (0,T) \times \Gamma_0. \end{aligned}$$

Following the model in [8, 1, 7], the velocity $\mathbf{u}$ and the pressure p of the fluid in the domain Q^η_T are described by the Navier-Stokes equations

$$\begin{array}{rcll} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - \operatorname{div}\sigma(\mathbf{u},p) &=& \mathbf{0} & (Q^\eta_T) \\ \operatorname{div}\mathbf{u} &=& 0 & (Q^\eta_T) \\ \mathbf{u} &=& \eta_t\mathbf{e}_2 & (\Sigma^{s,\eta}_T) \\ \mathbf{u} &=& \mathbf{0} & (\Sigma_T) \\ \mathbf{u}(0) &=& \mathbf{u}^0 & (\Omega_{\eta^0}) \end{array} \tag{1.2}$$

In this equation, the term $\sigma(\mathbf{u},p)$ is the Cauchy stress tensor defined by

$$\sigma(\mathbf{u},p) = -p\mathbf{I} + \nu\Big(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\Big).$$

The coefficient $\nu > 0$ is the viscosity of the fluid. Finally, $\mathbf{e}_1$ and $\mathbf{e}_2$ are the two vectors of $\mathbb{R}^2$

$$\mathbf{e}_1 = (1,0)^T, \qquad \mathbf{e}_2 = (0,1)^T.$$

Remark 1.1. *Due to the incompressibility condition of the fluid, solutions* $(\mathbf{u},p)$ *of system* (1.2) *and the Dirchlet boundary condition* $\eta_t\mathbf{e}_2$ *satisfy, for any time* t,

$$\int_{\Omega_{\eta(t)}} \operatorname{div}\mathbf{u}(t) = \int_{\partial\Omega_{\eta(t)}} \mathbf{u}(t)\cdot\mathbf{n}(t) = \int_{\Gamma^s_0} \eta_t(t) = 0.$$

The vector $\mathbf{n}(t)$ *is the unit normal to* $\partial\Omega_{\eta(t)}$ *outward* $\Omega_{\eta(t)}$. *It is fixed on* Γ *and is given on* $\Gamma^s_{\eta(t)}$ *by*

$$\mathbf{n}(t) = \frac{1}{\sqrt{1+\eta_x^2(t)}}\Big(-\eta_x(t)\mathbf{e}_1 + \mathbf{e}_2\Big).$$

Thus, we will consider functions η *in*

$$L^2_0(\Gamma^s_0) = \left\{\mu \in L^2(\Gamma^s_0) \text{ s.t. } \int_{\Gamma^s_0} \mu = 0\right\}.$$

We assume that the displacement of the beam is a Galerkin approximation of the Euler-Bernoulli beam model. Thus, the function η is of the form

$$\eta(t,x) = \sum_{k=1}^{N} q_k(t)\zeta_k(x), \text{ for } x \in (0,L) \text{ and } t \geq 0 \tag{1.3}$$

where N is a fixed integer greater than 1. The familly $(\zeta_k)_{k=1,\ldots,N}$ is a Hilbertian basis of $L^2_0(\Gamma^s_0)$ (see Remark 1.1). For each $k \geq 1$, ζ_k belongs to $\mathcal{C}^\infty(\Gamma^s_0;\mathbb{R})$ and satisfies

$$\zeta(x) = 0, \quad \zeta_x(x) = 0 \qquad \text{for} \quad x = 0, L.$$

The unknown $q(t)$ is a $N \times 1$ vector,

$$q(t) = (q_1(t), \ \cdots \ , q_N(t))^T,$$

which satisfies the following ordinary differential equation:

$$\begin{aligned} q''(t) + Aq(t) &= \Pi_N \Big[-\sigma(\mathbf{u},p)\Big(-\eta_x \mathbf{e}_1 + \mathbf{e}_2\Big)\cdot \mathbf{e}_2 \Big] \\ &= \left(\int_{\Gamma_0^s} -\sigma(\mathbf{u},p)\Big(-\eta_x \mathbf{e}_1 + \mathbf{e}_2\Big)\cdot \zeta_k \mathbf{e}_2 \right)^T_{k=1,\ldots,N} \\ \big(q(0), q'(0)\big) &= \big(q^0, q^1\big). \end{aligned} \tag{1.4}$$

In this equation, A is the symmetric positive matrix defined by

$$A = \left(\int_{\Gamma_0^s} \Big(\alpha \zeta_{k,xx}\zeta_{l,xx} + \beta \zeta_{k,x}\zeta_{l,x} \Big) \right)_{k,l=1,\ldots,N},$$

Π_N is the projection from $L^2_0(\Gamma_0^s)$ to $\mathbb{R}^N$. Then, Π_N satisfies, for every f in $L^2_0(\Gamma_0^s)$,

$$\Pi_N(f) = \left(\int_{\Gamma_0^s} \zeta_1 f, \ \cdots \ \int_{\Gamma_0^s} \zeta_N f \right)^T.$$

Introducing M the $\mathbb{R}^{2\times N}$ matrix, $M = \Big(\zeta_1 \mathbf{e}_2, \ldots, \zeta_N \mathbf{e}_2\Big) = \begin{pmatrix} 0 & \cdots & 0 \\ \zeta_1 & \cdots & \zeta_N \end{pmatrix}$, we have a quite simplier notation for the right-hand side of (1.4):

$$q''(t) + Aq(t) = -\int_{\Gamma_0^s} M^T \sigma(\mathbf{u},p)\Big(-\eta_x \mathbf{e}_1 + \mathbf{e}_2\Big).$$

The displacement we consider can be seen as a Galerkin approximation of the one in [1, 10, 7]. Indeed, let us introduce the following partial differential equation, called beam equation:

$$\begin{aligned} \eta_{tt} + \alpha M_s \eta_{xxxx} - \beta \eta_{xx} &= -\gamma_s \Big[\sigma(\mathbf{u},p)(-\eta_x \mathbf{e}_1 + \mathbf{e}_2)\cdot \mathbf{e}_2\Big] && (0,T)\times\Gamma_0^s \\ \eta &= 0 && (0,T)\times\{0,L\} \\ \eta_x &= 0 && (0,T)\times\{0,L\} \end{aligned} \tag{1.5}$$

The coefficients $\alpha > 0$ and $\beta \geq 0$ are respectively the rigidity and the stretching of the beam. The operator M_s is the projection from $L^2(\Gamma_0^s)$ onto $L^2_0(\Gamma_0^s)$ defined by

$$M_s \mu = \mu - \frac{1}{|\Gamma_0^s|}\int_{\Gamma_0^s} \mu, \qquad \forall \mu \in L^2(\Gamma_0^s).$$

We use the trace γ_s defined by

$$\gamma_s p = M_s(p_{|\Gamma_0^s}) = p_{|\Gamma_0^s} - \frac{1}{|\Gamma_0^s|}\int_{\Gamma_0^s} p_{|\Gamma_0^s} \qquad \forall p \in H^\sigma(\Omega_0) \text{ with } \sigma > 1/2.$$

Let us define the operator $(\mathcal{A}_{\alpha,\beta}, D(\mathcal{A}_{\alpha,\beta}))$ on $L^2_0(\Gamma_0^s)$ by

$$\begin{aligned} D(\mathcal{A}_{\alpha,\beta}) &= \Big\{ \mu \in H^4(\Gamma_0^s) \cap L^2_0(\Gamma_0^s) \text{ s.t. } \mu(x) = \mu_x(x) = 0 \text{ for } x = 0, L \Big\}, \\ \mathcal{A}_{\alpha,\beta}\mu &= \alpha M_s \mu_{xxxx} - \beta \mu_{xx} \text{ for all } \mu \in D(\mathcal{A}_{\alpha,\beta}). \end{aligned}$$

We can easily see that $(\mathcal{A}_{\alpha,\beta}, D(\mathcal{A}_{\alpha,\beta}))$ is a symmetric positive operator. We denote $\Big\{(\lambda_k, \zeta_k)\Big\}_{k\geq 1}$ its pairs of eigenvalues-eigenfunctions satisfying first $\zeta_k \in D(\mathcal{A}_{\alpha,\beta})$ for all $k \geq 1$ and second

$$\begin{aligned} \mathcal{A}_{\alpha,\beta}\zeta_k &= \lambda_k \zeta_k && \text{for all } k \geq 1, \\ \Big(\zeta_k, \zeta_l\Big)_{L^2(\Gamma_0^s)} &= 0 && \text{for } k,l \geq 1 \text{ s.t. } k \neq l, \\ \Big(\zeta_k, \zeta_l\Big)_{H^2(\Gamma_0^s)} &= \delta_{kl} && \text{for all } k,l \geq 1. \end{aligned}$$

Then, the family $(\zeta_k)_{k\geq 1}$ constitutes a Hilbertian basis of $L_0^2(\Gamma_0^s)$. Furthermore, each ζ_k for $k \geq 1$ belongs to $\mathcal{C}^\infty(\Gamma_0^s;\mathbb{R})$ as sums of exponential functions.

With a direct calculation, we can verify that the right-hand side of the beam equation (1.4) is

$$\sigma(\mathbf{u},p)\Big(-\eta_x\mathbf{e}_1+\mathbf{e}_2\Big)\cdot\mathbf{e}_2 = p - 2\nu u_{2,y} - \nu\eta_x\Big(u_{1,y}+u_{2,x}\Big).$$

Using the projection Π_N, it becomes

$$\Pi_N\Big[p-2\nu u_{2,y}\Big] - \nu\Pi_N\Big[\eta_x\Big(u_{1,y}+u_{2,x}\Big)\Big].$$

The first term is linear in the variables $(\mathbf{u},p,q)$ whereas the second is quadratic in the same variables. Then the finite dimensional beam equation is

$$\begin{array}{rcl} q''+Aq & = & \Pi_N\Big[p-2\nu u_{2,y}\Big] - \nu\Pi_N\Big[\eta_x\Big(u_{1,y}+u_{2,x}\Big)\Big], \\ (q(0),q'(0)) & = & (q^0,q^1). \end{array} \tag{1.6}$$

We set a control $\mathbf{c}$ in a subset ω of the fluid domain. In assumption (1.1), we can take ε such that the set ω will never touch the boundary $\Gamma^s_{\eta(t)}$. For that, let us suppose that

$$\sup_{(x,y)\in\omega} y \leq \varepsilon.$$

This is a physical issue because the domain ω is supposed to be in the fluid part of the domain and the control force cannot be out of the domain.

Denoting $Z(x)$ the $1\times N$ vector $Z(x)=\Big(\zeta_1(x),\ldots,\zeta_N(x)\Big)$, we have equivalently

$$\eta(t,x)=Z(x)q(t), \text{ for } x\in(0,L) \text{ and } t\geq 0.$$

The equality of the velocities on the boundary becomes $\mathbf{u}=\eta_t\mathbf{e}_2=Zq'\mathbf{e}_2$. Then, the equations of the fluid part are:

$$\begin{array}{rcll} \mathbf{u}_t+(\mathbf{u}\cdot\nabla)\mathbf{u}-\operatorname{div}\sigma(\mathbf{u},p) & = & \mathbf{c}\chi_\omega & (Q_T^\eta) \\ \operatorname{div}\mathbf{u} & = & 0 & (Q_T^\eta) \\ \mathbf{u} & = & Zq'\mathbf{e}_2 & (\Sigma_T^{s,\eta}) \\ \mathbf{u} & = & \mathbf{0} & (\Sigma_T) \\ \mathbf{u}(0) & = & \mathbf{u}^0 & (\Omega_{\eta^0}) \end{array} \tag{1.7}$$

The function χ_ω above is the indicator function of the domain ω.

1.2 Functional setting.

In the fixed domain Ω_0, we define the classic Hilbert space in two dimensions $\mathbf{L}^2(\Omega_0)=L^2(\Omega_0;\mathbb{R}^2)$ and in the same way the Sobolev spaces $\mathbf{H}^s(\Omega_0)=H^s(\Omega_0;\mathbb{R}^2)$. We denote

$$\mathbf{V}^\sigma(\Omega_0)=\Big\{\mathbf{u}\in\mathbf{H}^\sigma(\Omega_0)\ ;\ \operatorname{div}\mathbf{u}=0 \text{ in } \Omega_0\Big\}.$$

Then we define

$$\begin{array}{rcl} \mathbf{H}^{\sigma,\tau}(Q_T^0) & = & L^2(0,T;\mathbf{H}^\sigma(\Omega_0))\cap H^\tau(0,T;\mathbf{L}^2(\Omega_0)), \\ \mathbf{V}^{\sigma,\tau}(Q_T^0) & = & L^2(0,T;\mathbf{V}^\sigma(\Omega_0))\cap H^\tau(0,T;\mathbf{V}^0(\Omega_0)). \end{array}$$

We need a definition of Sobolev spaces in the time dependent domain $\Omega_{\eta(t)}$:

Definition 1.2. *We say that* $\mathbf{u}$ *belongs to* $H^\tau(\bigcup_{t\in(0,T)}\{t\}\times\mathbf{H}^\sigma(\Omega_{\eta(t)}))$ *(respectively to* $H^\tau(\bigcup_{t\in(0,T)}\{t\}\times\mathbf{V}^\sigma(\Omega_{\eta(t)})))$ *if*

- *for almost every* t *in* $(0,T)$*,* $\mathbf{u}(t)$ *is in* $\mathbf{H}^\sigma(\Omega_{\eta(t)})$ *(resp. in* $\mathbf{V}^\sigma(\Omega_{\eta(t)})$*),*
- $t\mapsto\|\mathbf{u}(t)\|_{\mathbf{H}^\sigma(\Omega_{\eta(t)})}$ *(resp.* $t\mapsto\|\mathbf{u}(t)\|_{\mathbf{V}^\sigma(\Omega_{\eta(t)})}$*) is in* $H^\tau(0,T;\mathbb{R})$*.*

We finally define

$$\mathbf{H}^{\sigma,\tau}\left(Q_T^{\eta}\right) = L^2\left(\bigcup_{t\in(0,T)}\{t\}\times\mathbf{H}^{\sigma}\left(\Omega_{\eta(t)}\right)\right)\bigcap H^{\tau}\left(\bigcup_{t\in(0,T)}\{t\}\times\mathbf{L}^{2}\left(\Omega_{\eta(t)}\right)\right),$$

$$\mathbf{V}^{\sigma,\tau}\left(Q_T^{\eta}\right) = L^2\left(\bigcup_{t\in(0,T)}\{t\}\times\mathbf{V}^{\sigma}\left(\Omega_{\eta(t)}\right)\right)\bigcap H^{\tau}\left(\bigcup_{t\in(0,T)}\{t\}\times\mathbf{V}^{0}\left(\Omega_{\eta(t)}\right)\right).$$

The pressure term p is defined in the Navier-Stokes equations up to a constant: only the derivatives of p appears in (1.7). Then, we define the space $\mathcal{H}^{\sigma}(\Omega_0)$ by

$$\mathcal{H}^{\sigma}(\Omega_0) = \left\{p \in H^{\sigma}(\Omega_0) \text{ such that } \int_{\Omega_0} p = 0\right\}.$$

We will look for p in $L^2\left(\bigcup_{t\in(0,T)}\{t\}\times\mathcal{H}^1\left(\Omega_{\eta(t)}\right)\right)$ (see Definition 1.2).

1.3 Main result.

The aim of this paper is to prove the following result of null controlability of the system (1.7)–(1.6):

Theorem 1.3. *Let $T > 0$. Let $(\mathbf{u}^0, q^0, q^1)$ be in $\mathbf{V}^1(\Omega_{\eta^0})\times\mathbb{R}^N\times\mathbb{R}^N$ satisfying the compatibility condition $\mathbf{u}^0 = Zq^1\mathbf{e}_2$ on $\Gamma^s_{\eta^0}$ and $\mathbf{u}^0 = \mathbf{0}$ on Γ. Then there exists $r > 0$ such that if*

$$\|\mathbf{u}^0\|_{\mathbf{V}^1(\Omega_{\eta^0})} + |q^0|_{\mathbb{R}^N} + |q^1|_{\mathbb{R}^N} < r,$$

then the system (1.7)–(1.6) is null controllable at time T in $(\mathbf{u}, q, q')$. That means exactly there exists $\mathbf{c} \in L^2(0,T;\mathbf{L}^2(\omega))$ such that

$$\mathbf{u}(T) = \mathbf{0}, \quad q(T) = 0 \quad \textit{and} \quad q'(T) = 0.$$

Like the other results of controllability of nonlinear coupled systems already mentioned in the introduction, the first step of the proof is to use a suitable change of variables to set the system in a fixed domain without changing the domain ω of the control. This change of variables and the equivalent system are introduced in the Section 1.4. Then, in section 2, we prove the null controllability for the linearized system with nonhomogeneous right-hand sides using a duality method and a Carleman estimate. The proof of the Carleman estimate is postponed to section 4. Section 3 is devoted to the proof of Theorem 1.3. It relies on a fixed point procedure.

1.4 The system in a fixed domain.

We suppose that the rectangle $\mathbf{R}_0 = (0, L)\times(0, 1)$ is included in the domain Ω_0, see Figure 1.

The change of variables is

$$\begin{array}{llll} \theta_t: & \Omega_{\eta(t)} & \longrightarrow & \Omega_0 \\ & (x,y) & \longmapsto & (x,z) \quad \text{with} \begin{cases} z = \varepsilon + (1-\varepsilon)\dfrac{y-\varepsilon}{1-\varepsilon+\eta(t,x)} & \text{if } 0 \le x \le L \text{ and } \varepsilon \le y < 1+\eta(t,x) \\ z = y & \text{otherwise.} \end{cases} \end{array}$$

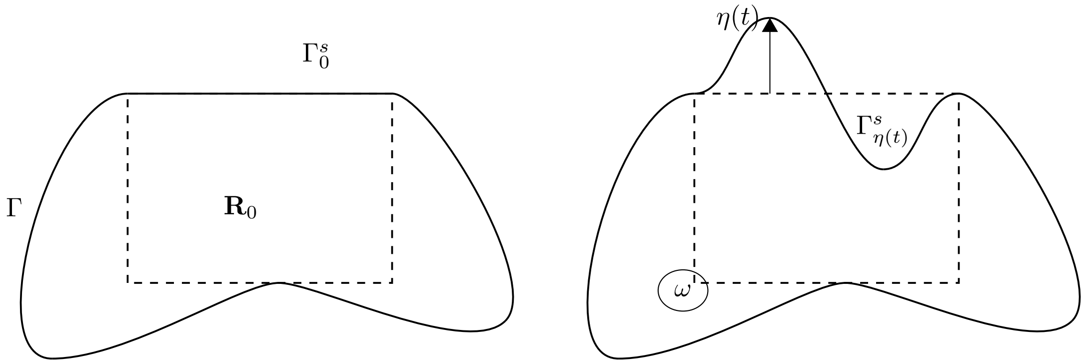

Figure 1: The domains Ω_0 (on the left), $\Omega_{\eta(t)}$(on the right) and $\mathbf{R}_0$.

Setting $\hat{f}(x,z) = f(x,y)$, we can calculate the derivatives of $f(x,y)$ using the derivatives of $\hat{f}(x,z)$ in $(0,L)\times(\varepsilon,1)$:

$$\left\{\begin{array}{rcl} f_t &=& \hat{f}_t - (z-\varepsilon)\dfrac{\eta_t}{1-\varepsilon+\eta}\hat{f}_z, \\ f_x &=& \hat{f}_x - (z-\varepsilon)\dfrac{\eta_x}{1-\varepsilon+\eta}\hat{f}_z, \\ f_y &=& \dfrac{1-\varepsilon}{1-\varepsilon+\eta}\hat{f}_z, \\ f_{xx} &=& \hat{f}_{xx} - 2(z-\varepsilon)\dfrac{\eta_x}{1-\varepsilon+\eta}\hat{f}_{xz} + \left((z-\varepsilon)\dfrac{\eta_x}{1-\varepsilon+\eta}\right)^2\hat{f}_{zz} - (z-\varepsilon)\dfrac{(1-\varepsilon+\eta)\eta_{xx}-\eta_x^2}{(1-\varepsilon+\eta)^2}\hat{f}_z, \\ f_{yy} &=& \dfrac{(1-\varepsilon)^2}{(1-\varepsilon+\eta)^2}\hat{f}_{zz}. \end{array}\right.$$

Now, we state the system satisfied by $\hat{\mathbf{u}}(x,z) = \mathbf{u}(x,y)$ and $\hat{p}(x,z) = p(x,y)$:

$$\begin{array}{rcll} \hat{\mathbf{u}}_t - \operatorname{div}\sigma(\hat{\mathbf{u}},\hat{p}) &=& \hat{\mathbf{c}}\chi_\omega + \mathbf{F}[\hat{\mathbf{u}},\hat{p},\eta] & (Q_T^0) \\ \operatorname{div}\hat{\mathbf{u}} &=& \operatorname{div}\mathbf{w}[\hat{\mathbf{u}},\eta] & (Q_T^0) \\ \hat{\mathbf{u}} &=& Zq'\mathbf{e}_2 & (\Sigma_T^{s,0}) \\ \hat{\mathbf{u}} &=& \mathbf{0} & (\Sigma_T) \\ \hat{\mathbf{u}}(0) &=& \hat{\mathbf{u}}^0 & (\Omega_0) \end{array}$$

with $\mathbf{F}[\hat{\mathbf{u}},\hat{p},\eta] = -(\hat{\mathbf{u}}\cdot\nabla)\hat{\mathbf{u}} = -(\mathbf{u}\cdot\nabla)\mathbf{u}$, $\hat{\mathbf{c}} = \mathbf{c}$ and $\mathbf{w}[\hat{\mathbf{u}},\eta] = \mathbf{0}$ for $(x,z)\in\Omega\setminus(0,L)\times(\varepsilon,1)$. For (x,z) in $(0,L)\times(\varepsilon,1)$, we have:

$$\begin{array}{rcl} \mathbf{F}(t,x,z) &=& \dfrac{1}{1-\varepsilon}\Bigg(-\eta\hat{\mathbf{u}}_t + \left[(z-\varepsilon)\eta_t + \nu(z-\varepsilon)\left(\dfrac{2\eta_x^2}{1-\varepsilon+\eta} - \eta_{xx}\right)\right]\hat{\mathbf{u}}_z \\ && + \nu\left\{-2(z-\varepsilon)\eta_x\hat{\mathbf{u}}_{xz} + \eta\hat{\mathbf{u}}_{xx} + \dfrac{(z-\varepsilon)^2\eta_x^2 - \eta(1-\varepsilon)}{1-\varepsilon+\eta}\hat{\mathbf{u}}_{zz}\right\} \\ && + ((z-\varepsilon)\eta_x\hat{p}_z - \eta\hat{p}_x)\mathbf{e}_1 - (1-\varepsilon+\eta)\hat{u}_1\hat{\mathbf{u}}_x + ((z-\varepsilon)\eta_x\hat{u}_1 - (1-\varepsilon)\hat{u}_2)\hat{\mathbf{u}}_z\Bigg) \end{array}$$

and

$$\mathbf{w}(t,x) = \frac{1}{1-\varepsilon}\left(-\eta\hat{u}_1\mathbf{e}_1 + (z-\varepsilon)\eta_x\hat{u}_1\mathbf{e}_2\right). \tag{1.8}$$

The change of variables gives us a new formula for the right-hand side of (1.6):

$$\Pi_N\left[\hat{p} - 2\nu\hat{u}_{2,z}\right] + h[\hat{\mathbf{u}},\eta]$$

where

$$h[\hat{\mathbf{u}},\eta] = \nu\Pi_N\left(\frac{\eta_x}{1+\eta}\hat{u}_{1,z} + \eta_x\hat{u}_{2,x} - \frac{\eta_x^2-2\eta}{1+\eta}\hat{u}_{2,z}\right). \tag{1.9}$$

With the identification (1.3), we can use the notation $h[\hat{\mathbf{u}},q]=h[\hat{\mathbf{u}},\eta]$ and the same for $\mathbf{F}[\hat{\mathbf{u}},\hat{p},q]$ and $\mathbf{w}[\hat{\mathbf{u}},q]$. To simplify the notation, we drop out the symbol $\hat{\cdot}$ and we get the following system:

$$\begin{array}{rcll}\mathbf{u}_t-\operatorname{div}\sigma(\mathbf{u},p) & = & \mathbf{c}\chi_\omega+\mathbf{F}[\mathbf{u},p,q] & (Q_T^0)\\ \operatorname{div}\mathbf{u} & = & \operatorname{div}\mathbf{w}[\mathbf{u},q] & (Q_T^0)\\ \mathbf{u} & = & Zq'\mathbf{e}_2 & (\Sigma_T^{s,0})\\ \mathbf{u} & = & \mathbf{0} & (\Sigma_T)\\ q''+Aq & = & \Pi_N\Big[p-2\nu u_{2,z}\Big]+h[\mathbf{u},q] & (0,T)\\ \big(\mathbf{u}(0),q(0),q'(0)\big) & = & \big(\mathbf{u}^0,q^0,q^1\big) & \end{array}. \tag{1.10}$$

A way to solve the system (1.10) is to find a equivalent problem with divergence free (see [1, 7]). Due to the expression of the nonhonmogeneous divergence term $\operatorname{div}\mathbf{w}$, we look for a solution $\mathbf{u}$ of (1.10) under the form $\mathbf{u}=\mathbf{v}+\mathbf{w}$. The new system in the variables $(\mathbf{v},p,q)$ is

$$\begin{array}{rcll}\mathbf{v}_t-\operatorname{div}\sigma(\mathbf{v},p) & = & \mathbf{c}\chi_\omega+\overline{\mathbf{F}}[\mathbf{u},p,q] & (Q_T^0)\\ \operatorname{div}\mathbf{v} & = & 0 & (Q_T^0)\\ \mathbf{v} & = & Zq'\mathbf{e}_2 & (\Sigma_T^{s,0})\\ \mathbf{v} & = & \mathbf{0} & (\Sigma_T)\\ q''+Aq & = & \Pi_N p+\overline{h}[\mathbf{u},q] & (0,T)\\ \big(\mathbf{v}(0),q(0),q'(0)\big) & = & \big(\mathbf{u}^0-\mathbf{w}(0),q^0,q^1\big) & \end{array}. \tag{1.11}$$

Indeed, the formula of $\mathbf{w}[\mathbf{u},q]$ gives us directly that $\mathbf{w}(0)=\frac{1}{1-\varepsilon}\left(-\eta^0u_1^0\mathbf{e}_1+(z-\varepsilon)\eta_x^0u_1^0\mathbf{e}_2\right)$ only depends on $(\mathbf{u}^0,q^0,q^1)$ and that $\mathbf{w}[\mathbf{u},q]_{|\Gamma}=\mathbf{0}$ for $(\mathbf{u},p,q)$ solution of the system (1.10). Furthermore, the term $\Pi_N\Big[-2\nu v_{2,z}\Big]$ does not appear in the right-hand side of $(1.11)_5$ because if $\mathbf{v}$ in $\mathbf{H}^{2,1}(Q_T^0)$ is solution of (1.11) then $\operatorname{div}\mathbf{v}=0$ and $v_1=0$ on Γ_0, which together give that $v_{2,z}=0$ on Γ_0^s.

In system (1.11), $\overline{\mathbf{F}}$ and $\overline{h}$ are defined by

$$\overline{\mathbf{F}}[\mathbf{u},p,q]=\mathbf{F}[\mathbf{u},p,q]+\nu\Delta\mathbf{w}[\mathbf{u},q]-\mathbf{w}[\mathbf{u},q]_t,\qquad \overline{h}[\mathbf{u},q]=h[\mathbf{u},q]-2\nu\Pi_N\Big[w_{2,z}[\mathbf{u},q]\Big] \tag{1.12}$$

with

$$\mathbf{v}=v_1\mathbf{e}_1+v_2\mathbf{e}_2\quad\text{and}\quad\mathbf{w}[\mathbf{u},q]=w_1[\mathbf{u},q]\mathbf{e}_1+w_2[\mathbf{u},q]\mathbf{e}_2.$$

From now on, we denote

$$\mathbf{v}^0=\mathbf{u}^0-\mathbf{w}[\mathbf{u},q](0). \tag{1.13}$$

On the other hand, we have to add a compatibility condition at time $t=0$ for $(\mathbf{v}^0,q^0,q^1)$:

$$\begin{array}{rcll}\operatorname{div}(\mathbf{v}^0) & = & 0 & (\Omega_0)\\ \mathbf{v}^0 & = & Zq^1\mathbf{e}_2 & (\Gamma_0^s)\\ \mathbf{v}^0 & = & \mathbf{0} & (\Gamma)\end{array}. \tag{1.14}$$

For $(\mathbf{u}^0,q^0,q^1)$ the compatibility conditions are

$$\begin{array}{rcll}\operatorname{div}\Big(\mathbf{u}^0+\frac{1}{1-\varepsilon}\left(Zq^0u_1^0\mathbf{e}_1-(z-\varepsilon)Z_xq^0u_1^0\mathbf{e}_2\right)\Big) & = & 0 & (\Omega_0)\\ \mathbf{u}^0 & = & Zq^1\mathbf{e}_2 & (\Gamma_0^s)\\ \mathbf{u}^0 & = & \mathbf{0} & (\Gamma)\end{array}. \tag{1.15}$$

2 Null controllability of the linearized system with nonhomogeneous right-hand sides.

Fixing intial data $(\mathbf{v}^0, q^0, q^1)$ and right-hand sides $(\overline{\mathbf{F}}, \overline{h})$, our goal in this section is to prove the null controllability of system (2.1).

$$\begin{array}{rcll}\mathbf{v}_t - \operatorname{div}\sigma(\mathbf{v},p) & = & \mathbf{c}\chi_\omega + \overline{\mathbf{F}} & (Q_T^0)\\ \operatorname{div}\mathbf{v} & = & 0 & (Q_T^0)\\ \mathbf{v} & = & Zq'\mathbf{e}_2 & (\Sigma_T^{s,0})\\ \mathbf{v} & = & \mathbf{0} & (\Sigma_T)\\ q''+Aq & = & \Pi_N p + \overline{h} & (0,T)\\ \big(\mathbf{v}(0), q(0), q'(0)\big) & = & \big(\mathbf{v}^0, q^0, q^1\big) & \end{array}. \tag{2.1}$$

This section is split into three parts. First, in section 2.1, we introduce an auxiliary linear system and we state a result of controllability for this system under some assumptions. In section 2.2, we set system (2.1) in the abstract general setting of the previous section. Then, in the last section, we prove the controllability of system (2.1).

2.1 An auxiliary result.

This part is adapted from [5]. We consider the following abstract linear system:

$$z'(t) = Az(t) + Bu(t) + Jf(t), \qquad z(0) = z^0. \tag{2.2}$$

Here, U, H, F are Hilbert spaces and A is an unbounded linear operator generator of an analytic semigroup on H denoted $(\mathrm{e}^{tA})_{t\geq 0}$. B and J are two linear continuous operators respectively from U into H and from F into H, z^0 is an element of H.

Let us introduce weight functions ρ_i $(i = 1, 2, 3)$ defined by

$$\rho_i : [0,T] \to \mathbb{R} \text{ continuous functions satisfying } \rho_i(T) = 0, \quad \rho_i(t) > 0 \; \forall t \in [0,T). \tag{2.3}$$

Then, we define three time-dependent weighted function spaces $\mathfrak{F}$, $\mathfrak{Z}$ and $\mathfrak{U}$ by

$$\begin{array}{rcl}\mathfrak{F} & = & \left\{ f \in L^2(0,T;F) \text{ s.t. } \rho_1^{-1} f \in L^2(0,T;F)\right\},\\ \mathfrak{Z} & = & \left\{ z \in L^2(0,T,H) \text{ s.t. } \rho_2^{-1} z \in L^2(0,T;H)\right\},\\ \mathfrak{U} & = & \left\{ u \in L^2(0,T,U) \text{ s.t. } \rho_3^{-1} u \in L^2(0,T;U)\right\}.\end{array}$$

In this general abstract setting, we prove the following lemma:

Lemma 2.1. *We have the equivalence between*

(i) For any ψ in $L^2(0,T;H)$, the solution ϕ of

$$-\phi'(t) = A^*\phi(t) + \psi(t), \qquad \phi(T) = 0 \tag{2.4}$$

satisfies the estimate

$$\|\phi(0)\|_H^2 + \int_0^T \rho_1^2(t)\|J^*\phi(t)\|_F^2 \leq C\left(\int_0^T \rho_2^2(t)\|\psi(t)\|_H^2 + \int_0^T \rho_3^2(t)\|B^*\phi(t)\|_U^2\right). \tag{2.5}$$

(ii) For any (z^0, f) in $H \times \mathfrak{F}$, there exists u in $\mathfrak{U}$ such that the solution z of (2.2) belongs to $\mathfrak{Z}$.

Proof. Remember that the general form of solution for system (2.2) can be written *via* the Duhamel formula

$$z(t) = \mathrm{e}^{tA}z^0 + \int_0^t \mathrm{e}^{(t-s)A}Bu(s)\mathrm{d}s + \int_0^t \mathrm{e}^{(t-s)A}Jf(s)\mathrm{d}s$$

which can also be written

$$z(t) - \int_0^t \mathrm{e}^{(t-s)A} Bu(s)\mathrm{d}s = \mathrm{e}^{tA} z^0 + \int_0^t \mathrm{e}^{(t-s)A} Jf(s)\mathrm{d}s.$$

We introduce two operators L_T and M_T as follows

$$\begin{array}{rccl} L_T: & H \times \mathfrak{F} & \longrightarrow & L^2(0,T;H) \\ & (z^0, f) & \longmapsto & \left(t \mapsto \mathrm{e}^{tA} z^0 + \displaystyle\int_0^t \mathrm{e}^{(t-s)A} Jf(s)\mathrm{d}s \right) \end{array}$$

and

$$\begin{array}{rccl} M_T: & \mathfrak{Z} \times \mathfrak{U} & \longrightarrow & L^2(0,T;H) \\ & (z, u) & \longmapsto & \left(t \mapsto z(t) - \displaystyle\int_0^t \mathrm{e}^{(t-s)A} Bu(s)\mathrm{d}s \right). \end{array}$$

Then, condition (ii) of the Lemma is equivalent to

$$\mathrm{Range}\, L_T \subset \mathrm{Range}\, M_T.$$

This last inclusion is equivalent to the existence of a constant $C > 0$ such that

$$\|L_T^* \psi\|_{H \times \mathfrak{F}'} \leq C \|M_T^* \psi\|_{\mathfrak{Z}' \times \mathfrak{U}'} \qquad \text{for all } \psi \in L^2(0,T;H). \tag{2.6}$$

The spaces $\mathfrak{F}'$, $\mathfrak{Z}'$ and $\mathfrak{U}'$ are the dual spaces of $\mathfrak{F}$, $\mathfrak{Z}$ and $\mathfrak{U}$:

$$\begin{array}{rcl} \mathfrak{F}' & = & \left\{ f \in L^2(0,T,F) \text{ s.t. } \rho_1 f \in L^2(0,T;F) \right\}, \\ \mathfrak{Z}' & = & \left\{ z \in L^2(0,T,H) \text{ s.t. } \rho_2 z \in L^2(0,T;H) \right\}, \\ \mathfrak{U}' & = & \left\{ u \in L^2(0,T,U) \text{ s.t. } \rho_3 u \in L^2(0,T;U) \right\} \end{array}$$

with the identifications $H \equiv H'$, $F' \equiv F$ and $U \equiv U'$.

By a simple calculation, we get, for ϕ solution of (2.4),

$$\begin{array}{rccl} L_T^*: & L^2(0,T;H) & \longrightarrow & H \times \mathfrak{F}', \\ & \psi & \longmapsto & (\phi(0), J^*\phi) \end{array} \qquad \begin{array}{rccl} M_T^*: & L^2(0,T;H) & \longrightarrow & \mathfrak{Z}' \times \mathfrak{U}'. \\ & \psi & \longmapsto & (\psi, B^*\phi) \end{array}$$

Then, (2.6) becomes

$$\|\phi(0)\|_H^2 + \int_0^T \rho_1^2(t) \|J^*\phi(t)\|_F^2 \leq C \left(\int_0^T \rho_2^2(t) \|\psi(t)\|_H^2 + \int_0^T \rho_3^2(t) \|B^*\phi(t)\|_U^2 \right),$$

which is exactly (2.5). □

Then, we have the following stronger result:

Theorem 2.2. *Under the hypothesis of Lemma 2.1, assume that (i) holds. Then we can define a linear operator U_T from $H \times \mathfrak{F}$ into $\mathfrak{U}$ by*

$$\begin{array}{rccl} U_T: & H \times \mathfrak{F} & \longrightarrow & \mathfrak{U} \\ & (z^0, f) & \longmapsto & u_{(z^0,f)}, \end{array}$$

such that the solution z of system (2.2) corresponding with the control $u_{(z^0,f)}$ belongs to $\mathfrak{Z}$.

Moreover, if z^0 belongs to $D((-A)^{1/2})$ and if there exists ρ_0 in $\mathcal{C}^2([0,T];\mathbb{R})$ such that

$$\begin{array}{l} \rho_0(t) \geq 0 \quad \forall t \in (0,T) \quad \text{and} \quad \rho_0(t) = 0 \Longleftrightarrow t = T, \\ \dfrac{\rho_i}{\rho_0} \in L^\infty(0,T) \text{ for } i = 1,2,3, \quad \dfrac{\rho_0' \rho_j}{\rho_0^2} \in L^\infty(0,T) \text{ for } j = 1 \text{ or } j = 2, \end{array} \tag{2.7}$$

then, z satisfies

$$\frac{z}{\rho_0} \in L^2(0,T;D(-A)) \cap H^1(0,T;H) \cap \mathcal{C}([0,T];D((-A)^{1/2})),$$

with the estimate

$$\left\| \frac{z}{\rho_0} \right\|_{L^2(0,T;D(-A)) \cap H^1(0,T;H) \cap \mathcal{C}([0,T];D((-A)^{1/2}))} \leq C \left(\|z^0\|_{D((-A)^{1/2})} + \|f\|_{\mathfrak{F}} \right).$$

Proof. We begin by proving the existence of the bounded linear operator U_T. Assuming condition (i) in Lemma 2.1, we know that there exists for any initial data z^0 in H and right-hand side f in $\mathfrak{F}$ at least a function u in $\mathfrak{U}$ such that z belongs to $\mathfrak{Z}$. Now, we consider the following functional

$$\mathcal{J}(z,u) = \frac{1}{2}\|z\|^2_{\mathfrak{F}} + \frac{1}{2}\|u\|^2_{\mathfrak{U}}.$$

Then, we can find among all the previous control u, the one minimizing this functional, with the corresponding z. Thanks to the observability inequality, a direct calculation gives that this control $\overline{u}$ satisfies the estimate

$$\|\overline{u}\|_{\mathfrak{U}} \leq C\Big(\|z^0\|_H + \|f\|_{\mathfrak{F}}\Big).$$

Denoting $\overline{u} = U_T(z^0, f)$, then U_T is a linear operator from $H \times \mathfrak{F}$ into $\mathfrak{U}$. Furthermore, it is bounded thanks to the previous inequality.

The second part relies on the following classical proposition:

Proposition 2.3. *Let $\mathcal{A} : D(\mathcal{A}) \subset \mathbb{X}$ into $\mathbb{X}$ where $\mathbb{X}$ is a Hilbert space and $\mathcal{A}$ an operator generator of an analytic semigroup on $D(\mathcal{A})$ with a compact resolvent in $\mathbb{X}$. If $\mathcal{Y}^0$ belongs to $D((-\mathcal{A})^{1/2})$ and $\mathcal{B}$ belongs to $L^2(0,T;\mathbb{X})$, then equation*

$$\mathcal{Y}'(t) = \mathcal{A}\mathcal{Y}(t) + \mathcal{B}(t), \qquad \mathcal{Y}(0) = \mathcal{Y}^0$$

admits a unique solution $\mathcal{Y}$ in $L^2(0,T;D(\mathcal{A})) \cap H^1(0,T;\mathbb{X}) \cap \mathcal{C}([0,T];D((-\mathcal{A})^{1/2}))$. Furthermore, we get the estimate

$$\|\mathcal{Y}\|_{L^2(0,T;D(\mathcal{A}))\cap H^1(0,T;\mathbb{X})\cap\mathcal{C}([0,T];D((-\mathcal{A})^{1/2}))} \leq C\Big(\|\mathcal{Y}^0\|_{D((-\mathcal{A})^{1/2})} + \|\mathcal{B}\|_{L^2(0,T;\mathbb{X})}\Big).$$

Because $u_{(z^0,f)}$, f and z^0 belongs respectively to $L^2(0,T;H)$, $L^2(0,T;F)$ and $D((-A)^{1/2})$, we can apply the previous proposition and we get that that the solution z of (2.2) belongs to $L^2(0,T;D(-A)) \cap H^1(0,T;H) \cap \mathcal{C}([0,T];D((-A)^{1/2}))$. Futhermore, dividing equation (2.2) by ρ_0, we obtain

$$\left(\frac{z}{\rho_0}\right)' = A\left(\frac{z}{\rho_0}\right) + \frac{f}{\rho_0} - \frac{\rho_0'}{\rho_0^2}z, \qquad \left(\frac{z}{\rho_0}\right)(0) = \frac{z^0}{\rho_0(0)}. \tag{2.8}$$

Then, we get that $\left(\frac{z}{\rho_0}\right)'$ belongs to $L^2(0,T;H)$ provided that $\frac{z}{\rho_0}$ belongs to $L^2(0,T;D(-A))$. From the previous lemma, we have $\frac{z}{\rho_2}$ in $L^2(0,T;H)$; second, from the choice of the function ρ_0, we have

$$-\frac{\rho_0'}{\rho_0^2}z = -\frac{\rho_0'\rho_2}{\rho_0^2}\frac{z}{\rho_2}$$

which belongs to $L^2(0,T;H)$. Then, applying Proposition 2.3 to system (2.8), we get that

$$\frac{z}{\rho_0} \in L^2(0,T;D(-A)) \cap H^1(0,T;H) \cap \mathcal{C}([0,T];D((-A)^{1/2}))$$

with the estimate

$$\left\|\frac{z}{\rho_0}\right\|_{L^2(0,T;D(-A))\cap H^1(0,T;H)\cap\mathcal{C}([0,T];D((-A)^{1/2}))} \leq C\Big(\|z^0\|_{D((-A)^{1/2})} + \|f\|_{\mathfrak{F}}\Big).$$

□

2.2 Equivalent system.

In this section, we fix the initial data $(\mathbf{v}^0, q^0, q^1)$ in $X^0_{\mathbf{cc}}$ defined by

$$X^0 = \mathbf{H}^1(\Omega_0) \times \mathbb{R}^N \times \mathbb{R}^N$$

and

$$X^0_{\mathbf{cc}} = \Big\{(\mathbf{z}^0, k^0, k^1) \in X^0 \text{ such that } (\mathbf{z}^0, k^0, k^1) \text{ verifies (1.14)}\Big\}.$$

The space X^0 is equipped with the norm

$$\|(\mathbf{z}^0, k^0, k^1)\|_{X^0} = \Big(\|\mathbf{z}^0\|^2_{\mathbf{H}^1(\Omega_0)} + |A^{1/2}k^0|^2_{\mathbb{R}^N} + |k^1|^2_{\mathbb{R}^N}\Big)^{1/2}.$$

The right-hand side $(\overline{\mathbf{F}}, \overline{h})$ belongs to the time-dependent weighted function space $\overline{\mathcal{W}}_T$ (see below). Let us define

$$\mathbb{V} = \mathbf{V}^0(\Omega_0) \times \mathbb{R}^N \times \mathbb{R}^N \tag{2.9}$$

equipped with the norm

$$\Big\|(\mathbf{v}, q, r)\Big\|^2_{\mathbb{V}} = \|\mathbf{v}\|^2_{\mathbf{L}^2(\Omega_0)} + |A^{1/2}q|^2_{\mathbb{R}^N} + |r|^2_{\mathbb{R}^N} \quad \text{for all } (\mathbf{v}, q, r) \in \mathbb{V}.$$

We introduce the spaces

$$\begin{aligned}
\overline{\mathcal{W}}_T &= \Big\{(\mathbf{G}, g) \in L^2(0, T; \mathbf{L}^2(\Omega_0) \times \mathbb{R}^N) \text{ s.t. } \rho_1^{-1}(\mathbf{G}, g) \text{ belongs to } L^2(0, T; \mathbf{L}^2(\Omega_0) \times \mathbb{R}^N)\Big\},\\
\mathcal{Z}_T &= \Big\{(\mathbf{z}, r) \text{ s.t. } (\mathbf{z}, r, r') \text{ and } \rho_2^{-1}(\mathbf{z}, r, r') \text{ are in } L^2(0, T; \mathbb{V})\Big\},\\
\mathcal{U}_T &= \Big\{\mathbf{d} \in L^2(0, T; \mathbf{L}^2(\omega)) \text{ s.t. } \rho_3^{-1}\mathbf{d} \text{ is in } L^2(0, T; \mathbf{L}^2(\omega))\Big\}.
\end{aligned}$$

These spaces are equipped with the norms

$$\begin{aligned}
\|(\mathbf{G}, g)\|_{\overline{\mathcal{W}}_T} &= \int_0^T \rho_1^{-2}(t)\Big[\|\mathbf{G}(t)\|^2_{\mathbf{L}^2(\Omega_0)} + |g(t)|^2_{\mathbb{R}^N}\Big]\mathrm{d}t \quad && \text{for all } (\mathbf{G}, g) \in \overline{\mathcal{W}}_T,\\
\|(\mathbf{z}, r)\|_{\mathcal{Z}_T} &= \int_0^T \rho_2^{-2}(t)\|(\mathbf{z}, r, r')(t)\|^2_{\mathbb{V}}\mathrm{d}t && \text{for all } (\mathbf{z}, r) \in \mathcal{Z}_T,\\
\|\mathbf{d}\|_{\mathcal{U}_T} &= \int_0^T \rho_3^{-2}(t)\|\mathbf{d}(t)\|^2_{\mathbf{L}^2(\omega)}\mathrm{d}t && \text{for all } \mathbf{d} \in \mathcal{U}_T.
\end{aligned}$$

We now write system (2.1) as a first order in time linear partial differential equation. Let us introduce the so-called Leray projection P from $\mathbf{L}^2(\Omega_0)$ in $\mathbf{V}^0_{\mathbf{n}}(\Omega_0)$ where

$$\mathbf{V}^0_{\mathbf{n}}(\Omega_0) = \Big\{\mathbf{u} \in \mathbf{L}^2(\Omega_0) \text{ such that } \mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \Gamma_0 \text{ and } \operatorname{div}\mathbf{u} = 0 \text{ in } \Omega_0\Big\}.$$

We split system (2.1) using the equality $\mathbf{v} = P\mathbf{v} + (I - P)\mathbf{v}$. Let us denote $\mathbf{v}_e = P\mathbf{v}$ and $\mathbf{v}_s = (I - P)\mathbf{v}$. Each part of the velocity field $\mathbf{v}$ is associated with a corresponding pressure term p_e and p_s. We have the following proposition:

Proposition 2.4. *System* (2.1) *can be splitted into two systems. One, system* (2.10)*, is an evolutionary system in the variables* $(\mathbf{v}_e, q_1, q_2)$ *(where* $q_1 = q$ *and* $q_2 = q'$*) and the other, system* (2.11)*, is a stationary system giving* $(\mathbf{v}_s, p_e, p_s)$ *as functions of* $(\mathbf{v}_e, q_1, q_2)$*. That is system* (2.1) *is equivalent to* (2.10)–(2.11) *(see the notation below):*

$$\begin{aligned}
\begin{pmatrix}\mathbf{v}_e\\ q_1\\ q_2\end{pmatrix}' &= K_s\begin{pmatrix} A_0 & 0 & (-A_0)PD_s\\ 0 & 0 & I_N\\ \nu\Pi_N\mathcal{N}(\Delta(\cdot)\cdot\mathbf{n}) & -A & 0\end{pmatrix}\begin{pmatrix}\mathbf{v}_e\\ q_1\\ q_2\end{pmatrix}\\
&\quad + K_s\left[\begin{pmatrix}P\overline{\mathbf{F}}\\ 0\\ \Pi_N\pi(\overline{\mathbf{F}}) + \overline{h}\end{pmatrix} + \begin{pmatrix}P(\mathbf{c}\chi_\omega)\\ 0\\ \Pi_N\pi_0(\mathbf{c}\chi_\omega)\end{pmatrix}\right]\\
(\mathbf{v}_e(0), q_1(0), q_2(0))^T &= (P\mathbf{v}^0, q^0, q^1)^T
\end{aligned}\tag{2.10}$$

and secondly

$$\begin{array}{rcll} \mathbf{v}_s & = & \nabla\mathcal{N}_s(Zq_2) & (Q_T^0) \\ p_e & = & \mathcal{N}(\Delta\mathbf{v}_e\cdot\mathbf{n}) & (Q_T^0) \\ p_s & = & \pi(\overline{\mathbf{F}})+\pi_0(\mathbf{c}\chi_\omega)-\mathcal{N}_s(Zq_2') & (Q_T^0) \\ p & = & p_e+p_s & (Q_T^0) \\ \mathbf{v} & = & \mathbf{v}_e+\mathbf{v}_s & (Q_T^0) \end{array} \tag{2.11}$$

Furthermore, system (2.10) *is exactly under the form of system* (2.2).

Proof. We use a method due to Raymond (see [9]). In particular, we adapt here the decomposition of a similar system made in [10]. We write it in this paper for sake of completness. From the Stokes system

$$\begin{array}{rcll} \mathbf{v}_t-\nu\Delta\mathbf{v}+\nabla p & = & \mathbf{c}\chi_\omega+\overline{\mathbf{F}} & (Q_T^0) \\ \operatorname{div}\mathbf{v} & = & 0 & (Q_T^0) \\ \mathbf{v} & = & Zq'\mathbf{e}_2 & (\Sigma_T^{s,0}) \\ \mathbf{v} & = & \mathbf{0} & (\Sigma_T) \\ \mathbf{v}(0) & = & \mathbf{v}^0 & (\Omega_0) \end{array},$$

we get the following equivalent system

$$\begin{array}{rcll} \mathbf{v}_{e,t}-\nu\Delta\mathbf{v}_e+\nabla p_e & = & P(\mathbf{c}\chi_\omega)+P\overline{\mathbf{F}} & (Q_T^0) \\ \mathbf{v}_e & = & -\gamma_\tau\mathbf{v}_s & (\Sigma_T^0) \\ \mathbf{v}_e(0) & = & P\mathbf{v}^0 & (\Omega_0) \\ \mathbf{v}_s & = & \nabla\mathcal{N}_s(Zq') & (Q_T^0) \\ p_s & = & \pi(\overline{\mathbf{F}})+\pi_0(\mathbf{c}\chi_\omega)-\mathcal{N}_s(Zq'') & (Q_T^0) \\ \mathbf{v} & = & \mathbf{v}_e+\mathbf{v}_s & (Q_T^0) \\ p & = & p_s+p_e & (Q_T^0) \end{array}. \tag{2.12}$$

In (2.12), we denote $\mathcal{N}_s(\cdot)=\mathcal{N}(\cdot\chi_{\Gamma_0^s})$ where $\mathcal{N}$ the operator from $H^\sigma(\Gamma_0)$ to $H^{\sigma+3/2}(\Omega_0)$ (for $\sigma\geq -1/2$) defined by $r=\mathcal{N}(j)$ for j in $H^\sigma(\Gamma_0)$ if and only if

$$\Delta r = 0 \quad \text{in } \Omega_0, \qquad \frac{\partial r}{\partial \mathbf{n}} = j \quad \text{on } \Gamma_0.$$

and π and π_0 are operators from $\mathbf{L}^2(\Omega_0)$ into $H^1(\Omega_0)$ defined by

$$\left\{\begin{array}{rcll} \Delta\pi(\overline{\mathbf{F}}) & = & \operatorname{div}\overline{\mathbf{F}} & (\Omega_0) \\ \dfrac{\partial\pi(\overline{\mathbf{F}})}{\partial\mathbf{n}} & = & \overline{\mathbf{F}}\cdot\mathbf{n} & (\Gamma_0) \end{array}\right. \quad\text{and}\quad \left\{\begin{array}{rcll} \Delta\pi_0(\mathbf{c}\chi_\omega) & = & \operatorname{div}(\mathbf{c}\chi_\omega) & (\Omega_0) \\ \dfrac{\partial\pi_0(\mathbf{c}\chi_\omega)}{\partial\mathbf{n}} & = & 0 & (\Gamma_0) \end{array}\right. . \tag{2.13}$$

We have an explicit formula for π and π_0:

$$\pi(\overline{\mathbf{F}})=-(-\Delta_D)^{-1}(\operatorname{div}\overline{\mathbf{F}})+\mathcal{N}((\overline{\mathbf{F}}+\nabla(-\Delta_D)^{-1}(\operatorname{div}\overline{\mathbf{F}}))\cdot\mathbf{n}),$$

$$\pi_0(\mathbf{c}\chi_\omega)=-(-\Delta_D)^{-1}(\operatorname{div}(\mathbf{c}\chi_\omega))+\mathcal{N}((\nabla(-\Delta_D)^{-1}(\operatorname{div}(\mathbf{c}\chi_\omega)))\cdot\mathbf{n}),$$

where $\pi_1=-(-\Delta_D)^{-1}(g)$ if and only if $\pi_1\in H_0^1(\Omega_0)$ and $\Delta\pi_1=g$ in Ω_0 for any $g\in H^{-1}(\Omega_0)$.

From the first equation in (2.12), we get that p_e satisfies for any time t in $(0,T)$:

$$\Delta p_e(t) = 0 \quad \text{in } \Omega_0, \qquad \frac{\partial p_e(t)}{\partial\mathbf{n}} = \nu\Delta\mathbf{v}_e(t) \quad \text{on } \Gamma_0,$$

that is $p_e=\nu\mathcal{N}(\Delta\mathbf{v}_e\cdot\mathbf{n})$.

In conclusion, $p=p_s+p_e$ is equal to

$$p=\pi(\overline{\mathbf{F}})+\pi_0(\mathbf{c}\chi_\omega)-\mathcal{N}_s(Zq'')+\nu\mathcal{N}(\Delta\mathbf{v}_e\cdot\mathbf{n}) \quad \text{in } \Omega_0.$$

Then the beam equation becomes

$$(I_N+\Pi_N\mathcal{N}_s(Z\,(\cdot)))q''+Aq=\nu\Pi_N\mathcal{N}(\Delta\mathbf{v}_e\cdot\mathbf{n})+\Pi_N\pi(\overline{\mathbf{F}})+\Pi_N\pi_0(\mathbf{c}\chi_\omega)+\overline{h}.$$

System (2.1) is equivalent to system

$$
\begin{array}{rcll}
\mathbf{v}_{e,t} - \Delta \mathbf{v}_e + \nabla p_e &=& P(\mathbf{c}\chi_\omega) + P\overline{\mathbf{F}} & (Q_T^0) \\
\mathbf{v}_e &=& -\gamma_\tau \mathbf{v}_s & (\Sigma_T^0) \\
\mathbf{v}_e(0) &=& P\mathbf{v}^0 & (\Omega_0) \\
\mathbf{v}_s &=& \nabla \mathcal{N}_s(Zq') & (Q_T^0) \\
p_s &=& \pi(\overline{\mathbf{F}}) - \mathcal{N}_s(Zq'') & (Q_T^0) \\
(I_N + \Pi_N \mathcal{N}_s(Z(\cdot)))q'' + Aq &=& \nu \Pi_N \mathcal{N}(\Delta \mathbf{v}_e \cdot \mathbf{n}) + \Pi_N \pi(\overline{\mathbf{F}}) + \Pi_N \pi_0(\mathbf{c}\chi_\omega) + \overline{h} & (0,T) \\
\big(q(0), q'(0)\big) &=& \big(q^0, q^1\big) & \\
\mathbf{v} &=& \mathbf{v}_e + \mathbf{v}_s & (Q_T^0) \\
p &=& p_s + p_e & (Q_T^0)
\end{array} . \qquad (2.14)
$$

From this system, we can obtain an evolution equation. Indeed, $(\mathbf{v}_e, q, q')$ is uncoupled to $(\mathbf{v}_s, p_e, p_s)$. Then, we have first, with obvious notation $q = q_1$ and $q' = q_2$:

$$
\begin{array}{rcl}
\begin{pmatrix} \mathbf{v}_e \\ q_1 \\ q_2 \end{pmatrix}' &=& K_s \begin{pmatrix} A_0 & 0 & (-A_0)PD_s \\ 0 & 0 & I_N \\ \nu \Pi_N \mathcal{N}(\Delta(\cdot)\cdot \mathbf{n}) & -A & 0 \end{pmatrix} \begin{pmatrix} \mathbf{v}_e \\ q_1 \\ q_2 \end{pmatrix} \\
& & + K_s \left[\begin{pmatrix} P\overline{\mathbf{F}} \\ 0 \\ \Pi_N \pi(\overline{\mathbf{F}}) + \overline{h} \end{pmatrix} + \begin{pmatrix} P(\mathbf{c}\chi_\omega) \\ 0 \\ \Pi_N \pi_0(\mathbf{c}\chi_\omega) \end{pmatrix} \right] \\
(\mathbf{v}_e(0), q_1(0), q_2(0))^T &=& (P\mathbf{v}^0, q^0, q^1)^T
\end{array} \qquad (2.15)
$$

and secondly

$$
\begin{array}{rcll}
\mathbf{v}_s &=& \nabla \mathcal{N}_s(Zq_2) & (Q_T^0) \\
p_e &=& \mathcal{N}(\Delta \mathbf{v}_e \cdot \mathbf{n}) & (Q_T^0) \\
p_s &=& \pi(\overline{\mathbf{F}}) + \pi_0(\mathbf{c}\chi_\omega) - \mathcal{N}_s(Zq_2') & (Q_T^0) \\
p &=& p_e + p_s & (Q_T^0) \\
\mathbf{v} &=& \mathbf{v}_e + \mathbf{v}_s & (Q_T^0)
\end{array} \qquad (2.16)
$$

where K_s an isomorphism from $\mathbf{V}^0_{\mathbf{n}}(\Omega_0) \times \mathbb{R}^N \times \mathbb{R}^N$ into itself defined by

$$
K_s = \begin{pmatrix} \mathbf{Id} & 0 & 0 \\ 0 & I_N & 0 \\ 0 & 0 & (I_N + \Pi_N \mathcal{N}_s(Z(\cdot)))^{-1} \end{pmatrix}, \qquad (2.17)
$$

A_0 is the Stokes operator defined by $D(A_0) = \mathbf{V}^2(\Omega_0) \cap \mathbf{V}^1_0(\Omega_0)$ in $\mathbf{V}^0_{\mathbf{n}}(\Omega_0)$ and $A_0 \mathbf{z}_e = \nu P \Delta \mathbf{z}_e$, for all $\mathbf{z}_e$ in $D(A_0)$. The operator D_s is a lifting of the nonhomogeneous Dirichlet condition $\mathbf{v} = Zq_2\mathbf{e}_2$ on Γ^s_0 defined from $\mathbb{R}^N$ into $\mathbf{V}^2(\Omega_0)$ for r in $\mathbb{R}^N$ by $\mathbf{z} = D_s r$ if and only if there exists a function ρ in $\mathcal{H}^1(\Omega_0)$ such that

$$
\begin{array}{rcll}
-\nu \Delta \mathbf{z} + \nabla \rho &=& \mathbf{0} & (\Omega_0) \\
\operatorname{div} \mathbf{z} &=& 0 & (\Omega_0) \\
\mathbf{z} &=& Zr\mathbf{e}_2 & (\Gamma^s_0) \\
\mathbf{z} &=& \mathbf{0} & (\Gamma)
\end{array} .
$$

We finally get that system (2.10)–(2.11) is equivalent to system (2.14), that is system (2.10)–(2.11) is equivalent to system (2.1).

We now can identify notations from (2.10) with those from the previous section. The Hilbert spaces H, U and F are now respectively

$$
\mathbb{V}_{\mathbf{n}} = \mathbf{V}^0_{\mathbf{n}}(\Omega_0) \times \mathbb{R}^N \times \mathbb{R}^N, \qquad \mathbf{L}^2(\omega) \quad \text{and} \quad \mathbf{L}^2(\Omega_0) \times \mathbb{R}^N.
$$

The operator A in (2.2) is remplaced by

$$
\mathcal{A} = K_s \begin{pmatrix} A_0 & 0 & (-A_0)PD_s \\ 0 & 0 & I_N \\ \nu \Pi_N \mathcal{N}(\Delta(\cdot)\cdot \mathbf{n}) & -A & 0 \end{pmatrix}
$$

which is defined from

$$D(\mathcal{A}) = \Big\{(\mathbf{z}_e, q_1, q_2) \in \mathbf{V}^2(\Omega_0) \cap \mathbf{V}^0_{\mathbf{n}}(\Omega_0) \times \mathbb{R}^N \times \mathbb{R}^N \text{ s.t. } \mathbf{z}_e = -\gamma_\tau \nabla \mathcal{N}_s(Zq_2) \text{ on } \Gamma_0\Big\}$$

in $\mathbb{V}_{\mathbf{n}}$. We have

$$\begin{array}{rccl} B: & \mathbf{L}^2(\omega) & \longrightarrow & \mathbb{V}_{\mathbf{n}} \\ & \mathbf{c}\chi_\omega & \longmapsto & \begin{pmatrix} P(\mathbf{c}\chi_\omega) \\ 0 \\ (I_N + \Pi_N \mathcal{N}_s(Z(\cdot)))^{-1}\Pi_N \pi_0(\mathbf{c}\chi_\omega) \end{pmatrix} \end{array}, \qquad J\begin{pmatrix} \overline{\mathbf{F}} \\ \overline{h} \end{pmatrix} = J_1 \overline{\mathbf{F}} + J_2 \overline{h}$$

with

$$\begin{array}{rccl} J_1: & \mathbf{L}^2(\Omega_0) & \longrightarrow & \mathbb{V}_{\mathbf{n}} \\ & \overline{\mathbf{F}} & \longmapsto & K_s\left(P\overline{\mathbf{F}}, 0, \Pi_N \pi_0(\overline{\mathbf{F}})\right)^T \end{array}, \qquad \begin{array}{rccl} J_2: & \mathbb{R}^N & \longrightarrow & \mathbb{V}_{\mathbf{n}} \\ & \overline{h} & \longmapsto & K_s\left(\mathbf{0}, 0, \overline{h}\right)^T \end{array}.$$

This gives, with $f = (\overline{\mathbf{F}}, \overline{h})$ in $\mathbf{L}^2(\Omega_0) \times \mathbb{R}^N$,

$$Jf = \Big(P\overline{\mathbf{F}}, 0, (I_N + \Pi_N \mathcal{N}_s(Z(\cdot)))^{-1}\Big[\Pi_N \pi(\overline{\mathbf{F}}) + \overline{h}\Big]\Big)^T.$$

Finally,

$$z = (\mathbf{v}_e, q_1, q_2)^T \qquad \text{and} \qquad z^0 = \left(P\mathbf{v}^0, q^0, q^1\right)^T.$$

□

2.3 Null Controllability of system (2.1).

We can now state the main result of this section:

Theorem 2.5. *Let $(\mathbf{v}^0, q^0, q^1)$ be in $X^0_{\mathbf{cc}}$. There exists a linear bounded operator $\overline{U}_T$ from $\mathbb{V} \times \overline{\mathcal{W}}_T$ into $L^2(0,T;\mathbf{L}^2(\omega))$ such that for all $(\overline{\mathbf{F}}, \overline{h})$ in $\overline{\mathcal{W}}_T$ the solution of system (2.1) associated with the function $\mathbf{c} = \overline{U}_T\Big((\mathbf{v}^0, q^0, q^1), (\overline{\mathbf{F}}, \overline{h})\Big)$ in the right-hand side belongs to $\mathcal{X}_T$ defined by*

$$\mathcal{X}_T = \left\{(\mathbf{x}, \pi, r) \in X_T\,; \rho_0^{-1}(\mathbf{x}, \pi, r) \in X_T\right\} \text{ equipped with the norm } \|(\mathbf{x}, \pi, r)\|_{\mathcal{X}_T} = \left\|\rho_0^{-1}(\mathbf{x}, \pi, r)\right\|_{X_T}$$

where $X_T = \mathbf{H}^{2,1}(Q^0_T) \times L^2(0,T;\mathcal{H}^1(\Omega_0)) \times H^2(0,T;\mathbb{R}^N)$. Furthermore, we have the estimate:

$$\|(\mathbf{v}, p, q)\|_{\mathcal{X}_T} \leq C\Big(\|(\mathbf{v}^0, q^0, q^1)\|_{X^0} + \|(\overline{\mathbf{F}}, \overline{h})\|_{\overline{\mathcal{W}}_T}\Big).$$

That is, system (2.1) is null controllable at time $T > 0$:

$$\mathbf{v}(T) = \mathbf{0} \textit{ in } \Omega_0, \quad q(T) = 0 \quad \textit{and} \quad q'(T) = 0.$$

The proof of this proposition relies on the two previous sections. First, thanks to section 2.2, system (2.1) is equivalent to system (2.10)–(2.11). Then, we can apply results of section 2.1 to system (2.10). Finally, this results and an observability inequality finish the proof.

First, we want to write Lemma 2.1 for system (2.10). Thus, we have to calculate the adjoint operators $\mathcal{A}^*$, B^* and J^*.

Lemma 2.6. *We define the bilinear form ϕ on $\mathbb{V}_{\mathbf{n}}$ by*

$$\phi\Big((\mathbf{v}_e, q_1, q_2), (\mathbf{y}_e, k_1, k_2)\Big) = (\mathbf{v}_e, \mathbf{y}_e)_{\mathbf{L}^2(\Omega_0)} + (A^{1/2}q_1, A^{1/2}k_1)_{\mathbb{R}^N} + (q_2, (I_n + \Pi_N \mathcal{N}_s(Z(\cdot)))k_2)_{\mathbb{R}^N},$$

for $(\mathbf{v}_e, q_1, q_2)$ and $(\mathbf{y}_e, k_1, k_2)$ in $\mathbb{V}_{\mathbf{n}}$. Then, ϕ is a scalar product on $\mathbb{V}_{\mathbf{n}}$. We still denote $\mathbb{V}_{\mathbf{n}}$ the space $\mathbb{V}_{\mathbf{n}}$ endowed with this scalar product. In the following, we set

$$\langle \cdot, \cdot \rangle_{\mathbb{V}_{\mathbf{n}}} = \phi(\cdot, \cdot).$$

Proof. We have to prove that the operator $\Pi_N \mathcal{N}_s(Z\cdot) : \mathbb{R}^N \to \mathbb{R}^N$ is symmetric and positive. Let us take q_2 and k_2 in $\mathbb{R}^N$, we calculate $(q_2, \Pi_N \mathcal{N}_s(Zk_2))_{\mathbb{R}^N}$. By definition, the function $a = \mathcal{N}_s(Zk_2)$ belongs to $H^2(\Omega_0)$ and satisfies $\left\{ \begin{array}{lll} \Delta a & = 0 & (\Omega_0) \\ \dfrac{\partial a}{\partial \mathbf{n}} & = Zk_2\chi_{\Gamma_0^s} & (\Gamma_0) \end{array} \right.$. In the same way, we denote $b = \mathcal{N}_s(Zq_2)$. First, with the previous notation

$$\begin{array}{rcl} (q_2, \Pi_N \mathcal{N}_s(Zk_2))_{\mathbb{R}^N} & = & (Zq_2, a)_{L^2(\Gamma_0^s)} \\ & = & \left(\dfrac{\partial b}{\partial \mathbf{n}}, a \right)_{L^2(\Gamma_0^s)} \end{array}.$$

Second, an integration by parts gives

$$(\Delta b, a)_{L^2(\Omega_0)} = -(\nabla b, \nabla a)_{\mathbf{L}^2(\Omega_0)} + \left(\frac{\partial b}{\partial \mathbf{n}}, a \right)_{L^2(\Gamma_0^s)} \quad \text{or} \quad (b, \Delta a)_{L^2(\Omega_0)} = -(\nabla b, \nabla a)_{\mathbf{L}^2(\Omega_0)} + \left(b, \frac{\partial a}{\partial \mathbf{n}} \right)_{L^2(\Gamma_0^s)}.$$

That is, because $\Delta a = 0$ and $\Delta b = 0$ in Ω_0,

$$\left(\frac{\partial b}{\partial \mathbf{n}}, a \right)_{L^2(\Gamma_0^s)} = (\nabla b, \nabla a)_{\mathbf{L}^2(\Omega_0)}, \qquad \left(b, \frac{\partial a}{\partial \mathbf{n}} \right)_{L^2(\Gamma_0^s)} = (\nabla b, \nabla a)_{\mathbf{L}^2(\Omega_0)}.$$

Putting all the calculations together, we get

$$\begin{array}{rcl} (q_2, \Pi_N \mathcal{N}_s(Zk_2))_{\mathbb{R}^N} & = & (\nabla b, \nabla a)_{\mathbf{L}^2(\Omega_0)} \\ & = & (\Pi_N \mathcal{N}_s(Zq_2), k_2)_{\mathbb{R}^N} \end{array}.$$

To prove the positivity, we calculate $(q_2, \Pi_N \mathcal{N}_s(Zq_2))_{\mathbb{R}^N}$ for q_2 in $\mathbb{R}^N$. With the previous equality, we obtain

$$(q_2, \Pi_N \mathcal{N}_s(Zq_2))_{\mathbb{R}^N} = \|\nabla b\|^2_{\mathbf{L}^2(\Omega_0)},$$

which concludes the proof. □

Proposition 2.7. - *The operator $\mathcal{A}$ is a generator of an analytic semigroup on $\mathbb{V}_\mathbf{n}$. Furthermore, it has a compact resolvent. The adjoint operator $\mathcal{A}^*$ is given by $D(\mathcal{A}^*) = D(\mathcal{A})$ and*

$$\mathcal{A}^* = \begin{pmatrix} \mathbf{Id} & 0 & 0 \\ 0 & I_N & 0 \\ 0 & 0 & (I_N + \Pi_N \mathcal{N}_s(Z(\cdot)))^{-1} \end{pmatrix} \begin{pmatrix} A_0 & 0 & (-A_0)PD_s \\ 0 & 0 & -I_N \\ \nu \Pi_N \mathcal{N}(\Delta(\cdot) \cdot \mathbf{n}) & A & 0 \end{pmatrix}.$$

- *The operator B^* is defined from $\mathbb{V}_\mathbf{n}$ into $\mathbf{L}^2(\omega)$ by*

$$B^* \begin{pmatrix} \mathbf{y}_e \\ r_1 \\ r_2 \end{pmatrix} = (\mathbf{y}_e + \nabla \mathcal{N}_s(Zr_2))\chi_\omega.$$

The operator J^* is defined from $\mathbb{V}_\mathbf{n}$ into $\mathbf{L}^2(\Omega_0) \times \mathbb{R}^N$ by

$$J^* \begin{pmatrix} \mathbf{y}_e \\ r_1 \\ r_2 \end{pmatrix} = \Big((\mathbf{y}_e + \nabla \mathcal{N}_s(Zr_2)), r_2 \Big).$$

Proof. The first point of the proof can be easily adapted from [10, Section 3.] and is left to the reader.

We now prove the second point. Let $\mathbf{d}$ be in $\mathbf{L}^2(\omega)$ and $(\mathbf{y}_e, r_1, r_2)$ be in $\mathbb{V}_\mathbf{n}$, then, by definition of B,

$$\left\langle B\mathbf{d}, \begin{pmatrix} \mathbf{y}_e \\ r_1 \\ r_2 \end{pmatrix} \right\rangle_{\mathbb{V}_\mathbf{n}} = (P\mathbf{d}, \mathbf{y}_e)_{\mathbf{V}^0_\mathbf{n}(\Omega_0)} + (\Pi_N \pi_0(\mathbf{d}), r_2)_{\mathbb{R}^N}.$$

By an integration by parts, we have

$$(\pi_0(\mathbf{d}), \Delta q)_{L^2(\Omega_0)} = -(\nabla \pi_0(\mathbf{d}), \nabla q)_{\mathbf{L}^2(\Omega_0)} + \left(\pi_0(\mathbf{d}), \frac{\partial q}{\partial \mathbf{n}}\right)_{L^2(\partial\Omega_0)}. \tag{2.18}$$

Denoting $q = \mathcal{N}_s(Zr_2)$, from equation (2.18), we obtain

$$(\Pi_N \pi_0(\mathbf{d}), r_2)_{\mathbb{R}^N} = (\pi_0(\mathbf{d}), Zr_2)_{L^2(\Omega_0)} = \left(\pi_0(\mathbf{d}), \frac{\partial q}{\partial \mathbf{n}}\right)_{L^2(\partial\Omega_0)} = (\nabla \pi_0(\mathbf{d}), \nabla q)_{\mathbf{L}^2(\Omega_0)}.$$

Then, setting $\mathbf{y} = \mathbf{y}_e + \nabla q$, we see that $\mathbf{y}$ is an element of $\mathbf{V}^0(\Omega)$ satisfying

$$\mathbf{y} \cdot \mathbf{n} = Zr_2 \text{ on } \Gamma_0^s, \qquad \mathbf{y} \cdot \mathbf{n} = 0 \text{ on } \Gamma_0^s$$

and furthermore, thanks to the definition of $\pi_0(\mathbf{d})$ (see (2.13)), we have $\mathbf{d} = P(\mathbf{d}) + \nabla \pi_0(\mathbf{d})$. Thus,

$$\begin{aligned}(\mathbf{y}, \mathbf{d})_{\mathbf{L}^2(\Omega_0)} &= (\mathbf{y}_e + \nabla q, P(\mathbf{d}) + \nabla \pi_0(\mathbf{d}))_{\mathbf{L}^2(\Omega_0)} \\ &= (\mathbf{y}_e, P(\mathbf{d}))_{\mathbf{L}^2(\Omega_0)} + (\nabla q, \nabla \pi_0(\mathbf{d}))_{\mathbf{L}^2(\Omega_0)} + (\mathbf{y}_e, \nabla \pi_0(\mathbf{d}))_{\mathbf{L}^2(\Omega_0)} + (\nabla q, P(\mathbf{d}))_{\mathbf{L}^2(\Omega_0)}.\end{aligned}$$

To conclude, we see that $\mathbf{y}_e$ and $P(\mathbf{d})$ belong to $\mathbf{V}^0_{\mathbf{n}}(\Omega_0)$ whereas ∇q and $\nabla \pi_0(\mathbf{d})$ belongs to $(\mathbf{V}^0_{\mathbf{n}}(\Omega_0))^{\perp}$. Then,

$$(\mathbf{y}, \mathbf{d})_{\mathbf{L}^2(\Omega_0)} = (\mathbf{y}_e, P(\mathbf{d}))_{\mathbf{L}^2(\Omega_0)} + (\nabla q, \nabla \pi_0(\mathbf{d}))_{\mathbf{L}^2(\Omega_0)}.$$

Finally, putting all this calculations together, we get

$$\left\langle B(\mathbf{d}), \begin{pmatrix} \mathbf{y}_e \\ r_1 \\ r_2 \end{pmatrix} \right\rangle_{\mathbb{V}_{\mathbf{n}}} = (\mathbf{d}, \mathbf{y})_{\mathbf{L}^2(\Omega_0)} = (\mathbf{d}, \mathbf{y})_{\mathbf{L}^2(\omega)}.$$

That is B^*, the adjoint operator of B, is defined from $\mathbb{V}_{\mathbf{n}}$ into $\mathbf{L}^2(\omega)$ by

$$B^* \begin{pmatrix} \mathbf{y}_e \\ r_1 \\ r_2 \end{pmatrix} = (\mathbf{y}_e + \nabla \mathcal{N}_s(Zr_2))\chi_\omega.$$

We directly deduce J^* from the calculations above. □

Then, we have the following proposition:

Proposition 2.8. *The two following statements are equivalent:*

(i) For all $(\mathbf{a}_e, b, c)$ in $L^2(0,T;\mathbb{V}_{\mathbf{n}})$, the solution $(\mathbf{y}_e, k_1, k_2)$ of equation

$$\begin{aligned} -\begin{pmatrix} \mathbf{y}_e \\ k_1 \\ k_2 \end{pmatrix}'(t) &= \mathcal{A}^* \begin{pmatrix} \mathbf{y}_e \\ k_1 \\ k_2 \end{pmatrix}(t) + \begin{pmatrix} \mathbf{a}_e \\ b \\ c \end{pmatrix}(t) \\ (\mathbf{y}_e(T), k_1(T), k_2(T))^T &= (\mathbf{0}, 0, 0)^T \end{aligned} \tag{2.19}$$

satisfies the inequality

$$\begin{aligned} &\|(\mathbf{y}_e(0), k_1(0), k_2(0))\|^2_{\mathbb{V}_{\mathbf{n}}} + \int_0^T \rho_1^2(t)\Big[\, \|\mathbf{y}_e + \nabla \mathcal{N}_s(Zk_2)\|^2_{\mathbf{L}^2(\Omega_0)} + |k_2|^2_{\mathbb{R}^N}\Big] \\ \leq\;& C\left(\int_0^T \rho_2^2(t)\, \|(\mathbf{a}_e(t), b(t), c(t))\|^2_{\mathbb{V}_{\mathbf{n}}} + \int_0^T \rho_3^2(t)\, \|\mathbf{y}_e + \nabla \mathcal{N}_s(Zk_2)\|^2_{\mathbf{L}^2(\omega)}\right). \end{aligned}$$

(ii) For all $\Big((P\mathbf{v}^0, q^0, q^1), (\overline{\mathbf{F}}, \overline{h})\Big)$ in $\mathbb{V}_{\mathbf{n}} \times \overline{\mathcal{W}}_T$, there exists a control $\mathbf{c}$ in $\mathcal{U}_T$ such that the solution $(\mathbf{v}_e, q_1, q_2)$ of (2.10) belongs to $\mathcal{Z}^e_T$ with

$$\mathcal{Z}^e_T = \Big\{(\mathbf{x}_e, r_1, r_2) \in L^2(0,T;\mathbb{V}_{\mathbf{n}}) \text{ s.t. } \rho_2^{-1}(\mathbf{x}_e, r_1, r_2) \in L^2(0,T;\mathbb{V}_{\mathbf{n}})\Big\}.$$

Using the same idea as in section 2.2, we get that there exists a pressure term π such that $(\mathbf{y},\pi,k_1,k_2)$ defined from $(\mathbf{y}_e,k_1,k_2)$ solution of (2.19) by $\mathbf{y}=\mathbf{y}_e+\nabla\mathcal{N}_s(Zk_2)$ is solution of the system

$$\begin{array}{rcll}
-\mathbf{y}_t-\operatorname{div}\sigma(\mathbf{y},\pi) & = & \mathbf{a} & (Q_T^0)\\
\operatorname{div}\mathbf{y} & = & 0 & (Q_T^0)\\
\mathbf{y} & = & 0 & (\Sigma_T)\\
\mathbf{y} & = & Zk_2\mathbf{e}_2 & (\Sigma_T^{s,0})\\
k_1' & = & k_2-b & (0,T)\\
k_2'+Ak_1 & = & -\Pi_N\pi-c & (0,T)\\
\big(\mathbf{y}(T),k_1(T),k_2(T)\big) & = & (\mathbf{0},0,0) &
\end{array}\tag{2.20}$$

with $\mathbf{a}=\mathbf{a}_e+\nabla\mathcal{N}_s(Zc)$. System (2.20) is exactly the adjoint of system (2.1). Furthermore, with the notation $\mathbf{y}=\mathbf{y}_e+\nabla\mathcal{N}_s(Zk_2)$ for $(\mathbf{y}_e,k_1,k_2)$ in $\mathbb{V}_\mathbf{n}$, we have first that $(\mathbf{y},k_1,k_2)$ belongs to $\mathbb{V}$ and second that

$$\begin{array}{rcl}
\Big\|(\mathbf{y}_e,k_1,k_2)\Big\|_{\mathbb{V}_\mathbf{n}}^2 & = & \|\mathbf{y}_e\|^2_{\mathbf{L}^2(\Omega_0)}+|A^{1/2}k_1|^2_{\mathbb{R}^N}+(k_2,(I_n+\Pi_N\mathcal{N}_s(Z\cdot))k_2)_{\mathbb{R}^N}\\
& = & \|\mathbf{y}\|^2_{\mathbf{L}^2(\Omega_0)}+|A^{1/2}k_1|^2_{\mathbb{R}^N}+|k_2|^2_{\mathbb{R}^N}\\
& = & \Big\|(\mathbf{y},k_1,k_2)\Big\|_{\mathbb{V}}^2
\end{array}$$

(see this calculation in the proof of Lemma 2.6 above).

Finally, Proposition 2.8 can be written in term of system (2.1) and its adjoint (2.20) as follows:

Proposition 2.9. *The two following statements are equivalent:*

(i) For all $(\mathbf{a},b,c)$ in $L^2(0,T;\mathbb{V})$, the solution $(\mathbf{y},\pi,k_1,k_2)$ of system (2.20) satisfies the inequality:

$$\begin{array}{l}
\Big\|\Big(\mathbf{y}(0),k_1(0),k_2(0)\Big)\Big\|_{\mathbb{V}}^2+\displaystyle\int_0^T\rho_1^2\Big[\|\mathbf{y}\|^2_{\mathbf{L}^2(\Omega_0)}+|k_2|^2_{\mathbb{R}^N}\Big]\\
\le\; C\left(\displaystyle\int_0^T\rho_2^2(t)\|(\mathbf{a}(t),b(t),c(t))\|^2_{\mathbb{V}}\,dt+\int_0^T\rho_3^2(t)\|\mathbf{y}(t)\|^2_{\mathbf{L}^2(\omega)}\right).
\end{array}$$

(ii) For all $(\mathbf{v}^0,q^0,q^1)$ in $\mathbb{V}$ and all $(\overline{\mathbf{F}},\overline{h})$ in $\overline{\mathcal{W}}_T$, there exists $\mathbf{c}$ in $\mathcal{U}_T$ such that the solution $(\mathbf{v},p,q)$ of system (2.1) satisfies $(\mathbf{v},q)\in\mathcal{Z}_T$.

We set here the result on the observability inequality.

Theorem 2.10. *We introduce the weight functions $(\rho_i)_{i=0,1,2,3}$*

$$\begin{array}{rcl}
\rho_0(t) & = & e^{-\frac{3s}{4}\delta^*(t)},\\
\rho_1(t) & = & (s\lambda)^{3/2}(\sigma^*(t))^{3/2}e^{-s\delta^*(t)},\\
\rho_2(t) & = & \lambda^{5/2}s^{15/4}(\sigma^*(t))^{15/4}e^{-s\delta^*(t)},\\
\rho_3(t) & = & \rho_2(t).
\end{array}\tag{2.21}$$

where σ^* and δ^* are given at the end of section 4. Then, there exists $C>0$ such that all the smooth solutions $(\mathbf{y},\pi,k_1,k_2)$ of system (2.20) with any right-hand side $(\mathbf{a},b,c)$ in $L^2(0,T;\mathbb{V})$ satisfy the inequality

$$\begin{array}{l}
\Big\|\Big(\mathbf{y}(0),k_1(0),k_2(0)\Big)\Big\|_{\mathbb{V}}^2+\displaystyle\int_0^T\rho_1^2\Big[\|\mathbf{y}\|^2_{\mathbf{L}^2(\Omega_0)}+|k_2|^2_{\mathbb{R}^N}\Big]\\
\le\; C\left(\displaystyle\int_0^T\rho_2^2(t)\|(\mathbf{a}(t),b(t),c(t))\|^2_{\mathbb{V}}\,dt+\int_0^T\rho_3^2(t)\|\mathbf{y}(t)\|^2_{\mathbf{L}^2(\omega)}\right)
\end{array}$$

for s and λ large enough ($s\ge\hat{s}$ and $\lambda\ge\hat{\lambda}$).

The proof is postponed to section 4 and relies on a Carleman inequality. Now, we are able to prove the main result of section 2.3.

Proof of Theorem 2.5. Thanks to Theorem 2.10, condition *(i)* of Proposition 2.9 is satisfied. Then, we can apply Theorem 2.2 to system (2.10). That is, there exists a bounded linear operator U_T^e from $\mathbb{V}_{\mathbf{n}} \times \overline{\mathcal{W}}_T$ into $\mathcal{U}_T$ such that the solution $(\mathbf{v}_e, q_1, q_2)$ of system (2.10) associated with $\mathbf{c} = U_T^e\Big((P\mathbf{v}^0, q^0, q^1), (\overline{\mathbf{F}}, \overline{h})\Big)$ belongs to $\mathcal{Z}_T^e$. Using (2.11), we get that $\mathbf{v}_s$ belongs to

$$\mathcal{Z}_T^s = \Big\{\mathbf{x}_s \in L^2(0,T;\mathbf{L}^2(\Omega_0)) \text{ s.t. } \rho_2^{-1}\mathbf{x}_s \in L^2(0,T;\mathbf{L}^2(\Omega_0))\Big\}.$$

This gives together that $(\mathbf{v}, q_1, q_2) \in \mathcal{Z}_T$. Then, denoting E_T the linear bounded operator from $\mathbb{V} \times \overline{\mathcal{W}}_T$ into $\mathbb{V}_{\mathbf{n}} \times \overline{\mathcal{W}}_T$ defined by

$$E_T\Big((\mathbf{v}^0, q^0, q^1), (\overline{\mathbf{F}}, \overline{h})\Big) = \Big((P\mathbf{v}^0, q^0, q^1), (\overline{\mathbf{F}}, \overline{h})\Big),$$

we get that $\overline{U}_T = U_T^e \circ E_T$ is the linear bounded operator of the proposition.

Furthermore, for $(\mathbf{v}^0, q^0, q^1)$ in $X^0_{\mathbf{cc}}$, we get that $(P\mathbf{v}^0, q^0, q^1)$ belongs to $D((-\mathcal{A})^{1/2}) = \mathbf{V}^1_{\mathbf{n}}(\Omega_0) \times \mathbb{R}^N \times \mathbb{R}^N$. Applying now the second point of Theorem 2.2 to system (2.10), we get that $\rho_0^{-1}(\mathbf{v}_e, q_1, q_2)$ belongs to

$$\begin{array}{rl} & L^2(0,T;D(-\mathcal{A})) \cap H^1(0,T;\mathbb{V}_{\mathbf{n}}) \cap \mathcal{C}([0,T];D((-\mathcal{A})^{1/2})) \\ = & \mathbf{V}^{2,1}(Q_T^0) \times H^1(0,T;\mathbb{R}^N) \times H^1(0,T;\mathbb{R}^N) \cap \mathcal{C}([0,T];\mathbf{V}^1_{\mathbf{n}}(\Omega_0) \times \mathbb{R}^N \times \mathbb{R}^N). \end{array}$$

Then, using (2.11), we get that $\rho_0^{-1}(\mathbf{v}_s, p_e, p_s)$ belongs to

$$\Big(\mathbf{H}^{2,1}(Q_T^0) \cap \mathcal{C}([0,T];\mathbf{H}^1(\Omega_0))\Big) \times \Big[L^2(0,T;H^1(\Omega_0))\Big]^2.$$

Finally, $\mathbf{v} = \mathbf{v}_e + \mathbf{v}_s$, $p = p_s + p_e$ and q satisfy

$$\begin{array}{rcl} \mathbf{v},\ \rho_0^{-1}\mathbf{v} & \in & \mathbf{H}^{2,1}(Q_T^0) \cap \mathcal{C}([0,T];\mathbf{H}^1(\Omega_0)), \\ p,\ \rho_0^{-1}p & \in & L^2(0,T;H^1(\Omega_0)), \\ q,\ q',\ \rho_0^{-1}q,\ \rho_0^{-1}q' & \in & H^1(0,T;\mathbb{R}^N). \end{array}$$

That is, thanks to the embedding $H^1(0,T;\mathbb{R}^N) \hookrightarrow \mathcal{C}([0,T];\mathbb{R}^N)$ and the definition of ρ_0 (especially, $\rho_0(T) = 0$), that we have the null controllability of system (2.1):

$$\mathbf{v}(T) = \mathbf{0}, \quad \text{in } \Omega_0 \qquad \text{and} \qquad q(T) = q'(T) = 0.$$

□

3 Proof of Theorem 1.3.

In this section, we prove Theorem 1.3. First, we use the previous section to prove the theorem in the cylinder $(0,T) \times \Omega_0$. Then, we will derive Theorem 1.3 from this result using the change of variables introduced in section 1.4.

3.1 In the cylinder $(0,T) \times \Omega_0$.

First, we begin by proving the null controllability of system

$$\begin{array}{rcll} \mathbf{u}_t - \operatorname{div}\sigma(\mathbf{u},p) & = & \mathbf{c}\chi_\omega + \mathbf{F} & (Q_T^0) \\ \operatorname{div}\mathbf{u} & = & \operatorname{div}\mathbf{w} & (Q_T^0) \\ \mathbf{u} & = & Zq'\mathbf{e}_2 & (\Sigma_T^{s,0}) \\ \mathbf{u} & = & \mathbf{0} & (\Sigma_T) \\ q'' + Aq & = & \Pi_N\Big[p - 2\nu u_{2,z}\Big] + h & (0,T) \\ \big(\mathbf{u}(0), q(0), q'(0)\big) & = & \big(\mathbf{u}^0, q^0, q^1\big) & \end{array} . \tag{3.1}$$

Because $\mathbf{u}^0$ is not divergence free (see (1.15)), we do not have $(\mathbf{u}^0, q^0, q^1)$ in the space $\mathbb{V}$. Thus, we introduce another Hilbert space

$$\mathbb{L} = \mathbf{L}^2(\Omega_0) \times \mathbb{R}^N \times \mathbb{R}^N.$$

In system (3.1), the right-hand side $(\mathbf{F}, \mathbf{w}, h)$ belongs to

$$\mathcal{W}_T = \Big\{(\mathbf{G}, \mathbf{z}, g) \in W_T \text{ s.t. } \rho_1^{-1}(\mathbf{G}, (-\Delta)\mathbf{z}, \mathbf{z}', g) \text{ belongs to } L^2(0,T; [\mathbf{L}^2(\Omega_0)]^3 \times \mathbb{R}^N)\Big\}$$

equipped with the norm

$$\|(\mathbf{G}, \mathbf{z}, g)\|_{\mathcal{W}_T} = \int_0^T \rho_1^{-2}(t)\Big[\|(\mathbf{G}(t), (-\Delta)\mathbf{z}(t), \mathbf{z}'(t))\|^2_{[\mathbf{L}^2(\Omega_0)]^3} + |g(t)|^2_{\mathbb{R}^N}\Big]\mathrm{d}t \quad \text{for all } (\mathbf{G}, \mathbf{z}, g) \in \mathcal{W}_T,$$

where

$$W_T = \Big\{(\mathbf{G}, \mathbf{z}, g) \in \mathbf{L}^2(Q_T^0) \times \mathbf{H}^{2,1}(Q_T^0) \times L^2(0,T;\mathbb{R}^N) \text{ such that } \mathbf{z} = \mathbf{0} \text{ on } \Gamma_0\Big\}.$$

Remark 3.1. *Conditions* $\frac{\rho_0' \rho_j}{\rho_0^2} \in L^\infty(0,T)$ *and* $\frac{\rho_i}{\rho_0} \in L^\infty(0,T)$ *in* (2.7) *for* $j = 1$ *or* $j = 2$ *give respectively the equivalence between*

$$\frac{\Delta \mathbf{w}}{\rho_1}, \frac{\mathbf{w}'}{\rho_1} \in \mathbf{L}^2(Q_T^0) \quad \textit{and} \quad \frac{\mathbf{w}}{\rho_0} \in \mathbf{H}^{2,1}(Q_T^0)$$

and

$$\frac{\Delta \mathbf{v}}{\rho_2}, \frac{\mathbf{v}'}{\rho_2} \in \mathbf{L}^2(Q_T^0) \quad \textit{and} \quad \frac{\mathbf{v}}{\rho_0} \in \mathbf{H}^{2,1}(Q_T^0).$$

Then, we have the following result:

Proposition 3.2. *Let* $(\mathbf{u}^0, q^0, q^1)$ *be in* X^0 *satisfying* (1.15). *There exists a linear bounded operator* U_T *from* $\mathbb{L} \times \mathcal{W}_T$ *into* $L^2(0,T;\mathbf{L}^2(\omega))$ *such that for all* $(\mathbf{F}, \mathbf{w}, h)$ *in* $\mathcal{W}_T$ *the solution of system* (3.1) *associated with the function* $\mathbf{c} = U_T\Big((\mathbf{u}^0, q^0, q^1), (\mathbf{F}, \mathbf{w}, h)\Big)$ *in the right-hand side belongs to* $\mathcal{X}_T$. *Furthermore, there exists a constant* $C_1 > 0$ *such that*

$$\|(\mathbf{u}, p, q)\|_{\mathcal{X}_T} \le C_1\Big(\|(\mathbf{u}^0, q^0, q^1)\|_{X^0} + \|(\mathbf{F}, \mathbf{w}, h)\|_{\mathcal{W}_T}\Big). \tag{3.2}$$

That is, system (3.1) *is null controllable at time* $T > 0$

$$\mathbf{u}(T) = \mathbf{0} \textit{ in } \Omega_0, \quad q(T) = 0 \quad \textit{and} \quad q'(T) = 0.$$

Proof. Let us define the operator K_T by

$$\begin{array}{cccc} K_T: & \mathbb{L} \times \mathcal{W}_T & \longrightarrow & \mathbb{V} \times \overline{\mathcal{W}}_T \\ & \Big((\mathbf{u}^0, q^0, q^1), (\mathbf{F}, \mathbf{w}, h)\Big) & \longmapsto & \Big((\mathbf{v}^0, q^0, q^1), (\overline{\mathbf{F}}, \overline{h})\Big) \end{array}$$

where $\mathbf{v}^0$ is defined by (see (1.13))

$$\mathbf{v}^0 = \mathbf{u}^0 - \mathbf{w}(0)$$

and $(\overline{\mathbf{F}}, \overline{h})$ are defined from $(\mathbf{F}, \mathbf{w}, h)$ as follow (see (1.12))

$$\overline{\mathbf{F}} = \mathbf{F} + \nu \Delta \mathbf{w} - \mathbf{w}_t, \qquad \overline{h} = h - 2\nu \Pi_N\Big[w_{2,z}\Big].$$

The operator K_T is clearly linear. Moreover it is bounded

$$\begin{aligned} \Big\|K_T\Big((\mathbf{u}^0, q^0, q^1), (\mathbf{F}, \mathbf{w}, h)\Big)\Big\|_{\mathbb{V} \times \overline{\mathcal{W}}_T} &\le C\Big(\|(\mathbf{u}^0, q^0, q^1)\|_{\mathbb{L}} + \|\mathbf{w}(0)\|_{\mathbf{L}^2(\Omega_0)} + \|(\overline{\mathbf{F}}, \overline{h})\|_{\overline{\mathcal{W}}_T}\Big) \\ &\le C\Big\|\Big((\mathbf{u}^0, q^0, q^1), (\mathbf{F}, \mathbf{w}, h)\Big)\Big\|_{\mathbb{L} \times \mathcal{W}_T}. \end{aligned}$$

Indeed, $\mathbf{w}$ belongs to $\mathbf{H}^{2,1}(Q_T^0) \hookrightarrow \mathcal{C}([0,T];\mathbf{H}^1(\Omega_0))$, then $\|\mathbf{w}(0)\|_{\mathbf{L}^2(\Omega_0)} \le C\|(\mathbf{F}, \mathbf{w}, h)\|_{\mathcal{W}_T}$.

Then, thanks to the existence of a bounded operator $\overline{U}_T$ from $\mathbb{V}\times\overline{\mathcal{W}}_T$ into $L^2(0,T;\mathbf{L}^2(\omega))$ used in Theorem 2.5, we get by composition a linear bounded operator U_T defined from $\mathbb{L}\times\mathcal{W}_T$ into $L^2(0,T;\mathbf{L}^2(\omega))$.

The fact that the solution $(\mathbf{u},p,q)$ of (3.1) associated to $\mathbf{c}=U_T\Big((\mathbf{u}^0,q^0,q^1),(\mathbf{F},\mathbf{w},h)\Big)$ belongs to $\mathcal{X}_T$ comes exactly from Theorem 2.5 and $\mathbf{u}=\mathbf{v}+\mathbf{w}$. Indeed, by construction, the solution $(\mathbf{v},p,q)$ of (2.1) corresponding with $(\mathbf{v}^0,q^0,q^1)$ and $(\overline{\mathbf{F}},\overline{h})$—both obtained from $(\mathbf{u}^0,q^0,q^1)$ and $(\mathbf{F},\mathbf{w},h)$—and associated to $\mathbf{c}=U_T\Big((\mathbf{u}^0,q^0,q^1),(\mathbf{F},\mathbf{w},h)\Big)=\overline{U}_T\Big((\mathbf{v}^0,q^0,q^1),(\overline{\mathbf{F}},\overline{h})\Big)$ belongs to $\mathcal{X}_T$. Moreover, as $\mathbf{w}$ and $\frac{\mathbf{w}}{\rho_0}$ belongs to $\mathbf{H}^{2,1}(Q_T^0)$ (see Remark 3.1 and the definition of ρ_0 in (2.7)), we have first $(\mathbf{u},p,q)=(\mathbf{v}+\mathbf{w},p,q)$ belongs to $\mathcal{X}_T$ with the expected estimate and second, thanks to $\mathbf{w}(T)=\mathbf{0}$, that

$$\mathbf{u}(T)=\mathbf{0}\text{ in }\Omega_0\qquad\text{and}\qquad q(T)=q'(T)=0.$$

□

From now on, the initial data $(\mathbf{u}^0,q^0,q^1)$ is fixed in X^0 and satisfies (1.15). The time $T>0$ is fixed too. We want to prove the controllability of the system written in the fixed domain (1.10). We use a fixed point procedure based on the result for the linearized system (3.1).

Lemma 3.3. *Let $(\mathbf{u},p,q)$ be the solution in $\mathcal{X}_T$ of the system (3.1) for the intial data $(\mathbf{u}^0,q^0,q^1)$ in X^0 satisfying (1.15) and right-hand sides $(\mathbf{F},\mathbf{w},h)$ in $\mathcal{W}_T$, then $(\overline{\mathbf{F}},\overline{\mathbf{w}},\overline{h})=(\mathbf{F}[\mathbf{u},p,q],\mathbf{w}[\mathbf{u},q],h[\mathbf{u},p,q])$ defined by (1.8) and (1.9) belongs to $\mathcal{W}_T$ and there exists a constant C_2 such that*

$$\|(\overline{\mathbf{F}},\overline{\mathbf{w}},\overline{h})\|_{\mathcal{W}_T}\leq C_2(1+\|(\mathbf{u},p,q)\|_{\mathcal{X}_T})\|(\mathbf{u},p,q)\|^2_{\mathcal{X}_T}.\tag{3.3}$$

Furthermore, let $(\mathbf{u}_i,p_i,q_i)$ $(i=1,2)$ be solutions in $\mathcal{X}_T$ of system (3.1) with the same initial data $(\mathbf{u}^0,q^0,q^1)$ in X^0 satisfying (1.15) and repectively right-hand sides $(\mathbf{F}_i,\mathbf{w}_i,h_i)$ $(i=1,2)$ in $\mathcal{W}_T$. If $(\mathbf{u}_i,p_i,q_i)$ $(i=1,2)$ satisfies for some $R>0$,

$$\|(\mathbf{u}_i,p_i,q_i)\|_{\mathcal{X}_T}\leq R,$$

then, we have the estimate

$$\|(\overline{\mathbf{F}}_1,\overline{\mathbf{w}}_1,\overline{h}_2)-(\overline{\mathbf{F}}_2,\overline{\mathbf{w}}_2,\overline{h}_2)\|_{\mathcal{W}_T}\leq C_2(1+R)R\|(\mathbf{u}_1,p_1,q_1)-(\mathbf{u}_2,p_2,q_2)\|_{\mathcal{X}_T}\tag{3.4}$$

where $(\overline{\mathbf{F}}_i,\overline{\mathbf{w}}_i,\overline{h}_i)=(\overline{\mathbf{F}}[\mathbf{u}_i,p_i,q_i],\overline{\mathbf{w}}[\mathbf{u}_i,q_i],\overline{h}[\mathbf{u}_i,q_i])$ $(i=1,2)$.

Proof. First, ρ_0 and ρ_2 defined in (2.21) satisfy $\frac{\rho_0^2}{\rho_2}\in L^\infty(0,T;\mathbb{R})$. Then, with this, the proof is a consequence of the definition of the right-hand sides $\overline{\mathbf{F}}$, $\overline{\mathbf{w}}$ in (1.8) and $\overline{h}$ in (1.9).The estimate of the $\mathcal{W}_T$-norm of $(\overline{\mathbf{F}},\overline{\mathbf{w}},\overline{h})$ is tedious but straightforward from Proposition 6.1 in [7]. □

Proposition 3.4. *Let $(\overline{\mathbf{u}},\overline{p},\overline{q})$ in $\mathcal{X}_T$ be a solution of the control problem of system (3.1) associated with $(\mathbf{u}^0,q^0,q^1)$, $(\mathbf{F},\mathbf{w},h)$ in $\mathcal{W}_T$ and the control $\mathbf{c}=U_T\Big((\mathbf{u}^0,q^0,q^1),(\mathbf{F},\mathbf{w},h)\Big)$ in $L^2(0,T;\mathbf{L}^2(\omega))$ (see Proposition 3.2). Then, system*

$$\begin{array}{rcll}\mathbf{u}_t-\operatorname{div}\sigma(\mathbf{u},p)&=&\mathbf{c}\chi_\omega+\mathbf{F}[\overline{\mathbf{u}},\overline{p},\overline{q}]&(Q_T^0)\\ \operatorname{div}\mathbf{u}&=&\operatorname{div}\mathbf{w}[\overline{\mathbf{u}},\overline{q}]&(Q_T^0)\\ \mathbf{u}&=&Zq'\mathbf{e}_2&(\Sigma_T^{s,0})\\ \mathbf{u}&=&\mathbf{0}&(\Sigma_T)\\ q''+Aq&=&\Pi_Np+h[\overline{\mathbf{u}},\overline{q}]&(0,T)\\ (\mathbf{u}(0),q(0),q'(0))&=&(\mathbf{u}^0,q^0,q^1)&\end{array}.\tag{3.5}$$

is null controllable at time T, that is there exists a control

$$\mathbf{c}=U_T\Big((\mathbf{u}^0,q^0,q^1),(\mathbf{F}[\overline{\mathbf{u}},\overline{p},\overline{q}],\mathbf{w}[\overline{\mathbf{u}},\overline{q}],h[\overline{\mathbf{u}},\overline{q}])\Big)$$

in $L^2(0,T;\mathbf{L}^2(\omega))$ such that the solution $(\mathbf{u},p,q)$ of system (3.5) corresponding with $\mathbf{c}$ belongs to $\mathcal{X}_T$ and satisfies

$$\mathbf{u}(T)=\mathbf{0}\ \textit{in}\ \Omega_0,\qquad q(T)=0,\qquad q'(T)=0.$$

Furthermore, the triplet $(\mathbf{u},p,q)$ *satisfies the estimate*

$$\|(\mathbf{u},p,q)\|^2_{\mathcal{X}_T} \leq C_1\Big(\|(\mathbf{u}^0,q^0,q^1)\|^2_{X^0} + C_2(1+\|(\overline{\mathbf{u}},\overline{p},\overline{q})\|_{\mathcal{X}_T})\|(\overline{\mathbf{u}},\overline{p},\overline{q})\|^2_{\mathcal{X}_T}\Big).$$

In other terms, we can contruct a mapping

$$\begin{array}{rccl} \mathcal{C}_T: & \mathcal{X}_T & \longrightarrow & \mathcal{X}_T \\ & (\overline{\mathbf{u}},\overline{p},\overline{q}) & \longmapsto & \mathcal{C}_T(\overline{\mathbf{u}},\overline{p},\overline{q}) = (\mathbf{u},p,q) \text{ is the solution of the control problem for system (3.5)} \end{array}$$

which satisfies the estimate

$$\|\mathcal{C}_T(\overline{\mathbf{u}},\overline{p},\overline{q})\|^2_{\mathcal{X}_T} \leq C_1\Big(\|(\mathbf{u}^0,q^0,q^1)\|^2_{X^0} + C_2(1+\|(\overline{\mathbf{u}},\overline{p},\overline{q})\|_{\mathcal{X}_T})\|(\overline{\mathbf{u}},\overline{p},\overline{q})\|^2_{\mathcal{X}_T}\Big). \tag{3.6}$$

Proof. The proof relies on Proposition 3.2 and estimate (3.3) in the previous lemma. The constants C_1 and C_2 are defined respectively in (3.2) and (3.3). □

We now are able to state the main result of this section:

Proposition 3.5. *Let* $(\mathbf{u}^0,q^0,q^1)$ *be in* X^0 *satisfying* (1.15). *Then, there exists* r *small enough such that, under condition*

$$\|(\mathbf{u}^0,q^0,q^1)\|_{X^0} \leq r,$$

system (1.10) *is null controllable at time* $T>0$*, that is there exists a control* $\mathbf{c}$ *in* $L^2(0,T;\mathbf{L}^2(\omega))$ *such that system* (1.10) *associated with this control* $\mathbf{c}$ *admits a solution* $(\mathbf{u},p,q)$ *in* $\mathcal{X}_T$ *satisfying*

$$\mathbf{u}(T)=\mathbf{0} \text{ in } \Omega_0, \qquad q(T)=0, \qquad q'(T)=0.$$

Proof. For $(\mathbf{u}^0,q^0,q^1)$ in X^0 as above, we denote $r=\|(\mathbf{u}^0,q^0,q^1)\|_{X^0}$ and $R=2C_1r$ (with C_1 defined in (3.2)). We choose r such that $C_2r(1+2C_1r)=1$ (with C_2 defined in (3.3)), that is

$$r=\frac{1}{2C_1^2C_2}\frac{1}{\sqrt{1+\frac{2}{C_1C_2}}}.$$

Then, we define a ball of the space $\mathcal{X}_T$ of radius R as follows:

$$\mathcal{X}_T^R=\Big\{(\mathbf{z},\rho,r)\in\mathcal{X}_T \text{ s.t. } \|(\mathbf{z},\rho,r)\|_{\mathcal{X}_T}\leq R\Big\}.$$

Then, $\mathcal{C}_T$ is a contraction mapping in $\mathcal{X}_T^R$. Indeed, for two triplets $(\mathbf{u}_i,p_i,q_i)$ in $\mathcal{X}_T$, by definition of $\mathcal{C}_T$, we get first that $\mathcal{C}_T(\mathbf{u}_i,p_i,q_i)$ $(i=1,2)$ is solution of the control problem of system (1.10) corresponding with initial data $(\mathbf{u}^0,q^0,q^1)$, right-hand sides $(\mathbf{F}[\mathbf{u}_i,p_i,q_i],\mathbf{w}[\mathbf{u}_i,q_i],h[\mathbf{u}_i,q_i])$ and the control $\mathbf{c}_i=U_T\Big((\mathbf{u}^0,q^0,q^1),(\mathbf{F}[\mathbf{u}_i,p_i,q_i],\mathbf{w}[\mathbf{u}_i,q_i],h[\mathbf{u}_i,q_i])\Big)$. This means that $\mathcal{C}_T(\mathbf{u}_i,p_i,q_i)$ $(i=1,2)$ satifies

$$\|\mathcal{C}_T(\mathbf{u}_i,p_i,q_i)\|_{\mathcal{X}_T}\leq\frac{R}{2}+\frac{R}{2}=R.$$

Furthermore, the difference $\mathcal{C}_T(\mathbf{u}_1,p_1,q_1)-\mathcal{C}_T(\mathbf{u}_2,p_2,q_2)$ satisifies by linearity system (1.10) with $(\mathbf{0},0,0)$ for initial data and $(\mathbf{F}_1,\mathbf{w}_1,h_2)-(\mathbf{F}_2,\mathbf{w}_2,h_2)$ for right-hand sides. Then, *via* the estimates (3.2) in Proposition 3.2 and (3.4) in Lemma 3.3 and the choice of r, we have

$$\|\mathcal{C}_T(\mathbf{u}_1,p_1,q_1)-\mathcal{C}_T(\mathbf{u}_2,p_2,q_2)\|_{\mathcal{X}_T}\leq\frac{1}{2}\|(\mathbf{u}_1,p_1,q_1)-(\mathbf{u}_2,p_2,q_2)\|_{\mathcal{X}_T}.$$

For r chosen as above, $\mathcal{C}_T$ is a contraction mapping from $\mathcal{X}_T^R$ into itself. Then, using the Picard-Banach fixed point theorem, this mapping admits a fixed point $(\tilde{\mathbf{u}},\tilde{p},\tilde{q})$ in $\mathcal{X}_T$ solution of the control problem (1.10) corresponding with initial data $(\mathbf{u}^0,q^0,q^1)$ in $X^0_{\mathbf{cc}}$, right-hand sides $(\mathbf{F}[\tilde{\mathbf{u}},\tilde{p},\tilde{q}],\mathbf{w}[\tilde{\mathbf{u}},\tilde{q}],h[\tilde{\mathbf{u}},\tilde{q}])$ and the control $\mathbf{c}=U_T\Big((\mathbf{u}^0,q^0,q^1),(\mathbf{F}[\tilde{\mathbf{u}},\tilde{p},\tilde{q}],\mathbf{w}[\tilde{\mathbf{u}},\tilde{q}],h[\tilde{\mathbf{u}},\tilde{q}])\Big)$. That is exactly $(\tilde{\mathbf{u}},\tilde{p},\tilde{q})$ is a solution of (1.10) in X_T and satisfies:

$$\tilde{\mathbf{u}}(T)=\mathbf{0} \text{ in } \Omega_0, \qquad \tilde{q}(T)=0 \quad \text{and} \quad \tilde{q}'(T)=0.$$

□

3.2 In the moving domain.

In this section, we have to check the conditions on the change of variables. That is we have to prove that the change of variables

$$\begin{array}{cccc} \phi_t : & \Omega_0 & \longrightarrow & \Omega_{\eta(t)} \\ & (x,z) & \longmapsto & (x,y) \end{array}$$

is well-defined as a $\mathcal{C}^1-$diffeomorphism from Ω_0 into $\Omega_{\eta(t)}$ for every t in $[0,T]$ and that condition (1.1) is checked. The regularity of q and of the functions ζ_k ($k=1,\dots,N$) gives together with the formula of change of variables in section 1.4 that ϕ_t is a $\mathcal{C}^1-$diffeomorphism. We just need to check assumption (1.1). Since $\eta(t,x)=Zq$, η would satisfy the hypothesis (1.1) if we have an estimate on q like

$$\|q\|_{L^\infty(0,T;\mathbb{R}^N)} \le \frac{1-\varepsilon}{3\|Z\|_{L^\infty(0,L)}}.$$

Indeed, the maximum of the function η in $\Sigma_T^{s,0}$ can be roughly bounded by

$$\|\eta\|_{L^\infty(\Sigma_T^{s,0})} \le \|Z\|_{L^\infty(\Gamma_0^s)}\|q\|_{L^\infty(0,T)}.$$

Then $1+\eta(t,x)\ge\varepsilon$ for $(t,x)\in\Sigma_T^{s,0}$ if $\|\eta\|_{L^\infty(\Sigma_T^{s,0})}\le 1-\varepsilon$. Because of the following estimate

$$\|q\|_{L^\infty(0,T;\mathbb{R}^N)} \le C\left\|\frac{q'}{\rho_0}\right\|_{H^1(0,T;\mathbb{R}^N)} \le C(\|(\mathbf{u}^0,q^0,q^1)\|_{X^0}+\|(\mathbf{F},\mathbf{w},h)\|_{\mathcal{W}_T}),$$

if both the conditions $\|(\mathbf{u}^0,q^0,q^1)\|_{X^0}\le r$ and $(\mathbf{F},\mathbf{w},h)\in\mathcal{W}_T$ such that

$$\|(\mathbf{F},\mathbf{w},h)\|_{\mathcal{W}_T}\le r$$

are satisfied then

$$\|q\|_{L^\infty(0,T;\mathbb{R}^N)} \le 2Cr \le \frac{2(1-\varepsilon)}{3\|Z\|_{L^\infty(0,L)}} \le \frac{1-\varepsilon}{\|Z\|_{L^\infty(0,L)}}$$

for r small enough and the hypothesis (1.1) is checked. That is, up to the change of parameter r_1 defined by

$$r_1 = \min\left(r, \frac{1}{C}\frac{1-\varepsilon}{3\|Z\|_{L^\infty(0,L)}}\right),$$

instead of r in the previous section, we have the result of Theorem 1.3 and in the same time the assumption 1.1 is checked.

To conclude, we can remark that the control $\mathbf{c}$ stated in Theorem 1.3 is exactely the one obtained by the fixed point procedure in section 3.1. Indeed, the change of variables does not change the subdomain ω where the control acts. In other words, we have, with obvious notations, $\phi_t(\mathbf{c})=\mathbf{c}$.

4 Proof of Theorem 2.10.

Our goal is to prove an observability inequality for the system

$$\begin{array}{rcll} -\mathbf{y}_t - \operatorname{div}\sigma(\mathbf{y},\pi) &=& \mathbf{a} & (Q_T^0) \\ \operatorname{div}\mathbf{y} &=& 0 & (Q_T^0) \\ \mathbf{y} &=& Zk_2\mathbf{e}_2 & (\Sigma_T^{s,0}) \\ \mathbf{y} &=& \mathbf{0} & (\Sigma_T) \\ k_1' &=& k_2 - b & (0,T) \\ k_2' &=& -Ak_1 - \Pi_N\pi - c & (0,T) \\ (\mathbf{y}(T),k_1(T),k_2(T)) &=& (\mathbf{0},0,0) & \end{array} \tag{4.1}$$

The proof of Theorem 2.10 is split into different steps. This steps can be found either in [13, 14] or in [4]. Let us detail the strategy of the proof.

Step 1. In section 4.1, we set a first Carleman estimate for the system.

Step 2. In sections 4.2 and 4.3, we get rid of the pressure term and local integral terms of the right-hand side of the previous Carleman estimate *via* the method of Fernandez-Cara, Guerrero, Imanuvilov and Puel in [4] itself using [6].

Step 3. Following [14], we get rid of the integral in k_1 in the right-hand side of the Carleman estimate obtained in the previous step (see section 4.4).

Step 4. In section 4.5, we derive the observability inequality from the last Carleman estimate.

The different steps above are very classic in the proof of Carleman estimates. They can be found in details in the papers cited above, especially in [4, 14]. More precisely, step 1 can be adapted from [4, section 2], [14, section 3] or [5, section 3]. Steps 2 and 4 are derived from [4] respectively from *Steps 3, 4 & 5* in section 2 and from the beginning of section 3. As already mentionned, step 3 is directly adapted from [14, sections 6 & 7].

We begin with some notations. Let ϕ be a $\mathcal{C}^2(\overline{\Omega_0})$ function satisfying

$$\begin{array}{rclll} \phi(x) & > & 0, & \text{for all } x\in\overline{\Omega_0}, & |\nabla\phi(x)| > 0 \quad \text{for all } x\in\overline{\Omega_0\setminus\omega_0}, \\ \phi(x) & = & C & \text{for all } x\in\Gamma, & \partial_{\mathbf{n}}\phi(x) \leq 0 \quad \text{for all } x\in\Gamma_0, \\ \partial_{\mathbf{n}}\phi(x) & = & -1, & \text{for all } x\in\Gamma_0^s, & \Delta\phi(x) = \mathbf{0} \quad \text{for all } x\in\Gamma_0^s. \end{array} \tag{4.2}$$

We define for a large parameter $\lambda\geq 1$, the functions

$$\begin{array}{rcll} \xi(x,t) & = & \dfrac{\mathrm{e}^{\lambda(\phi+m\|\phi\|_\infty)}}{t^k(T-t)^k}, & m>1 \\ \kappa(x) & = & \mathrm{e}^{\lambda m K_1}-\mathrm{e}^{\lambda(\phi(x)+m\|\phi\|_\infty)}, & \forall x\in\overline{\Omega_0}, \end{array}$$

where $K_1>0$ is a constant such that $K_1\geq 2\|\phi\|_\infty$. We set next $\varphi_\lambda(x,t)=\frac{\kappa(x)}{t^k(T-t)^k}$ and $\rho(x,t)=\mathrm{e}^{\varphi_\lambda(x,t)}$ where k is a constant number such that $k\geq 2$. The number k will be fixed to 4 in section 4.3, following [4, 14, 5].

Let us define $\mathbf{z}(x,t)=\rho^{-s}(x,t)\mathbf{y}(x,t)$. System (4.1) written in the variables $(\mathbf{z},\pi,k_1,k_2)$ is

$$\begin{array}{rcll} M_1\mathbf{z}+M_2\mathbf{z} & = & \mathbf{f}_s & (Q_T^0) \\ \operatorname{div}\mathbf{z} & = & -s\nabla\varphi_\lambda\cdot\mathbf{z} & (Q_T^0) \\ \mathbf{z} & = & \rho^{-s}Zk_2\mathbf{e}_2 & (\Sigma_T^{s,0}) \\ \mathbf{z} & = & \mathbf{0} & (\Sigma_T) \\ k_1' & = & k_2-b & (0,T) \\ k_2'+Ak_1 & = & \Pi_N\pi-c & (0,T) \\ (\mathbf{z}(0),k_1(0),k_2(0))=(\mathbf{z}(T),k_1(T),k_2(T)) & = & (\mathbf{0},0,0) & \end{array} \tag{4.3}$$

with

$$\begin{array}{rclcrcl} M_1\mathbf{z} & = & \mathbf{z}'-2s\nu\nabla\varphi_\lambda\cdot\nabla\mathbf{z} & \text{and} & M_2\mathbf{z} & = & s\varphi_\lambda'\mathbf{z}-\nu\Delta\mathbf{z}-s^2\nu|\nabla\varphi_\lambda|^2\mathbf{z}, \\ \mathbf{f}_s & = & \rho^{-s}\mathbf{a}-\rho^{-s}\nabla\pi+s\nu(\Delta\varphi_\lambda)\mathbf{z}. & & & & \end{array} \tag{4.4}$$

Indeed, the calculation of $\rho^{-s}\Big(\partial_t-\nu\Delta\Big)\rho^s\mathbf{z}=-\rho^{-s}\nabla\pi+\rho^{-s}\mathbf{a}$ give the differents terms above from

$$\rho^{-s}\Delta(\rho^s\mathbf{z})=s^2|\nabla\varphi_\lambda|^2\mathbf{z}+s\Delta\varphi_\lambda\mathbf{z}+2s\nabla\mathbf{z}\nabla\varphi_\lambda+\Delta\mathbf{z} \qquad \text{and} \qquad \rho^{-s}\partial_t(\rho^s\mathbf{z})=s\partial_t\varphi_\lambda\mathbf{z}+\mathbf{z}'.$$

4.1 First Carleman Estimate.

After some calculations, and using the estimate

$$\int_0^T\rho_\Gamma^{-2s}(|k_2'|^2_{\mathbb{R}^N}+|A^{1/2}k_1|^2_{\mathbb{R}^N})\leq C\left\{\int_0^T\rho_\Gamma^{-2s}\,|\Pi_N\pi|^2_{\mathbb{R}^N}+\int_0^T\rho_\Gamma^{-2s}(|A^{1/2}k_1|^2_{\mathbb{R}^N}+|c|^2_{\mathbb{R}^N})\right\},$$

we obtain the following Carleman estimate:

Theorem 4.1. *For λ large enough, there is $s_0(\lambda) > 0$ such that for all $s \geq s_0(\lambda)$ and for all the solutions $(\mathbf{z}, k_1, k_2)$ of (4.3), we have*

$$
\begin{aligned}
& s^{-1}\int_{Q_T^0} \xi^{-1}\Big(|\mathbf{z}'|^2 + |\Delta \mathbf{z}|^2\Big) + \int_{Q_T^0} |M_1\mathbf{z}|^2 + \int_{Q_T^0} |M_2\mathbf{z}|^2 + \int_{Q_T^0} \rho^{-2s}|\nabla\pi|^2 \\
& + s\lambda^2\int_{Q_T^0} \xi|\nabla \mathbf{z}|^2 + s^3\lambda^4\int_{Q_T^0} \xi^3|\mathbf{z}|^2 + s^3\lambda^3\int_{\Sigma_T^{s,0}} \xi^3\rho^{-2s}|Zk_2|^2 + \int_0^T \rho_\Gamma^{-2s}\Big(|k_2'|^2_{\mathbb{R}^N} + |A^{1/2}k_1|^2_{\mathbb{R}^N}\Big) \\
\leq\ & C\left[\int_{Q_T^0} \rho^{-2s}|\nabla\pi|^2 + s^3\lambda^4\int_{\omega_1\times(0,T)} \xi^3|\mathbf{z}|^2 + \int_0^T \rho_\Gamma^{-2s}(|A^{1/2}k_1|^2_{\mathbb{R}^N} + |c|^2_{\mathbb{R}^N}) + \int_0^T \rho_\Gamma^{-2s}\,|\Pi_N\pi|^2_{\mathbb{R}^N}\right]
\end{aligned} \tag{4.5}
$$

where $\omega_0 \subset\subset \omega_1 \subset\subset \Omega_0$.

4.2 First treatment of the pressure term integral.

We need to get rid of the term $\int_{Q_T^0} \rho^{-2s}|\nabla\pi|^2$ in the righ-hand side of the previous inequality. We follow the idea of [6, 4]. First, we consider ω_2 such that $\omega_1 \subset\subset \omega_2 \subset\subset \omega$. We take π (defined up to an additive constant) such that $\int_{\omega_2} \pi(t) = 0$ for almost every t in $(0,T)$. Then, after some calculations and using the equality $\nabla\pi = \mathbf{y}_t + \nu\Delta\mathbf{y} + \mathbf{a}$, we have the following inequality:

$$
\begin{aligned}
I(s,\lambda;\xi) \ \leq\ & C\Bigg(s^3\lambda^4\int_{\omega_2\times(0,T)} \xi^3|\mathbf{z}|^2 + s^{5/2}\int_0^T (\xi^*)^3 \mathrm{e}^{-2s\varphi_\lambda^*}(|A^{1/2}k_1|^2_{\mathbb{R}^N} + |k_2|^2_{\mathbb{R}^N}) \\
& + \int_0^T \rho_2^2(t)\|(\mathbf{a}(t), b(t), c(t))\|^2_{\mathbb{V}} + \int_{(0,T)\times\omega_2} s^2\lambda^2\hat{\xi}^2\mathrm{e}^{-2s\hat{\varphi}_\lambda}(|\mathbf{a}|^2 + |\Delta\mathbf{y}|^2 + |\mathbf{y}'|^2)\Bigg)
\end{aligned}
$$

where $I(s,\lambda;\xi)$ is the left-hand side of inequality (4.5), namely

$$
\begin{aligned}
I(s,\lambda;\xi) \ =\ & s^{-1}\int_{Q_T^0} \xi^{-1}\Big(|\mathbf{z}'|^2 + |\Delta\mathbf{z}|^2\Big) + \int_{Q_T^0} |M_1\mathbf{z}|^2 + \int_{Q_T^0} |M_2\mathbf{z}|^2 + \int_{Q_T^0} \rho^{-2s}|\nabla\pi|^2 \\
& + s\lambda^2\int_{Q_T^0} \xi|\nabla\mathbf{z}|^2 + s^3\lambda^4\int_{Q_T^0} \xi^3|\mathbf{z}|^2 + s^3\lambda^3\int_{\Sigma_T^{s,0}} \xi^3\rho^{-2s}|Zk_2|^2 + \int_0^T \rho_\Gamma^{-2s}\Big(|k_2'|^2_{\mathbb{R}^N} + |A^{1/2}k_1|^2_{\mathbb{R}^N}\Big).
\end{aligned}
$$

4.3 Estimates of the local integrals of $\Delta\mathbf{y}$ and $\mathbf{y}'$.

The next steps consist in estimating the two local integrals in the right-hand side of the previous inequality. From now on, we fix $k = 4$ as in [4, 14, 5]. We denote $\hat{\theta}(t) = s\lambda\hat{\xi}\mathrm{e}^{-s\hat{\varphi}_\lambda}$. First, we have

$$
\int_{(0,T)\times\omega_2} |\hat{\theta}|^2|\Delta\mathbf{y}|^2 \leq \int_{(0,T)\times\omega_3} |\hat{\theta}'(t)|^2|\mathbf{y}|^2 + \int_{(0,T)\times\omega_3} |\hat{\theta}(t)|^2(|\mathbf{a}|^2 + |\mathbf{y}|^2)
$$

for ω_3 such that $\omega_2 \subset\subset \omega_3 \subset\subset \omega$. Second

$$
\begin{aligned}
\int_{(0,T)\times\omega_2} |\hat{\theta}|^2|\mathbf{y}'|^2 \ \leq\ & C\Bigg(\int_{(0,T)\times\omega_2} \lambda^2 s^{9/2}(\xi^*)^{9/2}\mathrm{e}^{-2s\varphi_\lambda^*}|\mathbf{y}|^2 + \lambda^5\|(s\xi^*)^{15/4}\mathrm{e}^{-s\varphi_\lambda^*}\mathbf{y}\|^2_{\mathbf{L}^2(\omega_2\times(0,T))} \\
& + \lambda^5\int_0^T (s\xi^*)^{15/2}\mathrm{e}^{-2s\varphi_\lambda^*}\|(\mathbf{a}, b, c)\|^2_{\mathbb{V}} + \int_0^T \lambda^{-1}s^{3/2}(\xi^*)^{3/2}\mathrm{e}^{-2s\varphi_\lambda^*}\|(\mathbf{y}, k_1, k_2)\|^2_{\mathbb{V}} \\
& + \int_0^T \lambda^{-1}s^{-1}\hat{\xi}^{-1}\mathrm{e}^{-2s\hat{\varphi}_\lambda}\|(\mathbf{y}', k_1', k_2')\|^2_{\mathbb{V}}\Bigg).
\end{aligned}
$$

Combining all the previous estimates, we get that

$$
\begin{aligned}
I(s,\lambda;\xi) \ \leq\ & C\Bigg(\int_{(0,T)\times\omega_2} \lambda^5(s\xi^*)^{15/2}\mathrm{e}^{-2s\varphi_\lambda^*}|\mathbf{y}|^2 + \int_{(0,T)\times\omega_2} \lambda^5(s\xi^*)^{15/2}\mathrm{e}^{-2s\varphi_\lambda^*}\|(\mathbf{a}, b, c)\|^2_{\mathbb{V}} \\
& + \int_0^T \lambda^{-1}s^{3/2}(\xi^*)^{3/2}\mathrm{e}^{-2s\varphi_\lambda^*}\|(\mathbf{y}, k_1, k_2)\|^2_{\mathbb{V}} + \int_0^T \lambda^{-1}s^{-1}\hat{\xi}^{-1}\mathrm{e}^{-2s\hat{\varphi}_\lambda}\|(\mathbf{y}', k_1', k_2')\|^2_{\mathbb{V}} \\
& + s^{5/2}\int_0^T (\xi^*)^3\mathrm{e}^{-2s\varphi_\lambda^*}\Big(|A^{1/2}k_1|^2_{\mathbb{R}^N} + |k_2|^2_{\mathbb{R}^N}\Big)\Bigg).
\end{aligned} \tag{4.6}
$$

The terms in the second line and the one depending on k_2 in the last line of the right-hand side of (4.6) can be absorbed in the left-hand side because of the factor λ^{-1} and estimates on the derivatives of $(\mathbf{y}, k_1, k_2)$ in Theorem 2.5.

Remember that $\mathbf{y} = \mathrm{e}^{s\varphi_\lambda}\mathbf{z}$, we can rewrite inequality $I(s,\lambda,\xi)$ in terms of $\mathbf{y}$ as follow

$$\begin{aligned} I(s,\lambda;\xi) \quad &= \quad s^{-1}\int_{Q_T^0} \xi^{-1}\rho^{-2s}\Big(|\mathbf{y}'|^2 + |\Delta \mathbf{y}|^2\Big) + \int_{Q_T^0} \rho^{-2s}|\nabla\pi|^2 + s\lambda^2\int_{Q_T^0} \xi\rho^{-2s}|\nabla \mathbf{y}|^2 \\ &+ s^3\lambda^4\int_{Q_T^0} \xi^3\rho^{-2s}|\mathbf{y}|^2 + s^3\lambda^3\int_{\Sigma_T^{s,0}} \xi^3\rho_\Gamma^{-2s}|Zk_2|^2 + \int_0^T \rho_\Gamma^{-2s}\Big(|k_2'|^2_{\mathbb{R}^N} + |A^{1/2}k_1|^2_{\mathbb{R}^N}\Big). \end{aligned} \tag{4.7}$$

Finally, we can sum up all the previous results in the following proposition:

Proposition 4.2. *For λ large enough, there is $s_0(\lambda) > 0$ such that for all $s \geq s_0(\lambda)$ and for all the solutions $(\mathbf{z}, k_1, k_2)$ of (4.3), we have*

$$\begin{aligned} I(s,\lambda;\xi) \quad &\leq \quad C\Bigg(\int_{(0,T)\times\omega_2} \lambda^5(s\xi^*)^{15/2}e^{-2s\varphi_\lambda^*}|\mathbf{y}|^2 + \int_0^T \lambda^5(s\xi^*)^{15/2}e^{-2s\varphi_\lambda^*}\|(\mathbf{a},b,c)\|^2_{\mathbb{V}} \\ &+ s^{5/2}\int_0^T (\xi^*)^3e^{-2s\varphi_\lambda^*}|A^{1/2}k_1|^2_{\mathbb{R}^N}\Bigg) \end{aligned} \tag{4.8}$$

where $I(s,\lambda,\xi)$ has been redefined in (4.7).

4.4 Treatment of the integral of k_1.

Here, we estimate the term $s^{5/2}\int_0^T(\xi^*)^3\mathrm{e}^{-2s\varphi_\lambda^*}|A^{1/2}k_1|^2_{\mathbb{R}^N}$. Using the same idea as in [14, sections 6 & 7], that is the finite dimensional setting of the beam equation, we get the following last Carleman estimate:

$$I(s,\lambda;\xi) \leq C\Bigg(\int_{(0,T)\times\omega_2} \lambda^5(s\xi^*)^{15/2}\mathrm{e}^{-2s\varphi_\lambda^*}|\mathbf{y}|^2 + \int_0^T \lambda^5(s\xi^*)^{15/2}\mathrm{e}^{-2s\varphi_\lambda^*}\|(\mathbf{a},b,c)\|^2_{\mathbb{V}}\Bigg)$$

for s and λ large enough.

4.5 From the Carleman estimate to the observability inequality.

We introduce here a piecewise continuous function l defined in $[0,T]$ by

$$l(t) = \begin{cases} T^2/4 & \text{if} \quad t \in [0,T/2], \\ t(T-t) & \text{if} \quad t \in [T/2,T]. \end{cases}$$

which gives us two new weight functions $\delta(x,t) = \frac{\kappa(x)}{l^k(t)}$ and $\sigma(x,t) = \frac{\mathrm{e}^{\lambda(\phi(x)+m\|\phi\|_\infty)}}{l^k(t)}$.

We use here the energy estimates for the system (4.1). Namely, we have

$$\begin{aligned} &\|\mathbf{y}\|^2_{L^\infty(0,T;\mathbf{L}^2(\Omega_0))} + \|A^{1/2}k_1\|^2_{L^\infty(0,T;\mathbb{R}^N)} + \|k_2\|^2_{L^\infty(0,T;\mathbb{R}^N)} + 2\nu\|\nabla\mathbf{y}\|^2_{L^2(Q_T^0)} \\ \leq \quad &C\Big(\|\mathbf{a}\|^2_{\mathbf{L}^2(Q_T^0)} + \|A^{1/2}b\|^2_{L^2(0,T;\mathbb{R}^N)} + \|c\|^2_{L^2(0,T;\mathbb{R}^N)}\Big). \end{aligned} \tag{4.9}$$

That is, using the notation of the space $\mathbb{V}$ defined in (2.9), we have

$$\|(\mathbf{y},k_1,k_2)\|^2_{L^\infty(0,T;\mathbb{V})} + 2\nu\|\nabla\mathbf{y}\|^2_{L^2(Q_T^0)} \leq C\Big(\|(\mathbf{a},b,c)\|^2_{\mathbf{L}^2(0,T;\mathbb{V})}\Big).$$

We introduce a weight function θ in $\mathcal{C}^1([0,T];\mathbb{R})$ satisfying

$$\theta \equiv 1 \text{ in } [0,T/2], \quad \theta \equiv 0 \text{ in } [3T/4,T] \quad \text{and } |\theta'| \leq 1/T.$$

Let us now consider the system satisfied by $(\theta\mathbf{y}, \theta\pi, k_1, k_2) = (\mathbf{y}^*, \pi^*, k_1, k_2)$:

$$\begin{array}{rcll} -\mathbf{y}_t^* - \operatorname{div}\sigma(\mathbf{y}^*, \pi^*) & = & \theta\mathbf{a} - \theta'\mathbf{y} & (Q_T^0) \\ \operatorname{div}\mathbf{y}^* & = & 0 & (Q_T^0) \\ \mathbf{y}^* & = & \theta Z k_2 \mathbf{e}_2 & (\Sigma_T^{s,0}) \\ \mathbf{y}^* & = & \mathbf{0} & (\Sigma_T) \\ \theta k_1' & = & \theta k_2 - \theta b & (0,T) \\ \theta k_2' + \theta A k_1 & = & -\Pi_N \pi^* - \theta c & (0,T) \\ (\mathbf{y}(T), k_1(T), k_2(T)) & = & (\mathbf{0}, 0, 0) & \end{array} \tag{4.10}$$

By some integrations by parts, and thanks to (4.9), we get the energy estimate of system (4.10):

$$\left\|\Big(\mathbf{y}, k_1, k_2\Big)\right\|^2_{L^2(0,T/2;\mathbb{V})} + \left\|\Big(\mathbf{y}, k_1, k_2\Big)\right\|^2_{L^\infty(0,T/2;\mathbb{V})} + \nu\|\nabla\mathbf{y}\|^2_{L^2(0,T/2;\mathbf{L}^2(\Omega_0))} \leq C\left\|\Big(\mathbf{a}, b, c\Big)\right\|^2_{L^2(0,3T/4;\mathbb{V})}. \tag{4.11}$$

Because the weigths δ and σ are constant in time on $[0, T/2]$ and the weights in s and λ are bigger in the right-hand side than in the left-hand side, this gives in particular,

$$\begin{array}{rl} & \left\|\Big(\mathbf{y}(0), k_1(0), k_2(0)\Big)\right\|^2_{\mathbb{V}} + s^3\lambda^4 \displaystyle\int_0^{T/2}\int_{\Omega_0} \mathrm{e}^{-2s\delta}\sigma^3|\mathbf{y}|^2 \\ & +s\lambda^2 \displaystyle\int_0^{T/2}\int_{\Omega_0} \mathrm{e}^{-2s\delta}\sigma|\nabla\mathbf{y}|^2 + s^3\lambda^3\int_0^{T/2} \mathrm{e}^{-2s\delta}\sigma^3|k_2|^2_{\mathbb{R}^N} \\ \leq & C\left[\displaystyle\int_0^{T/2} \lambda^5(s\sigma^*)^{15/2}\mathrm{e}^{-2s\delta^*}\|(\mathbf{a}, b, c)\|^2_{\mathbb{V}}\right]. \end{array} \tag{4.12}$$

On the other hand, the Carleman estimate (4.8) in Proposition 4.2 gives, because $\delta = \varphi_\lambda$ and $\xi = \sigma$ for t in $[T/2, T]$, the same result:

$$\begin{array}{rl} & s\lambda^2 \displaystyle\int_{T/2}^T\int_{\Omega_0} \sigma|\nabla\mathbf{y}|^2\mathrm{e}^{-2s\delta} + s^3\lambda^4\int_{T/2}^T\int_{\Omega_0}\sigma^3|\mathbf{y}|^2\mathrm{e}^{-2s\delta} + s^3\lambda^3\int_{T/2}^T\int_{\Gamma_0^s}\sigma^3\mathrm{e}^{-2s\delta}|k_2|^2_{\mathbb{R}^N} \\ \leq & C\left(\displaystyle\int_{T/2}^T\int_{\omega_2}\lambda^5(s\sigma^*)^{15/2}\mathrm{e}^{-2s\delta^*}|\mathbf{y}|^2 + \int_{T/2}^T \lambda^5(s\sigma^*)^{15/2}\mathrm{e}^{-2s\delta^*}\|(\mathbf{a}, b, c)\|^2_{\mathbb{V}}\right). \end{array} \tag{4.13}$$

Finally, adding inequalities (4.12) and (4.13), we get the expected observability inequality

$$\begin{array}{rl} & \left\|\Big(\mathbf{y}(0), k_1(0), k_2(0)\Big)\right\|^2_{\mathbb{V}} + s^3\lambda^3\displaystyle\int_0^T \sigma^{*3}(t)\mathrm{e}^{-2s\delta^*(t)}\Big(\|\mathbf{y}(t)\|^2_{\mathbf{L}^2(\Omega_0)} + |k_2|^2_{\mathbb{R}^N}\Big) \\ \leq & C\left(\displaystyle\int_0^T \lambda^5(s\sigma^*(t))^{15/2}\mathrm{e}^{-2s\delta^*(t)}\|(\mathbf{a}(t), b(t), c(t))\|^2_{\mathbb{V}} + \int_0^T \lambda^5(s\sigma^*(t))^{15/2}\mathrm{e}^{-2s\delta^*(t)}\|\mathbf{y}\|^2_{\mathbf{L}^2(\omega_2)}\right). \end{array}$$

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